Chapter 1. Electric Charge Interaction

This chapter reviews the basics of electrostatics – the description of interactions between stationary (or relatively slowly moving) electric charges. Much of this material should be known to the reader from their undergraduate studies;[1] *because of that, the explanations are very brief.*

1.1. The Coulomb law

A quantitative discussion of classical electrodynamics, starting from the electrostatics, requires common agreement on the meaning of the following notions:[2]

- *electric charges* q_k, as revealed, most explicitly, by observation of *electrostatic interaction* between the charged particles;

- *point charges* – the charged particles so small that their position in space, for the given problem, may be completely described (in the given reference frame) by their radius-vectors $\mathbf{r}_k$; and

- *electric charge conservation* – the fact that the algebraic sum of all charges q_k inside any closed volume is conserved unless the charged particles cross the volume's border.

I will assume that these notions are well known to the reader. Using them, the *Coulomb law*[3] for the interaction of two stationary point charges may be formulated as follows:

Coulomb law

$$\mathbf{F}_{kk'} = \kappa q_k q_{k'} \frac{\mathbf{r}_k - \mathbf{r}_{k'}}{\left|\mathbf{r}_k - \mathbf{r}_{k'}\right|^3} \equiv \kappa \frac{q_k q_{k'}}{R_{kk'}^2} \mathbf{n}_{kk'}, \quad \text{with } \mathbf{R}_{kk'} \equiv \mathbf{r}_k - \mathbf{r}_{k'}, \quad \mathbf{n}_{kk'} \equiv \frac{\mathbf{R}_{kk'}}{R_{kk'}}, \quad R_{kk'} \equiv \left|\mathbf{R}_{kk'}\right|, \tag{1.1}$$

where $\mathbf{F}_{kk'}$ denotes the electrostatic (*Coulomb*) force exerted on the charge number k by the charge number k', separated from it by distance $R_{kk'}$ – see Fig. 1.

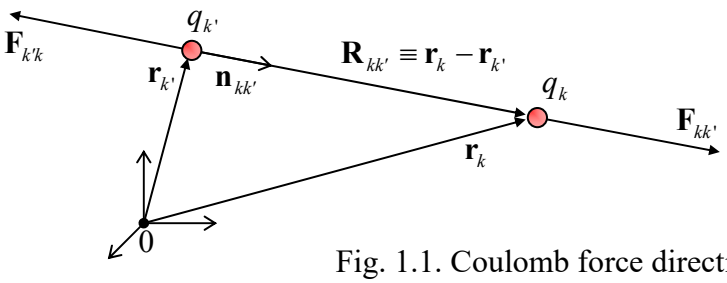


Fig. 1.1. Coulomb force directions (for the case $q_k q_{k'} > 0$).

[1] For remedial reading, I can recommend, for example, D. Griffiths, *Introduction to Electrodynamics*, 4th ed., Pearson, 2015.

[2] On top of the more general notions of the *classical Newtonian space*, *point particles* and *forces*, as used in classical mechanics – see, e.g., CM Sec. 1.1.

[3] Formulated in 1785 by Charles-Augustin de Coulomb, on the basis of his earlier experiments, in turn rooted in prior studies of electrostatic phenomena, with notable contributions by William Gilbert, Otto von Guericke, Charles François de Cisternay Du Fay, Benjamin Franklin, and Henry Cavendish.

I am confident that this law is very familiar to the reader, but a few comments may still be due:

(i) Flipping the indices k and k', we see that Eq. (1) complies with the 3rd Newton law: the reciprocal force is equal in magnitude but opposite in direction: $\mathbf{F}_{k'k} = -\mathbf{F}_{kk'}$.

(ii) Since the vector $\mathbf{R}_{kk'} \equiv \mathbf{r}_k - \mathbf{r}_{k'}$, by its definition, is directed from point $\mathbf{r}_{k'}$ toward point $\mathbf{r}_k$ (Fig. 1), Eq. (1) correctly describes the experimental fact that charges of the same sign (i.e. with $q_k q_{k'} > 0$) repulse, while those with opposite signs ($q_k q_{k'} < 0$) attract each other.

(iii) In some textbooks, the Coulomb law (1) is given with the qualifier "in free space" or "in vacuum". However, actually, Eq. (1) remains valid even in the presence of any other charges – for example, of internal charges in a quasi-continuous medium that may surround the two charges (number k and k') under consideration. The confusion stems from the fact, to be discussed in detail in Chapter 3 below, that in some cases it is convenient to *formally* represent the effect of the other charges as an *effective* (rather than actual!) modification of the Coulomb law.

(iv) The constant κ in Eq. (1) depends on the system of units we use. In the *Gaussian* units, κ is set to 1, for the price of introducing a special unit of charge (the *statcoulomb*) that would make experimental data compatible with Eq. (1) if the force $\mathbf{F}_{kk'}$ is measured in the Gaussian units (*dynes*). On the other hand, in the *International System* ("SI") of units, the charge's unit is one *coulomb* (abbreviated C), and κ is different from 1:

$$\kappa\big|_{\rm SI} = \frac{1}{4\pi\varepsilon_0}, \qquad (1.2)$$

κ in SI units

where $\varepsilon_0 \approx 8.854\times10^{-12}$ is called the *electric constant*.[4]

Unfortunately, the continuing struggle between zealous proponents of these two systems of units bears all the not-so-nice features of a religious war, with a similarly slim chance for any side to win it in any foreseeable future. In my humble view, each of these systems has its advantages and handicaps (to be noted on several occasions below), and every educated physicist should have no problem with using any of them. Following insisting recommendations of international scientific unions, I am using the SI units throughout my series. However, for the readers' convenience, in this course (where the difference between the Gaussian and SI systems is especially significant) I will write the most important formulas with the constant (2) clearly displayed – for example, the combination of Eqs. (1) and (2) as

$$\mathbf{F}_{kk'} = \frac{1}{4\pi\varepsilon_0} q_k q_{k'} \frac{\mathbf{r}_k - \mathbf{r}_{k'}}{\left|\mathbf{r}_k - \mathbf{r}_{k'}\right|^3}, \qquad (1.3)$$

so the formal transfer to the Gaussian units may be performed just by dropping the front fraction. (In the rare cases when the transfer is not obvious, I will duplicate formulas in the Gaussian units.)

Besides Eq. (3), another key experimental law of electrostatics is the *linear superposition principle*: the electrostatic forces exerted on some point charge (say, q_k) by other charges add up as vectors, forming the net force

[4] Since 2018, one coulomb is defined, in the "legal" metrology, as a certain exactly fixed number of the fundamental electric charges e, and the "legal" SI value of ε_0 is not more exactly equal to $10^7/4\pi c^2$ (where c is the speed of light) as it was before, but remains extremely close to that fraction, with the relative difference of the order of 10^{-10} – see appendix *UCA*: *Selected Units and Constants*. In this series, this minute difference is ignored.

$$\mathbf{F}_k = \sum_{k' \neq k} \mathbf{F}_{kk'}, \tag{1.4}$$

where the summation is extended over all charges but q_k, and the partial force $\mathbf{F}_{kk'}$ is described by Eq. (3). The fact that the sum is restricted to $k' \neq k$ means that a *point charge, in statics, does not interact with itself*. This fact may look obvious from Eq. (3), whose right-hand side diverges at $\mathbf{r}_k \to \mathbf{r}_{k'}$, but becomes less evident (though still true) in quantum mechanics – where the charge of even an elementary particle is effectively spread around some volume, together with the particle's wavefunction.[5]

Now we may combine Eqs. (3) and (4) to get the following expression for the net force $\mathbf{F}$ acting on a *probe charge* q located at point $\mathbf{r}$:

$$\mathbf{F}(\mathbf{r}) = q \frac{1}{4\pi\varepsilon_0} \sum_{\mathbf{r}_{k'} \neq \mathbf{r}} q_{k'} \frac{\mathbf{r} - \mathbf{r}_{k'}}{\left|\mathbf{r} - \mathbf{r}_{k'}\right|^3}. \tag{1.5}$$

This equality implies that it makes sense to introduce the notion of the *electric field* (as an entity independent of q), whose distribution in space is characterized by the following vector:

Electric field: definition

$$\boxed{\mathbf{E}(\mathbf{r}) \equiv \frac{\mathbf{F}(\mathbf{r})}{q},} \tag{1.6}$$

formally called the *electric field strength* – but much more frequently, just the "electric field". In these terms, Eq. (5) becomes

Electric field of point charges

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{\mathbf{r}_{k'} \neq \mathbf{r}} q_{k'} \frac{\mathbf{r} - \mathbf{r}_{k'}}{\left|\mathbf{r} - \mathbf{r}_{k'}\right|^3}. \tag{1.7}$$

Being just convenient is electrostatics, the notion of the field becomes unavoidable for the description of time-dependent phenomena (such as electromagnetic waves, see Chapter 7 and on), where the electromagnetic field shows up as a specific form of matter, different from the usual "material" particles – even though quantum electrodynamics (to be reviewed in QM Chapter 9) offers their joint description.

Many real-world problems involve multiple point charges located so closely that it is possible to approximate them with a continuous charge distribution. Indeed, let us consider a group of many ($dN >> 1$) close charges, located at points $\mathbf{r}_{k'}$, all within an elementary volume d^3r'. For relatively distant field observation points, with $|\mathbf{r} - \mathbf{r}_{k'}| >> dr'$, the geometrical factor in the corresponding terms of Eq. (7) is essentially the same. As a result, these charges may be treated as a single elementary charge $dQ(\mathbf{r}')$. Since at $dN >> 1$, this elementary charge is proportional to the elementary volume d^3r', we can define the local 3D *charge density* $\rho(\mathbf{r}')$ by the following relation:

$$\rho(\mathbf{r}')d^3r' \equiv dQ(\mathbf{r}') \equiv \sum_{r_{k'} \in d^3r'} q_{k'}, \tag{1.8}$$

and rewrite Eq. (7) as an integral (over the volume containing all essential charges):

Electric field of continuous charge

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{\left|\mathbf{r} - \mathbf{r}'\right|^3} d^3r'.} \tag{1.9}$$

[5] Note that some widely used approximations, e.g., the density functional theory (DFT) of multiparticle systems, essentially violate this law, thus limiting their accuracy and applicability – see, e.g., QM Sec. 8.4.

Note that for a continuous, smooth charge density $\rho(\mathbf{r}')$, the integral in Eq. (9) does not diverge at $\mathbf{R} \equiv \mathbf{r} - \mathbf{r}' \to 0$, because in this limit, the fraction under the integral increases as R^{-2}, i.e. slower than the decrease of the elementary volume d^3r', proportional to R^3.

Let me emphasize the dual use of Eq. (9). In the case when $\rho(\mathbf{r})$ is a continuous function representing the average charge defined by Eq. (8), Eq. (9) is not valid at distances $|\mathbf{r} - \mathbf{r}_{k'}|$ of the order of the distance between the adjacent point charges, i.e. does not describe rapid variations of the electric field at these distances. Such *approximate*, smoothly changing field $\mathbf{E}(\mathbf{r})$, is called *macroscopic*; we will repeatedly return to this notion in the following chapters. On the other hand, Eq. (9) may be also used for the description of the *exact* (frequently called *microscopic*) field of discrete point charges, by employing the notion of Dirac's delta function, which is the mathematical description of a very sharp function equal to zero everywhere but one point, and still having a finite integral (equal to 1).[6] Indeed, in this formalism, a set of point charges $q_{k'}$ located in points $\mathbf{r}_{k'}$ may be represented by the pseudo-continuous density

$$\rho(\mathbf{r}') = \sum_{k'} q_{k'} \delta(\mathbf{r}' - \mathbf{r}_{k'}). \tag{1.10}$$

Plugging this expression into Eq. (9), we return to its exact, discrete version (7). In this sense, Eq. (9) is exact, and we may use it as the general expression for the electric field.

1.2. The Gauss law

Due to the extension of Eq. (9) to point ("discrete") charges, it may seem that we do not need anything besides it to solve any problem of electrostatics. In practice, however, this is not quite true – first of all, because the direct use of Eq. (9) frequently leads to complex calculations. Indeed, let us try to solve a problem that is conceptually very simple: find the electric field induced by a spherically symmetric charge distribution with density $\rho(r')$ – see Fig. 2.

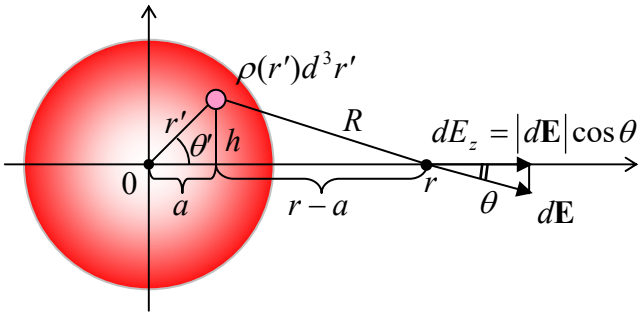


Fig. 1.2. One of the simplest problems of electrostatics: the electric field produced by a spherically-symmetric charge distribution.

We may immediately use the problem's symmetry to argue that the electric field should be also spherically symmetric, with only one component in the spherical coordinates: $\mathbf{E}(\mathbf{r}) = E(r)\mathbf{n}_r$, where $\mathbf{n}_r \equiv \mathbf{r}/r$ is the unit vector in the direction of the field observation point **r.** Taking this direction for the polar axis of a spherical coordinate system, we can use the evident axial symmetry of the system to reduce Eq. (9) to

[6] See, e.g., MA Sec. 14. The 2D (*areal*) charge density σ and the 1D (*linear*) density λ may be defined absolutely similarly to the 3D (*volumic*) density ρ: $dQ = \sigma d^2r$, $dQ = \lambda dr$. Note that the approximations in that either $\sigma \neq 0$ or $\lambda \neq 0$ imply that ρ is formally infinite at the charge location; for example, the model in that a plane $z = 0$ is charged with areal density $\sigma \neq 0$, means that $\rho = \sigma\delta(z)$, where $\delta(z)$ is Dirac's delta function.

$$E = \frac{1}{4\pi\varepsilon_0} 2\pi \int_0^{\pi} \sin\theta' d\theta' \int_0^{\infty} r'^2 dr' \frac{\rho(r')}{R^2} \cos\theta, \tag{1.11}$$

where θ, θ', and R are the geometrical parameters marked in Fig. 2. Since θ and R may be readily expressed via r' and θ', using the auxiliary parameters a and h,

$$\cos\theta = \frac{r-a}{R}, \qquad R^2 = h^2 + (r - r'\cos\theta)^2, \qquad \text{where } a \equiv r'\cos\theta', \quad h \equiv r'\sin\theta', \tag{1.12}$$

Eq. (11) may be eventually reduced to an explicit integral over r' and θ', and worked out analytically, but that would require some effort.

For other problems, the integral (9) may be much more complicated, defying an analytical solution. One could argue that with the present-day abundance of computers and numerical algorithm libraries, one can always resort to numerical integration. This argument may be enhanced by the fact that numerical *integration* is based on the replacement of the required integral by a discrete sum, and the summation is much more robust to the (unavoidable) rounding errors than the finite-difference schemes typical for the numerical solution of *differential* equations. These arguments, however, are only partly justified, since in many cases the numerical approach runs into a problem sometimes called the *curse of dimensionality* – the exponential dependence of the number of needed calculations on the number of independent parameters of the problem.[7] Thus, despite the proliferation of numerical methods in physics, analytical results have an everlasting value, and we should try to get them whenever we can. For our current problem of finding the electric field generated by a fixed set of electric charges, large help may come from the so-called *Gauss law*.

To derive it, let us consider a single point charge q inside a smooth closed surface S (Fig. 3), and calculate the product $E_n d^2r$, where d^2r is an elementary area of the surface (which may be well approximated with a plane fragment of that area), and $E_n \equiv \mathbf{E}\cdot\mathbf{n}$ is the component of the electric field at that point, normal to the plane.

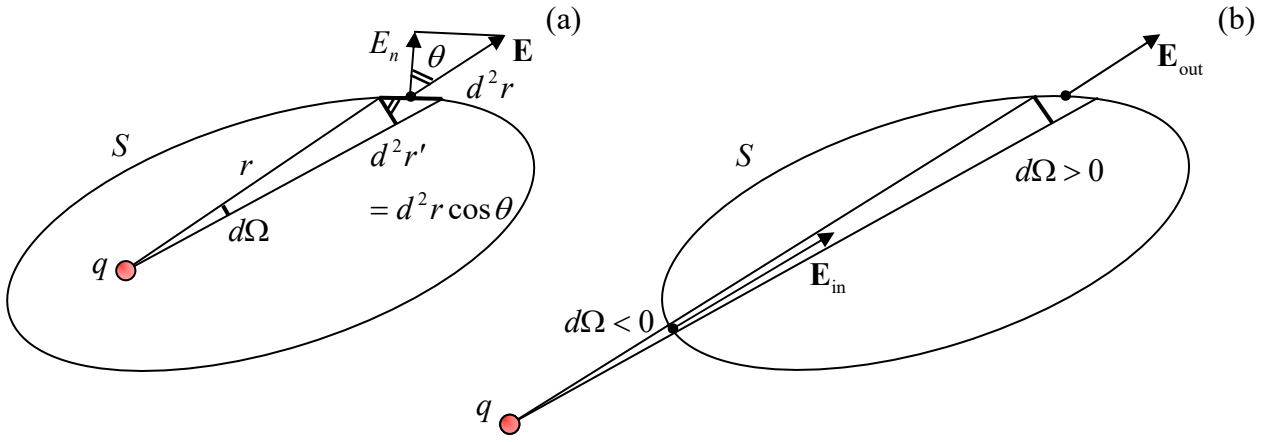


Fig. 1.3. Deriving the Gauss law: a point charge q (a) inside the volume V, and (b) outside of that volume.

This component may be calculated as $E\cos\theta$, where θ is the angle between the vector $\mathbf{E}$ and the unit vector $\mathbf{n}$ normal to the surface. Now let us notice that the product $\cos\theta\, d^2r$ is nothing more than the

[7] For a more detailed discussion of this problem, see, e.g., CM Sec. 5.8.

area d^2r' of the projection of d^2r onto the plane normal to the vector **r** connecting the charge q with the considered point of the surface (Fig. 3), because the angle between the elementary areas d^2r' and d^2r is also equal to θ. Using the Coulomb law for **E**, we get

$$E_n d^2 r = E\cos\theta\, d^2 r = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} d^2 r'. \tag{1.13}$$

But the ratio d^2r'/r^2 is nothing more than the elementary solid angle $d\Omega$ under which the areas d^2r' and d^2r are seen from the charge point, so $E_n d^2r$ may be represented just as a product of $d\Omega$ by a constant $(q/4\pi\varepsilon_0)$. Summing these products over the whole surface, we get

$$\oint_S E_n d^2 r = \frac{q}{4\pi\varepsilon_0}\oint_S d\Omega \equiv \frac{q}{\varepsilon_0}, \tag{1.14}$$

since the full solid angle equals 4π. (The integral on the left-hand side of this relation is called the *flux of electric field* through the surface S.)

Relation (14) expresses the Gauss law for one point charge. However, it is only valid if the charge is located *inside* the volume V limited by the surface S. To find the flux created by a charge located *outside* of this volume, we still can use Eq. (13), but have to be careful with the signs of the elementary contributions $E_n dA$. Let us use the common convention to direct the unit vector **n** out of the closed volume we are considering (the so-called *outer normal*), so the elementary product $E_n d^2r = (\mathbf{E}\cdot\mathbf{n})d^2r$ and hence $d\Omega = E_n d^2r'/r^2$ is positive if the vector **E** is pointing out of the volume (like in the example shown in Fig. 3a and at the upper-right area in Fig. 3b), and negative in the opposite case (for example, at the lower-left area in Fig. 3b). As the latter panel shows, if the charge is located outside of the volume, for each positive contribution $d\Omega$ there is always an equal and opposite contribution to the integral. As a result, at the integration over the solid angle, the positive and negative contributions cancel exactly, so

$$\oint_S E_n d^2 r = 0. \tag{1.15}$$

The real power of the Gauss law is revealed by its generalization to the case of several, especially many charges. Since the calculation of flux is a linear operation, the linear superposition principle (4) means that the flux created by several charges is equal to the (algebraic) sum of individual fluxes from each charge, for which either Eq. (14) or Eq. (15) are valid, depending on whether the charge is in or out of the volume. As a result, for the total flux, we get:

$$\oint_S E_n d^2 r = \frac{Q_V}{\varepsilon_0} \equiv \frac{1}{\varepsilon_0}\sum_{\mathbf{r}_j \in V} q_j \equiv \frac{1}{\varepsilon_0}\int_V \rho(\mathbf{r}')d^3r', \tag{1.16}$$

Gauss law

where Q_V is the net charge inside volume V. This is the full version of the Gauss law.[8]

In order to appreciate the problem-solving power of the law, let us revisit the problem shown in Fig. 2, i.e. the field of a spherical charge distribution. Due to its symmetry, which had already been discussed above, if we apply Eq. (16) to a sphere of a certain radius r, the electric field has to be normal

[8] The law is named after the famed Carl Gauss (1777-1855), even though it was first formulated earlier (in 1773) by Joseph-Louis Lagrange who was also the father-founder of analytical mechanics – see, e.g., CM Chapter 2.

to the sphere at each point (i.e., $E_n = E$), and its magnitude has to be the same at all points: $E_n = E(r)$. As a result, the flux calculation is elementary:

$$\oint E_n d^2 r = 4\pi r^2 E(r). \tag{1.17}$$

Now applying the Gauss law (16), we get:

$$4\pi r^2 E(r) = \frac{1}{\varepsilon_0} \int_{r'<r} \rho(r') d^3 r' = \frac{4\pi}{\varepsilon_0} \int_0^r r'^2 \rho(r') dr', \tag{1.18}$$

so, finally,

$$E(r) = \frac{1}{r^2 \varepsilon_0} \int_0^r r'^2 \rho(r') dr' \equiv \frac{1}{4\pi\varepsilon_0} \frac{Q_r}{r^2}, \tag{1.19}$$

where Q_r is the full charge inside the sphere of radius r:

$$Q_r \equiv \int_{r'<r} \rho(r') d^3 r' = 4\pi \int_0^r \rho(r') r'^2 dr'. \tag{1.20}$$

In particular, this formula shows that the field *outside* of a sphere of a finite radius R is exactly the same as if all its charge $Q = Q(R)$ is concentrated in the sphere's center. (Note that this important result is only valid for a spherically symmetric charge distribution.) For the field *inside* the sphere, finding the electric field still requires the explicit integration (20), but this 1D integral is much simpler than the 2D integral (11), and in some important cases may be readily worked out analytically. For example, if the charge Q is uniformly distributed inside a sphere of radius R,

$$\rho(r') = \rho = \frac{Q}{V} = \frac{Q}{(4\pi/3)R^3}, \tag{1.21}$$

then the integration is elementary:

$$E(r) = \frac{\rho}{r^2 \varepsilon_0} \int_0^r r'^2 dr' = \frac{\rho r}{3\varepsilon_0} = \frac{1}{4\pi\varepsilon_0} \frac{Qr}{R^3}. \tag{1.22}$$

We see that in this case, the field is growing linearly from the center to the sphere's surface, and only at $r > R$ starts to decrease in agreement with Eq. (19) with constant $Q(r) = Q$. Note also that the electric field is continuous for all r (including $r = R$) – as for all systems with finite volumic density,

In order to underline the importance of the last condition, let us consider one more elementary but very important example of Gauss law's application. Let a thin plane sheet (Fig. 4) be charged uniformly, with a finite *areal* density σ = const. In this case, it is fruitful to use the Gauss volume in the form of a planar "pillbox" of thickness $2z$ (where z is the Cartesian coordinate perpendicular to the plane) and certain area A – see the dashed lines in Fig. 4. Due to the symmetry of the problem, it is evident that the electric field should be: (i) directed along the z-axis, (ii) constant on each of the upper and bottom sides of the pillbox, (iii) equal and opposite on these sides, and (iv) parallel to the side surfaces of the box. As a result, the full electric field flux through the pillbox's surface is just $2AE(z)$, so the Gauss law (16) yields $2AE(z) = Q_A/\varepsilon_0 \equiv \sigma A/\varepsilon_0$, and we get a very simple but important formula

$$E(z) = \frac{\sigma}{2\varepsilon_0} = \text{const}. \tag{1.23}$$

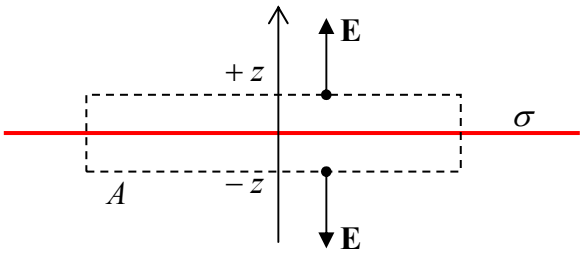


Fig. 1.4. The electric field of a charged plane.

Notice that, somewhat counter-intuitively, the field magnitude does not depend on the distance from the charged plane. From the point of view of the Coulomb law (5), this result may be explained as follows: the farther the observation point from the plane, the weaker the effect of each elementary charge, $dQ = \sigma d^2 r$, but the more such elementary charges give contributions to the z-component of vector $\mathbf{E}$, because they are "seen" from the observation point at relatively small angles to the z-axis.

Note also that though the magnitude $E \equiv |\mathbf{E}|$ of this electric field is constant, its component E_n normal to the plane (for our coordinate choice, E_z) changes its sign at the plane, experiencing a *discontinuity* (jump) equal to

$$\Delta E_z \equiv E_z(z=+0) - E_z(z=-0) = \frac{\sigma}{\varepsilon_0}. \tag{1.24}$$

This jump disappears if the surface is not charged. Returning for a split second to our charged sphere problem (Fig. 2), solving it we have considered the volumic charge density ρ to be finite everywhere, including the sphere's surface, so on it $\sigma = 0$, and the electric field should be continuous – as it is.

Admittedly, the *integral form* (16) of the Gauss law is immediately useful only for highly symmetrical geometries, such as in the two problems discussed above. However, it may be recast into an alternative, *differential form* whose field of useful applications is much wider. This form may be obtained from Eq. (16) using the *divergence theorem* of the vector algebra, which is valid for any space-differentiable vector, in particular $\mathbf{E}$, and for the volume V limited by any closed surface S:[9]

$$\oint_S E_n d^2 r = \int_V (\nabla \cdot \mathbf{E}) d^3 r, \tag{1.25}$$

where ∇ is the *del* (or "nabla") *operator* of spatial differentiation.[10] Combining Eq. (25) with the Gauss law (16), we get

$$\int_V \left(\nabla \cdot \mathbf{E} - \frac{\rho}{\varepsilon_0} \right) d^3 r = 0. \tag{1.26}$$

For a given spatial distribution of electric charge (and hence of its electric field), this equation should be valid for any choice of the volume V. This can hold only if the function under the integral vanishes at each point, i.e. if[11]

[9] See, e.g., MA Eq. (12.2). Note also that the scalar product under the volumic integral in Eq. (25) is nothing else than the divergence of the vector $\mathbf{E}$ – see, e.g., MA Eq. (8.4), hence the theorem's name.

[10] See, e.g., MA Secs. 8-10.

[11] In the Gaussian units, just as in the initial Eq. (6), ε_0 has to be replaced with $1/4\pi$, so the Maxwell equation (27) looks like $\nabla \cdot \mathbf{E} = 4\pi\rho$, while Eq. (28) stays the same.

Inhomogeneous Maxwell equation for **E**

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}. \tag{1.27}$$

Note that in sharp contrast with the integral form (16), Eq. (27) is *local*: it relates the electric field's divergence to the charge density *at the same point*. This equation, being the differential form of the Gauss law, is frequently called one of the famed *Maxwell equations*[12] – to be discussed again and again later in this course.

In the mathematical terminology, Eq. (27) is *inhomogeneous*, because it has a right-hand side independent (at least explicitly) of the field **E** that it describes. Another, *homogeneous* Maxwell equation's "embryo" (this one valid for the stationary case only!) may be obtained by noticing that the curl of the point charge's field, and hence that of *any* system of charges, equals zero:[13]

Homogeneous Maxwell equation for **E**

$$\nabla \times \mathbf{E} = 0. \tag{1.28}$$

(We will arrive at two other Maxwell equations, for the magnetic field, in Chapter 5, and then generalize all the equations to their full, time-dependent form at the end of Chapter 6. However, Eq. (27) will stay the same.)

Just to get a better gut feeling of Eq. (27), let us apply it to the same example of a uniformly charged sphere (Fig. 2). Vector algebra tells us that the divergence of a spherically symmetric vector function $\mathbf{E}(\mathbf{r}) = E(r)\mathbf{n}_r$ may be simply expressed in spherical coordinates:[14] $\nabla\cdot\mathbf{E} = [d(r^2E)/dr]/r^2$. As a result, Eq. (27) yields a linear ordinary differential equation for the scalar function $E(r)$:

$$\frac{1}{r^2}\frac{d}{dr}(r^2E) = \begin{cases} \rho/\varepsilon_0, & \text{for } r \le R, \\ 0, & \text{for } r \ge R, \end{cases} \tag{1.29}$$

which may be readily integrated on each of these segments:

$$E(r) = \frac{1}{\varepsilon_0}\frac{1}{r^2} \times \begin{cases} \rho\int r^2 dr = \rho r^3/3 + c_1, & \text{for } r \le R, \\ c_2, & \text{for } r \ge R. \end{cases} \tag{1.30}$$

To determine the integration constant c_1, we can use the following boundary condition: $E(0) = 0$. (It follows from the problem's spherical symmetry: in the center of the sphere, the electric field has to vanish, because otherwise, where would it be directed?) This requirement gives $c_1 = 0$. The second constant, c_2, may be found from the continuity condition $E(R - 0) = E(R + 0)$, which has already been discussed above, giving $c_2 = \rho R^3/3 \equiv Q/4\pi$. As a result, we arrive at our previous results (19) and (22).

We can see that in this particular, highly symmetric case, using the differential form of the Gauss law is a bit more complex than its integral form. (For our second example, shown in Fig. 4, it would be even less natural.) However, Eq. (27) and its generalizations are more convenient for asymmetric charge

[12] Named after the genius of classical electrodynamics and statistical physics, James Clerk Maxwell (1831-1879).

[13] This follows, for example, from the direct application of MA Eq. (10.11) to any spherically-symmetric vector function of type $\mathbf{f}(\mathbf{r}) = f(r)\mathbf{n}_r$ (in particular, to the electric field of a point charge placed at the origin), giving $f_\theta = f_\varphi = 0$ and $\partial f_r/\partial\theta = \partial f_r/\partial\varphi = 0$ so all components of the vector $\nabla\times\mathbf{f}$ vanish. Since nothing prevents us from placing the reference frame's origin at the point charge's location, this result remains valid for any position of the charge.

[14] See, e.g., MA Eq. (10.10) for the particular case $\partial/\partial\theta = \partial/\partial\varphi = 0$.

distributions, and are invaluable in cases where the distribution $\rho(\mathbf{r})$ is not known *a priori* and has to be found in a self-consistent way. (We will start discussing such cases in the next chapter.)

1.3. Scalar potential and electric field energy

One more help for solving problems of electrostatics (and electrodynamics as a whole) may be obtained from the notion of the *electrostatic potential*, which is just the electrostatic potential energy U of a probe point charge q placed into the field in question, normalized by its charge:

$$\phi \equiv \frac{U}{q}. \tag{1.31}$$

Electro-static potential

As we know from classical mechanics,[15] the notion of U (and hence ϕ) makes the most sense for the case of *potential forces* – for example, those depending just on the particle's position. Eqs. (6) and (9) show that stationary electric fields fall into this category. For such a field, the potential energy may be defined as a scalar function $U(\mathbf{r})$ that allows the force to be calculated as its gradient (with the opposite sign):

$$\mathbf{F} = -\nabla U\,. \tag{1.32}$$

Dividing both sides of this equation by the probe charge, and using Eqs. (6) and (31), we get[16]

$$\mathbf{E} = -\nabla\phi\,. \tag{1.33}$$

Electrostatic field as a gradient

To calculate the scalar potential, let us start from the simplest case of a single point charge q placed at the origin. For it, Eq. (7) takes the simple form

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} q \frac{\mathbf{r}}{r^3} \equiv \frac{1}{4\pi\varepsilon_0} q \frac{\mathbf{n}_r}{r^2}. \tag{1.34}$$

It is straightforward to verify that the last fraction in the last form of Eq. (34) is equal to $-\nabla(1/r)$.[17] Hence, according to the definition (33), for this particular case

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}. \tag{1.35}$$

Potential of a point charge

(In the Gaussian units, this result is spectacularly simple: $\phi = q/r$.) Note that we could add an arbitrary constant to this potential (and indeed to *any* other distribution of ϕ discussed below) without changing the field, but it is convenient to define the potential energy so it would approach zero at infinity.

In order to justify the introduction and the forthcoming exploration of U and ϕ, let me demonstrate (I hope, unnecessarily :-) how useful the notions are, on a very simple example. Let two similar charges q be launched from afar, with the same initial speed $v_0 << c$ each, straight toward each other (i.e. with the zero impact parameter) – see Fig. 5. Since, according to the Coulomb law, the

[15] See, e.g., CM Sec. 1.4.

[16] Eq. (28) could be also derived from this relation because according to vector algebra, any gradient field has no curl – see, e.g., MA Eq. (11.1).

[17] This may be done either by Cartesian components or using the well-known expression $\nabla f = (df/dr)\mathbf{n}_r$ valid for any spherically-symmetric scalar function $f(r)$ – see, e.g., MA Eq. (10.8) for the particular case $\partial/\partial\theta = \partial/\partial\varphi = 0$.

charges repel each other with increasing force, they will stop at some minimum distance r_{min} from each other, and then fly back. We could of course find r_{min} directly from the Coulomb law. However, for that, we would need to write the 2nd Newton law for each particle (actually, due to the problem symmetry, they would be similar), then integrate them over time to find the particle velocity v as a function of distance, and only then recover r_{min} from the requirement $v = 0$.

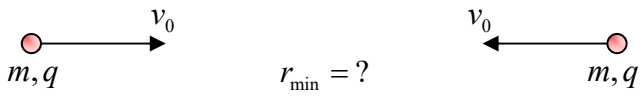


Fig. 1.5. A simple problem of charged particle motion.

The notion of potential allows this problem to be solved in one line. Indeed, in the field of potential forces, the system's total energy $\mathcal{E} = T + U \equiv T + q\phi$ is conserved. In our non-relativistic case $v << c$, the kinetic energy T is just $mv^2/2$. Hence, equating the total energy of two particles at the points $r = \infty$ and $r = r_{min}$, and using Eq. (35) for ϕ, we get

$$2\frac{mv_0^2}{2} + 0 = 0 + \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r_{min}}, \tag{1.36}$$

immediately giving us the final answer: $r_{min} = q^2/4\pi\varepsilon_0 m v_0^2$. So, the notion of scalar potential is indeed very useful.

With this motivation, let us calculate ϕ for an arbitrary configuration of charges. For a single charge in an arbitrary position (say, at point $\mathbf{r}_{k'}$), $r \equiv |\mathbf{r}|$ in Eq. (35) should be evidently replaced with $|\mathbf{r} - \mathbf{r}_{k'}|$. Now, the linear superposition principle (3) allows for an easy generalization of this formula to the case of an arbitrary set of discrete charges,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{\mathbf{r}_{k'}\neq\mathbf{r}}\frac{q_{k'}}{|\mathbf{r} - \mathbf{r}_{k'}|}. \tag{1.37}$$

Finally, using the same arguments as in Sec. 1, we can use this result to argue that in the case of an arbitrary continuous charge distribution

Potential of a charge distribution

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}d^3r'. \tag{1.38}$$

Again, Dirac's delta function allows using the last equation to recover Eq. (37) for discrete charges as well, so Eq. (38) may be considered as the general expression for the electrostatic potential.

For most practical calculations, using this expression and then applying Eq. (33) to the result, is preferable to using Eq. (9), because ϕ is a scalar, while $\mathbf{E}$ is a 3D vector, mathematically equivalent to three scalars. Still, this approach may lead to technical problems similar to those discussed in Sec. 2. For example, applying it to the spherically symmetric distribution of charge (Fig. 2), we get the integral

$$\phi = \frac{1}{4\pi\varepsilon_0}2\pi\int_0^{\pi}\sin\theta' d\theta'\int_0^{\infty}r'^2 dr'\frac{\rho(r')}{R}\cos\theta, \tag{1.39}$$

which is not much simpler than Eq. (11).

The situation may be much improved by recasting Eq. (38) into a differential form. For that, it is sufficient to plug the definition of ϕ, Eq. (33), into Eq. (27):

$$\nabla \cdot (-\nabla \phi) = \frac{\rho}{\varepsilon_0}. \tag{1.40}$$

The left-hand side of this equation is nothing else than the Laplace operator of ϕ (with the minus sign), so we get the famous *Poisson equation*[18] for the electrostatic potential:

$$\boxed{\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}}. \tag{1.41}$$

Poisson equation for ϕ

(In the Gaussian units, the Poisson equation is $\nabla^2\phi = -4\pi\rho$.) This differential equation is so convenient for applications that even its particular case for $\rho = 0$,

$$\boxed{\nabla^2 \phi = 0}, \tag{1.42}$$

Laplace equation for ϕ

has earned a special name – the *Laplace equation*.[19]

In order to get a gut feeling of the Poisson equation's value as a problem-solving tool, let us return to the spherically-symmetric charge distribution (Fig. 2) with a constant charge density ρ. Exploiting this symmetry, we can represent the potential as $\phi(r)$, and hence use the following simple expression for its Laplace operator:[20]

$$\nabla^2 \phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right), \tag{1.43}$$

so for the points inside the charged sphere ($r \le R$) the Poisson equation yields

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = -\frac{\rho}{\varepsilon_0}, \qquad \text{i.e. } \frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = -\frac{\rho}{\varepsilon_0} r^2. \tag{1.44}$$

Integrating the last form of the equation over r once, with the natural boundary condition $d\phi/dr\big|_{r=0} = 0$ (because of the condition $E(0) = 0$, which has been discussed above), we get

$$\frac{d\phi}{dr}(r) = -\frac{\rho}{r^2 \varepsilon_0}\int_0^r r'^2 dr' = -\frac{\rho r}{3\varepsilon_0} \equiv -\frac{1}{4\pi\varepsilon_0}\frac{Qr}{R^3}. \tag{1.45}$$

Since this derivative is nothing more than $-E(r)$, in this formula we can readily recognize our previous result (22). Now we may like to carry out the second integration to calculate the potential itself:

$$\phi(r) = -\frac{Q}{4\pi\varepsilon_0 R^3}\int_0^r r' dr' + c_1 = -\frac{Qr^2}{8\pi\varepsilon_0 R^3} + c_1. \tag{1.46}$$

[18] Named after Siméon Denis Poisson (1781-1840), also famous for the *Poisson distribution* – one of the central results of the probability theory – see, e.g., SM Sec. 5.2.

[19] Named after the famous mathematician (and astronomer) Pierre-Simon Laplace (1749-1827) who, together with Alexis Clairault, is credited for the development of the very concept of potential.

[20] See, e.g., MA Eq. (10.8) for $\partial/\partial\theta = \partial/\partial\varphi = 0$.

Before making any judgment on the integration constant c_1, let us solve the Poisson equation (in this case, just the Laplace equation) for the range outside the sphere ($r > R$):

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 0 \cdot \tag{1.47}$$

Its first integral,

$$\frac{d\phi}{dr}(r) = \frac{c_2}{r^2}, \tag{1.48}$$

also gives the electric field (with the minus sign). Now using Eq. (45) and requiring the field to be continuous at $r = R$, we get

$$\frac{c_2}{R^2} = -\frac{Q}{4\pi\varepsilon_0 R^2}, \qquad \text{i.e. } \frac{d\phi}{dr}(r) = -\frac{Q}{4\pi\varepsilon_0 r^2}, \tag{1.49}$$

in an evident agreement with Eq. (19). Integrating this result again,

$$\phi(r) = -\frac{Q}{4\pi\varepsilon_0}\int\frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0 r} + c_3, \qquad \text{for } r > R, \tag{1.50}$$

we can select $c_3 = 0$, so $\phi(\infty) = 0$, in accordance with the usual (though not compulsory) convention. Now we can finally determine the constant c_1 in Eq. (46) by requiring that this equation and Eq. (50) give the same value of ϕ at the boundary $r = R$. (According to Eq. (33), if the potential had a jump, the electric field at that point would be infinite.) The final answer may be represented as

$$\phi(r) = \frac{Q}{4\pi\varepsilon_0 R}\left(\frac{R^2 - r^2}{2R^2} + 1\right), \qquad \text{for } r \le R. \tag{1.51}$$

This calculation shows that using the Poisson equation to find the electrostatic potential distribution for highly symmetric problems may be a bit more cumbersome than directly finding the electric field – say, from the Gauss law. However, we will repeatedly see below that if the electric charge distribution is not fixed in advance, using Eq. (41) may be the only practicable way to proceed.

Returning now to the general theory of electrostatic phenomena, let us calculate the potential energy U of an arbitrary system of point electric charges q_k. Despite the apparently simple relation (31) between U and ϕ, the result is not that straightforward. Indeed, let us assume that the charge distribution has a finite spatial extent, so at large distances from it (formally, at $\mathbf{r} = \infty$) the electric field tends to zero, so the electrostatic potential tends to a constant. Selecting this constant, for convenience, to equal zero, we may calculate U as a sum of the energy increments ΔU_k created by bringing the charges, one by one, from infinity to their final positions $\mathbf{r}_k$ – see Fig. 6.[21] According to the integral form of Eq. (32), such a contribution is

$$\Delta U_k = -\int_{\infty}^{\mathbf{r}_k}\mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = -q_k\int_{\infty}^{\mathbf{r}_k}\mathbf{E}(\mathbf{r})\cdot d\mathbf{r} \equiv q_k\phi(\mathbf{r}_k), \tag{1.52}$$

[21] Indeed, by the very definition of the potential energy of a system, it should not depend on the way we are arriving at its final configuration.

where $\mathbf{E}(\mathbf{r})$ is the total electric field, and $\phi(\mathbf{r})$ is the total electrostatic potential during this process, besides the field created by the very charge q_k that is being moved.

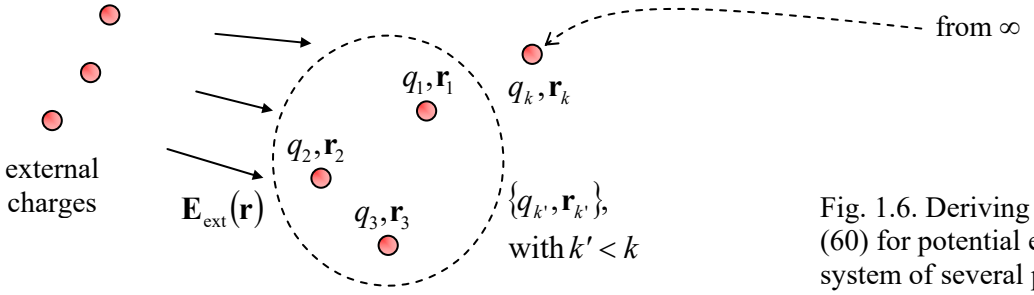


Fig. 1.6. Deriving Eqs. (55) and (60) for potential energies of a system of several point charges.

This expression shows that the increment ΔU_k, and hence the total potential energy U, depends on the source of the electric field $\mathbf{E}$. If the field is dominated by an *external field* $\mathbf{E}_{\text{ext}}$, induced by some *external* charges, not being a part of the charge configuration under our analysis (whose energy we are calculating, see Fig. 6), then the spatial distribution $\phi(\mathbf{r})$ is determined by this field, i.e. does not depend on how many charges we have already brought in, so Eq. (52) is reduced to

$$\Delta U_k = q_k \phi_{\text{ext}}(\mathbf{r}_k), \quad \text{where } \phi_{\text{ext}}(\mathbf{r}) \equiv -\int_{\infty}^{\mathbf{r}} \mathbf{E}_{\text{ext}}(\mathbf{r}') \cdot d\mathbf{r}'. \tag{1.53}$$

Summing up these contributions, we get what is called the charge system's energy *in* the external field:[22]

$$U_{\text{ext}} \equiv \sum_k \Delta U_k = \sum_k q_k \phi_{\text{ext}}(\mathbf{r}_k). \tag{1.54}$$

Now repeating the argumentation that has led us to Eq. (9), we see that for a continuously distributed charge, this sum turns into an integral:

$$\boxed{U_{\text{ext}} = \int \rho(\mathbf{r}) \phi_{\text{ext}}(\mathbf{r}) d^3 r}. \tag{1.55}$$

Energy: external field

(As was discussed above, using the delta-functional representation of point charges, we may always return from here to Eq. (54), so Eq. (55) may be considered as a final, universal result.)

The result is different in the opposite limit when the electric field $\mathbf{E}(\mathbf{r})$ is created only by the very charges whose energy we are calculating. In this case, $\phi(\mathbf{r}_k)$ in Eq. (52) is the potential created only by the charges with numbers $k' = 1, 2, \ldots, (k-1)$ that are already in place when the k^{th} charge is moved in (in Fig. 6, the charges inside the dashed boundary), and we may use the linear superposition principle to write

$$\Delta U_k = q_k \sum_{k'<k} \phi_{k'}(\mathbf{r}_k), \qquad \text{so that } U = \sum_k U_k = \sum_{\substack{k,k' \\ (k'<k)}} q_k \phi_{k'}(\mathbf{r}_k). \tag{1.56}$$

This result is so important that it is worthy of rewriting in several other forms. First, we may use Eq. (35) to represent Eq. (56) in a more symmetric form:

[22] An alternative, perhaps more accurate term for U_{ext} is the energy of the system's *interaction with* the external field.

$$U = \frac{1}{4\pi\varepsilon_0} \sum_{\substack{k,k' \\ (k'<k)}} \frac{q_k q_{k'}}{|\mathbf{r}_k - \mathbf{r}_{k'}|}. \tag{1.57}$$

The expression under this sum is evidently symmetric with respect to the index swap, so it may be extended into a different form,

$$U = \frac{1}{4\pi\varepsilon_0} \frac{1}{2} \sum_{\substack{k',k \\ (k'\neq k)}} \frac{q_k q_{k'}}{|\mathbf{r}_k - \mathbf{r}_{k'}|}, \tag{1.58}$$

where the interaction between each couple of charges is described by two equal terms under the sum, and the front coefficient ½ is used to compensate for this *double-counting*. The convenience of the last form is that it may be readily generalized to the continuous case:

$$U = \frac{1}{4\pi\varepsilon_0} \frac{1}{2} \int d^3r \int d^3r' \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{1.59}$$

(As before, in this case, the restriction expressed in the discrete charge case as $k \neq k'$ is not important, because if the charge density is a continuous function, the integral (59) does not diverge at point $\mathbf{r} = \mathbf{r}'$.)

To represent this result in one more form, let us notice that according to Eq. (38), the inner integral over r' in Eq. (59), divided by $4\pi\varepsilon_0$, is just the full electrostatic potential at point $\mathbf{r}$, and hence

Energy: charge interaction

$$\boxed{U = \frac{1}{2} \int \rho(\mathbf{r})\phi(\mathbf{r}) d^3r}. \tag{1.60}$$

For the discrete charge case, this result is

$$U = \frac{1}{2} \sum_k q_k \phi(\mathbf{r}_k), \tag{1.61}$$

but here it is important to remember that the "full" potential's value $\phi(\mathbf{r}_k)$ should exclude the (infinite) contribution from the point charge k itself. Comparing the last two formulas with Eqs. (54) and (55), we see that the electrostatic energy of charge interaction within the system, as expressed via the charge-by-potential product, is twice less than that of the energy of charge interaction with a fixed ("external") field. This is the result of the fact that in the case of mutual interaction of the charges, the electric field $\mathbf{E}$ in the basic Eq. (52) is proportional to the charge's magnitude, rather than constant.[23]

Now we are ready to address an important conceptual question: can we locate this interaction energy in space? This task may seem trivial: Eqs. (58)-(61) seem to imply that non-zero contributions to U come only from the regions where the electric charges are located. However, one of the most beautiful features of physics is that sometimes completely different interpretations of the same mathematical result are possible. To get an alternative view of our current result, let us write Eq. (60) for a volume V so large that the electric field on the limiting surface S is negligible, and plug into it the charge density expressed from the Poisson equation (41):

$$U = -\frac{\varepsilon_0}{2} \int_V \phi \nabla^2 \phi \, d^3r. \tag{1.62}$$

[23] The nature of this additional factor ½ is absolutely the same as in the well-known formula $U = (\frac{1}{2})\kappa x^2$ for the potential energy of an elastic spring providing the returning force $F = -\kappa x$, proportional to its displacement x from the equilibrium position.

This expression may be integrated by parts as[24]

$$U = -\frac{\varepsilon_0}{2}\left[\oint_S \phi(\nabla\phi)_n\, d^2r - \int_V (\nabla\phi)^2\, d^3r\right]. \tag{1.63}$$

According to our condition of negligible field $\mathbf{E} = -\nabla\phi$ at the surface, the first integral vanishes, and we get a very important formula

$$U = \frac{\varepsilon_0}{2}\int (\nabla\phi)^2\, d^3r = \frac{\varepsilon_0}{2}\int E^2\, d^3r. \tag{1.64}$$

This result, if represented in the following equivalent form:[25]

$$U = \int u(\mathbf{r})\, d^3r, \quad \text{with } u(\mathbf{r}) \equiv \frac{\varepsilon_0}{2}E^2(\mathbf{r}), \tag{1.65}$$

Energy: electric field

certainly invites an interpretation very much different than Eq. (60): it is natural to consider $u(\mathbf{r})$ as the *spatial density of the electric field energy*, which is continuously distributed over all the space where the field exists – rather than just its part where the charges are located.

Let us have a look at how these two alternative pictures work for our testbed problem, a uniformly charged sphere. If we start with Eq. (60), we may limit the integration by the sphere volume ($0 \le r \le R$) where $\rho \ne 0$. Using Eq. (51), and the spherical symmetry of the problem (giving $d^3r = 4\pi r^2 dr$), we get

$$U = \frac{1}{2}4\pi\int_0^R \rho\phi\, r^2 dr = \frac{1}{2}4\pi\rho\frac{Q}{4\pi\varepsilon_0 R}\int_0^R\left(\frac{R^2 - r^2}{2R^2} + 1\right)r^2 dr = \frac{6}{5}\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{2R}. \tag{1.66}$$

On the other hand, if we use Eq. (65), we need to integrate the energy density everywhere, i.e. both inside and outside of the sphere:

$$U = \frac{\varepsilon_0}{2}4\pi\left(\int_0^R E^2 r^2 dr + \int_R^\infty E^2 r^2 dr\right). \tag{1.67}$$

Using Eqs. (19) and (22) for, respectively, the external and internal regions, we get

$$U = \frac{\varepsilon_0}{2}4\pi\left[\int_0^R\left(\frac{Qr}{4\pi\varepsilon_0}\right)^2 r^2 dr + \int_R^\infty\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2 r^2 dr\right] = \left(\frac{1}{5}+1\right)\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{2R}. \tag{1.68}$$

This is (fortunately :-) the same answer as given by Eq. (66), but to some extent, Eq. (68) is more informative because it shows how exactly the electric field's energy is distributed between the interior and exterior of the charged sphere.[26]

---

[24] This transformation follows from the divergence theorem MA (12.2) applied to the vector function $\mathbf{f} = \phi\nabla\phi$, taking into account the differentiation rule MA Eq. (11.4a): $\nabla\cdot(\phi\nabla\phi) = (\nabla\phi)\cdot(\nabla\phi) + \phi\nabla\cdot(\nabla\phi) = (\nabla\phi)^2 + \phi\nabla^2\phi$.

[25] In the Gaussian units, the standard replacement $\varepsilon_0 \to 1/4\pi$ turns the last of Eqs. (65) into $u(\mathbf{r}) = E^2/8\pi$.

[26] Note that $U \to \infty$ at $R \to 0$. Such divergence appears at the application of Eq. (65) to any point charge. Since it does not affect the force acting on the charge, the divergence does not create any technical difficulty for analysis of charge statics or non-relativistic dynamics, but it points to a possible conceptual problem of classical electrodynamics as a whole at describing point charges. This issue will be discussed at the very end of the course (Sec. 10.6).

We see that, as we could expect, within the realm of *electrostatics*, Eqs. (60) and (65) are equivalent. However, when we examine *electrodynamics* (in Chapter 6 and beyond), we will see that the latter equation is more general and that it is more adequate to associate the electric energy with the field itself rather than its sources – in our current case, the electric charges.

Finally, let us calculate the potential energy of a system of charges in the general case when both the internal interaction of the charges and their interaction with an external field are important. One might fancy that such a calculation should be very hard since, in both ultimate limits, when one of these interactions dominates, we have gotten different results. However, once again we get help from the almighty linear superposition principle: in the general case, for the total electric field we may write

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{\text{int}}(\mathbf{r}) + \mathbf{E}_{\text{ext}}(\mathbf{r}), \qquad \phi(\mathbf{r}) = \phi_{\text{int}}(\mathbf{r}) + \phi_{\text{ext}}(\mathbf{r}), \tag{1.69}$$

where the index "int" now marks the field induced by the charge system under analysis, i.e. the variables participating (without indices) in Eqs. (56)-(65). Now let us imagine that our system is being built up in the following way: first, the charges are brought together at $\mathbf{E}_{\text{ext}} = 0$, giving the potential energy U_{int} expressed by Eq. (60), and then $\mathbf{E}_{\text{ext}}$ is slowly increased. Evidently, the energy contribution from the latter process cannot depend on the internal interaction of the charges, and hence may be expressed in the form (55). As a result, the total potential energy[27] is the sum of these two components:

$$U = U_{\text{int}} + U_{\text{ext}} = \frac{1}{2}\int \rho(\mathbf{r})\phi_{\text{int}}(\mathbf{r})d^3r + \int \rho(\mathbf{r})\phi_{\text{ext}}(\mathbf{r})d^3r\,. \tag{1.70}$$

Now making the transition from the potentials to the fields, absolutely similar to that performed in Eqs. (62)-(65), we may rewrite this expression as

$$U = \int u(\mathbf{r})d^3r, \qquad \text{with } u(\mathbf{r}) \equiv \frac{\varepsilon_0}{2}\left[E_{\text{int}}^2(\mathbf{r}) + 2\mathbf{E}_{\text{int}}(\mathbf{r})\cdot\mathbf{E}_{\text{ext}}(\mathbf{r})\right]. \tag{1.71}$$

One might think that this result, more general than Eq. (65) and perhaps less familiar to the reader, is something entirely new; however, it is not. Indeed, let us add to, and subtract $E_{\text{ext}}^2(\mathbf{r})$ from the sum in the brackets, and use Eq. (69) for the total electric field $\mathbf{E}(\mathbf{r})$; then Eq. (71) takes the form

$$U = \frac{\varepsilon_0}{2}\int E^2(\mathbf{r})d^3r - \frac{\varepsilon_0}{2}\int E_{\text{ext}}^2(\mathbf{r})d^3r\,. \tag{1.72}$$

Hence, in the most important case when we are using the potential energy to analyze the statics and dynamics of a system of charges in a fixed external field, i.e. when the second term on the right-hand side of Eq. (72) may be considered as a constant, we may still use for U an expression similar to the familiar Eq. (65), but with the field $\mathbf{E}(\mathbf{r})$ being the sum (69) of the internal and external fields.

Let us see how this works in a very simple situation. A uniform external electric field $\mathbf{E}_{\text{ext}}$ is applied normally to a very broad, plane layer that contains a very large and equal number of free electric charges of both signs – see Fig. 7. What is the equilibrium distribution of the charges over the layer?

[27] This total U (or rather its part dependent on our system of charges) is sometimes called the *Gibbs potential energy* of the system. (I will discuss this notion in detail in Sec. 3.5.)

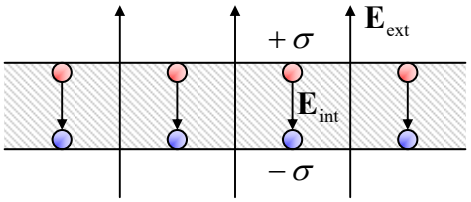


Fig. 1.7. A simple model of the electric field screening in a conductor. Here (and in all figures below) the red and blue colors are used to denote the opposite charge signs.

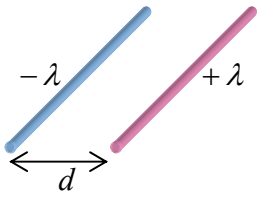
Since any area-uniform distribution of the charge inside the layer does not affect the field (and hence its energy) outside it, and the equilibrium distribution has to minimize the total potential energy of the system, Eq. (72) immediately gives the answer: the distribution should provide $\mathbf{E} \equiv \mathbf{E}_{int} + \mathbf{E}_{ext} = 0$ inside the whole layer – the effect called the electric field *screening*. The only way to ensure this equality is to have enough free charges of opposite signs residing on the layer's surfaces to induce a uniform field $\mathbf{E}_{int} = -\mathbf{E}_{ext}$, exactly compensating the external field at each point inside the layer – see Fig. 7. According to Eq. (24), the areal density of these surface charges should equal $\pm\sigma$, with $\sigma = E_{ext}/\varepsilon_0$. This is a rudimentary but reasonable model of conductors' *polarization* – to be discussed in detail in the next chapter.

1.4. Exercise problems

1.1. Calculate the electric field of a thin, long, straight filament, electrically charged with a constant linear density λ, by using two approaches:

(i) directly from the Coulomb law, and
(ii) from the Gauss law.

1.2. Two thin, straight, parallel filaments separated by distance d carry equal and opposite uniformly distributed charges with linear density λ – see the figure on the right. Calculate the force (per unit length) of the Coulomb interaction of the filaments. Compare its functional dependence on d with the Coulomb law for two point charges, and interpret their difference.



1.3. Calculate the electric field of the following spherically symmetric charge distribution: $\rho(r) = \rho_0 \exp\{-\lambda r\}$.

1.4. A sphere of radius R, whose volume had been charged with a constant density ρ, is split with a very narrow planar gap passing through its center. Calculate the force of the mutual electrostatic repulsion of the resulting two hemispheres.

1.5. A thin spherical shell of radius R, which had been charged with a constant areal density σ, is split into two equal halves with a very thin planar cut passing through the sphere's center. Calculate the force of electrostatic repulsion between the resulting hemispheric shells, and compare the result with that of the previous problem.

1.6. Calculate the spatial distribution of the electrostatic potential created by a straight thin filament of a finite length $2l$, charged with a constant linear density λ, and explore the result in the limits of very small and very large distances from the filament.

1.7. A thin planar sheet, perhaps of an irregular shape, carries an electric charge with a constant areal density σ.

(i) Express the electric field's component normal to the plane, at a certain distance from it, via the solid angle Ω at which the sheet is visible from the observation point.

(ii) Use the result to calculate the field in the center of a cube with one face charged with a constant density σ.

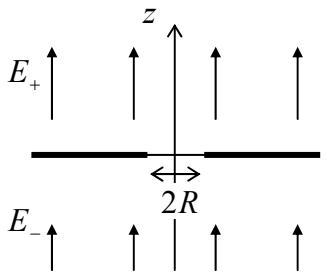
1.8. Can one create, in an extended region of space, electrostatic fields with the Cartesian components proportional to the following products of the Cartesian coordinates $\{x, y, z\}$:

(i) $\{yz, xz, xy\}$,

(ii) $\{xy, xy, yz\}$?

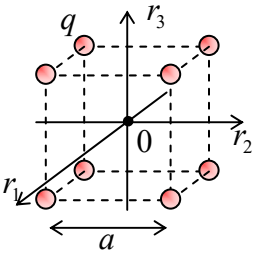
1.9. Distant sources have been used to create different uniform electrostatic fields in two half-spaces:

$$\mathbf{E}(\mathbf{r})\big|_{r>>R} = \mathbf{n}_z \times \begin{cases} E_+, & \text{at } z < 0, \\ E_-, & \text{at } z > 0, \end{cases}$$



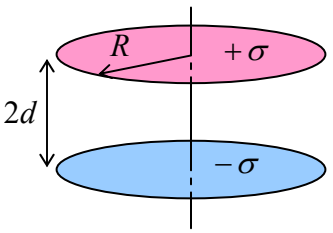
except for a transitional region of scale R near the origin, where the field is perturbed but still axially symmetric. (As will be discussed in the next chapter, this may be done, for example, using a thin conducting membrane with a round hole of radius R in it – see the figure on the right.) Prove that such field may serve as an electrostatic lens for charged particles flying along the z-axis, at distances $\rho << R$ from it, and calculate the focal distance f of this lens. Spell out the conditions of validity of your result.

1.10. Eight equal point charges q are located in the corners of a cube of side a. Calculate all Cartesian components E_j of the electric field, and their spatial derivatives $\partial E_j/\partial r_{j'}$, in the cube's center, where r_j are the Cartesian coordinates oriented along the cube's sides – see the figure on the right. Are all of your results valid for the center of a planar square, with four equal charges at its corners?



1.11. By a direct calculation, find the average electric potential of a spherical surface of radius R, created by a point charge q located at a distance $r > R$ from the sphere's center. Use the result to prove the following general *mean value theorem*: the electric potential at any point is always equal to its average value on any spherical surface with the center at that point while containing no electric charges inside it.

1.12. Two similar thin, circular, coaxial disks of radius R, separated by distance $2d$, are uniformly charged with equal and opposite areal densities $\pm\sigma$ – see the figure on the right. Calculate and sketch the distribution of the electrostatic potential and the electric field of the disks along their common axis.



1.13. The electrostatic potential, created by some electric charge distribution, is

$$\phi(\mathbf{r}) = C\left(\frac{1}{r} + \frac{1}{2r_0}\right)\exp\left\{-\frac{r}{r_0}\right\},$$

where C and r_0 are constants, and $r \equiv |\mathbf{r}|$ is the distance from the origin. Calculate the charge distribution in space.

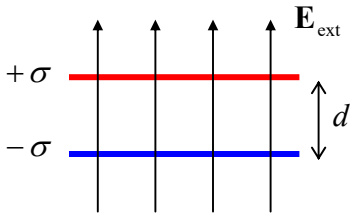
1.14. A thin, flat, rectangular sheet of size $a \times b$ is electrically charged with a constant areal density σ. Without an explicit calculation of the spatial distribution $\phi(\mathbf{r})$ of the electrostatic potential induced by this charge, find the ratio of its values in the center and in the corners of the rectangle.

Hint: Consider partitioning the rectangle into several similar parts and using the linear superposition principle.

1.15. Calculate the electrostatic energy per unit area of the system of two thin, parallel planes with equal and opposite charges of a constant areal density σ, separated by distance d.

1.16. The system analyzed in the previous problem (two thin, parallel, oppositely charged planes) is now placed into an external, uniform, normal electric field $E_{ext} = \sigma/\varepsilon_0$ – see the figure on the right. Find the force (per unit area) acting on each plane, by two methods:

(i) directly from the electric field distribution, and
(ii) from the potential energy of the system.



1.17. Explore the relationship between the Laplace equation (42) and the minimum of the electrostatic field energy (65).

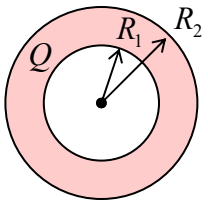
1.18. Prove the following *reciprocity theorem of electrostatics*:[28] if two spatially-confined charge distributions $\rho_1(\mathbf{r})$ and $\rho_2(\mathbf{r})$ create, respectively, electrostatic potentials $\phi_1(\mathbf{r})$ and $\phi_2(\mathbf{r})$, then

$$\int \rho_1(\mathbf{r})\phi_2(\mathbf{r})d^3r = \int \rho_2(\mathbf{r})\phi_1(\mathbf{r})d^3r\,.$$

Hint: Consider the integral $\int \mathbf{E}_1 \cdot \mathbf{E}_2 d^3r$.

1.19. Calculate the energy of the electrostatic interaction of two spheres, of radii R_1 and R_2, each with a spherically symmetric charge distribution, separated by distance $d > R_1 + R_2$.

1.20. Calculate the electrostatic energy U of a (generally, thick) spherical shell, with charge Q uniformly distributed through its volume – see the figure on the right. Interpret the dependence of U on the inner cavity's radius R_1, at fixed Q and R_2.



[28] This is only the simplest one of several reciprocity theorems in electromagnetism – see, e.g., Sec. 6.8 below.