Chapter 5. Magnetism

Even though this chapter addresses a completely new type of electric charge interaction, its discussion (for the stationary case) will take not too much time/space, because it recycles many ideas and methods of electrostatics, though with a twist or two.

5.1. Magnetic interaction of currents

DC currents in conductors usually leave them *electroneutral*, $\rho(\mathbf{r}) = 0$, with very good precision, because even a minute imbalance of positive and negative charge density results in extremely strong Coulomb forces that restore the electroneutrality by a very fast additional shift of free charge carriers. This is why let us start the discussion of magnetism from the simplest case of two spatially separated, dc-current-carrying, electroneutral conductors (Fig. 1).

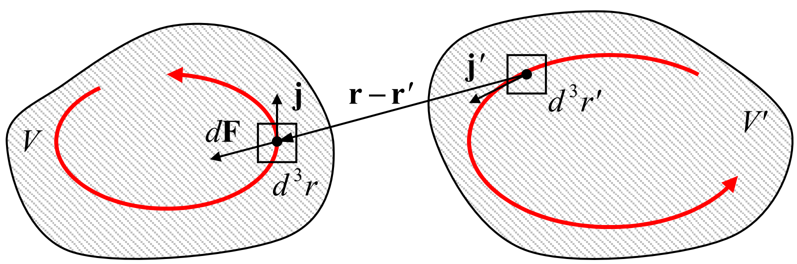

Fig. 5.1. Magnetic interaction of two currents.

According to the Coulomb law, there is no electrostatic force between them. However, several experiments carried out in 1820[1] proved that there is a different, *magnetic* interaction between the currents. In the present-day notation, the results of all such experiments may be summarized with just one formula, in SI units expressed as

Magnetic force

$$\mathbf{F} = -\frac{\mu_0}{4\pi}\int_V d^3r \int_{V'} d^3r' \left[\mathbf{j}(\mathbf{r})\cdot\mathbf{j}'(\mathbf{r}')\right]\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}. \qquad (5.1)$$

Here the coefficient $\mu_0/4\pi$ (where μ_0 is called either the *magnetic constant* or the *free space permeability*) equals *almost exactly* 10^{-7} SI units, with the product $\varepsilon_0\mu_0$ equal to *exactly* $1/c^2$.[2]

Note a close similarity of this expression to the Coulomb law (1.1) rewritten for the interaction of two continuously distributed charges, with the account of the linear superposition principle (1.4):

Electric force

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\int_V d^3r \int_{V'} d^3r' \, \rho(\mathbf{r})\rho'(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}. \qquad (5.2)$$

[1] Most notably, by Hans Christian Ørsted who discovered the effect of electric currents on magnetic needles, and André-Marie Ampère who extended this work by finding the magnetic interaction between two currents.

[2] For details, see Appendix *UCA: Selected Units and Constants*. In the Gaussian units, the coefficient $\mu_0/4\pi$ in Eq. (1) and beyond is replaced with $1/c^2$.

Besides the different coefficient and a different sign, the "only" difference of Eq. (1) from Eq. (2) is the scalar product of the current densities, evidently necessary because of their vector character. We will see soon that this difference brings certain complications in applying the approaches discussed in the previous chapters, to magnetostatics.

Before going to their discussion, let us have one more glance at the coefficients in Eqs. (1) and (2). To compare them, let us consider two objects with *uncompensated* charge distributions $\rho(\mathbf{r})$ and $\rho'(\mathbf{r})$, moving parallel to each other with certain velocities $\mathbf{v}$ and $\mathbf{v}'$, as measured in the same inertial ("laboratory") reference frame. In this case, $\mathbf{j}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}$, so $\mathbf{j}(\mathbf{r})\cdot\mathbf{j}'(\mathbf{r}) = \rho(\mathbf{r})\rho'(\mathbf{r})vv'$, and the integrals in Eqs. (1) and (2) become functionally similar, differing only by the factor

$$\frac{F_{\text{magnetic}}}{F_{\text{electric}}} = -\frac{\mu_0 vv'}{4\pi} \Big/ \frac{1}{4\pi\varepsilon_0} \equiv -\frac{vv'}{c^2}. \tag{5.3}$$

(The last expression is valid in any consistent system of units.) We immediately see that magnetism is an essentially relativistic phenomenon, very weak in comparison with the electrostatic interaction at the human scale velocities, $v << c$, and may dominate only if the latter interaction vanishes – as it does in electroneutral systems.[3] The discovery and initial studies[4] of such a subtle, relativistic phenomenon as magnetism were much facilitated by the relative abundance of natural *ferromagnets*: materials with a spontaneous magnetic polarization, whose strong magnetic field is due to relativistic effects (such as spin) inside the constituent atoms – see Sec. 5 below.

Also, Eq. (3) points to an interesting paradox. Consider two electron beams moving parallel to each other, with the same velocity v with respect to a lab reference frame. Then, according to Eq. (3), the net force of their total (electric plus magnetic) interaction is proportional to $(1 - v^2/c^2)$, tending to zero in the limit $v \to c$. However, in the reference frame moving together with the electrons, they are not moving at all, i.e. $v = 0$. Hence, from the point of view of such a moving observer, the electron beams should interact only electrostatically, with a repulsive force independent of the velocity v. Historically, this had been one of several paradoxes that led to the development of special relativity; its resolution will be discussed in Chapter 9 devoted to this theory.

Returning to Eq. (1), in some simple cases the double integration in it may be carried out analytically. First of all, let us simplify this expression for the case of two thin, long conductors ("wires") separated by a distance much larger than their thickness. In this case, we may integrate the products $\mathbf{j}d^3r$ and $\mathbf{j}'d^3r'$ over the wires' cross-sections first, neglecting the corresponding change of the factor $(\mathbf{r} - \mathbf{r}')$. Since the integrals of the current density over the cross-sections of the wires are just the currents I and I' flowing in the wires, and cannot change along their lengths (say, l and l', respectively), they may be taken out of the remaining integrals, reducing Eq. (1) to

$$\mathbf{F} = -\frac{\mu_0 II'}{4\pi} \oint_l \oint_{l'} (d\mathbf{r} \cdot d\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{5.4}$$

[3] An important case when the electroneutrality may not hold is the motion of electrons in free space. (However, in this case, the electron speed is often comparable with the speed of light, so the magnetic forces may be comparable in strength with electrostatic forces, and hence important.) Minor local violations of electroneutrality also play an important role in some semiconductor devices – see, e.g., SM Chapter 6.

[4] The first detailed book on this subject, *De Magnete* by William Gilbert (a.k.a. Gilberd), was published as early as 1600.

As the simplest example, consider two straight, parallel wires (Fig. 2) separated by distance d, both with length $l >> d$.

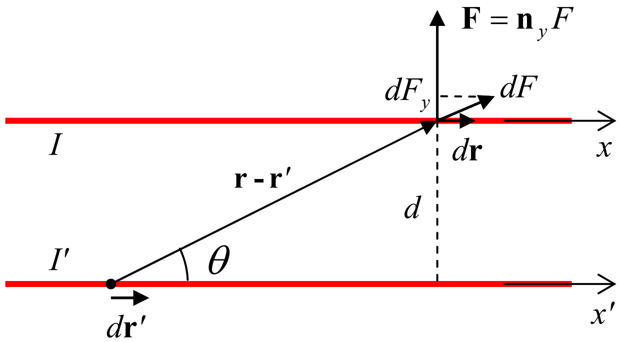

Fig. 5.2. The magnetic force between two straight parallel currents.

Due to the symmetry of this system, the vector of the magnetic interaction force has to:

(i) lie in the same plane as the currents, and
(ii) be normal to the wires – see Fig. 2.

Hence we may limit our calculations to just one component of the force – normal to the wires. Using the fact that with the coordinate choice shown in Fig. 2, the scalar product $d\mathbf{r}\cdot d\mathbf{r}'$ is just $dxdx'$, we get

$$F = -\frac{\mu_0 II'}{4\pi}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx' \frac{\sin\theta}{d^2 + (x-x')^2} = -\frac{\mu_0 II'}{4\pi}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx' \frac{d}{\left[d^2 + (x-x')^2\right]^{3/2}}. \tag{5.5}$$

Now introducing, instead of x', a new, dimensionless variable $\xi \equiv (x - x')/d$, we may reduce the internal integral to a table one, which we have already encountered in this course:

$$F = -\frac{\mu_0 II'}{4\pi d}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} \frac{d\xi}{\left(1+\xi^2\right)^{3/2}} = -\frac{\mu_0 II'}{2\pi d}\int_{-\infty}^{+\infty} dx . \tag{5.6}$$

The integral over x formally diverges, but it gives a finite interaction force *per unit length* of the wires:

$$\frac{F}{l} = -\frac{\mu_0 II'}{2\pi d} . \tag{5.7}$$

Note that the force drops rather slowly (only as $1/d$) as the distance d between the wires is increased, and is attractive (rather than repulsive as in the Coulomb law) if the currents are of the same sign.

This is an important result,[5] but again, the problems so simply solvable are few and far between, and it is intuitively clear that we would strongly benefit from the same approach as in electrostatics, i.e., from decomposing Eq. (1) into a product of two factors via the introduction of a suitable field. Such decomposition may be done as follows:

Lorentz force: current

$$\mathbf{F} = \int_V \mathbf{j}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) d^3 r , \tag{5.8}$$

[5] In particular, until very recently (2018), Eq. (7) was used for the legal definition of the SI unit of current, the ampere (A), via the SI unit of force (the newton, N), with the coefficient μ_0 considered exactly fixed. (A brief description of the recent changes in legal metrology is given in Appendix UCA.)

where the vector **B** is called the *magnetic field*.[6] In the case when it is induced by the current **j**':

Biot-Savart law

$$\mathbf{B}(\mathbf{r}) \equiv \frac{\mu_0}{4\pi}\int_{V'} \mathbf{j}'(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}d^3r'. \quad (5.9)$$

The last relation is called the *Biot-Savart law*,[7] while the force **F** expressed by Eq. (8) is sometimes called the *Lorentz force*.[8] However, more frequently the latter term is reserved for the full force,

Lorentz force: particle

$$\mathbf{F} = q\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right), \quad (5.10)$$

exerted by electric and magnetic fields field on a point charge q, moving with velocity **v**.[9]

Now we have to prove that the new formulation, given by Eqs. (8)-(9), is equivalent to Eq. (1). At first glance, this seems unlikely. Indeed, first of all, Eqs. (8) and (9) involve vector products, while Eq. (1) is based on a scalar product. More profoundly, in contrast to Eq. (1), Eqs. (8) and (9) do *not* satisfy the 3rd Newton's law applied to elementary current components $\mathbf{j}d^3r$ and $\mathbf{j}'d^3r'$, if these vectors are not parallel to each other. Indeed, consider the situation shown in Fig. 3.

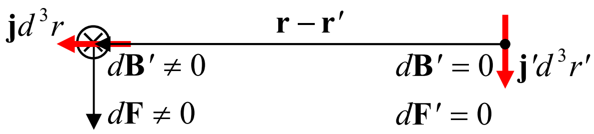

Fig. 5.3. The apparent violation of the 3rd Newton law in magnetism.

Here the vector **j**' is perpendicular to the vector (**r** – **r**'), and hence, according to Eq. (9), produces a non-zero contribution $d\mathbf{B}'$ to the magnetic field directed (in Fig. 3) normally to the plane of the drawing, i.e. perpendicular to the vector **j**. Hence, according to Eq. (8), this field provides a non-zero contribution to **F**. On the other hand, if we calculate the reciprocal force **F**' by swapping the prime indices in Eqs. (8) and (9), the latter equation immediately shows that $d\mathbf{B}(\mathbf{r}') \propto \mathbf{j}\times(\mathbf{r}'-\mathbf{r}) = 0$, because the two operand vectors are parallel – see Fig. 3 again. Hence, the current component $\mathbf{j}'d^3\mathbf{r}'$ does exert a force on its counterpart, while $\mathbf{j}d^3\mathbf{r}$ does not.

[6] The SI unit of the magnetic field is called *tesla* (T) – after Nikola Tesla, a pioneer of electrical engineering. In the Gaussian units, the already discussed constant $1/c^2$ in Eq. (1) is equally divided between Eqs. (8) and (9), so in them both, the constant before the integral is $1/c$. The resulting Gaussian unit of the field **B** is called *gauss* (G); taking into account the difference of units of electric charge and length, and hence of the current density, 1 G equals exactly 10^{-4} T. Note also that in some textbooks, especially old ones, **B** is called either the *magnetic induction* or the *magnetic flux density*, while the term "magnetic field" is reserved for the field **H** that will be introduced in Sec. 5 below.

[7] Named after Jean-Baptiste Biot and Félix Savart who made several key contributions to the theory of magnetic interactions – in the same notorious 1820.

[8] Named after Hendrik Antoon Lorentz, famous mostly for his numerous contributions to the development of special relativity – see Chapter 9 below. To be fair, the magnetic part of the Lorentz force was implicitly described in a much earlier (1865) paper by J. C. Maxwell and then spelled out by Oliver Heaviside (another genius of electrical engineering – and mathematics!) in 1889, i.e. also before the 1895 work by H. Lorentz.

[9] From the magnetic part of Eq. (10), Eq. (8) may be derived by the elementary summation of all forces acting on $n >> 1$ particles in a unit volume, with $\mathbf{j} = qn\mathbf{v}$ – see the footnote on Eq. (4.13a). On the other hand, the reciprocal derivation of Eq. (10) from Eq. (8) with $\mathbf{j} = q\mathbf{v}\delta(\mathbf{r}-\mathbf{r}_0)$, where $\mathbf{r}_0$ is the current particle's position (so $d\mathbf{r}_0/dt = \mathbf{v}$), requires certain mathematical care and will be performed in Chapter 9.

Despite this apparent problem, let us still go ahead and plug Eq. (9) into Eq. (8):

$$\mathbf{F} = \frac{\mu_0}{4\pi}\int_V d^3r \int_{V'} d^3r' \, \mathbf{j}(\mathbf{r}) \times \left(\mathbf{j}'(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3} \right). \tag{5.11}$$

This double vector product may be transformed into two scalar products, using the vector algebraic identity called the *bac minus cab rule*, **a**×(**b**×**c**) = **b**(**a**·**c**) – **c**(**a**·**b**).[10] Applying this relation, with **a** = **j**, **b** = **j**', and **c** = **R** ≡ **r** – **r**', to Eq. (11), we get

$$\mathbf{F} = \frac{\mu_0}{4\pi}\int_{V'} d^3r' \mathbf{j}'(\mathbf{r}') \left(\int_V d^3r \frac{\mathbf{j}(\mathbf{r}) \cdot \mathbf{R}}{R^3} \right) - \frac{\mu_0}{4\pi}\int_V d^3r \int_{V'} d^3r' \, \mathbf{j}(\mathbf{r}) \cdot \mathbf{j}'(\mathbf{r}') \frac{\mathbf{R}}{R^3}. \tag{5.12}$$

The second term on the right-hand side of this equality coincides with the right-hand side of Eq. (1), while the first term equals zero because its internal integral vanishes. Indeed, we may break the volumes V and V' into narrow *current tubes* – the stretched elementary volumes whose walls are not crossed by current lines (so on their walls, $j_n = 0$). As a result, the elementary current in each tube, $dI = jdA = jd^2r$, is the same along its length, and, just as in a thin wire, $\mathbf{j}d^2r$ may be replaced with $dId\mathbf{r}$, with the vector $d\mathbf{r}$ directed along **j**. Because of this, each tube's contribution to the internal integral in the first term of Eq. (12) may be represented as

$$dI\oint_l d\mathbf{r} \cdot \frac{\mathbf{R}}{R^3} = -dI\oint_l d\mathbf{r} \cdot \nabla \frac{1}{R} = -dI\oint_l dr \frac{\partial}{\partial r}\frac{1}{R}, \tag{5.13}$$

where the operator ∇ acts in the **r**-space, and the integral is taken along the tube's length l. Due to the current continuity expressed by Eq. (4.6), each loop should follow a closed contour, and an integral of a full differential of some scalar function (in our case, of $1/R$) along such contour equals zero.

So we have recovered Eq. (1). Returning for a minute to the paradox illustrated in Fig. 3, we may conclude that the apparent violation of the 3rd Newton law was the artifact of our interpretation of Eqs. (8) and (9) as the sums of independent elementary components. In reality, due to the dc current continuity, these components are *not* independent. For the whole currents, Eqs. (8)-(9) do obey the 3rd law – as follows from their already proved equivalence to Eq. (1).

Thus it is possible to break the magnetic interaction into two effects: the induction of the magnetic field **B** by one current (in our notation, **j**'), and the effect of this field on the other current (**j**). Now comes an additional experimental fact: other elementary components $\mathbf{j}d^3r'$ of the current **j**(**r**) also contribute to the magnetic field (9) acting on the component $\mathbf{j}d^3r$.[11] This fact allows us to drop the prime sign after **j** in Eq. (9), and rewrite Eqs. (8) and (9) as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{V'} \mathbf{j}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3} d^3r', \tag{5.14}$$

$$\mathbf{F} = \int_V \mathbf{j}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) d^3r. \tag{5.15}$$

[10] See, e.g., MA Eq. (7.5).

[11] Just as in electrostatics, one needs to exercise due caution in transforming these expressions for the limit of discrete classical particles, and extended wavefunctions in quantum mechanics, to avoid the (non-existing) magnetic interaction of a charged particle with itself.

Again, the field *observation* point **r** and the field *source* point **r***'* have to be clearly distinguished. We immediately see that these expressions are close to, but still different from the corresponding relations of the electrostatics, namely Eq. (1.9) and the distributed-charge version of Eq. (1.6):

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\oint_{V'} \rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}d^3r' , \tag{5.16}$$

$$\mathbf{F} = \oint_V \rho(\mathbf{r})\,\mathbf{E}(\mathbf{r})d^3r \ . \tag{5.17}$$

(Note that the sign difference has disappeared, at the cost of the replacement of scalar-by-vector multiplications in electrostatics with cross-products of vectors in magnetostatics.)

For the frequent particular case of a thin wire of length l', Eq. (14) may be re-written as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint_{l'} d\mathbf{r}'\times\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}. \tag{5.18}$$

Let us see how this formula works for the simplest case of a straight wire (Fig. 4a). The magnetic field contributions $d\mathbf{B}$ due to all small fragments $d\mathbf{r}'$ of the wire's length are directed along the same line (perpendicular to both the wire and the shortest distance d from the observation point to the wire's line), and its magnitude is

$$dB = \frac{\mu_0 I}{4\pi}\frac{dx'}{\left|\mathbf{r}-\mathbf{r}'\right|^2}\sin\theta = \frac{\mu_0 I}{4\pi}\frac{dx'}{\left(d^2+x^2\right)}\frac{d}{\left(d^2+x^2\right)^{1/2}}. \tag{5.19}$$

Summing up all such elementary contributions, we get

$$B = \frac{\mu_0 I\rho}{4\pi}\int_{-\infty}^{\infty}\frac{dx}{(x^2+d^2)^{3/2}} = \frac{\mu_0 I}{2\pi d}. \tag{5.20}$$

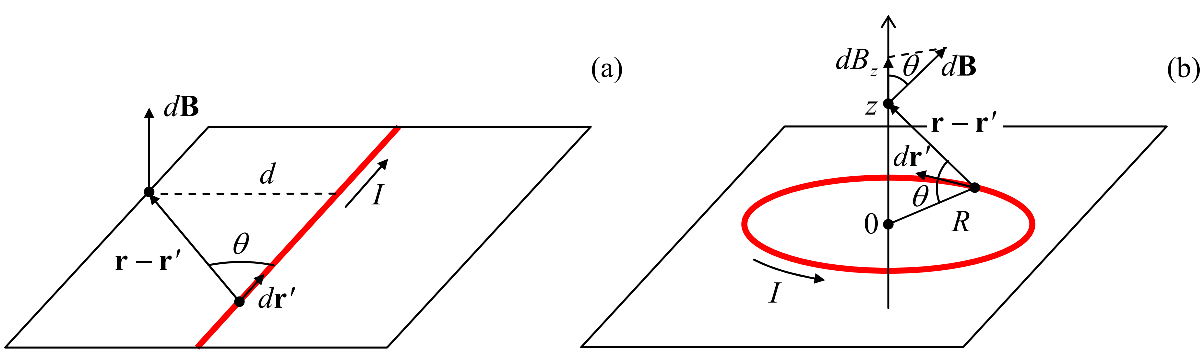

Fig. 5.4. Calculating magnetic fields: (a) of a straight current, and (b) of a current loop.

This is a simple but important result. (Note that it is only valid for very long ($l >> d$), straight wires.) It is especially crucial to note the "vortex" character of the field: its lines go around the wire, forming rings with the centers on the current line. This is in sharp contrast to the electrostatic field lines, which can only begin and end on electric charges and never form closed loops (otherwise the Coulomb force $q\mathbf{E}$ would not be conservative). In the magnetic case, the vortex structure of the *field* may be reconciled with the potential character of the magnetic *forces*, which is evident from Eq. (1), due to the vector products in Eqs. (14)-(15).

Now we may readily use Eq. (15), or rather its thin-wire version

$$\mathbf{F} = I\oint_l d\mathbf{r} \times \mathbf{B}(\mathbf{r}), \tag{5.21}$$

to apply Eq. (20) to the two-wire problem (Fig. 2). Since for the second wire, the vectors d**r** and **B** are perpendicular to each other, we immediately arrive at our previous result (7), which was obtained directly from Eq. (1).

The next important example of the application of the Biot-Savart law (14) is the magnetic field at the axis of a circular current loop (Fig. 4b). Due to the problem's symmetry, the net field **B** has to be directed along the axis, but each of its elementary components d**B** is tilted by the angle $\theta = \tan^{-1}(z/R)$ to this axis, so its axial component is

$$dB_z = dB\cos\theta = \frac{\mu_0 I}{4\pi}\frac{dr'}{R^2+z^2}\frac{R}{\left(R^2+z^2\right)^{1/2}}. \tag{5.22}$$

Since the denominator of this expression remains the same for all wire components dr', the integration over $\mathbf{r}'$ is easy ($\int dr' = 2\pi R$), giving finally

$$B = \frac{\mu_0 I}{2}\frac{R^2}{\left(R^2+z^2\right)^{3/2}}. \tag{5.23}$$

Note that the magnetic field in the loop's center (i.e., for $z = 0$),

$$B = \frac{\mu_0 I}{2R}, \tag{5.24}$$

is π times higher than that due to a similar current in a straight wire, at the distance $d = R$ from it. This difference is readily understandable, since all elementary components of the loop are at the same distance R from the observation point, while in the case of a straight wire, all its points but one are separated from the observation point by distances larger than d.

Another notable fact is that at large distances ($z^2 >> R^2$), the field (23) is proportional to z^{-3}:

$$B \approx \frac{\mu_0 I}{2}\frac{R^2}{|z|^3} \equiv \frac{\mu_0}{4\pi}\frac{2m}{|z|^3}, \qquad \text{with } m \equiv IA, \tag{5.25}$$

where $A = \pi R^2$ is the loop area. Comparing this expression with Eq. (3.13), for the particular case $\theta = 0$, we see that such field is similar to that of an electric dipole (at least along its direction), with the replacement of the electric dipole moment magnitude p with the m so defined – besides the front factor. Indeed, such a plane current loop is the simplest example of a system whose field, at distances much larger than R, is that of a *magnetic dipole*, with a *dipole moment* **m** – the notions to be discussed in much more detail in Sec. 4 below.

5.2. Vector potential and the Ampère law

The reader could see that the calculations of the magnetic field using Eq. (14) or (18) are still somewhat cumbersome even for the very simple systems we have examined. As we saw in Chapter 1, similar calculations in electrostatics, at least for several important highly symmetric systems, could be substantially simplified using the Gauss law (1.16). A similar relation exists in magnetostatics as well, but has a different form, due to the vortex character of the magnetic field.

To derive it, let us notice that in an analogy with the scalar case, the vector product under the integral (14) may be transformed as

$$\frac{j(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}=\nabla\times\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{5.26}$$

where the operator ∇ acts in the **r**-space. (This equality may be readily verified by Cartesian components, noticing that the current density is a function of **r**' and hence its components are independent of **r**.) Plugging Eq. (26) into Eq. (14), and moving the operator ∇ out of the integral over **r**', we see that the magnetic field may be represented as the curl of another vector field – the so-called *vector potential*, defined as:[12]

$$\mathbf{B}(\mathbf{r})\equiv\nabla\times\mathbf{A}(\mathbf{r}), \tag{5.27}$$

and in our current case equal to

Vector potential

$$\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int_{V'}\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'. \tag{5.28}$$

Please note a beautiful analogy between Eqs. (27)-(28) and, respectively, Eqs. (1.33) and (1.38).[13] This analogy implies that the vector potential **A** plays, for the magnetic field, essentially the same role as the scalar potential ϕ plays for the electric field (hence the name "potential"), with due respect to the vortex character of **B**. This notion will be discussed in more detail below.

Now let us see what equations we may get for the spatial derivatives of the magnetic field. First, vector algebra says that the divergence of any curl is zero.[14] In application to Eq. (27), this means that

No magnetic monopoles

$$\nabla\cdot\mathbf{B}=0. \tag{5.29}$$

Comparing this equation with Eq. (1.27), we see that Eq. (29) may be interpreted as the absence of a magnetic analog of an electric charge, on which magnetic field lines could originate or end. Numerous searches for such hypothetical magnetic charges, called *magnetic monopoles*, using very sensitive and sophisticated experimental setups,[15] have not given any reliable evidence of their existence in Nature.

Proceeding to the alternative, vector derivative of the magnetic field, i.e. to its curl, and using Eq. (28), we obtain

$$\nabla\times\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\nabla\times\left(\nabla\times\int_{V'}\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'\right). \tag{5.30}$$

This expression may be simplified by using the following general vector identity:[16]

$$\nabla\times(\nabla\times\mathbf{c})=\nabla(\nabla\cdot\mathbf{c})-\nabla^2\mathbf{c}, \tag{5.31}$$

applied to vector $\mathbf{c}(\mathbf{r})\equiv\mathbf{j}(\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|$:

[12] In the Gaussian units, Eq. (27) remains the same, and hence in Eq. (28), $\mu_0/4\pi$ is replaced with $1/c$.
[13] In Eq. (1.38), there was no real need for the additional clarification provided by the integration volume label *V'*.
[14] See, e.g., MA Eq. (11.2).
[15] For a recent example, see B. Acharya *et al*., *Nature* **602**, 63 (2022).
[16] See, e.g., MA Eq. (11.3).

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \nabla \int_{V'} \mathbf{j}(\mathbf{r}') \cdot \nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} d^3r' - \frac{\mu_0}{4\pi} \int_{V'} \mathbf{j}(\mathbf{r}') \nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} d^3r' . \tag{5.32}$$

As was already discussed during our study of electrostatics in Sec. 3.1,

$$\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta(\mathbf{r}-\mathbf{r}') , \tag{5.33}$$

so the last term of Eq. (32) is just $\mu_0\mathbf{j}(\mathbf{r})$. On the other hand, inside the first integral, we can replace ∇ with $(-\nabla')$, where prime means the differentiation in the space of the radius vectors $\mathbf{r}'$. Integrating that term by parts, we get

$$\nabla \times \mathbf{B} = -\frac{\mu_0}{4\pi} \nabla \oint_{S'} j_n(\mathbf{r}') \frac{1}{|\mathbf{r}-\mathbf{r}'|} d^2r' + \nabla \int_{V'} \frac{\nabla' \cdot \mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r' + \mu_0 \mathbf{j}(\mathbf{r}) . \tag{5.34}$$

Applying this equality to the volume V' limited by a surface S' either sufficiently distant from the field concentration or with no current crossing it, we may neglect the first term on the right-hand side of Eq. (34), while the second term always equals zero in statics, due to the dc charge continuity – see Eq. (4.6). As a result, we arrive at a very simple differential equation[17]

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} . \tag{5.35}$$

This is (the dc form of) the inhomogeneous Maxwell equation – which in magnetostatics plays a role similar to Eq. (1.27) in electrostatics. Let me display, for the first time in this course, this fundamental system of equations (at this stage, for statics only), and give the reader a minute to stare, in silence, at their beautiful symmetry – which has inspired so much of the later development of physics:

Maxwell equations: statics

$$\nabla \times \mathbf{E} = 0, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{j},$$
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad \nabla \cdot \mathbf{B} = 0. \tag{5.36}$$

Their only asymmetry, two zeros on the right-hand sides (for the magnetic field's divergence and electric field's curl), is due to the absence in the Nature of magnetic monopoles and their currents. I will discuss these equations in more detail in Sec. 6.7, after the first two equations (for the fields' curls) have been generalized to their full, time-dependent versions.

Returning now to our current, more mundane but important task of calculating the magnetic field induced by simple current configurations, we can benefit from an integral form of Eq. (35). For that, let us integrate this equation over an arbitrary surface S limited by a closed contour C, and apply to the result the Stokes theorem.[18] The resulting expression,

Ampère law

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 \oint_S j_n d^2r \equiv \mu_0 I , \tag{5.37}$$

where I is the net electric current crossing surface S, is called the *Ampère law*.

[17] As in all earlier formulas for the magnetic field, in the Gaussian units, the coefficient μ_0 in this relation is replaced with $4\pi/c$.

[18] See, e.g., MA Eq. (12.1) with $\mathbf{f} = \mathbf{B}$.

As the first example of its application, let us return to the current in a straight wire (Fig. 4a). With the Ampère law in our arsenal, we can readily pursue an even more ambitious goal than that achieved in the previous section, namely to calculate the magnetic field both outside and inside of a wire of an arbitrary radius R, with an arbitrary (albeit axially-symmetric) current distribution $j(\rho)$ – see Fig. 5.

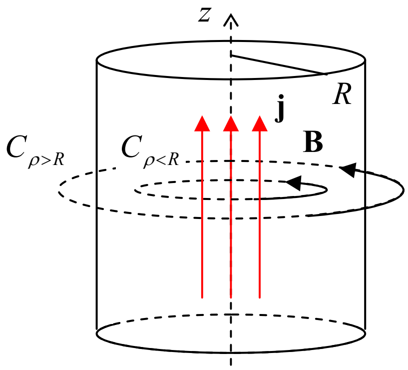

Fig. 5.5. The simplest application of the Ampère law: the magnetic field of a straight current.

Selecting the Ampère-law contour C in the form of a ring of some radius ρ in the plane normal to the wire's axis z, we have $\mathbf{B}\cdot d\mathbf{r} = B\rho\, d\varphi$, where φ is the azimuthal angle, so Eq. (37) yields:

$$2\pi\,\rho B(\rho) = \mu_0 \times \begin{cases} 2\pi \int_0^\rho j(\rho')\rho' d\rho', & \text{for } \rho \le R, \\ 2\pi \int_0^R j(\rho')\rho'\, d\rho' \equiv I, & \text{for } \rho \ge R. \end{cases} \qquad (5.38)$$

Thus we have not only recovered our previous result (20), with the notation replacement $d \to \rho$, in a much simpler way but also have been able to calculate the magnetic field's distribution inside the wire. In the most common particular case when the current is uniformly distributed along its cross-section, $j(\rho) = \text{const}$, the first of Eqs. (38) immediately yields $B \propto \rho$ for $\rho \le R$.

Another important system is a straight, long *solenoid* (Fig. 6a), with dense winding: $n^2A >> 1$, where n is the number of wire turns per unit length, and A is the area of the solenoid's cross-section.

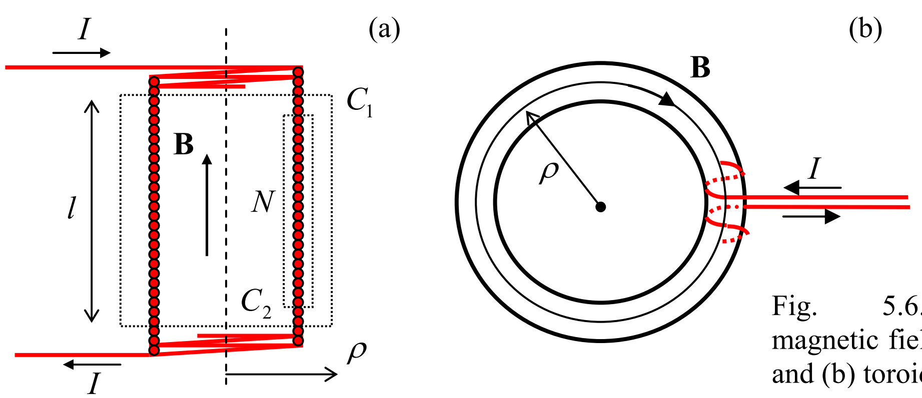

Fig. 5.6. Calculating magnetic fields of (a) straight and (b) toroidal solenoids.

From the symmetry of this problem, the longitudinal (in Fig. 6a, vertical) component B_z of the magnetic field may only depend on the distance ρ of the observation point from the solenoid's axis. First taking a plane Ampère contour C_1, with both long sides outside the solenoid, we get $B_z(\rho_2) - B_z(\rho_1) = 0$, because the total current piercing the contour equals zero. This is only possible if $B_z = 0$ at any ρ outside of the solenoid, provided that it is infinitely long.[19] With this result on hand, from the Ampère law applied to the contour C_2, we get the following relation for the only (z-) component of the internal field:

$$Bl = \mu_0 NI \,, \tag{5.39}$$

where N is the number of wire turns passing through the contour of length l. This means that regardless of the exact position of the internal side of the contour, the result is the same:

$$B = \mu_0 \frac{N}{l} I \equiv \mu_0 nI \,. \tag{5.40}$$

Thus, the field inside an infinitely long solenoid (with an arbitrary shape of its cross-section) is uniform; in this sense, a long solenoid is a magnetic analog of a wide plane capacitor, explaining why this system is so widely used in physical experiment.

As should be clear from its derivation, the obtained results, especially that the field outside of the solenoid equals zero, are conditional on the solenoid length being very large in comparison with its lateral size. (From Eq. (25), we may predict that for a solenoid of a finite length l, the close-range external field is a factor of $\sim A/l^2$ lower than the internal one.) A much better suppression of such "fringe" fields may be obtained using *toroidal solenoids* (Fig. 6b). The application of the Ampère law to this geometry shows that in the limit of dense winding ($N >> 1$), there is no fringe field at all (for any relation between the two radii of the torus), while inside the solenoid, at distance ρ from the system's axis,

$$B = \frac{\mu_0 NI}{2\pi\rho} \,. \tag{5.41}$$

We see that a possible drawback of this system for practical applications is that the internal field does depend on ρ, i.e. is not quite uniform; however, if the torus is relatively thin, this deficiency is minor.

Next let us discuss a very important question: how can we solve the problems of magnetostatics for systems whose low symmetry does not allow getting easy results from the Ampère law? (The examples are of course too numerous to list; for example, we cannot use this approach even to reproduce Eq. (23) for a round current loop.) From the deep analogy with electrostatics, we may expect that in this case, we could calculate the magnetic field by solving a certain boundary problem for the field's potential – in our current case, the vector potential **A** defined by Eq. (28). However, despite the similarity of this formula and Eq. (1.38) for ϕ, which was noticed above, there is an additional issue we should tackle in the magnetic case – besides the obvious fact that calculating the vector potential distribution means determining three scalar functions (say, A_x, A_y, and A_z), rather than just one (ϕ).

To reveal the issue, let us plug Eq. (27) into Eq. (35):

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{j} \,, \tag{5.42}$$

and then apply to the left-hand side of this equation the same identity (31). The result is

[19] Applying the Ampère law to a circular contour of radius ρ, coaxial with the solenoid, we see that the field outside (but not inside!) it has an azimuthal component B_φ, similar to that of the straight wire (see Eq. (38) above) and hence (at $N >> 1$) much weaker than the longitudinal field inside the solenoid – see Eq. (40).

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{j}. \tag{5.43}$$

On the other hand, as we know from electrostatics (please compare Eqs. (1.38) and (1.41)), the vector potential $\mathbf{A}(\mathbf{r})$ given by Eq. (28) has to satisfy a simpler ("vector-Poisson") equation

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{j},} \tag{5.44}$$

Poisson equation for **A**

which is just a set of three usual Poisson equations for each Cartesian component of **A**.

To resolve the difference between these results, let us note that Eq. (43) is reduced to Eq. (44) if $\nabla\cdot\mathbf{A} = 0$. In this context, let us discuss what discretion we have in the choice of the potential. In electrostatics, we may add, to the scalar function ϕ' that satisfies Eq. (1.33) for the given field **E**, not only an arbitrary constant but even an arbitrary function of time:

$$-\nabla\left[\phi' + f(t)\right] = -\nabla\phi' = \mathbf{E}. \tag{5.45}$$

Similarly, using the fact that the curl of the gradient of any scalar function equals zero,[20] we may add to any vector function $\mathbf{A}'$ that satisfies Eq. (27) for the given field **B**, not only any constant but even a gradient of an arbitrary scalar function $\chi(\mathbf{r}, t)$, because

$$\nabla \times (\mathbf{A}' + \nabla\chi) = \nabla \times \mathbf{A}' + \nabla \times (\nabla\chi) = \nabla \times \mathbf{A}' = \mathbf{B}. \tag{5.46}$$

Such additions, which keep the fields intact, are called *gauge transformations*.[21] Let us see what such a transformation does to $\nabla\cdot\mathbf{A}'$:

$$\nabla \cdot (\mathbf{A}' + \nabla\chi) = \nabla \cdot \mathbf{A}' + \nabla^2\chi. \tag{5.47}$$

For any choice of such a function $\mathbf{A}'$, we can always choose the function χ in such a way that it satisfies the Poisson equation $\nabla^2\chi = -\nabla\cdot\mathbf{A}'$, and hence makes the divergence of the transformed vector potential, $\mathbf{A} \equiv \mathbf{A}' + \nabla\chi$, equal to zero everywhere,

$$\boxed{\nabla \cdot \mathbf{A} = 0,} \tag{5.48}$$

Coulomb gauge

thus reducing Eq. (43) to Eq. (44).

To summarize, the set of distributions $\mathbf{A}'(\mathbf{r})$ that satisfy Eq. (27) for a given field $\mathbf{B}(\mathbf{r})$, is not limited to the vector potential $\mathbf{A}(\mathbf{r})$ given by Eq. (44), but is reduced to it upon the additional *Coulomb gauge condition* (48). However, as we will see in a minute, even this condition still leaves some degrees of freedom in the choice of the vector potential. To illustrate this fact, and also to get a better gut feeling of the vector potential's distribution in space, let us calculate $\mathbf{A}(\mathbf{r})$ for two very basic cases.

First, let us revisit the straight wire problem shown in Fig. 5. As Eq. (28) shows, in this case the vector potential **A** has just one component (along the axis z). Moreover, due to the problem's axial symmetry, its magnitude may only depend on the distance from the axis: $\mathbf{A} = \mathbf{n}_z A(\rho)$. Hence, the gradient of **A** is directed across the z-axis, so Eq. (48) is satisfied at all points. For our symmetry ($\partial/\partial\varphi = \partial/\partial z = 0$), the Laplace operator, written in cylindrical coordinates, has just one term,[22] reducing Eq. (44) to

[20] See, e.g., MA Eq. (11.1).

[21] The use of the term "gauge" (originally meaning "a measure" or "a scale") in this context is purely historic, so the reader should not try to find too much hidden sense in it.

[22] See, e.g., MA Eq. (10.3).

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dA}{d\rho}\right) = -\mu_0 j(\rho). \tag{5.49}$$

Multiplying both sides of this equation by ρ and integrating them over the coordinate once, we get

$$\rho\frac{dA}{d\rho} = -\mu_0 \int_0^\rho j(\rho')\rho' d\rho' + \text{const}. \tag{5.50}$$

Since in the cylindrical coordinates, for our symmetry, $B = -dA/d\rho$,[23] Eq. (50) is nothing else than our old result (38) for the magnetic field.[24] However, let us continue the integration, at least for the region outside the wire, where the function $A(\rho)$ depends only on the full current I rather than on the current distribution. Dividing both parts of Eq. (50) by ρ, and integrating them over this argument again, we get

$$A(\rho) = -\frac{\mu_0 I}{2\pi}\ln\rho + \text{const}, \qquad \text{where } I = 2\pi\int_0^R j(\rho)\rho d\rho, \quad \text{for } \rho \geq R. \tag{5.51}$$

As a reminder, we had similar logarithmic behavior for the electrostatic potential outside a uniformly charged straight line. This is natural because the Poisson equations for both cases are similar.

Now let us find the vector potential for the long solenoid (Fig. 6a), with its uniform magnetic field. Since Eq. (28) tells us that the vector **A** should follow the direction of the inducing current, we may start by looking for it in the form $\mathbf{A} = \mathbf{n}_\varphi A(\rho)$. (This is especially natural if the solenoid's cross-section is circular.) With this orientation of **A**, the same general expression for the curl operator in cylindrical coordinates yields $\nabla\times\mathbf{A} = \mathbf{n}_z(1/\rho)d(\rho A)/d\rho$. According to Eq. (27), this expression should be equal to **B** – in our current case to $\mathbf{n}_z B$, with a constant B – see Eq. (40). Integrating this equality, and selecting such integration constant that $A(0)$ is finite, we get

$$A(\rho) = \frac{B\rho}{2}, \qquad \text{i.e. } \mathbf{A} = \frac{B\rho}{2}\mathbf{n}_\varphi. \tag{5.52}$$

Plugging this result into the general expression for the Laplace operator in the cylindrical coordinates,[25] we see that the Poisson equation (44) with $\mathbf{j} = 0$ (i.e. the Laplace equation) is satisfied again – which is natural since, for this distribution, the Coulomb gauge condition (48) is satisfied: $\nabla\cdot\mathbf{A} = 0$.

However, Eq. (52) is not the unique (or even the simplest) vector potential that gives the same uniform field $\mathbf{B} = \mathbf{n}_z B$. Indeed, using the well-known expression for the curl operator in Cartesian coordinates,[26] it is straightforward to check that each of the vector functions $\mathbf{A}' = \mathbf{n}_y Bx$ and $\mathbf{A}'' = -\mathbf{n}_x By$ also has the same curl, and also satisfies the Coulomb gauge condition (48).[27] If such solutions do not look very natural because of their anisotropy in the [x, y] plane, please consider the fact that they represent the uniform magnetic field regardless of its source – for example, regardless of the shape of the long solenoid's cross-section. Such choices of the vector potential may be very convenient for some

[23] See, e.g., MA Eq. (10.5) with $\partial/\partial\varphi = \partial/\partial z = 0$.

[24] Since the magnetic field at the wire's axis has to be zero (otherwise, being normal to the axis, where would it be directed?), the integration constant in Eq. (50) has to equal zero.

[25] See, e.g., MA Eq. (10.6).

[26] See, e.g., MA Eq. (8.5).

[27] The axially symmetric vector potential (52) is just a weighed sum of these two functions: $\mathbf{A} = (\mathbf{A}' + \mathbf{A}'')/2$.

problems, for example for the quantum-mechanical analysis of the 2D motion of a charged particle in the perpendicular magnetic field, giving the famous Landau energy levels.[28]

5.3. Magnetic energy, flux, and inductance

Considering the currents flowing in a system as generalized coordinates, the magnetic forces (1) between them are their unique functions, and in this sense, the energy U of their magnetic interaction may be considered the potential energy of the system. The apparent (but somewhat deceptive) way to derive an expression for this energy is to use the analogy between Eq. (1) and its electrostatic analog, Eq. (2). Indeed, Eq. (2) may be transformed into Eq. (1) with just three replacements:

(i) $\rho(\mathbf{r})\rho'(\mathbf{r}')$ should be replaced with $[\mathbf{j}(\mathbf{r})\cdot\mathbf{j}'(\mathbf{r}')]$,

(ii) ε_0 should be replaced with $1/\mu_0$, and

(iii) the sign before the double integral has to be replaced with the opposite one.

Hence we may avoid repeating the calculation made in Chapter 1, by making these replacements in Eq. (1.59), which gives the electrostatic potential energy of the system with $\rho(\mathbf{r})$ and $\rho'(\mathbf{r}')$ describing the same charge distribution, i.e. with $\rho'(\mathbf{r}) = \rho(\mathbf{r})$, to get the following expression for the magnetic potential energy in the system with, similarly, $\mathbf{j}'(\mathbf{r}) = \mathbf{j}(\mathbf{r})$:[29]

$$U_j = -\frac{\mu_0}{4\pi}\frac{1}{2}\int d^3r\int d^3r'\,\frac{\mathbf{j}(\mathbf{r})\cdot\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{5.53}$$

But this is *not* the unique answer! Indeed, Eq. (53) describes the proper potential energy of the system (in particular, giving the correct result for the current interaction forces) only in the case when the interacting currents are fixed – just as Eq. (1.59) is adequate when the interacting charges are fixed. Here comes a substantial difference between electrostatics and magnetostatics: due to the fundamental fact of electric charge conservation (already discussed in Secs. 1.1 and 4.1), keeping electric charges fixed does not require external work, while the maintenance of currents generally does. As a result, Eq. (53) describes the energy of the magnetic interaction *plus* of the system keeping the currents constant – or rather of its part depending on the system under our consideration. In this situation, using the terminology already used in Sec. 3.5 (see also a general discussion in CM Sec. 1.4.), U_j may be called the Gibbs potential energy of our magnetic system.

Now to exclude from U_j the contribution due to the interaction with the current-supporting system(s), i.e. calculate the potential energy U of our system as such, we need to know this contribution. The simplest way to do this is to use the *Faraday induction law* that describes this interaction and will be discussed at the beginning of the next chapter. This is why let me postpone the derivation until that point, and for now, ask the reader to believe me that the removal of the interaction leads to an expression similar to Eq. (53), but with the opposite sign:

$$U = \frac{\mu_0}{4\pi}\frac{1}{2}\int d^3r\int d^3r'\,\frac{\mathbf{j}(\mathbf{r})\cdot\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{5.54}$$

Magnetic interaction energy

[28] See, e.g., QM Sec. 3.2.

[29] Just as in electrostatics, for the interaction of two *independent* current distributions $\mathbf{j}(\mathbf{r})$ and $\mathbf{j}'(\mathbf{r}')$, the factor ½ should be dropped.

I will prove this result in Sec. 6.2, but actually, this sign dichotomy should not be quite surprising to the attentive reader, in the context of a similar duality of Eqs. (3.73) and (3.81) for the electrostatic energies including and excluding the interaction with the field source.

Due to the importance of Eq. (54), let us rewrite it in several other forms, convenient for various applications. First of all, just as in electrostatics, it may be recast into a potential-based form. Indeed, with the definition (28) of the vector potential **A**(**r**), Eq. (54) becomes

$$U=\frac{1}{2}\int \mathbf{j}(\mathbf{r})\cdot\mathbf{A}(\mathbf{r})d^3r\,. \tag{5.55}$$

This formula, which is a clear magnetic analog of Eq. (1.60) of electrostatics, is very popular among field theorists, because it is very handy for their manipulations; it is also useful for some practical applications. However, for many calculations, it is more convenient to have a direct expression of the energy via the magnetic field. Again, this may be done very similarly to what had been done for electrostatics in Sec. 1.3, i.e. by plugging into Eq. (55) the current density expressed from Eq. (35) and then transforming it as[30]

$$U=\frac{1}{2}\int \mathbf{j}\cdot\mathbf{A}d^3r=\frac{1}{2\mu_0}\int \mathbf{A}\cdot(\nabla\times\mathbf{B})d^3r=\frac{1}{2\mu_0}\int \mathbf{B}\cdot(\nabla\times\mathbf{A})d^3r-\frac{1}{2\mu_0}\int\nabla\cdot(\mathbf{A}\times\mathbf{B})d^3r\,. \tag{5.56}$$

Now using the divergence theorem, the second integral may be transformed into a surface integral of $(\mathbf{A}\times\mathbf{B})_n$. According to Eqs. (27)-(28) if the current distribution **j**(**r**) is localized, this vector product drops, at large distances, faster than $1/r^2$, so if the integration volume is large enough, the surface integral is negligible. In the remaining first integral in Eq. (56), we may use Eq. (27) to rewrite $\nabla\times\mathbf{A}$ as **B**. As a result, we get a very simple and fundamental formula.

$$U=\frac{1}{2\mu_0}\int B^2d^3r\,. \tag{5.57a}$$

Just as with the electric field, this expression may be interpreted as a volume integral of the *magnetic energy density u*:

Magnetic field energy

$$U=\int u(\mathbf{r})d^3r,\quad \text{with}\ \ u(\mathbf{r})\equiv\frac{1}{2\mu_0}\mathbf{B}^2(\mathbf{r}), \tag{5.57b}$$

clearly similar to Eq. (1.65).[31] Again, the conceptual choice between the spatial localization of magnetic energy – either at the location of electric currents only, as implied by Eqs. (54) and (55), or in all regions where the magnetic field exists, as apparent from Eq. (57b), cannot be done within the framework of magnetostatics, and only the electrodynamics gives a decisive preference for the latter choice.

For the practically important case of currents flowing in several thin wires, Eq. (54) may be first integrated over the cross-section of each wire, just as was done at the derivation of Eq. (4). As before, since the integral of the current density over the k^{th} wire's cross-section is just the current I_k in the wire, and cannot change along its length, it may be taken from the remaining integrals, giving

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[30] For that, we may use MA Eq. (11.7) with **f** = **A** and **g** = **B**, giving $\mathbf{A}\cdot(\nabla\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \nabla\cdot(\mathbf{A}\times\mathbf{B})$.

[31] The transfer to the Gaussian units in Eqs. (57) may be accomplished by the usual replacement $\mu_0 \to 4\pi$, thus giving, in particular, $u = B^2/8\pi$.

$$U = \frac{\mu_0}{4\pi}\frac{1}{2}\sum_{k,k'} I_k I_{k'} \oint_{l_k}\oint_{l_{k'}} \frac{d\mathbf{r}_k \cdot d\mathbf{r}_{k'}}{|\mathbf{r}_k - \mathbf{r}_{k'}|}, \qquad (5.58)$$

where l_k is the full length of the k^{th} wire loop. Note that Eq. (58) is valid if all currents I_k are independent of each other, because the double sum counts each current pair twice, compensating the coefficient ½ in front of the sum. It is useful to decompose this relation as

$$U = \frac{1}{2}\sum_{k,k'} I_k I_{k'} L_{kk'}, \qquad (5.59)$$

where the coefficients $L_{kk'}$ are independent of the currents:

$$L_{kk'} \equiv \frac{\mu_0}{4\pi} \oint_{l_k}\oint_{l_{k'}} \frac{d\mathbf{r}_k \cdot d\mathbf{r}_{k'}}{|\mathbf{r}_k - \mathbf{r}_{k'}|}, \qquad (5.60)$$

Mutual inductance coefficients

The coefficient $L_{kk'}$ with $k \neq k'$, is called the *mutual inductance* between current the k^{th} and k'^{th} loops, while the diagonal coefficient $L_k \equiv L_{kk}$ is called the *self-inductance* (or just *inductance*) of the k^{th} loop.[32] From the symmetry of Eq. (60) with respect to the index swap, $k \leftrightarrow k'$, it is evident that the matrix of coefficients $L_{kk'}$ is symmetric:[33]

$$L_{kk'} = L_{k'k}, \qquad (5.61)$$

so for the practically most important case of two interacting currents I_1 and I_2, Eq. (59) reads

$$U = \frac{1}{2}L_1 I_1^2 + M I_1 I_2 + \frac{1}{2}L_2 I_2^2, \qquad (5.62)$$

where $M \equiv L_{12} = L_{21}$ is the *mutual inductance coefficient*.

These formulas clearly show the importance of the self- and mutual inductances, so I will demonstrate their calculation for at least a few basic geometries. Before doing that, however, let me recast Eq. (58) into one more form that may facilitate such calculations. Namely, let us notice that for the magnetic field induced by current I_k in a thin wire, Eq. (28) is reduced to

$$\mathbf{A}_k(\mathbf{r}) = \frac{\mu_0}{4\pi} I_k \int_{l'} \frac{d\mathbf{r}_k}{|\mathbf{r} - \mathbf{r}_k|}, \qquad (5.63)$$

so Eq. (58) may be rewritten as

$$U = \frac{1}{2}\sum_{k,k'} I_k \oint_{l_k} \mathbf{A}_{k'}(\mathbf{r}_k)\cdot d\mathbf{r}_{k'}. \qquad (5.64)$$

But according to the same Stokes theorem that was used earlier in this chapter to derive the Ampère law, and Eq. (27), the integral in Eq. (64) is nothing else than the *magnetic field's flux* Φ (more frequently called just the *magnetic flux*) through a surface S limited by the contour l :

[32] As evident from Eq. (60), these coefficients depend only on the geometry of the system. Moreover, in the Gaussian units, in which Eq. (60) is valid without the factor $\mu_0/4\pi$, the inductance coefficients have the dimension of length (centimeters). The SI unit of inductance is called the *henry*, abbreviated H – after Joseph Henry, who in particular discovered the effect of electromagnetic induction (see Sec. 6.1) independently of Michael Faraday.

[33] Note that the matrix of the mutual inductances $L_{jj'}$ is very similar to the matrix of *reciprocal* capacitance coefficients $p_{kk'}$ – for example, compare Eq. (62) with Eq. (2.21).

Magnetic flux

$$\oint_l \mathbf{A}(\mathbf{r})\cdot d\mathbf{r} = \int_S (\nabla\times A)_n d^2r = \int_S B_n d^2 r \equiv \Phi \tag{5.65}$$

– in the particular case of Eq. (64), the flux $\Phi_{kk'}$ of the field induced by the k' th current through the loop of the k^{th} current.[34] As a result, Eq. (64) may be rewritten as

$$U = \frac{1}{2}\sum_{k,k'} I_k \Phi_{kk'} . \tag{5.66}$$

Comparing this expression with Eq. (59), we see that

$$\Phi_{kk'} \equiv \int_{S_k} (\mathbf{B}_{k'})_n d^2 r = L_{kk'} I_{k'}, \tag{5.67}$$

This expression not only gives us one more means for calculating the coefficients $L_{kk'}$, but also shows their physical sense: the mutual inductance characterizes what part of the magnetic flux (colloquially, "what fraction of field lines") induced by the current $I_{k'}$ pierces the k^{th} loop's area S_k – see Fig. 7.

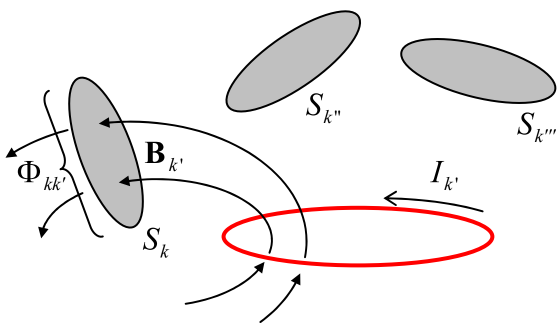

Fig. 5.7. The physical sense of the mutual inductance coefficient $L_{kk'} \equiv \Phi_{kk'}/I_{k'}$ – schematically.

Due to the linear superposition principle, the total flux piercing the k^{th} loop may be represented as

Magnetic flux from currents

$$\Phi_k \equiv \sum_{k'} \Phi_{kk'} = \sum_{k'} L_{kk'} I_{k'} \tag{5.68}$$

For example, for the system of two currents, this expression is reduced to a clear analog of Eqs. (2.19):

$$\begin{aligned}\Phi_1 &= L_1 I_1 + M I_2,\\ \Phi_2 &= M I_1 + L_2 I_2 .\end{aligned} \tag{5.69}$$

For the even simpler case of a single current,

Φ of a single current

$$\Phi = LI , \tag{5.70}$$

so the magnetic energy of the current may be represented in several equivalent forms:

[34] The SI unit of magnetic flux is called *weber*, abbreviated Wb – after Wilhelm Edward Weber (1804-1891), who in particular co-invented (with Carl Gauss) the electromagnetic telegraph. More importantly for this course, in 1856 he was the first (together with Rudolf Kohlrausch) to notice that the value of (in modern terms) $1/(\varepsilon_0\mu_0)^{1/2}$, derived from electrostatic and magnetostatic measurements, coincides with the independently measured speed of light c. This observation gave an important motivation for Maxwell's theory.

$$U = \frac{L}{2} I^2 = \frac{1}{2} I\Phi = \frac{1}{2L}\Phi^2 . \qquad (5.71)$$

U of a single current

These relations, similar to Eqs. (2.14)-(2.15) of electrostatics, show that the self-inductance L of a current loop may be considered a measure of the system's magnetic energy. However, as we will see in Sec. 6.1, this measure is adequate only if the flux Φ, rather than the current I, is fixed.

Now we are well equipped for the calculation of inductance coefficients for particular systems, having three options. The first one is to use Eq. (60) directly.[35] The second one is to calculate the magnetic field energy from Eq. (57) as the function of all currents I_k in the system, and then use Eq. (59) to find all coefficients $L_{kk'}$. For example, for a system with just one current, Eq. (71) yields

$$L = \frac{U}{I^2/2} . \qquad (5.72)$$

Finally, if the system consists of thin wires, so the loop areas S_k and hence the fluxes $\Phi_{kk'}$ are well defined, we may calculate them from Eq. (65), and then use Eq. (67) to find the inductances.

Usually, the third option is simpler, but the first two may be very useful even for thin-wire systems, especially if the notion of magnetic flux in them is not quite apparent. As an important example, let us find the self-inductance of a long solenoid – see Fig. 6a again. We have already calculated the magnetic field inside it – see Eq. (40) – so, due to the field uniformity, the magnetic flux piercing each turn of the wire is just

$$\Phi_1 = BA = \mu_0 nIA , \qquad (5.73)$$

where A is the area of the solenoid's cross-section – for example πR^2 for a round solenoid, though Eq. (40), and hence Eq. (73) are valid for cross-sections of any shape. Comparing Eqs. (73) with Eq. (70), one might wrongly conclude that $L = \Phi_1/I = \mu_0 nA$ (WRONG!), i.e. that the solenoid's inductance is independent of its length. Actually, the magnetic flux Φ_1 pierces *each* wire turn, so the total flux through the *whole* current loop, consisting of N turns, is

$$\Phi = N\Phi_1 = \mu_0 n^2 lAI , \qquad (5.74)$$

and the correct expression for the long solenoid's self-inductance is

$$L = \frac{\Phi}{I} = \mu_0 n^2 lA \equiv \frac{\mu_0 N^2 A}{l} , \qquad (5.75)$$

L of a solenoid

i.e. at fixed A and l, the inductance scales as N^2, not as N. Since this reasoning may seem not quite evident, it is prudent to verify this result by using Eq. (72), with the full magnetic energy inside the solenoid (neglecting minor fringe field contributions), given by Eq. (57) with $\mathbf{B}$ = const within the internal volume $V = lA$, and zero outside of it:

$$U = \frac{1}{2\mu_0} B^2 Al = \frac{1}{2\mu_0} (\mu_0 nI)^2 Al \equiv \mu_0 n^2 lA \frac{I^2}{2} . \qquad (5.76)$$

Plugging this relation into Eq. (72) immediately confirms the result (75).

[35] Numerous applications of that *Neumann formula* (derived in 1845 by F. Neumann) to electrical engineering problems may be found, for example, in the classical text by F. Grover, *Inductance Calculations*, Dover, 1946.

This energy-based approach becomes virtually inevitable for continuously distributed currents. As an example, let us calculate the self-inductance L of a long coaxial cable with the cross-section shown in Fig. 8, [36] and the full current in the outer conductor equal and opposite to that (I) in the inner conductor.

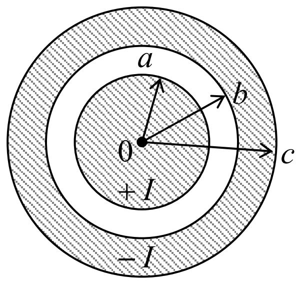

Fig. 5.8. The cross-section of a coaxial cable.

Let us assume that the current is uniformly distributed over the cross-sections of both conductors. (As we know from the previous chapter, this is indeed the case if both the internal and external conductors are made of a uniform resistive material.) First, we should calculate the radial distribution of the magnetic field – which has only one, azimuthal component because of the axial symmetry of the problem. This distribution may be immediately found by applying the Ampère law (37) to circular contours of radii ρ within four different ranges:

$$2\pi\rho B = \mu_0 I\big|_{\text{piercing the contour}} = \mu_0 I \times \begin{cases} \rho^2/a^2, & \text{for } \rho < a, \\ 1, & \text{for } a < \rho < b, \\ (c^2-\rho^2)/(c^2-b^2), & \text{for } b < \rho < c, \\ 0, & \text{for } c < \rho. \end{cases} \tag{5.77}$$

Now, an easy integration yields the magnetic energy per unit length of the cable:

$$\begin{aligned}\frac{U}{l} &= \frac{1}{2\mu_0}\int B^2 d^2r = \frac{\pi}{\mu_0}\int_0^\infty B^2 \rho d\rho = \frac{\mu_0 I^2}{4\pi}\left[\int_0^a \left(\frac{\rho}{a^2}\right)^2 \rho d\rho + \int_a^b \left(\frac{1}{\rho}\right)^2 \rho d\rho + \int_b^c \left(\frac{c^2-\rho^2}{\rho(c^2-b^2)}\right)^2 \rho d\rho\right] \\ &= \frac{\mu_0}{2\pi}\left[\ln\frac{b}{a} + \frac{c^2}{c^2-b^2}\left(\frac{c^2}{c^2-b^2}\ln\frac{c}{b} - \frac{1}{2}\right)\right]\frac{I^2}{2}.\end{aligned} \tag{5.78}$$

From here, and Eq. (72), we get the final answer:

$$\frac{L}{l} = \frac{\mu_0}{2\pi}\left[\ln\frac{b}{a} + \frac{c^2}{c^2-b^2}\left(\frac{c^2}{c^2-b^2}\ln\frac{c}{b} - \frac{1}{2}\right)\right]. \tag{5.79}$$

Note that for the particular case of a thin outer conductor, $c - b \ll b$, this expression reduces to

$$\frac{L}{l} \approx \frac{\mu_0}{2\pi}\left(\ln\frac{b}{a} + \frac{1}{4}\right), \tag{5.80}$$

where the first term in the parentheses is due to the contribution of the magnetic field energy in the free space between the conductors. This distinction is important for some applications because in

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[36] As a reminder, the mutual capacitance C between the conductors of such a system was calculated in Sec. 2.3.

superconductor cables, as well as the normal-metal cables at high frequencies (to be discussed in the next chapter), the field does not penetrate the conductor's bulk, so Eq. (80) is valid without the last term ¼ in the parentheses – for any $b < c$.

As the last example, let us calculate the *mutual* inductance between a long straight wire and a round wire loop adjacent to it (Fig. 9), neglecting the thickness of both wires.

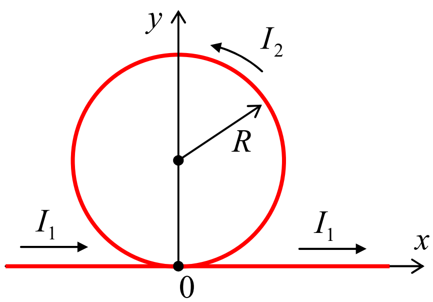

Fig. 5.9. An example of the mutual inductance calculation.

Here there is no problem with using the last approach discussed above, based on the direct calculation of the magnetic flux. Indeed, as was discussed in Sec. 1, the field $\mathbf{B}_1$ induced by the current I_1 at any point of the round loop is normal to its plane – e.g., to the plane of the drawing of Fig. 9. In the Cartesian coordinates shown in that figure, Eq. (20) reads $B_1 = \mu_0 I_1/2\pi y$, giving the following magnetic flux through the loop:

$$\Phi_{21} = \frac{\mu_0 I_1}{2\pi}\int_{-R}^{+R} dx \int_{R-(R^2-x^2)^{1/2}}^{R+(R^2-x^2)^{1/2}} \frac{dy}{y} = \frac{\mu_0 I_1}{\pi}\int_0^R \ln\frac{R+(R^2-x^2)^{1/2}}{R-(R^2-x^2)^{1/2}}dx \equiv \frac{\mu_0 I_1 R}{\pi}\int_0^1 \ln\frac{1+(1-\xi^2)^{1/2}}{1+(1-\xi^2)^{1/2}}d\xi. \quad (5.81)$$

This is a table integral equal to π,[37] so $\Phi_{21} = \mu_0 I_1 R$, and the final answer for the mutual inductance $M \equiv L_{12} = L_{21} = \Phi_{21}/I_1$ is finite (and very simple):

$$M = \mu_0 R\,, \quad (5.82)$$

despite the magnetic field's divergence at the lowest point of the loop ($y = 0$).

Note that in contrast with the finite *mutual* inductance of this system, the *self*-inductances of both its wires are formally infinite in the thin-wire limit – see, e.g., Eq. (80), which, in the limit $b/a >> 1$, describes a thin straight wire. However, since this divergence is very weak (logarithmic), it is quenched by any deviation from this perfectly axial geometry. For example, a fair estimate of the inductance of a wire of a large but finite length $l >> a$ may be obtained from Eq. (80) by the replacement of b with l:

$$L \approx \frac{\mu_0 l}{2\pi}\ln\frac{l}{a}\,. \quad (5.83)$$

(Note, however, that the exact result depends on where from/to the current flows beyond that segment.) It turns out that a similar approximate result, with l replaced with $2\pi R$ in the front factor, and with R under the logarithm, is valid for the self-inductance of a round loop with $a << R$. (A proof of this fact is a very useful exercise, highly recommended to the reader.)

[37] See, e.g., MA Eq. (6.13), with $a = 1$.

5.4. Magnetic dipole moment, and magnetic dipole media

The most natural way of the magnetic media description parallels that for dielectrics in Chapter 3, and is based on the properties of *magnetic dipoles* – the notion close (but not identical!) to that of the electric dipoles discussed in Sec. 3.1. To introduce this notion quantitatively, let us consider, just as in Sec. 3.1, a spatially-localized system with a current distribution $\mathbf{j}(\mathbf{r}')$, whose magnetic field is measured at relatively large distances $r >> r'$ (Fig. 10).

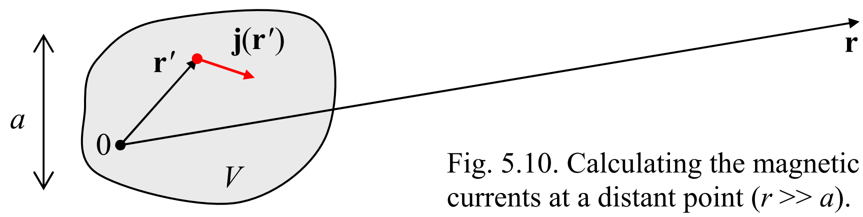

Fig. 5.10. Calculating the magnetic field of localized currents at a distant point ($r >> a$).

Applying the truncated Taylor expansion (3.5) of the fraction $1/|\mathbf{r} - \mathbf{r}'|$ to the vector potential given by Eq. (28), we get

$$\mathbf{A}(\mathbf{r}) \approx \frac{\mu_0}{4\pi}\left[\frac{1}{r}\int_V \mathbf{j}(\mathbf{r}')d^3r' + \frac{1}{r^3}\int_V (\mathbf{r}\cdot\mathbf{r}')\mathbf{j}(\mathbf{r}')d^3r'\right]. \quad (5.84)$$

Now, due to the vector character of this potential, we have to depart somewhat from the approach of Sec. 3.1 and use the following vector algebra identity:[38]

$$\int_V \left[f(\mathbf{j}\cdot\nabla g) + g(\mathbf{j}\cdot\nabla f)\right]d^3r = 0, \quad (5.85)$$

that is valid for any pair of smooth (differentiable) scalar functions $f(\mathbf{r})$ and $g(\mathbf{r})$, and any vector function $\mathbf{j}(\mathbf{r})$ that, as the dc current density, satisfies the continuity condition $\nabla\cdot\mathbf{j} = 0$ and whose normal component vanishes on the surface of the volume V. First, let us use Eq. (85) with f equal to 1, and g equal to any Cartesian component of the radius-vector $\mathbf{r}$: $g = r_l$ ($l = 1, 2, 3$). Then it yields

$$\int_V (\mathbf{j}\cdot\mathbf{n}_l)d^3r = \int_V j_l d^3r = 0, \quad (5.86)$$

so for the vector as the whole

$$\int_V \mathbf{j}(\mathbf{r})d^3r = 0, \quad (5.87)$$

showing that the first term on the right-hand side of Eq. (84) equals zero. Next, let us use Eq. (85) again, but now with $f = r_l$, $g = r_{l'}$ (where $l, l' = 1, 2, 3$); then it yields

$$\int_V \left(r_l j_{l'} + r_{l'} j_l\right)d^3r = 0, \quad (5.88)$$

so the l^{th} Cartesian component of the second integral in Eq. (84) may be transformed as

[38] See, e.g., MA Eq. (12.3) with the additional condition $j_n|_S = 0$, pertinent to space-restricted currents.

$$\int_V (\mathbf{r}\cdot\mathbf{r}')j_l d^3r' = \int_V \sum_{l'=1}^{3} r_{l'}r'_{l'}j_l d^3r' = \frac{1}{2}\sum_{l'=1}^{3} r_{l'}\int_V (r'_{l'}j_l + r'_{l'}j_l)d^3r'$$
$$= \frac{1}{2}\sum_{l'=1}^{3} r_{l'}\int_V (r'_{l'}j_l - r'_{l}j_{l'})d^3r' = -\frac{1}{2}\left[\mathbf{r}\times\int_V (\mathbf{r}'\times\mathbf{j})d^3r'\right]_l . \quad (5.89)$$

As a result, Eq. (84) may be rewritten as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3}, \quad (5.90)$$

Magnetic dipole and its potential

where the vector **m**, defined as[39]

$$\mathbf{m} \equiv \frac{1}{2}\int_V \mathbf{r}\times\mathbf{j}(\mathbf{r})\, d^3r, \quad (5.91)$$

is called the *magnetic dipole moment* of a field source – which itself, within the long-range approximation (90), is called the *magnetic dipole*.

Note a close analogy between the **m** defined by Eq. (91), and the orbital[40] angular momentum of a non-relativistic particle with mass m_k:

$$\mathbf{L}_k \equiv \mathbf{r}_k \times \mathbf{p}_k = \mathbf{r}_k \times m_k \mathbf{v}_k, \quad (5.92)$$

where $\mathbf{p}_k = m_k\mathbf{v}_k$ is its linear momentum. Indeed, for a continuum of such particles with equal electric charges q, distributed with spatial density n, we have $\mathbf{j} = qn\mathbf{v}$, and Eq. (91) yields

$$\mathbf{m} = \int_V \frac{1}{2}\mathbf{r}\times\mathbf{j}d^3r = \int_V \frac{nq}{2}\mathbf{r}\times\mathbf{v}\,d^3r, \quad (5.93)$$

while the total angular momentum of such a system of particles of equal masses m_0, is

$$\mathbf{L} = \int_V nm_0\mathbf{r}\times\mathbf{v}d^3r,$$

so we get a very straightforward relation

$$\mathbf{m} = \frac{q}{2m_0}\mathbf{L}. \quad (5.95)$$

m vs. **L**

For the orbital motion, this classical relation survives in quantum mechanics for linear operators, and hence for eigenvalues of the observables. Since the orbital angular momentum is quantized in the units of the Planck constant $\hbar$, the orbital magnetic moment of an electron is always a multiple of the so-called *Bohr magneton*

$$\mu_B \equiv \frac{e\hbar}{2m_e}, \quad (5.96)$$

Bohr magneton

where m_e is the free electron mass.[41] However, for particles with spin, such a universal relation between the vectors **m** and **L** is no longer valid. For example, the electron's spin $s = ½$ gives a contribution of $\hbar/2$ to its mechanical angular momentum, but a contribution very close to μ_B to its magnetic moment.

[39] In the Gaussian units, the definition (91) is kept valid "as is", so Eq. (90) is stripped of the factor $\mu_0/4\pi$.

[40] This adjective is used, especially in quantum mechanics, to distinguish the motion of a particle as a whole (not necessarily along a closed orbit!) from its intrinsic angular momentum, the spin – see, e.g., QM Chapters 3-6.

[41] In the SI units, $m_e \approx 0.91\times10^{-30}$ kg, so $\mu_B \approx 0.93\times10^{-23}$ J/T.

The next important example of a magnetic dipole is a *planar* thin-wire loop, limiting area A (of arbitrary shape), and carrying current I, for which $\mathbf{m}$ has a surprisingly simple form,

$$\mathbf{m} = I\mathbf{A}, \tag{5.97}$$

where the modulus of the vector $\mathbf{A}$ equals the loop's area A, and its direction is normal to the loop's plane. This formula may be readily proved by noticing that if we select the coordinate frame origin on the plane of the loop (Fig. 11), then the elementary component of the magnitude of the integral (91),

$$dm = \frac{1}{2}\left|\oint_C \mathbf{r}\times I\, d\mathbf{r}\right| \equiv I\oint_C \left|\frac{1}{2}\mathbf{r}\times d\mathbf{r}\right| = I\oint_C \frac{1}{2} r^2 d\varphi\,, \tag{5.98}$$

is just the elementary area $dA = (1/2)rd(r\varphi) = r^2 d\varphi/2$ – the equality already used in CM Eq. (3.40).

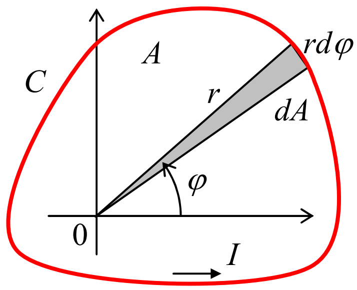

Fig. 5.11. Calculating the magnetic dipole moment of a planar current loop.

The comparison of Eqs. (96) and (97) allows a useful estimate of atomic currents, by finding what current I should flow in a circular loop of the atomic size scale (the Bohr radius) $r_B \sim 0.5\times10^{-10}$ m, i.e. of area $A \sim 10^{-20}$ m^2, to produce a magnetic moment of the order of μ_B.[42] The result is surprisingly macroscopic: $I \sim 1$ mA – quite comparable to the current driving the sound in your phone's earbuds. Though due to the quantum-mechanical spread of electron wavefunctions, this estimate should not be taken too literally, it is very useful for getting a gut feeling of how significant the atomic magnetism is, and hence why ferromagnets may provide such strong magnetic fields.

After these illustrations, let us return to the discussion of the general Eq. (90). Plugging it into (also general) Eq. (27), we may calculate the magnetic field of a magnetic dipole:[43]

[42] Another way to arrive at the same estimate is to take $I \sim ef = e\omega/2\pi$ with $\omega \sim 10^{16}$ s^{-1} being the typical frequency of radiation due to atomic interlevel quantum transitions.

[43] Similarly to the situation with the electric dipoles (see Eq. (3.24) and its discussion), it may be shown that the magnetic field of any closed current loop (or any system of such loops) satisfies the following equality:

$$\int_V \mathbf{B}(\mathbf{r})d^3r = (2/3)\mu_0\mathbf{m}\,,$$

where the integral is over any sphere confining all the currents. On the other hand, as we know from Sec. 3.1, for a field with the structure (99), derived from the long-range approximation (90), such an integral vanishes. As a result, to get a coarse-grain description of the magnetic field of a small system located at $r = 0$, that would give the correct average value of the magnetic field, Eq. (99) should be modified as follows:

$$\mathbf{B}_{cg}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{3\mathbf{r}(\mathbf{r}\cdot\mathbf{m}) - \mathbf{m}r^2}{r^5} + \frac{8\pi}{3}\mathbf{m}\delta(\mathbf{r})\right),$$

in a conceptual (though not quantitative) similarity to Eq. (3.25).

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3\mathbf{r}(\mathbf{r}\cdot\mathbf{m}) - \mathbf{m}r^2}{r^5}. \tag{5.99}$$

Magnetic dipole's field

The structure of this formula *exactly* replicates that of Eq. (3.13) for the electric dipole field – including the sign). Because of this similarity, the energy of a dipole of a fixed magnitude m in an external field, and hence the torque and the force exerted on it by a fixed external field, are given by expressions fully similar to those for an electric dipole – see Eqs. (3.15)-(3.19):[44]

Magnetic dipole in external field

$$U = -\mathbf{m}\cdot\mathbf{B}_{\text{ext}}, \tag{5.100}$$

and as a result,

$$\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}_{\text{ext}}, \tag{5.101}$$

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}_{\text{ext}}). \tag{5.102}$$

Now let us consider a system of many magnetic dipoles (e.g., atoms or molecules), distributed in space with an atomic-scale-averaged density n. Then we can use Eq. (90) generalized in an evident way for an arbitrary position $\mathbf{r}'$ of the dipole, and the linear superposition principle, to calculate the macroscopic vector potential $\mathbf{A}$:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|^3}d^3r', \tag{5.103}$$

where $\mathbf{M} \equiv n\mathbf{m}$ is the *magnetization*: the average magnetic moment per unit volume. Transforming this integral absolutely similarly to how Eq. (3.27) had been transformed into Eq. (3.29), we get:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}d^3r'. \tag{5.104}$$

Comparing this result with Eq. (28), we see that $\nabla\times\mathbf{M}$ is equivalent, in its magnetic effect, to the density $\mathbf{j}_{\text{ef}}$ of a certain effective "magnetization current". Just as the electric-polarization charge ρ_{ef} discussed in Sec. 3.2 (see Fig. 3.4), the vector $\mathbf{j}_{\text{ef}} = \nabla\times\mathbf{M}$ may be interpreted as the uncompensated part of the loop currents representing single magnetic dipoles $\mathbf{m}$ – see Fig. 12. Note, however, that since the atomic magnetic dipoles may be due to particles' spins, rather than the actual electric currents due to the orbital motion, the magnetization current's nature is not as direct as that of the polarization charge.

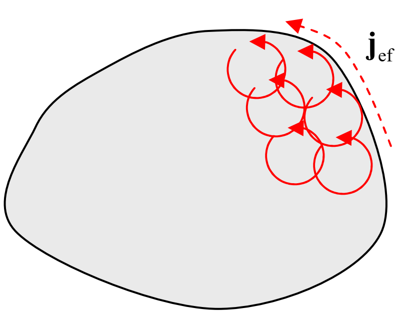

Fig. 5.12. A cartoon illustrating the physical nature of the effective magnetization current $\mathbf{j}_{\text{ef}} = \nabla\times\mathbf{M}$.

[44] Note that the fixation of m and $\mathbf{B}_{\text{ext}}$ effectively means that the currents producing them are fixed – please have one more look at Eqs. (35) and (97). As a result, Eq. (100) is a particular case of Eq. (53) rather than (54) – hence the minus sign.

Now, using Eq. (28) to add the possible contribution from the *stand-alone currents* **j** not included in the currents of microscopic magnetic dipoles, we get the general expression for the vector potential of the macroscopic field:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\left[\mathbf{j}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')\right]}{\left|\mathbf{r}-\mathbf{r}'\right|} d^3r' . \tag{5.105}$$

Repeating the calculations that have led us from Eq. (28) to the Maxwell equation (35), with the account of the magnetization current term, for the macroscopic magnetic field **B** we get

$$\nabla \times \mathbf{B} = \mu_0\left(\mathbf{j} + \nabla \times \mathbf{M}\right). \tag{5.106}$$

Following the same reasoning as in Sec. 3.2, we may recast this equation as

$$\nabla \times \mathbf{H} = \mathbf{j}, \tag{5.107}$$

where the field defined as

Magnetic field **H**

$$\mathbf{H} \equiv \frac{\mathbf{B}}{\mu_0} - \mathbf{M}, \tag{5.108}$$

for historic reasons (and very unfortunately) is also called the *magnetic field*.[45] This is why it is crucial to remember that the physical sense of field **H** is very much different from field **B**. To understand this difference better, let us use Eq. (107) to bring Eqs. (3.32), (3.36), (29), and (107) together, writing them as the following system of *macroscopic Maxwell equations* (again, so far for the stationary case $\partial/\partial t = 0$):[46]

Stationary macroscopic Maxwell equations

$$\begin{aligned} \nabla \times \mathbf{E} &= 0, \qquad & \nabla \times \mathbf{H} &= \mathbf{j}, \\ \nabla \cdot \mathbf{D} &= \rho, \qquad & \nabla \cdot \mathbf{B} &= 0. \end{aligned} \tag{5.109}$$

These equations clearly show that the roles of the vector fields **D** and **H** are very similar: they both may be called "would-be fields" – meaning the fields that *would be* induced by the stand-alone charges ρ and currents **j**, if the medium had not modified them by its dielectric and magnetic polarization.

Despite this similarity, let me note an important difference of signs in the relation (3.33) between **E, D**, and **P**, on one hand, and the relation (108) between **B**, **H**, and **M**, on the other hand. This is *not* just a matter of definition. Indeed, due to the similarity of Eqs. (3.15) and (100), including similar signs, the electric and magnetic fields both try to orient the corresponding dipole moments along the field. Hence, in the media that allow such an orientation (and as we will see momentarily, for magnetic media it is not always the case), the induced polarizations **P** and **M** are directed along, respectively, the vectors **E** and **B** of the genuine (though macroscopic, i.e. atomic-scale-averaged) fields. According to Eq. (3.33), if the would-be field **D** is fixed – say, by a fixed stand-alone charge distribution $\rho(\mathbf{r})$ – such a polarization *reduces* the electric field $\mathbf{E} = (\mathbf{D} - \mathbf{P})/\varepsilon_0$. On the other hand, Eq. (108) shows that in a magnetic media with a fixed would-be field **H**, the magnetic polarization making **M** parallel to **B**,

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[45] This confusion is exacerbated by the fact that in Gaussian units, Eq. (108) has the form $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$, and hence the fields **B** and **H** have the same dimensionality (and are formally equal in free space!) – though the unit of **H** has a different name (*oersted*, abbreviated as Oe). Mercifully, in the SI units, the dimensionality of **B** and **H** is different, with the unit of **H** called the *ampere per meter*.

[46] Let me remind the reader once again that in contrast with the system (36) of the Maxwell equations for the genuine (microscopic) fields, the right-hand sides of Eqs. (109) represent only the stand-alone charges and currents, not included in the microscopic electric and magnetic dipoles.

enhances the magnetic field $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$. This difference may be traced back to the sign difference in the basic relations (1) and (2), i.e. to the fundamental fact that the electric charges of the same sign repulse, while the currents of the same direction attract each other.

5.5. Magnetic materials

In order to form a complete system, sufficient for the calculation of all fields from given $\rho(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$, the macroscopic Maxwell equations (109) have to be complemented with the constitutive relations describing the medium: $\mathbf{D} \leftrightarrow \mathbf{E}$, $\mathbf{j} \leftrightarrow \mathbf{E}$, and $\mathbf{B} \leftrightarrow \mathbf{H}$. The first two of them were discussed, in brief, in the last two chapters; let us proceed to the last one.

A major difference between the dielectric and magnetic constitutive relations $\mathbf{D}(\mathbf{E})$ and $\mathbf{B}(\mathbf{H})$ is that while a dielectric medium always *reduces* the external field, magnetic media may *either reduce or enhance* it. To quantify this fact, let us consider the most common case – *linear magnetic materials* in that $\mathbf{M}$ (and hence $\mathbf{H}$) is proportional to $\mathbf{B}$. For isotropic materials, this proportionality is characterized by a scalar – either the *magnetic permeability* μ defined by the following relation:

$$\mathbf{B} \equiv \mu \mathbf{H}, \tag{5.110}$$

Magnetic permeability

or the *magnetic susceptibility*[47] defined as

$$\mathbf{M} = \chi_m \mathbf{H}. \tag{5.111}$$

Magnetic susceptibility

Plugging these relations into Eq. (108), we see that these two parameters are not independent, but are related as

$$\mu = (1 + \chi_m)\mu_0. \tag{5.112}$$

χ_m vs. μ

Note that despite the superficial similarity between Eqs. (110)-(112) and the corresponding relations (3.43)-(3.47) for linear dielectrics:

$$\mathbf{D} = \varepsilon \mathbf{E}, \qquad \mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}, \qquad \varepsilon = (1 + \chi_e)\varepsilon_0, \tag{5.113}$$

there is an important conceptual difference between them. Namely, while the vector $\mathbf{E}$ on the right-hand sides of Eqs. (113) is the actual (though macroscopic) electric field, the vector $\mathbf{H}$ on the right-hand side of Eqs. (110)-(111) represents a "would-be" magnetic field, in most aspects similar to $\mathbf{D}$ rather than $\mathbf{E}$ – see, for example, Eqs. (109). This historic difference in the traditional form of the constitutive relations for the electric and magnetic fields is not without its physical reasons. Most experiments with electric and magnetic materials are performed by placing their samples into nearly uniform electric and magnetic fields, and the simplest systems for their implementation are, respectively, plane capacitors (Fig. 2.3) and long solenoids (Fig. 6). The field in the former system may be most conveniently

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[47] According to Eqs. (110) and (112), i.e. in the SI units, χ_m is dimensionless, while μ has the same dimensionality as μ_0. In the Gaussian units, μ is dimensionless: $(\mu)_{\text{Gaussian}} = (\mu)_{\text{SI}}/\mu_0$, and χ_m is also introduced differently, as $\mu = 1 + 4\pi\chi_m$, Hence, just as for the electric susceptibilities, these dimensionless coefficients are different in the two systems: $(\chi_m)_{\text{SI}} = 4\pi(\chi_m)_{\text{Gaussian}}$. Note also that χ_m is formally called the *volumic* magnetic susceptibility, to distinguish it from the *atomic* (or "molecular") susceptibility χ defined by a similar relation, $\langle \mathbf{m} \rangle \equiv \chi \mathbf{H}$, where $\mathbf{m}$ is the induced magnetic moment of a single dipole – e.g., an atom. (χ is an analog of the electric atomic polarizability α – see Eq. (3.48) and its discussion.) In a dilute medium, i.e. in the absence of substantial dipole-dipole interactions, $\chi_m = n\chi$, where n is the dipole density.

controlled by fixing the voltage V between its plates, which is proportional to the electric field **E**. On the other hand, the field provided by the solenoid may be fixed by the current I in it, and according to Eq. (107), the field proportional to this stand-alone current is **H**, rather than **B**.[48]

Table 1 lists the approximate magnetic susceptibility values for several materials. It shows that in contrast to linear dielectrics whose susceptibility χ_e is always positive, i.e. the dielectric constant $\kappa = \chi_e + 1$ is always larger than 1 (see Table 3.1), linear magnetic materials may be either *paramagnets* (with $\chi_m > 0$, i. e. $\mu > \mu_0$) or *diamagnets* (with $\chi_m < 0$, i.e. $\mu < \mu_0$).

Table 5.1. Susceptibility $(\chi_m)_{SI}$ of a few representative and/or important magnetic materials[(a)]

"Mu-metal" (75% Ni + 15% Fe + a few %% of Cu and Mo)	~20,000[(b)]
Permalloy (80% Ni + 20% Fe)	~8,000[(b)]
"Electrical" (or "transformer") steel (Fe + a few %% of Si)	~4,000[(b)]
Nickel	~100
Aluminum	$+2\times10^{-5}$
Oxygen (at ambient conditions)	$+0.2\times10^{-5}$
Water	-0.9×10^{-5}
Diamond	-2×10^{-5}
Copper	-7×10^{-5}
Bismuth (the strongest non-superconducting diamagnet)	-17×10^{-5}

[(a)]The table does not include bulk superconductors, which may be described, in a so-called *coarse-scale approximation*, as perfect diamagnets (with **B** = 0, i.e. formally with $\chi_m = -1$ and $\mu = 0$), though the actual physics of this phenomenon is different – see Sec. 6.3 below.

[(b)] The exact values of $\chi_m >> 1$ for soft ferromagnetic materials (see, e.g., the upper three rows of the table) depend not only on their composition but also on their thermal processing ("annealing"). Moreover, due to unintentional vibrations, the extremely high values of χ_m of such materials may decay with time, though they may be restored to the original values by new annealing. The reason for such behavior is discussed in the text below.

The reason for this difference is that in dielectrics, two different polarization mechanisms (schematically illustrated by Fig. 3.7) lead to the same sign of the average polarization – see the discussion in Sec. 3.3. One of these mechanisms, illustrated by Fig. 3.7b, i.e. the ordering of spontaneous dipoles by the applied field, is also possible for magnetization – for the atoms and molecules with spontaneous internal magnetic dipoles of magnitude $m_0 \sim \mu_B$, due to their net spins. Again, in the absence of an external magnetic field the spins, and hence the dipole moments $\mathbf{m}_0$ may be disordered, but according to Eq. (100), the external magnetic field tends to align the dipoles along its direction. As a result, the average direction of the spontaneous elementary moments $\mathbf{m}_0$, and hence the direction of the arising magnetization **M**, is the same as that of the microscopic field **B** at the points of the dipole location (i.e., for a diluted media, of $\mathbf{H} \approx \mathbf{B}/\mu_0$), resulting in a positive susceptibility χ_m, i.e. in the paramagnetism, such as that of oxygen and aluminum – see Table 1.

[48] This fact also explains the misleading term "magnetic field" for **H**.

However, in contrast to the electric polarization of atoms/molecules with no spontaneous electric dipoles, which gives the same sign of $\chi_e \equiv \kappa - 1$ (see Fig. 3.7a and its discussion), the magnetic materials with no spontaneous atomic magnetic dipole moments have $\chi_m < 0$ – the effect called the *orbital* (or "Larmor"[49]) *diamagnetism*. As the simplest model of this effect, let us consider the orbital motion of an atomic electron about an atomic nucleus as that of a classical particle of mass m_0, with an electric charge q, about an immobile attracting center. As classical mechanics tells us, the central attractive force does not change the particle's angular momentum $\mathbf{L} \equiv m_0\mathbf{r}\times\mathbf{v}$, but the applied magnetic field $\mathbf{B}$ (that may be taken uniform on the atomic scale) does, due to the torque (101) it exerts on the magnetic moment (95):

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} = \mathbf{m}\times\mathbf{B} = \frac{q}{2m_0}\mathbf{L}\times\mathbf{B}. \tag{5.114}$$

The diagram in Fig. 13 shows that in the limit of a relatively weak field, when the magnitude of the angular momentum $\mathbf{L}$ may be considered constant, this equation describes the rotation (called the *torque-induced precession*[50]) of the vector $\mathbf{L}$ about the direction of the vector $\mathbf{B}$, with the angular frequency $\boldsymbol{\Omega} = -q\mathbf{B}/2m_0$, independent of the angle θ. According to Eqs. (91) and (114), the resulting additional (field-induced) magnetic moment $\Delta\mathbf{m} \propto q\boldsymbol{\Omega} \propto -q^2\mathbf{B}/m_0$ has, irrespectively of the sign of q, a direction *opposite* to the field. Hence, according to Eq. (111) with $\mathbf{H} \approx \mathbf{B}/\mu_0$, the susceptibility $\chi_m \propto \chi \equiv \Delta\mathbf{m}/\mathbf{H}$ is indeed negative. (Let me leave its quantitative estimate within this classical model for the reader's exercise.) The quantum-mechanical treatment confirms this qualitative picture of the Larmor diamagnetism, giving only quantitative corrections to the classical result for χ_m.[51]

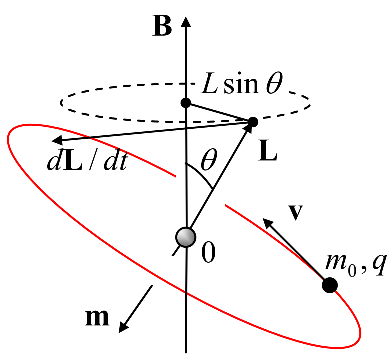

Fig. 5.13. The torque-induced precession of a classical charged particle in a magnetic field.

A simple estimate (also left for the reader's exercise) shows that in atoms with spontaneous non-zero net spins, the magnetic dipole orientation mechanism prevails over the orbital diamagnetism, so the materials incorporating such atoms usually exhibit net paramagnetism – see Table 1. Due to possible strong quantum interaction between the spin dipole moments, the magnetism of such materials is rather complex, with numerous interesting phenomena and elaborate theories. Unfortunately, all this physics is well outside the framework of this course, and I have to refer the interested reader to special literature,[52] but still will mention some key facts.

[49] Named after Sir Joseph Larmor who was the first (in 1897) to describe this effect mathematically.

[50] For a detailed discussion of this effect see, e.g., CM Sec. 4.5.

[51] See, e.g., QM Sec. 6.4. Quantum mechanics also explains why in most common (s-) ground states, the average contribution (95) of the orbital angular momentum $\mathbf{L}$ to the net vector $\mathbf{m}$ vanishes.

[52] See, e.g., D. J. Jiles, *Introduction to Magnetism and Magnetic Materials*, 2nd ed., CRC Press, 1998, or R. C. O'Handley, *Modern Magnetic Materials*, Wiley, 1999.

Most importantly, a sufficiently strong magnetic dipole-dipole interaction may lead to their spontaneous ordering, even in the absence of the applied field. This ordering may correspond to either parallel alignment of the dipoles (*ferromagnetism*) or anti-parallel alignment of the adjacent dipoles (*antiferromagnetism*). Evidently, the external effects of ferromagnetism are stronger, because this phase corresponds to a substantial spontaneous magnetization **M** even in the absence of an external magnetic field. (The corresponding magnitude of $\mathbf{B} = \mu_0\mathbf{M}$ is called the *remanence field*, B_R.) The direction of the vector $\mathbf{B}_R$ may be switched by the application of an external magnetic field, with a magnitude above a certain value H_C called *coercivity*, leading to the well-known hysteretic loops on the [H, B] plane (see Fig. 14 for a typical example) – similar to those in ferroelectrics, already discussed in Sec. 3.3.

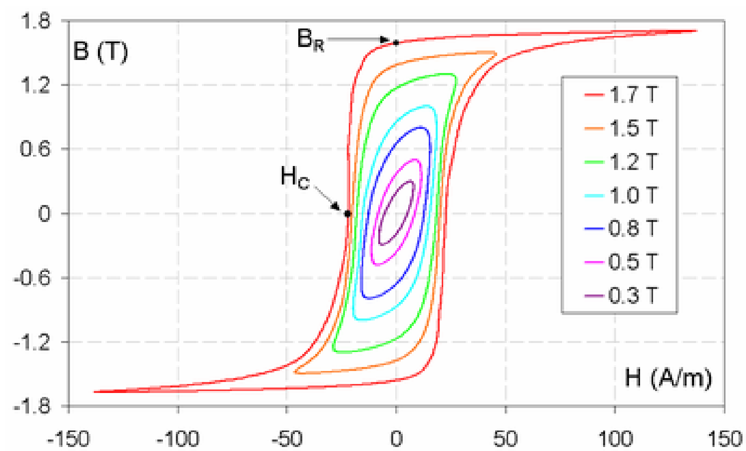

Fig. 5.14. Experimental magnetization curves of specially processed (cold-rolled) electrical steel – a solid solution of ~10% C and ~6% Si in Fe. (Reproduced from www.thefullwiki.org/Hysteresis under the Creative Commons BY-SA 3.0 license.)

Just as the ferroelectrics, the ferromagnets may also be *hard* or *soft* – in the magnetic rather than mechanical sense. In hard ferromagnets (also called *permanent magnets*), the dipole interaction is so strong that B stays close to B_R in all applied fields below H_C, so the hysteretic loops are virtually rectangular. Hence, in lower fields, the magnetization **M** of a permanent magnet may be considered constant, with the magnitude B_R/μ_0. Such hard ferromagnetic materials (notably, rare-earth compounds such as $SmCo_5$, Sm_2Co_{17}, and especially $Nd_2Fe_{14}B$), with high remanence fields (~1 T) and high coercivity (~10^6 A/m), have numerous practical applications.[53] Let me give just two, most important examples.

First, permanent magnets are the core components of most *electric motors*. By the way, this venerable (~150-years-old) technology is currently experiencing a quiet revolution, driven mostly by the electric car development. In the most advanced type of motors, called *permanent-magnet synchronous machines* (PMSM), the remanence magnetic field B_R of a permanent-magnet rotating part of the machine (called the *rotor*) interacts with the magnetic field of ac currents passed through wire windings in the external, static part of the motor (called the *stator*). The resulting torque may drive the rotor to extremely high speeds, exceeding 10,000 rotations per minute, enabling the motor to deliver several kilowatts of mechanical power from each kilogram of its mass.

As the second important example, despite the decades of the exponential (*Moore's-law*) progress of semiconductor electronics, most computer data storage systems (e.g., in data centers) are still based

[53] Currently, the neodymium-iron-boron compound holds nearly 95% percent of the world permanent-magnet application market, due to its combination of high B_R and H_C with lower fabrication costs.

on *hard disk drives* whose active media are submicron-thin layers of hard ferromagnets, with the data bits stored in the form of the direction of the remanent magnetization of small film spots. This technology has reached fantastic sophistication, with the recorded data density of the order of 10^{12} bits per square inch.[54] (Only recently it started to be seriously challenged by *solid-state drives* based on the floating-gate semiconductor memories already mentioned in Chapter 3.)[55]

In contrast, in soft ferromagnets, with their lower magnetic dipole interactions, the magnetization is constant only inside each of the spontaneously formed magnetic domains, while the volume and shape of the domains are affected by the applied magnetic field. As a result, the hysteresis loop's shape of soft ferromagnets is dependent on the cycled field's amplitude and cycling history – see Fig. 14. At high fields, their **B** (and hence **M**) is driven into saturation, with $B \approx B_R$, but at low fields, they behave essentially as linear magnetics with very high values of χ_m and hence μ – see the top rows of Table 1. (The magnetic domain interaction, and hence the low-field susceptibility of such soft ferromagnets are highly dependent on the material's fabrication technology and its post-fabrication thermal and mechanical treatments.) Due to these high values of μ, soft ferromagnets, especially iron and its alloys (e.g., various special steels), are extensively used in electrical engineering – for example in the cores of transformers – see the next section.

Due to the relative weakness of the magnetic dipole interaction in some materials, their ferromagnetic ordering may be destroyed by thermal fluctuations, if the temperature is increased above some value called the *Curie temperature* T_C, specific for each material. The transition between the ferromagnetic and paramagnetic phases at $T = T_C$ is a classical example of a *continuous phase transition*, with the average polarization **M** playing the role of the so-called *order parameter* that (in the absence of external fields) becomes different from zero only at $T < T_C$, increasing gradually at the further temperature reduction.[56]

5.6. Systems with magnetic materials

Just as the electrostatics of linear dielectrics, the magnetostatics is very simple in the particular case when all essential stand-alone currents are embedded into a linear magnetic medium with a constant permeability μ. Indeed, let us assume that we know the solution $\mathbf{B}_0(\mathbf{r})$ of the magnetic pair of

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[54] "*A magnetic head slider* [the read/write head – KKL] *flying over a disk surface with a flying height of 25 nm with a relative speed of 20 meters/second* [all realistic parameters – KKL] *is equivalent to an aircraft flying at a physical spacing of 0.2 µm at 900 kilometers/hour.*" B. Bhushan, as quoted in the (generally good) book by G. Hadjipanayis, *Magnetic Storage Systems Beyond 2000*, Springer, 2001.

[55] The high-frequency properties of hard ferromagnets are also very non-trivial. For example, according to Eq. (101), an external magnetic field $\mathbf{B}_{ext}$ exerts torque $\boldsymbol{\tau} = \mathbf{M}\times\mathbf{B}_{ext}$ on the spontaneous magnetic moment **M** of a unit volume of a ferromagnet. In some nearly-isotropic, mechanically fixed ferromagnetic samples, this torque causes the precession, around the direction of $\mathbf{B}_{ext}$ (very similar to that illustrated in Fig. 13), of not the sample as such, but of the magnetization **M** inside it, with a certain frequency ω_r. If the frequency ω of an additional ac field becomes very close to ω_r, its absorption sharply increases – the so-called *ferromagnetic resonance*. Moreover, if ω is somewhat higher than ω_r, the effective magnetic permeability $\mu(\omega)$ of the material for the ac field may become *negative*, enabling a series of interesting effects and practical applications. Very unfortunately, I could not find time for their discussion in this series and have to refer the interested reader to literature, for example the monograph by A. Gurevich and G. Melkov, *Magnetization Oscillations and Waves*, CRC Press, 1996.

[56] In this series, a quantitative discussion of such transitions is given in SM Chapter 4.

the genuine ("microscopic") Maxwell equations (36) in free space, i.e. when the genuine current density **j** coincides with that of stand-alone currents. Then the macroscopic Maxwell equations (109) and the linear constitutive equation (110) are satisfied with the pair of functions

$$\mathbf{H}(\mathbf{r}) = \frac{\mathbf{B}_0(\mathbf{r})}{\mu_0}, \qquad \mathbf{B}(\mathbf{r}) = \mu\mathbf{H}(\mathbf{r}) = \frac{\mu}{\mu_0}\mathbf{B}_0(\mathbf{r}). \tag{5.115}$$

Hence the only effect of the complete filling of a fixed-current system with a uniform, linear magnetic medium is the change of the magnetic field **B** at all points by the same constant factor $\mu/\mu_0 \equiv 1 + \chi_m$, which may be either larger or smaller than 1. (As a reminder, a similar filling of a system of fixed stand-alone charges with a uniform, linear dielectric always leads to a reduction of the electric field **E** by a factor of $\varepsilon/\varepsilon_0 \equiv 1 + \chi_e$ – the difference whose physics was already discussed at the end of Sec. 4.)

However, this simple result is generally invalid in the case of nonuniform (or piecewise-uniform) magnetic samples. To analyze this case, let us first integrate the macroscopic Maxwell equation (107) along a closed contour C limiting a smooth surface S. Now using the Stokes theorem just as at the derivation of Eq. (37), we get the macroscopic version of the Ampère law (37):

Macroscopic Ampère law

$$\oint_C \mathbf{H}\cdot d\mathbf{r} = I. \tag{5.116}$$

Let us apply this relation to a sharp boundary between two regions with different magnetic materials, with no stand-alone currents on the interface, similarly to how this was done for the field **E** in Sec. 3.4 – see Fig. 3.5. The result is similar as well:

$$H_\tau = \text{const}. \tag{5.117}$$

On the other hand, the integration of the Maxwell equation (29) over a Gaussian pillbox enclosing a border fragment (again just as shown in Fig. 3.5 for the field **D**) yields a result similar to Eq. (3.35):

$$B_n = \text{const}. \tag{5.118}$$

For linear magnetic media, with $\mathbf{B} = \mu\mathbf{H}$, the latter boundary condition is reduced to

$$\mu H_n = \text{const}. \tag{5.119}$$

Let us use these boundary conditions, first of all, to see what happens with a long cylindrical sample of a uniform magnetic material, placed parallel to a uniform external magnetic field $\mathbf{B}_0$ – see Fig. 15. Such a sample cannot noticeably disturb the field in the free space outside it, at most of its length: $\mathbf{B}_{ext} = \mathbf{B}_0$, $\mathbf{H}_{ext} = \mu_0\mathbf{B}_{ext} = \mu_0\mathbf{B}_0$. Now applying Eq. (117) to the dominating surfaces of the sample, we get $\mathbf{H}_{int} = \mathbf{H}_0$.[57] For a linear magnetic material, these relations yield $\mathbf{B}_{int} = \mu\mathbf{H}_{int} = (\mu/\mu_0)\,\mathbf{B}_0$.[58] For the high-$\mu$ media, this means that $B_{int} >> B_0$. This effect may be vividly represented as the concentration of the magnetic field lines in high-μ samples – see Fig. 15 again. (The concentration affects the external field

[57] The independence of **H** on magnetic properties of the sample in this geometry explains why this field's magnitude is commonly used as the argument in the plots like Fig. 14: such measurements are typically carried out by placing an elongated sample of the material under study into a long solenoid with a controllable current I, so according to Eq. (116), $H_0 = nI$, regardless of the sample.

[58] The reader is highly encouraged to carry out a similar analysis of the fields inside narrow gaps cut in a linear magnetic material, similar to that carried in Sec. 3.3 out for linear dielectrics – see Fig. 3.6 and its discussion.

distribution only at distances of the order of $(\mu/\mu_0)\, t << l$ near the sample's ends.) Such concentration is widely used in such practically important devices as transformers, in which two multi-turn coils are wound on a ring-shaped (e.g., toroidal, see Fig. 6b) core made of a soft ferromagnetic material (such as the *transformer steel*, see Table 1) with $\mu >> \mu_0$. This minimizes the number of "stray" field lines, and makes the magnetic flux Φ piercing each wire turn of either coil virtually the same – the equality important for the secondary voltage induction – see the next chapter.

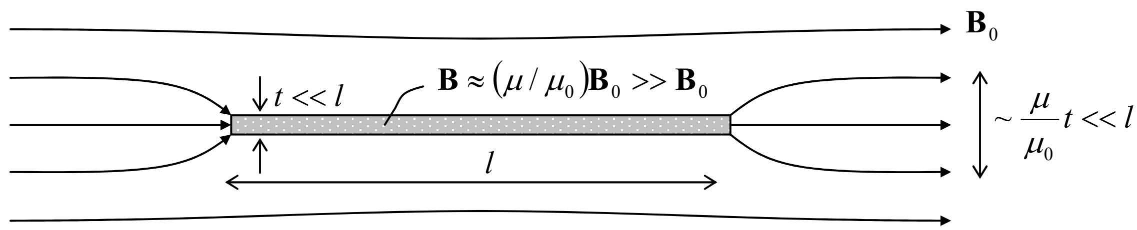

Fig. 5. 15. Magnetic field concentration in long, high-μ magnetic samples (schematically).

Samples of other geometries may create strong perturbations of the external field, extended to distances of the order of the sample's dimensions. To analyze such problems, we may benefit from a simple, partial differential equation for a scalar function, e.g., the Laplace equation, because in Chapter 2 we have learned how to solve it for many simple geometries. In magnetostatics, the introduction of a scalar potential is generally impossible due to the vortex-like magnetic field lines. However, if there are no stand-alone currents within the region we are interested in, then the macroscopic Maxwell equation (107) for the field **H** is reduced to $\nabla \times \mathbf{H} = 0$, similar to Eq. (1.28) for the electric field, showing that we may introduce the scalar potential of the magnetic field, ϕ_m, using a relation similar to Eq. (1.33):

$$\mathbf{H} = -\nabla \phi_{\mathrm{m}}. \tag{5.120}$$

Combining it with the homogenous Maxwell equation (29) for the magnetic field, $\nabla\cdot\mathbf{B} = 0$, and Eq. (110) for a linear magnetic material, we arrive at a single differential equation, $\nabla\cdot(\mu\nabla\phi_m) = 0$. For a uniform medium ($\mu(\mathbf{r})$ = const), it is reduced to our beloved Laplace equation:

$$\nabla^2 \phi_{\mathrm{m}} = 0. \tag{5.121}$$

Moreover, Eqs. (117) and (119) give us very familiar boundary conditions: the first of them

$$\frac{\partial \phi_{\mathrm{m}}}{\partial \tau} = \text{const}, \tag{5.122a}$$

being equivalent to

$$\phi_{\mathrm{m}} = \text{const}, \tag{5.122b}$$

while the second one giving

$$\mu \frac{\partial \phi_{\mathrm{m}}}{\partial n} = \text{const}. \tag{5.123}$$

Indeed, these boundary conditions are absolutely similar for (3.37) and (3.56) of electrostatics, with the replacement $\varepsilon \to \mu$.[59]

[59] This similarity may seem strange because earlier we have seen that the parameter μ is physically more similar to $1/\varepsilon$. The reason for this paradox is that in magnetostatics, the magnetic potential ϕ_m is traditionally used to

Let us analyze the geometric effects on magnetization, first using the (too?) familiar structure: a sphere, made of a linear magnetic material, placed into a uniform external field $\mathbf{H}_0 \equiv \mathbf{B}_0/\mu_0$. Since the differential equation and the boundary conditions are similar to those of the corresponding electrostatics problem (see Fig. 3.11 and its discussion), we can use the above analogy to reuse the solution we already have – see Eqs. (3.63). Just as in the electric case, the field outside the sphere, with

$$\left(\phi_\text{m}\right)_{r>R} = H_0\left(-r + \frac{\mu-\mu_0}{\mu+2\mu_0}\frac{R^3}{r^2}\right)\cos\theta, \tag{5.124}$$

is a sum of the uniform external field $\mathbf{H}_0$, with the potential $-H_0 r\cos\theta \equiv -H_0 z$, and the dipole field (99) with the following induced magnetic dipole moment of the sphere:[60]

$$\mathbf{m} = 4\pi\frac{\mu-\mu_0}{\mu+2\mu_0}R^3\mathbf{H}_0. \tag{5.125}$$

On the contrary, the internal field is perfectly uniform, and directed along the external one:

$$\left(\phi_\text{m}\right)_{r<R} = -H_0\frac{3\mu_0}{\mu+2\mu_0}r\cos\theta, \quad \text{so that} \quad \frac{H_\text{int}}{H_0} = \frac{3\mu_0}{\mu+2\mu_0}, \qquad \frac{B_\text{int}}{B_0} = \frac{\mu H_\text{int}}{\mu_0 H_0} = \frac{3\mu}{\mu+2\mu_0}. \tag{5.126}$$

Note that the field $\mathbf{H}_\text{int}$ inside the sphere is *not* equal to the applied external field $\mathbf{H}_0$. This example shows that the interpretation of $\mathbf{H}$ as the "would-be" magnetic field generated by external stand-alone currents $\mathbf{j}$ should not be exaggerated by saying that its distribution is independent of the magnetic bodies in the system. In the limit $\mu >> \mu_0$, Eqs. (126) yield $H_\text{int}/H_0 << 1$, $B_\text{int}/H_0 = 3\mu_0$, the factor 3 being specific for the particular geometry of the sphere. If a sample is strongly stretched along the applied field, with its length l much larger than the scale t of its cross-section, this geometric effect is gradually decreased, and B_int tends to its value $\mu H_0 >> B_0$, as was discussed above – see Fig. 15.

Now let us calculate the field distribution in a similar, but slightly more complex (and practically important) system: a round cylindrical shell, made of a linear magnetic material, placed into a uniform external field $\mathbf{H}_0$ normal to its axis – see Fig. 16.

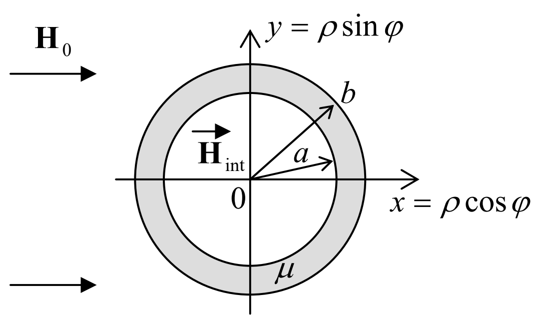

Fig. 5.16. Cylindrical magnetic shield.

describe the "would-be field" $\mathbf{H}$, while in electrostatics, the potential ϕ describes the actual electric field $\mathbf{E}$. (This tradition persists from the days when $\mathbf{H}$ was perceived as a genuine magnetic field.)

[60] To derive Eq. (125), we may either calculate the gradient of the ϕ_m given by Eq. (124), or use the similarity of Eqs. (3.13) and (99), to derive from Eq. (3.17) a similar expression for the magnetic dipole's potential:

$$\phi_\text{m} = \frac{1}{4\pi}\frac{m\cos\theta}{r^2}.$$

Now comparing this formula with the second term of Eq. (124), we immediately get Eq. (125).

Since there are no stand-alone currents in the region of our interest, we can again represent the field **H**(**r**) by the gradient of the magnetic potential ϕ_m – see Eq. (120). Inside each of three constant-μ regions, i.e. at $\rho < b$, $a < \rho < b$, and $b < \rho$ (where ρ is the 2D distance from the cylinder's axis), the potential obeys the Laplace equation (121). In the convenient, polar coordinates (see Fig. 16), we may, guided by the general solution (2.112) of the Laplace equation and our experience in its application to axially-symmetric geometries, look for ϕ_m in the following form:

$$\phi_m = \begin{cases} \left(-H_0\rho + b_1'/\rho\right)\cos\varphi, & \text{for } b \leq \rho, \\ \left(a_1\rho + b_1/\rho\right)\cos\varphi, & \text{for } a \leq \rho \leq b, \\ -H_{\text{int}}\rho\cos\varphi, & \text{for } \rho \leq a\,. \end{cases} \tag{5.127}$$

Plugging this solution into the boundary conditions (122)-(123) at both interfaces ($\rho = b$ and $\rho = a$), we get the following system of four equations:

$$\begin{aligned} &-H_0 b + b_1'/b = a_1 b + b_1/b, && \left(a_1 a + b_1/a\right) = -H_{\text{int}}a, \\ &\mu_0\left(-H_0 - b_1'/b^2\right)H_0 = \mu\left(a_1 - b_1/b^2\right), && \mu\left(a_1 - b_1/a^2\right) = -\mu_0 H_{\text{int}}, \end{aligned} \tag{5.128}$$

for four unknown coefficients a_1, b_1, b_1', and H_{int}. Solving the system, we get, in particular:

$$\frac{H_{\text{int}}}{H_0} = \frac{\alpha_c - 1}{\alpha_c - (a/b)^2}, \qquad \text{with } \alpha_c \equiv \left(\frac{\mu + \mu_0}{\mu - \mu_0}\right)^2. \tag{5.129}$$

According to these formulas, at $\mu > \mu_0$, the field in the free space inside the cylinder is lower than the external field. This fact allows using such structures, made of high-μ materials such as permalloy (see Table 1), for passive shielding[61] from unintentional magnetic fields (e.g., the Earth's field) – the task very important for the design of many physical experiments. As Eq. (129) shows, the larger is μ, the closer is α_c to 1, and the smaller is the ratio H_{int}/H_0, i.e. the better is the shielding, for the same a/b ratio. On the other hand, for a given magnetic material, i.e. for a fixed parameter α_c, the shielding is improved by making the ratio $a/b < 1$ smaller, i.e. the shield thicker. On the other hand, as Fig. 16 shows, smaller a leaves less space for the shielded samples, calling for a compromise.

Note that in the limit $\mu/\mu_0 \to \infty$, both Eq. (126) and Eq. (129), describing different geometries, yield $H_{\text{int}}/H_0 \to 0$. Indeed, as it follows from Eq. (119), in this limit, the field **H** tends to zero inside magnetic samples of virtually any geometry. (The formal exception is the longitudinal cylindrical geometry shown in Fig. 15, with $t/l \to 0$, where $\mathbf{H}_{\text{int}} = \mathbf{H}_0$ for any finite μ, but even in it, the last equality holds only if $t/l \ll \mu_0/\mu$.)

Now let us discuss a curious (and practically important) approach to systems with relatively thin, closed magnetic cores made of several sections of high-μ magnetic materials, with the cross-section areas A_k much smaller than the squared lengths l_k of the sections – see Fig. 17. If all $\mu_k \gg \mu_0$, virtually all field lines are confined to the interior of the core. Then, applying the macroscopic Ampère law (116) to a contour C that follows a magnetic field line inside the core (see, for example, the dashed line in Fig. 17), we get the following approximate expression (exactly valid only in the limit μ_k/μ_0, $l_k^2/A_k \to \infty$):

[61] Another approach to the undesirable magnetic fields' reduction is the "active shielding" – the external field's compensation with the counter-field induced by controlled currents in specially designed wire coils.

$$\oint_C H_l dl \approx \sum_k l_k H_k \equiv \sum_k l_k \frac{B_k}{\mu_k} = NI \,. \qquad (5.130)$$

However, since the magnetic field lines stay in the core, the magnetic flux $\Phi_k \approx B_k A_k$ should be the same ($\equiv \Phi$) for each section, so $B_k = \Phi/A_k$. Plugging this condition into Eq. (130), we get

Magnetic Ohm law and reluctance

$$\Phi = \frac{NI}{\sum_k \mathscr{R}_k}, \qquad \text{where } \mathscr{R}_k \equiv \frac{l_k}{\mu_k A_k} \,. \qquad (5.131)$$

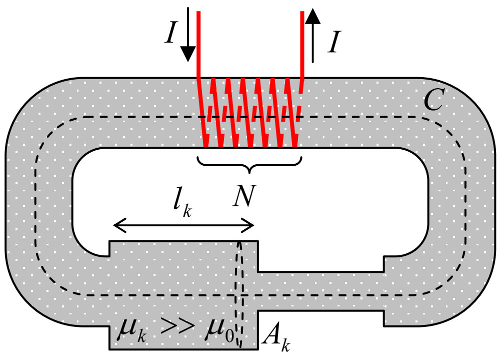

Fig. 5.17. Deriving the "magnetic Ohm law" (131).

Note a close analogy of the first of these equations with the usual Ohm law for several resistors connected in series, with the magnetic flux playing the role of electric current, while the product NI, the role of the voltage applied to the chain of resistors. This analogy is fortified by the fact that the second of Eqs. (131) is similar to the expression for the resistance $R = l/\sigma A$ of a long, uniform conductor, with the magnetic permeability μ playing the role of the electric conductivity σ. (To sound similar, but still different from the resistance R, the parameter $\mathscr{R}$ is called *reluctance*.) This is why Eq. (131) is called the *magnetic Ohm law*; it is very useful for approximate analyses of systems like ac transformers, magnetic energy storage systems, etc.

Now let me proceed to a brief discussion of systems with permanent magnets. First of all, using the definition (108) of the field **H**, we may rewrite the Maxwell equation (29) for the field **B** as

$$\nabla \cdot \mathbf{B} \equiv \mu_0 \nabla \cdot (\mathbf{H} + \mathbf{M}) = 0, \qquad \text{i.e. as } \nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M} \,, \qquad (5.132)$$

While this relation is general, it is especially convenient in permanent magnets, where the magnetization vector **M** may be approximately considered field-independent.[62] In this case, Eq. (132) for **H** is an exact analog of Eq. (1.27) for **E**, with the fixed term $-\nabla\cdot\mathbf{M}$ playing the role of the fixed charge density (more exactly, of ρ/ε_0). For the scalar potential ϕ_m, defined by Eq. (120), this gives the Poisson equation

$$\nabla^2 \phi_m = \nabla \cdot \mathbf{M} \,, \qquad (5.133)$$

similar to those solved, for quite a few geometries, in the previous chapters.

In the case when **M** is not only field-independent but also uniform inside a permanent magnet's volume, then the right-hand sides of Eqs. (132) and (133) vanish both inside the volume and in the

[62] Note that in this approximation, there is no difference between the *remanence magnetization* $M_R \equiv B_R/\mu_0$, of the magnet and its *saturation magnetization* $M_S \equiv \lim_{H\to\infty}[B(H)/\mu_0 - H]$.

surrounding free space, and give a non-zero effective charge only on the magnet's surface. Integrating Eq. (132) along a short path normal to the surface and crossing it, we get the following boundary conditions:

$$\Delta H_n \equiv \left(H_n\right)_{\text{in free space}} - \left(H_n\right)_{\text{in magnet}} = M_n \equiv M\cos\theta, \tag{5.134}$$

where θ is the angle between the magnetization vector and the outer normal to the magnet's surface. This relation is an exact analog of Eq. (1.24) for the normal component of the field **E**, with the effective surface charge density (or rather σ/ε_0) equal to $M\cos\theta$.

This analogy between the magnetic field induced by a fixed, constant magnetization and the electric field induced by surface electric charges enables one to reuse the solutions of quite a few problems considered in Chapters 1-3. Leaving a few such problems for the reader's exercise (see Sec. 7), let me demonstrate the power of this analogy on just two examples specific to magnetic systems. First, let us calculate the force necessary to detach the flat ends of two long, uniform rod magnets, of length l and cross-section area $A << l^2$, with the saturated remanent magnetization $\mathbf{M}_0$ directed along their length – see Fig. 18.

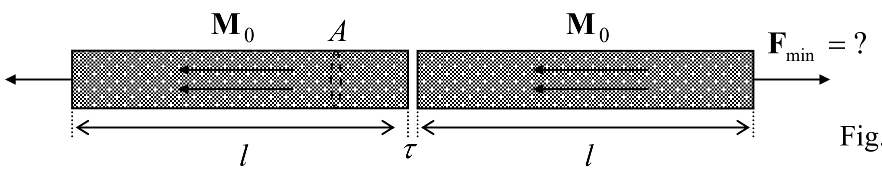

Fig. 5.18. Detaching two magnets.

Let us assume we have succeeded in separating the magnets by an infinitesimal distance $\tau << A^{1/2}$, l. Then, according to Eqs. (133)-(134), the distribution of the magnetic field near this small gap should be similar to that of the electric field in a system of two equal by opposite surface charges with the surface density σ proportional to M_0. From Chapters 1-3, we know the properties of such a system very well: within the gap, the field is virtually constant, uniform, proportional to σ, and independent of τ. For its magnitude in the magnetic case, Eq. (134) gives simply $H = M_0$, and hence $B = \mu_0 M_0$. (Just outside of the gap, the field is very low, because due to the condition $A << l^2$, the effect of the similar effective charges at the "outer" ends of the rods on the field near the gap t is negligible.)

From here, we can readily calculate $F_{\min}$ as the force exerted by this field on the effective surface "charges". However, it is even easier to find it from the following energy argument. Since the magnetic field energy localized inside the magnets and near their outer ends cannot depend on τ, this small detachment may only alter the energy inside the gap. For this part of the energy, Eq. (57) yields:

$$\Delta U = \frac{B^2}{2\mu_0} V = \frac{\left(\mu_0 M_0\right)^2}{2\mu_0} A\tau . \tag{5.135}$$

The gradient of this potential energy is equal to the attraction force $\mathbf{F} = -\nabla(\Delta U)$, trying to reduce ΔU by decreasing the gap, with the following magnitude:

$$\left|F\right| = \frac{\partial\left(\Delta U\right)}{\partial\tau} = \frac{\mu_0 M_0^2 A}{2} . \tag{5.136}$$

The magnet detachment requires an equal and opposite external force. For a typical permanent magnet, with $\mu_0 M_0 \approx B_R \sim 1$T, the force corresponds to a ratio $|F|/A$ close to 4×10^5 Pa, a few times the normal atmospheric pressure.

Now let us consider the situation when similar long permanent magnets (such as the *magnetic needles* used in magnetic compasses) are separated, in otherwise free space, by a larger distance $d >> A^{1/2}$ – see Fig. 19. For each needle (Fig. 19a), of a length $l >> A^{1/2}$, the right-hand side of Eq. (133) is substantially different from zero only in two relatively small areas at the needle's ends. Integrating the equation over each area, we see that at distances $r >> A^{1/2}$ from each end, we may reduce Eq. (132) to

$$\nabla \cdot \mathbf{H} = \frac{q_\mathrm{m}}{\mu_0}\left[\delta(\mathbf{r}-\mathbf{r}_+) - \delta(\mathbf{r}-\mathbf{r}_-)\right], \qquad \text{i.e. } \nabla \cdot \mathbf{B} = q_\mathrm{m}\left[\delta(\mathbf{r}-\mathbf{r}_+) - \delta(\mathbf{r}-\mathbf{r}_-)\right], \tag{5.137}$$

where $\mathbf{r}_\pm$ are the ends' positions, and $q_\mathrm{m} \equiv \mu_0 M_0 A$, with A being the needle's cross-section area.[63] This expression for $\mathbf{B}$ is completely similar to Eq. (3.32) for the electric displacement $\mathbf{D}$, for the particular case of two equal and opposite point charges, i.e. with $\rho = q[\delta(\mathbf{r} - \mathbf{r}_+) - \delta(\mathbf{r} - \mathbf{r}_+)]$, with the only replacement $q \to q_\mathrm{m}$. Since we know the resulting electric field all too well (see, e.g., Eq. (1.7) for $\mathbf{E} \equiv \mathbf{D}/\varepsilon_0$), we may immediately write a similar expression for the field $\mathbf{H}$:

$$\mathbf{H}(\mathbf{r}) = \frac{q_\mathrm{m}}{4\pi\mu_0}\left(\frac{\mathbf{r}-\mathbf{r}_+}{\left|\mathbf{r}-\mathbf{r}_+\right|^3} - \frac{\mathbf{r}-\mathbf{r}_-}{\left|\mathbf{r}-\mathbf{r}_-\right|^3}\right). \tag{5.138}$$

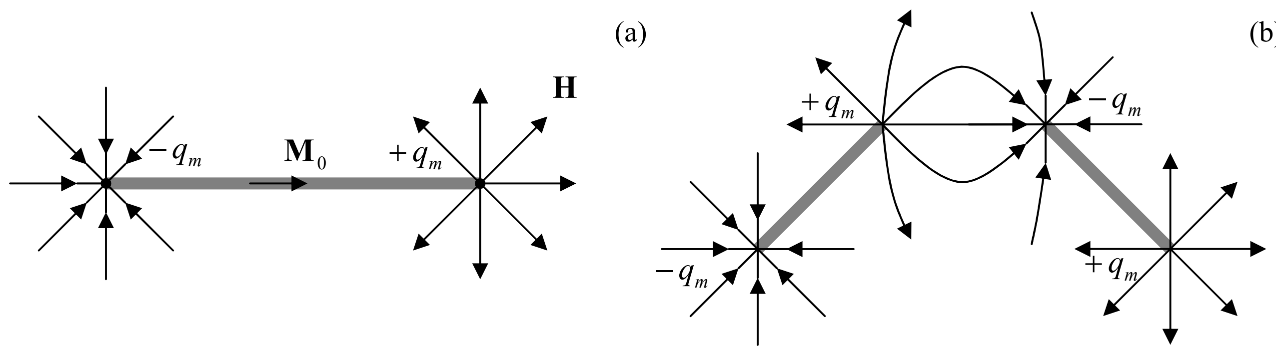

Fig. 5.19. (a) "Magnetic charges" at the ends of a thin permanent-magnet needle and (b) the result of its breaking into two parts (schematically).

The resulting magnetic field $\mathbf{H}(\mathbf{r})$ exerts on another "magnetic charge" q'_m, located at some point $\mathbf{r}'$, the force $\mathbf{F} = q'_\mathrm{m}\mathbf{H}(\mathbf{r}')$.[64] Hence if two ends of different needles are separated by an intermediate distance R ($A^{1/2} << R << l$, see Fig. 19b), we may neglect one term in Eq. (138), and get the following "magnetic Coulomb law" for the interaction of the nearest ends:

$$\mathbf{F} = \pm\frac{q_\mathrm{m} q'_\mathrm{m}}{4\pi\mu_0}\frac{\mathbf{R}}{R^3}. \tag{5.139}$$

The "only" (but conceptually, crucial!) difference between this interaction and that of the electric point charges is that the two "magnetic charges" (quasi-monopoles) of a magnetic needle cannot be fully separated. For example, if we break a needle in the middle in an attempt to bring its two ends further apart, two new "point charges" appear – see Fig. 19b.

[63] Note that the constant coefficient in the definition of q_m, and hence in Eqs. (138)-(139), is the matter of convention. The above choice makes the free-space Maxwell equations $\nabla\cdot\mathbf{D} = \rho$ and $\nabla\cdot\mathbf{B} = \rho_\mathrm{m}$ (where ρ and ρ_m are the volumic densities of the electric and magnetic charges) pleasantly symmetric.

[64] This expression is the magnetic analog of the basic equation $\mathbf{F} = q'_\mathrm{e}\mathbf{E}(\mathbf{r}')$ for the electric charges.

There are several solid-state systems where more flexible structures, similar in their magnetostatics to the needles, may be implemented. First of all, certain ("type-II") superconductors may carry so-called *Abrikosov vortices* – flexible tubes with field-suppressed superconductivity inside, each carrying one quantum $\Phi_0 = \pi\hbar/e \approx 2\times10^{-15}$ Wb of the magnetic flux. Ending on superconductor's surfaces, these tubes let their magnetic field lines spread into the surrounding free space, essentially forming magnetic monopole analogs – of course, with equal and opposite "magnetic charges" q_m on each end of the tube – just as Fig. 19a shows. Such flux tubes are not only flexible but also stretchable, resulting in several peculiar effects – see Sec. 6.4 for more detail. Another recently found example of such paired quasi-monopoles is *spin chains* in the so-called *spin ices* – crystals with paramagnetic ions arranged into a specific (pyrochlore) lattice – such as dysprosium titanate $Dy_2Ti_2O_7$.[65] Let me emphasize again that any reference to magnetic monopoles in such systems should not be taken literally.

In order to complete this section (and this chapter), let me briefly discuss the magnetic field energy U, for the simplest case of systems with linear magnetic materials. In this case, we still may use Eq. (55), but if we want to operate only with macroscopic fields, and hence only stand-alone currents, we should repeat the manipulations that have led us to Eq. (57), using **j** not from Eq. (35), but from Eq. (107). As a result, instead of Eq. (57) we get

$$U = \int_V u(\mathbf{r})d^3r, \quad \text{with } u = \frac{\mathbf{B}\cdot\mathbf{H}}{2} = \frac{B^2}{2\mu} = \frac{\mu H^2}{2}, \tag{5.140}$$

Magnetic field energy: linear medium

This result is evidently similar to Eq. (3.73) of electrostatics.

As a simple but important example of its application, let us again consider a long solenoid (Fig. 6a), but now filled with a linear magnetic material with permeability μ. Using the macroscopic Ampère law (116), just as we used Eq. (37) for the derivation of Eq. (40), we get

$$H = In, \quad \text{and hence } B = \mu In, \tag{5.141}$$

where $n \equiv N/l$, just as in Eq. (40), is the winding density, i.e. the number of wire turns per unit length. (At $\mu = \mu_0$, we immediately return to that old result.) Now we may plug Eq. (141) into Eq. (140) to calculate the magnetic energy stored in the solenoid:

$$U = uV = \frac{\mu H^2}{2} lA = \frac{\mu(nI)^2 lA}{2}, \tag{5.142}$$

and then use Eq. (72) to calculate its self-inductance:[66]

$$L = \frac{U}{I^2/2} = \mu n^2 lA \tag{5.143}$$

We see that $L \propto \mu V$, so filling a solenoid with a high-μ material may allow making it more compact while preserving the same value of inductance. In addition, as the discussion of Fig. 15 has shown, such filling reduces the fringe fields near the solenoid's ends, which may be detrimental for some applications, especially in physical experiments striving for high measurement precision.

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[65] See, e.g., L. Jaubert and P. Holdworth, *J. Phys. – Cond. Matt*. **23**, 164222 (2011), and references therein.

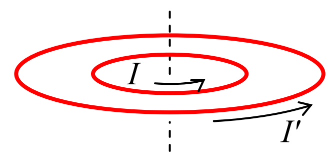
[66] Admittedly, we could get the same result simpler, just by arguing that since the magnetic material fills the whole volume of a substantial magnetic field in this system, the filling simply increases the vector **B** at all points, and hence its flux Φ, and hence $L \equiv \Phi/I$ by the factor μ/μ_0 in comparison with the free-space value (75).

However, we still need to explore the issue of magnetic energy beyond Eq. (140), not only to get a general expression for it in materials with an arbitrary dependence **B**(**H**), but also to finally prove Eq. (54) and explore its relation with Eq. (53). I will do this at the beginning of the next chapter.

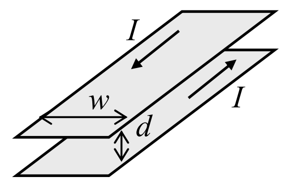
5.7. Exercise problems

5.1. DC current I flows around a thin wire loop bent into the form of a plane equilateral triangle with side a. Calculate the magnetic field in the center of the loop.

5.2. A circular wire loop, carrying a fixed dc current, is placed inside a similar but larger loop, carrying a fixed current in the same direction – see the figure on the right. Use semi-quantitative arguments to analyze the mechanical stability of the coaxial and coplanar position of the inner loop with respect to its possible angular, axial, and lateral displacements relative to the outer loop.



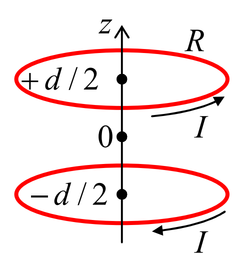
5.3. Two planar, parallel, long, thin conducting strips of width w, separated by distance d, carry equal but oppositely directed currents I – see the figure on the right. Calculate the magnetic field in the plane located in the middle between the strips, assuming that the flowing currents are uniformly distributed across the strip widths.



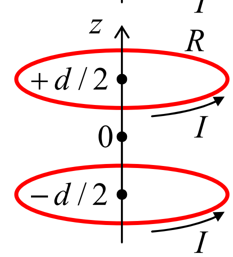
5.4. For the system studied in the previous problem, but now only in the limit $d << w$, calculate:

(i) the distribution of the magnetic field in space,
(ii) the vector potential of the field,
(iii) the magnetic force (per unit length) exerted on each strip, and
(iv) the magnetic energy and self-inductance of the loop formed by the strips (per unit length).

5.5. Calculate the magnetic field distribution near the center of the system of two similar, plane, round, coaxial wire coils, carrying equal but oppositely directed currents – see the figure on the right.

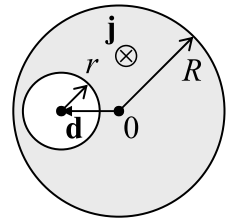


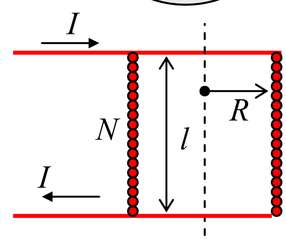
5.6. The two-coil-system considered in the previous problem now carries equal and *similarly* directed currents – see the figure on the right.[67] Calculate what should be the ratio d/R for the second derivative $\partial^2 B_z/\partial z^2$ to equal zero at $z = 0$.



[67] This *Helmholtz coils* system, producing a highly uniform field near its center, is broadly used in physical experiment.

5.7. DC current of a constant density j flows along a round cylindrical wire of radius R, with a round cylindrical cavity of radius r cut in it. The cavity's axis is parallel to that of the wire but offset from it by a distance $d < R - r$ (see the figure on the right). Calculate the magnetic field inside the cavity.





5.8. Calculate the magnetic field's distribution along the axis of a straight solenoid (see Fig. 6a, partly reproduced on the right) of a finite length l, and a round cross-section of radius R. Assume that the solenoid has many ($N >> 1, l/R$) wire turns, uniformly distributed along its length.

5.9. A thin round disk of radius R, carrying an electric charge of a constant areal density σ, rotates about its axis with a constant angular velocity ω. Calculate:

(i) the magnetic field on the disk's axis,
(ii) the magnetic moment of the disk,

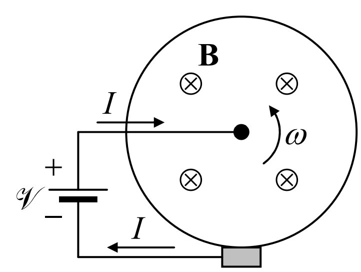
and relate these results.

5.10. A thin spherical shell of radius R, with charge Q uniformly distributed over its surface, rotates about its diameter with a constant angular velocity ω. Calculate the distribution of the magnetic field everywhere in space.

5.11. A sphere of radius R, made of an insulating material with a uniform electric charge density ρ, rotates about its diameter with a constant angular velocity ω. Calculate the magnetic field distribution inside the sphere and outside it.

5.12. A conducting sphere with no total electric charge is rotated about its diameter with a constant angular velocity ω, in a uniform constant external magnetic field **B** directed along the rotation axis. Assuming that the sphere's contribution to the magnetic field is negligibly small, calculate the stationary distribution of the electric charge density inside the sphere and on its surface, and the electrostatic potential both inside and outside the sphere. Quantify the above assumption.

5.13.* The simplest version of the famous *homopolar* (or "unipolar") *motor* is a thin round conducting disk, placed into a uniform magnetic field normal to its plane, with dc current passed between the disk's center and a sliding electrode ("brush") on its rim – see the figure on the right.

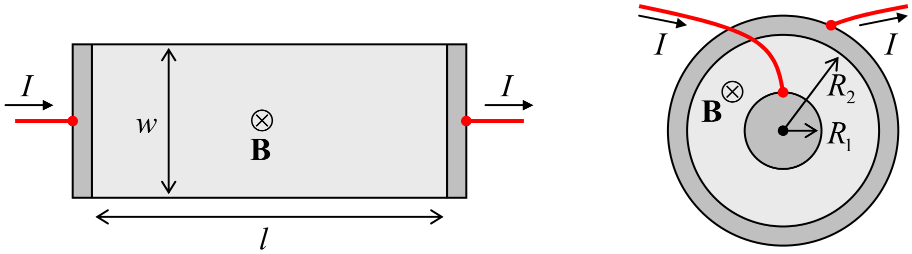


(i) Express the torque rotating the disk via its radius R, the magnetic field **B**, and the current I.

(ii) If the disk is allowed to rotate about its axis, and the motor is driven by a battery with e.m.f. $\mathscr{V}$, calculate its stationary angular velocity ω, neglecting friction and the electric circuit's resistance.

(iii) Now assuming that the current's path (battery + wires + contacts + disk itself) has a non-zero resistance $\mathscr{R}$, derive and solve the equation for the time evolution of ω, and analyze the solution.

5.14. The reader is hopefully familiar with the classical Hall effect in the usual rectangular *Hall bar* geometry – see the left panel of the figure below. However, the effect takes a different form in the so-called *Corbino disk* – see the right panel of the figure below. (Dark shading shows electrodes, with no appreciable resistance.) Analyze the effect in both geometries, assuming that in both cases, the conductors are thin and planar, have a constant Ohmic conductivity σ and charge carrier density n, and that the applied magnetic field **B** is uniform and normal to conductors' planes.



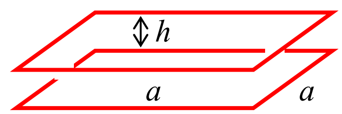
5.15. A wire with a round cross-section of radius a has been bent into a round loop of radius R >> a. Prove the formula for its self-inductance, which was mentioned at the end of Sec. 5.3 of the lecture notes: $L = \mu_0 R \ln(cR/a)$, with $c \sim 1$.

5.16. Prove that:

(i) the self-inductance L of a current loop cannot be negative, and

(ii) any inductance coefficient $L_{kk'}$, defined by Eq. (60), cannot be larger than $(L_{kk}L_{k'k'})^{1/2}$.

5.17. Calculate the mutual inductance of two similar thin-wire square-shaped loops offset by distance h in the direction normal to their planes – see the figure on the right.



5.18.* Estimate the values of magnetic susceptibility due to

(i) orbital diamagnetism, and

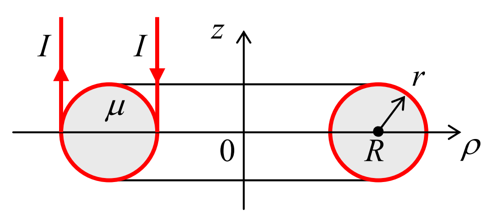
(ii) spin paramagnetism,

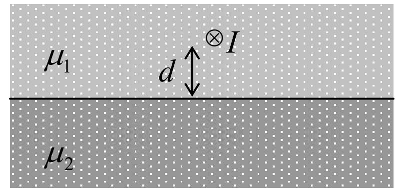
for a medium with negligible interactions between the induced molecular dipoles. Compare the results.

Hints: For Task (i), you may use the classical model described by Eq. (114) – see Fig. 13. For Task (ii), assume the ordering of spontaneous magnetic dipoles $\mathbf{m}_0$, with a fixed magnitude m_0 of the order of the Bohr magneton μ_B, similar to the one sketched for electric dipoles in Fig. 3.7a.

5.19.* Use the classical picture of the orbital ("Larmor") diamagnetism, discussed in Sec. 5, to calculate its (small) contribution $\Delta\mathbf{B}(0)$ to the magnetic field **B** felt by an atomic nucleus, treating the electrons of the atom as a spherically symmetric cloud with an electric charge density $\rho(r)$. Express the result via the value $\phi(0)$ of the electrostatic potential of the electron cloud, and use this expression for a crude numerical estimate of the ratio $\Delta B(0)/B$ for the hydrogen atom.

5.20. Calculate the self-inductance of a toroidal solenoid with a round cross-section of radius $r \sim R$, with N >> 1, R/r wire turns uniformly distributed along the perimeter, and filled with a linear magnetic material of permeability μ.



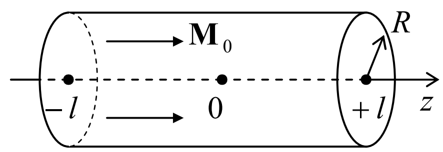


5.21. A long, straight, thin wire carrying current I runs parallel to the plane boundary between two uniform, linear magnetic media – see the figure on the right. Calculate the magnetic field everywhere in the system, and the force (per unit length) exerted on the wire.

5.22. Solve the magnetic shielding problem similar to that discussed in Sec. 5.6 of the lecture notes, but for a spherical rather than cylindrical shell, with the same central cross-section as shown in Fig. 16. Compare the efficiency of those two shields, for the same shell's permeability μ, and the same b/a ratio.

5.23. Calculate the magnetic field's distribution around a spherical permanent magnet with uniform magnetization $\mathbf{M}_0$ = const.

5.24. A limited volume V is filled with a magnetic material with field-independent magnetization $\mathbf{M}(\mathbf{r})$. Write explicit expressions for the magnetic field induced by the magnetization and its potential, and recast these expressions into the forms that are more convenient when $\mathbf{M}(\mathbf{r}) = \mathbf{M}_0$ = const throughout the volume.



5.25. Use the results of the previous problem to calculate the distribution of the magnetic field $\mathbf{H}$ along the axis of a straight permanent magnet of length $2l$ and a round cross-section of radius R, with a uniform magnetization $\mathbf{M}_0$ parallel to the axis – see the figure on the right.

5.26. A flat end of a long straight permanent magnet, similar to that considered in the previous problem but with an arbitrary cross-section of area A, is stuck to a flat surface of a large sample of a linear magnetic material with a very high permeability $\mu >> \mu_0$. Calculate the normally directed force needed to detach them.

5.27. A permanent magnet with a uniform magnetization $\mathbf{M}_0$ has the form of a spherical shell with an internal radius R_1 and an external radius $R_2 > R_1$. Calculate the magnetic field inside the shell.

5.28. A very broad film of thickness $2t$ is permanently magnetized normally to its plane, with a periodic checkerboard pattern, with the square of area $a \times a$:

$$\mathbf{M}\Big|_{|z|<t} = \mathbf{n}_z M(x,y), \qquad \text{with } M(x,y) = M_0 \,\mathrm{sgn}\left(\cos\frac{\pi x}{a}\ \cos\frac{\pi y}{a}\right).$$

Calculate the magnetic field's distribution in space.

5.29.* Based on the discussion of the quadrupole electrostatic lens in Sec. 2.4, suggest the permanent-magnet systems that may similarly focus particles moving close to the system's axis, for the cases when each particle carries:

(i) an electric charge,

(ii) no net electric charge, but a spontaneous magnetic dipole moment $\mathbf{m}$ of a certain orientation.