Chapter 1. Review of Fundamentals

After a brief discussion of the title and contents of the course, this introductory chapter reviews the basic notions and facts of the non-relativistic classical mechanics, that are supposed to be known to the reader from their undergraduate studies.[1] *Due to this reason, the discussion is very short.*

1.0. Terminology: Mechanics and dynamics

A more fair title for this course would be *Classical Mechanics and Dynamics*, because the notions of mechanics and dynamics, though much intertwined, are still somewhat different. The term *mechanics*, in its narrow sense, means the derivation of equations of motion of point-like particles and their systems (including solids and fluids), the solution of these equations, and an interpretation of the results. *Dynamics* is a more ambiguous term; it may mean, in particular:

(i) the part of physics that deals with motion (in contrast to *statics*);
(ii) the part of physics that deals with reasons for motion (in contrast to *kinematics*);
(iii) the part of mechanics that focuses on its two last tasks, i.e. the solution of the equations of motion and discussion of the results.[2]

Because of this ambiguity, after some hesitation, I have opted to use the traditional name *Classical Mechanics*, with the word *Mechanics* in its broad sense that includes (similarly to *Quantum Mechanics* and *Statistical Mechanics*) studies of dynamics of some non-mechanical systems as well.

1.1. Kinematics: Basic notions

The basic notions of kinematics may be defined in various ways, and some mathematicians pay much attention to alternative systems of axioms and the relations between them. In physics, we typically stick to less rigorous ways (in order to proceed faster to solving particular problems) and end debating any definition as soon as "everybody in the room" agrees that we are all speaking about the same thing – at least in the context in which they are being discussed. Let me hope that the following notions used in classical mechanics do satisfy this criterion in our "room":

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[1] The reader is advised to perform (perhaps after reading this chapter as a reminder) a self-check by solving a few problems of those listed in Sec. 1.6. If the results are not satisfactory, it may make sense to start with some remedial reading. For that, I could recommend, e.g., J. Marion and S. Thornton, *Classical Dynamics of Particles and Systems,* 5th ed., Saunders, 2003; and D. Morin, *Introduction to Classical Mechanics*, Cambridge U., 2008.

[2] The reader may have noticed that the last definition of dynamics is suspiciously close to the part of mathematics devoted to differential equation analysis; what is the difference? An important bit of philosophy: physics may be defined as an art (and a bit of science :-) of describing Mother Nature by mathematical means; hence in many cases the approaches of a mathematician and a physicist to a problem are very similar. The main difference between them is that physicists try to express the results of their analyses in terms of the properties of the *systems* under study, rather than the *functions* describing them, and as a result develop a sort of intuition ("gut feeling") about how other similar systems may behave, even if their exact equations of motion are somewhat different – or not known at all. The intuition so developed has enormous heuristic power, and most discoveries in physics have been made through gut-feeling-based insights rather than by plugging one formula into another one.

(i) All the *Euclidean geometry* notions, including the *point*, the *straight line*, the *plane*, etc.[3]

(ii) *Reference frames*: platforms for observation and mathematical description of physical phenomena. A reference frame includes a *coordinate system* used for measuring the point's position (namely, its *radius vector* **r** that connects the coordinate origin to the point – see Fig. 1) and a clock that measures *time t*. A coordinate system may be understood as a certain method of expressing the radius vector **r** of a point as a set of its *scalar coordinates*. The most important of such systems (but by no means the only one) are the *Cartesian* (orthogonal, linear) *coordinates*[4] r_j of a point, in which its radius vector may be represented as the following sum:

$$\mathbf{r} = \sum_{j=1}^{3} \mathbf{n}_j r_j , \qquad (1.1)$$

Cartesian coordinates

where $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ are unit vectors directed along the coordinate axis – see Fig. 1.[5]

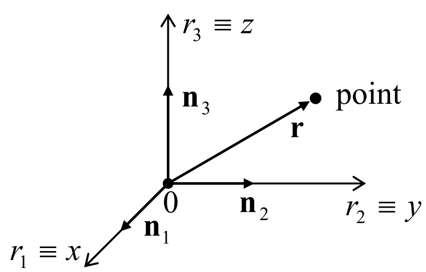

Fig. 1.1. Cartesian coordinates of a point.

(iii) The *absolute* ("Newtonian") *space/time*,[6] which does not depend on the matter distribution. The space is assumed to have the *Euclidean metric*, which may be expressed as the following relation between the length *r* of any radius vector **r** and its Cartesian coordinates:

$$r^2 \equiv |\mathbf{r}|^2 = \sum_{j=1}^{3} r_j^2 , \qquad (1.2)$$

Euclidean metric

while time *t* is assumed to run similarly in all reference frames. These assumptions are critically revised in the relativity theory (which, in this series, is discussed only starting from EM Chapter 9.)

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[3] All these notions are of course abstractions: *simplified models* of the real objects existing in Nature. But please always remember that *any* quantitative statement made in physics (e.g., a formula) may be strictly valid only for an approximate model of a physical system. (The reader should not be disheartened too much by this fact: experiments show that many models make extremely precise predictions of the behavior of the real systems.)

[4] In this series, the Cartesian coordinates (introduced in 1637 by René Descartes, a.k.a. Cartesius) are denoted either as either $\{r_1, r_2, r_3\}$ or $\{x, y, z\}$, depending on convenience in each particular case. Note that axis numbering is important for operations like the vector ("cross") product; the "correct" (meaning generally accepted) numbering order is such that the rotation $\mathbf{n}_1 \rightarrow \mathbf{n}_2 \rightarrow \mathbf{n}_3 \rightarrow \mathbf{n}_1\ldots$ looks counterclockwise if watched from a point with all $r_j > 0$ – like the one shown in Fig. 1.

[5] Note that representation (1) is also possible for locally orthogonal but *curvilinear* (for example, polar/cylindrical and spherical) coordinates, which will be extensively used in this series. However, such coordinates are not Cartesian, and for them some of the relations given below are invalid – see, e.g., MA Sec. 10.

[6] These notions were formally introduced by Sir Isaac Newton in his main work, the three-volume *Philosophiae Naturalis Principia Mathematica* published in 1686-1687, but are rooted in earlier ideas by Galileo Galilei, published in 1632.

(iv) The (instant) *velocity* of the point,

Velocity

$$\mathbf{v}(t) \equiv \frac{d\mathbf{r}}{dt} \equiv \dot{\mathbf{r}}, \tag{1.3}$$

and its *acceleration*:

Acceleration

$$\mathbf{a}(t) \equiv \frac{d\mathbf{v}}{dt} \equiv \dot{\mathbf{v}} = \ddot{\mathbf{r}}. \tag{1.4}$$

(v) *Transfer between reference frames*. The above definitions of vectors **r**, **v**, and **a** depend on the chosen reference frame (are "reference-frame-specific"), and we frequently need to relate those vectors as observed in different frames. Within Euclidean geometry, the relation between the radius vectors in two frames with the corresponding axes parallel at the moment of interest (Fig. 2), is very simple:

Radius vector's transformation

$$\mathbf{r}\big|_{\text{in } 0'} = \mathbf{r}\big|_{\text{in } 0} + \mathbf{r}_0\big|_{\text{in } 0'}. \tag{1.5}$$

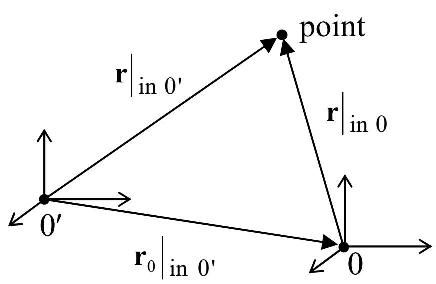

Fig. 1.2. Transfer between two reference frames.

If the frames move versus each other by *translation* only (no mutual rotation!), similar relations are valid for the velocities and accelerations as well:

$$\mathbf{v}\big|_{\text{in } 0'} = \mathbf{v}\big|_{\text{in } 0} + \mathbf{v}_0\big|_{\text{in } 0'}, \tag{1.6}$$

$$\mathbf{a}\big|_{\text{in } 0'} = \mathbf{a}\big|_{\text{in } 0} + \mathbf{a}_0\big|_{\text{in } 0'}. \tag{1.7}$$

Note that in the case of mutual rotation of the reference frames, the transfer laws for velocities and accelerations are more complex than those given by Eqs. (6) and (7). Indeed, in this case, notions like $\mathbf{v}_0\big|_{\text{in } 0'}$ are not well defined: different points of an imaginary rigid body connected to frame 0 may have different velocities when observed in frame 0'. It will be more natural for me to discuss these more general relations at the end of Chapter 4 devoted to rigid body motion.

(vi) A *particle* (or "point particle"): a localized physical object whose size is negligible, and whose shape is irrelevant *to the given problem*. Note that the last qualification is extremely important. For example, the size and shape of a spaceship are not too important for the discussion of its orbital motion but are paramount when its landing procedures are being developed. Since classical mechanics neglects the quantum mechanical uncertainties,[7] in it, the position of a particle at any particular instant t may be identified with a single geometrical point, i.e. with a single radius vector $\mathbf{r}(t)$. The formal final goal of classical mechanics is finding the *laws of motion* $\mathbf{r}(t)$ of all particles in the given problem.

[7] This approximation is legitimate when the product of the coordinate and momentum scales of the particle motion is much larger than Planck's constant $\hbar \sim 10^{-34}$ J·s. More detailed conditions of the classical mechanics' applicability depend on a particular system – see, e.g., the QM part of this series.

1.2. Dynamics: Newton's laws

Generally, the classical dynamics is fully described (in addition to the kinematic relations discussed above) by three *Newton's laws*. In contrast to the impression some textbooks on theoretical physics try to create, these laws are experimental in nature, and cannot be derived from *purely* theoretical arguments.

I am confident that the reader of these notes is already familiar with Newton's laws,[8] in some formulation. Let me note only that in some formulations, the *1st Newton's law* looks just like a particular case of the 2nd law – when the net force acting on a particle equals zero. To avoid this duplication, the 1st law may be formulated as the following postulate:

> There exists at least one reference frame, called *inertial*, in which any *free particle* (i.e. a particle fully isolated from the rest of the Universe) moves with $\mathbf{v}$ = const, i.e. with $\mathbf{a} = 0$.

1st Newton's law

Note that according to Eq. (7), this postulate immediately means that there is also an infinite number of inertial reference frames – because all frames 0' moving without rotation or acceleration relative to the postulated inertial frame 0 (i.e. having $\mathbf{a}_0\big|_{\text{in } 0'} = 0$) are also inertial.

On the other hand, the 2nd and 3rd Newton's laws may be postulated *together* in the following elegant way. Each particle, say number k, may be characterized by a scalar constant (called *mass* m_k), such that at any interaction of N particles (isolated from the rest of the Universe), in any inertial system,

$$\mathbf{P} \equiv \sum_{k=1}^{N} \mathbf{p}_k \equiv \sum_{k=1}^{N} m_k \mathbf{v}_k = \text{const.} \tag{1.8}$$

Total momentum and its conservation

(Each component of this sum,

$$\mathbf{p}_k \equiv m_k \mathbf{v}_k, \tag{1.9}$$

Particle's momentum

is called the *mechanical momentum*[9] of the corresponding particle, while the sum $\mathbf{P}$, the *total momentum* of the system.)

Let us apply this postulate to just two interacting particles. Differentiating Eq. (8) written for this case, over time, we get

$$\dot{\mathbf{p}}_1 = -\dot{\mathbf{p}}_2. \tag{1.10}$$

Let us give the derivative $\dot{\mathbf{p}}_1$ (which is a vector) the name of the *force* $\mathbf{F}$ exerted on particle 1. In our current case, when the only possible source of the force is particle 2, it may be denoted as $\mathbf{F}_{12}$: $\dot{\mathbf{p}}_1 \equiv \mathbf{F}_{12}$. Similarly, $\mathbf{F}_{21} \equiv \dot{\mathbf{p}}_2$, so Eq. (10) becomes the *3rd Newton's law*

$$\mathbf{F}_{12} = -\mathbf{F}_{21}. \tag{1.11}$$

3rd Newton's law

Plugging Eq. (1.9) into these force definitions, and differentiating the products $m_k\mathbf{v}_k$, taking into account that particle masses are constants,[10] we get that for the k and k' taking any of values 1, 2,

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[8] Due to the genius of Sir Isaac, these laws were formulated in the same *Principia* (1687), well ahead of the physics of his time.

[9] The more extended term *linear momentum* is typically used only in cases when there is a chance of confusion with the *angular momentum* of the same particle/system – see below. The present-day definition of linear momentum and the term itself belong to John Wallis (1670), but the concept may be traced back to more vague notions of several previous scientists – all the way back to at least a 570 AD work by John Philoponus.

$$m_k \dot{\mathbf{v}}_k \equiv m_k \mathbf{a}_k = \mathbf{F}_{kk'}, \qquad \text{where } k' \neq k.. \tag{1.12}$$

Now, returning to the general case of several interacting particles, and making an additional (but very natural) assumption that all partial forces $\mathbf{F}_{kk'}$ acting on particle k add up as vectors, we may generalize Eq. (12) into the *2nd Newton's law*

2nd Newton's law

$$m_k \mathbf{a}_k \equiv \dot{\mathbf{p}}_k = \sum_{k' \neq k} \mathbf{F}_{kk'} \equiv \mathbf{F}_k, \tag{1.13}$$

that allows a clear interpretation of the mass as a measure of a particle's *inertia*.

As a matter of principle, if the dependence of all pair forces $\mathbf{F}_{kk'}$ of particle positions (and generally of time as well) is known, Eq. (13) augmented with the kinematic relations (2) and (3) allows calculation of the laws of motion $\mathbf{r}_k(t)$ of all particles of the system. For example, for one particle the 2nd law (13) gives an ordinary differential equation of the second order:

$$m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, t), \tag{1.14}$$

which may be integrated – either analytically or numerically.

In certain cases, this is very simple. As an elementary example, for local motions with $\Delta r << r$, Newton's gravity force[11]

Newton's gravity law

$$\mathbf{F} = -G\frac{mm'}{R^3}\mathbf{R} \tag{1.15}$$

(where $\mathbf{R} \equiv \mathbf{r} - \mathbf{r}'$ is the distance between particles of masses m and m')[12] may be approximated as

Uniform gravity field

$$\mathbf{F} = m\mathbf{g}, \tag{1.16}$$

with the vector $\mathbf{g} \equiv -(Gm'/R^3)\mathbf{R}$ being constant.[13] As a result, m in Eq. (13) cancels, it is reduced to just $\ddot{\mathbf{r}} = \mathbf{g}$ = const, and may be easily integrated twice:

$$\dot{\mathbf{r}}(t) \equiv \mathbf{v}(t) = \int_0^t \mathbf{g}\, dt' + \mathbf{v}(0) = \mathbf{g}t + \mathbf{v}(0), \qquad \mathbf{r}(t) = \int_0^t \mathbf{v}(t')dt' + \mathbf{r}(0) = \mathbf{g}\frac{t^2}{2} + \mathbf{v}(0)t + \mathbf{r}(0), \tag{1.17}$$

thus giving the generic solution to all those undergraduate problems on the projectile motion, which should be so familiar to the reader.

[10] Note that this may not be true for composite bodies of varying total mass M (e.g., rockets emitting jets, see Problem 11), in these cases the momentum's derivative may differ from $M\mathbf{a}$.

[11] Introduced in the same famous *Principia*!

[12] The fact that the masses participating in Eqs. (14) and (16) are equal, the so-called *weak equivalence principle*, is actually highly nontrivial, but has been repeatedly verified with gradually improved relative accuracy, starting from $\sim 10^{-3}$ in Isaac Newton's own experimentation and all the way down to 1.5×10^{-15} from recent satellite experiments – see P. Touboul *et al.*, *Phys. Rev. Lett.* **129**, 121102 (2022).

[13] Of course, the most important particular case of Eq. (16) is the gravity field near the Earth's surface. In this case, using the fact that Eq. (15) remains valid for the gravity field created by a spherically uniform sphere, we get $g = GM_E/R_E^2$, where M_E and R_E are the Earth's mass and radius. Plugging in their values, $M_E \approx 5.97\times10^{24}$ kg and $R_E \approx 6.37\times10^6$ m, we get $g \approx 9.82$ m/s^2. The experimental value of g varies from 9.78 to 9.83 m/s^2 at various locations on the surface (due to the deviations of Earth's shape from a sphere, and the location-dependent effect of the centrifugal "inertial force" – see Sec. 4.5 below), with an average value of approximately 9.807 m/s^2.

All this looks (and indeed is) very simple, but in most other cases, Eq. (13) leads to more complex calculations. As an example, let us think about how would we use it to solve another simple problem: a bead of mass m sliding, without friction, along a round ring of radius R in a gravity field obeying Eq. (16) – see Fig. 3. (This system is equivalent to the usual *point pendulum*, i.e. a point mass suspended from point 0 on a light rod or string, and constrained to move in one vertical plane.)

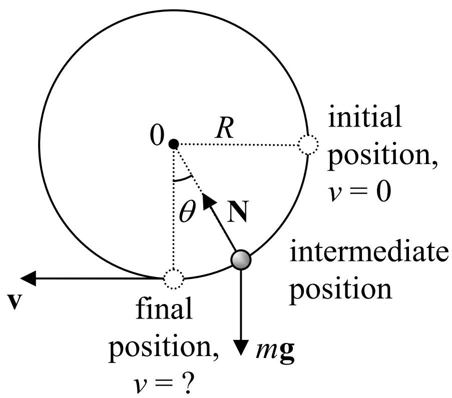

Fig. 1.3. A bead sliding along a vertical ring.

Suppose we are only interested in the bead's velocity v at the lowest point after it has been dropped from the rest at the rightmost position. If we want to solve this problem using only the Newton laws, we have to take the following steps:

(i) consider the bead in an arbitrary intermediate position on a ring, described, for example by the angle θ shown in Fig. 3;

(ii) draw all the forces acting on the particle – in our current case, the gravity force $m\mathbf{g}$ and the reaction force $\mathbf{N}$ exerted by the ring – see Fig. 3 above

(iii) write the Cartesian components of the 2nd Newton's law (14) for the bead acceleration: $ma_x = N_x$, $ma_y = N_y - mg$,

(iv) recognize that in the absence of friction, the force $\mathbf{N}$ should be normal to the ring, so that we can use two additional equations, $N_x = -N\sin\theta$ and $N_y = N\cos\theta$;

(v) eliminate unknown variables N, N_x, and N_y from the resulting system of four equations, thus getting a single second-order differential equation for one variable, for example, θ:

$$mR\ddot{\theta} = -mg\sin\theta\,; \tag{1.18}$$

(vi) use the mathematical identity $\ddot{\theta} \equiv d\left(\dot{\theta}^2/2\right)/d\theta$ to integrate this equation over θ once to get an expression relating the velocity $\dot{\theta}$ and the angle θ; and, finally,

(vii) using our specific initial condition ($\dot{\theta} = 0$ at $\theta = \pi/2$), find the final velocity as $v = R\dot{\theta}$ at $\theta = 0$.

All this is very much doable, but please agree that the procedure it too cumbersome for such a simple problem. Moreover, in many other cases even writing equations of motion along relevant coordinates is very complex, and any help the general theory may provide is highly valuable. In many cases, such help is given by *conservation laws*; let us review the most general of them.

1.3. Conservation laws

(i) *Energy* conservation is arguably the most general law of physics, but in mechanics, it takes a more humble form of *mechanical energy conservation*, which has limited applicability. To derive it, we first have to define the *kinetic energy* of a particle as[14]

Kinetic energy

$$T \equiv \frac{m}{2}v^2, \tag{1.19}$$

and then recast its differential as[15]

$$dT \equiv d\left(\frac{m}{2}v^2\right) \equiv d\left(\frac{m}{2}\mathbf{v}\cdot\mathbf{v}\right) = m\mathbf{v}\cdot d\mathbf{v} = m\frac{d\mathbf{r}\cdot d\mathbf{v}}{dt} = d\mathbf{r}\cdot\frac{d\mathbf{p}}{dt}. \tag{1.20}$$

Now plugging in the momentum's derivative from the 2nd Newton's law, $d\mathbf{p}/dt = \mathbf{F}$, where $\mathbf{F}$ is the full force acting on the particle, we get $dT = \mathbf{F}\cdot d\mathbf{r}$. The integration of this equality along the particle's trajectory connecting some points A and B gives the formula that is sometimes called the *work-energy principle*:

Work-energy principle

$$\Delta T \equiv T(\mathbf{r}_B) - T(\mathbf{r}_A) = \int_A^B \mathbf{F}\cdot d\mathbf{r}, \tag{1.21}$$

where the integral on the right-hand side is called the *work* of the force $\mathbf{F}$ on the path from A to B.

The next step may be made only for a *potential* (also called "conservative") force that may be represented as the (minus) gradient of some scalar function $U(\mathbf{r})$, called the *potential energy*.[16] The vector operator ∇ (called either *del* or *nabla*) of spatial differentiation[17] allows a very compact expression of this fact:

Force vs potential energy

$$\mathbf{F} = -\nabla U. \tag{1.22}$$

For example, for the uniform gravity field (16),

$$U = mgh + \text{const}, \tag{1.23}$$

where h is the vertical coordinate directed "up" – opposite to the direction of the vector $\mathbf{g}$.

Integrating the tangential component F_τ of the vector $\mathbf{F}$ given by Eq. (22), along an arbitrary path connecting the points A and ***B***, we get

$$\int_A^B F_\tau dr \equiv \int_A^B \mathbf{F}\cdot d\mathbf{r} = U(\mathbf{r}_A) - U(\mathbf{r}_B), \tag{1.24}$$

[14] In such quantitative form, the kinetic energy was introduced (under the name "living force") by Gottfried Leibniz and Johann Bernoulli (circa 1700), though its main properties (21) and (27) had not been clearly revealed until an 1829 work by Gaspard-Gustave de Coriolis. The modern term "kinetic energy" was coined only in 1849-1851 by Lord Kelvin (born William Thomson).

[15] In these notes, $\mathbf{a}\cdot\mathbf{b}$ denotes the scalar (or "dot-") product of vectors $\mathbf{a}$ and $\mathbf{b}$ – see, e.g., MA Eq. (7.1).

[16] Note that because of its definition via the gradient, the potential energy is only defined up to an arbitrary additive constant. This notion had been used already by G. Leibniz, though the term we are using for it nowadays was introduced much later (in the mid-19th century) by William Rankine.

[17] Its basic properties are listed in MA Sec. 8.

i.e. work of potential forces may be represented as the difference of values of the function $U(\mathbf{r})$ in the initial and final points of the path. (Note that according to Eq. (24), the work of a potential force on any closed path, with $\mathbf{r}_A = \mathbf{r}_B$, is zero.)

Now returning to Eq. (21) and comparing it with Eq. (24), we see that

$$T(\mathbf{r}_B) - T(\mathbf{r}_A) = U(\mathbf{r}_A) - U(\mathbf{r}_B), \quad \text{i.e. } T(\mathbf{r}_A) + U(\mathbf{r}_A) = T(\mathbf{r}_A) + U(\mathbf{r}_A), \tag{1.25}$$

so the *total mechanical energy* E, defined as

Total mechanical energy

$$E \equiv T + U, \tag{1.26}$$

is indeed conserved:

Mechanical energy: conservation

$$E(\mathbf{r}_A) \equiv E(\mathbf{r}_B), \tag{1.27}$$

but for conservative forces only. (Non-conservative forces may change E by either transferring energy from its mechanical form to another form, e.g., to heat in the case of friction, or by pumping the energy into the system under consideration from another, "external" system.)

Mechanical energy conservation allows us to return for just a second to the problem shown in Fig. 3 and solve it in one shot by writing Eq. (27) for the initial and final points:[18]

$$0 + mgR = \frac{m}{2}v^2 + 0. \tag{1.28}$$

The (elementary) solution of Eq. (28) for v immediately gives us the desired answer. Let me hope that the reader agrees that this way of problem's solution is much simpler, and I have earned their attention to discuss other conservation laws – which may be equally effective.

(ii) *Linear momentum*. The conservation of the full linear momentum of any system of particles isolated from the rest of the world was already discussed in the previous section, and may serve as the basic postulate of classical dynamics – see Eq. (8). In the case of one free particle, the law is reduced to the trivial result $\mathbf{p}$ = const, i.e. $\mathbf{v}$ = const. If a system of N particles *is* affected by external forces $\mathbf{F}^{(\text{ext})}$, we may write

$$\mathbf{F}_k = \mathbf{F}_k^{(\text{ext})} + \sum_{k=1}^{N} \mathbf{F}_{kk'}. \tag{1.29}$$

If we sum up the resulting Eqs. (13) for all particles of the system then, due to the 3rd Newton's law (11) valid for any indices $k \neq k'$, the contributions of all internal forces $\mathbf{F}_{kk'}$ to the resulting double sum on the right-hand side cancel, and we get the following equation:

System's momentum evolution

$$\dot{\mathbf{P}} = \mathbf{F}^{(\text{ext})}, \quad \text{where } \mathbf{F}^{(\text{ext})} \equiv \sum_{k=1}^{N} \mathbf{F}_k^{(\text{ext})}. \tag{1.30}$$

It tells us that the translational motion of the system as a whole is similar to that of a single particle, under the effect of the *net external force* $\mathbf{F}^{(\text{ext})}$. As a simple sanity check, if the external forces have a zero sum, we return to the postulate (8). Just one reminder: Eq. (30), as its precursor Eq. (13), is only valid in an inertial reference frame.

[18] Here the arbitrary constant in Eq. (23) is chosen so that the potential energy is zero at the final point.

I hope that the reader knows numerous examples of the application of the linear momentum's conservation law, including all these undergraduate problems on car collisions, where the large collision forces are typically not known so the direct application of Eq. (13) to each car is impracticable.

Angular momentum: definition

(iii) The *angular momentum* of a particle[19] is defined as the following vector:[20]

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}, \tag{1.31}$$

where **a**×**b** means the vector (or "cross-") product of the vector operands.[21] Differentiating Eq. (31) over time, we get

$$\dot{\mathbf{L}} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}}. \tag{1.32}$$

In the first product, $\dot{\mathbf{r}}$ is just the velocity vector **v**, parallel to the particle momentum $\mathbf{p} = m\mathbf{v}$, so this term vanishes since the vector product of any two parallel vectors equals zero. In the second product, $\dot{\mathbf{p}}$ is equal to the full force **F** acting on the particle, so Eq. (32) is reduced to

Angular momentum: evolution

$$\dot{\mathbf{L}} = \boldsymbol{\tau}, \tag{1.33}$$

where the vector

Torque

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F}, \tag{1.34}$$

is called the *torque* exerted by force **F**.[22] (Note that the torque is reference-frame specific – and again, the frame has to be inertial for Eq. (33) to be valid, because we have used Eq. (13) for its derivation.) For an important particular case of a *central* force **F** that is directed along the radius vector **r** of a particle, the torque vanishes, so (in that particular reference frame only!) the angular momentum is conserved:

Angular momentum: conservation

$$\mathbf{L} = \text{const.} \tag{1.35}$$

For a system of N particles, the total angular momentum is naturally defined as

System's angular momentum: definition

$$\mathbf{L} \equiv \sum_{k=1}^{N} \mathbf{L}_k . \tag{1.36}$$

Differentiating this equation over time, using Eq. (33) for each $\dot{\mathbf{L}}_k$, and again partitioning each force per Eq. (29), we get

$$\dot{\mathbf{L}} = \sum_{\substack{k,k'=1\\ k'\neq k}}^{N} \mathbf{r}_k \times \mathbf{F}_{kk'} + \boldsymbol{\tau}^{(\text{ext})}, \quad \text{where } \boldsymbol{\tau}^{(\text{ext})} \equiv \sum_{k=1}^{N} \mathbf{r}_k \times \mathbf{F}_k^{(\text{ext})}. \tag{1.37}$$

The first (double) sum may be always divided into pairs of the type $(\mathbf{r}_k \times \mathbf{F}_{kk'} + \mathbf{r}_{k'} \times \mathbf{F}_{k'k})$. With a natural assumption of the central forces, $\mathbf{F}_{kk'} \parallel (\mathbf{r}_k - \mathbf{r}_{k'})$, each of these pairs equals zero. Indeed, in this case,

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[19] Here we imply that the internal motions of the particle, including its rotation about its axis, are negligible. (Otherwise, it could not be represented by a point, as was postulated in Sec. 1.)

[20] This explicit definition of angular momentum (in different mathematical forms, and under the name of "moment of rotational motion") appeared in scientific publications only in the 1740s, though the fact of its conservation (35) in the field of central forces, in the form of the 2nd Kepler law (see Fig. 3.4 below), had been proved already by I. Newton in his *Principia*.

[21] See, e.g., MA Eq. (7.3).

[22] Alternatively, especially in mechanical engineering, torque is called the *force moment*. This notion may be traced all the way back to Archimedes' theory of levers developed in the 3rd century BC.

each component of the pair is a vector perpendicular to the plane containing the positions of both particles and the reference frame origin, i.e. to the plane of the drawing in Fig. 4.

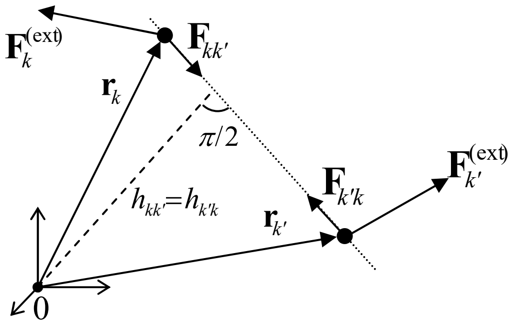

Fig. 1.4. Internal and external forces, and the internal torque cancellation in a system of two particles.

Also, due to the 3rd Newton's law (11), these two forces are equal and opposite, and the magnitude of each term in the sum may be represented as $|F_{kk'}|\, h_{kk'}$, with equal "lever arms" $h_{kk'} = h_{k'k}$. As a result, each sum $(\mathbf{r}_k\times\mathbf{F}_{kk'} + \mathbf{r}_{k'}\times\mathbf{F}_{k'k})$, and hence the whole double sum in Eq. (37) vanish, and it is reduced to a very simple result,

System's angular momentum: evolution

$$\dot{\mathbf{L}} = \boldsymbol{\tau}^{(\text{ext})}, \tag{1.38}$$

which is similar to Eq. (33) for a single particle, and is the angular analog of Eq. (30).

In particular, Eq. (38) shows that if the full external torque $\boldsymbol{\tau}^{(\text{ext})}$ vanishes for some reason (e.g. if the system of particles is isolated from the rest of the Universe), the conservation law (35) is valid for the full angular momentum $\mathbf{L}$ even if its individual components $\mathbf{L}_k$ are not conserved due to inter-particle interactions.

Please note again that since the conservation laws may be derived from Newton's laws (as was done above), they do not introduce anything new to the dynamics of any system. Indeed, from the mathematical point of view, the conservation laws discussed above are just the first integrals of the second-order differential equations of motion following from Newton's laws. However, for a physicist, thinking about particular systems in terms of the conserved (or potentially conserved) quantities frequently provides decisive clues on their dynamics.

1.4. Potential energy and equilibrium

Another important role of the potential energy U, especially for dissipative systems whose total mechanical energy E is *not* conserved because it may be drained to the environment, is finding the positions of equilibrium (sometimes called the *fixed points*) of the system and analyzing their stability with respect to small perturbations. For a single particle, this is very simple: the force (22) vanishes at each extremum (either minimum or maximum) of the potential energy.[23] (Of those fixed points, only the minimums of $U(\mathbf{r})$ are stable – see Sec. 3.2 below for a discussion of this point.)

A slightly more subtle case is a particle with an internal potential energy $U(\mathbf{r})$, subjected to an *additional* external force $\mathbf{F}^{(\text{ext})}(\mathbf{r})$. In this case, the stable equilibrium is reached at the minimum of not the function $U(\mathbf{r})$, but of what is sometimes called the *Gibbs potential energy*

[23] Assuming that the additional, non-conservative forces (such as viscosity) responsible for the mechanical energy drain, vanish at equilibrium – as they typically do. (The static friction is one counter-example.)

Gibbs' potential energy

$$U_{\rm G}(\mathbf{r}) \equiv U(\mathbf{r}) - \int^{\mathbf{r}} \mathbf{F}^{(\rm ext)}(\mathbf{r}') \cdot d\mathbf{r}', \qquad (1.39)$$

which is defined, just as $U(\mathbf{r})$ is, to an arbitrary additive constant.[24] The proof of Eq. (39) is very simple: in an extremum of this function, the total force acting on the particle,

$$\mathbf{F}^{(\rm tot)} = \mathbf{F} + \mathbf{F}^{(\rm ext)} \equiv -\nabla U + \nabla \int^{\mathbf{r}} \mathbf{F}^{(\rm ext)}(\mathbf{r}') \cdot d\mathbf{r}' \equiv -\nabla U_{\rm G} \qquad (1.40)$$

vanishes, as it is necessary for equilibrium.

Physically, the difference $U_{\rm G} - U$ specified by Eq. (39) is the **r**-dependent part of the potential energy $U^{(\rm ext)}$ of the external system responsible for the force $\mathbf{F}^{(\rm ext)}$, so $U_{\rm G}$ is just the total potential energy $U + U^{(\rm ext)}$, excluding its part that does not depend on **r** and hence is irrelevant for the analysis. According to the 3rd Newton's law, the force exerted by the particle on the external system equals $(-\mathbf{F}^{(\rm ext)})$, so its work (and hence the change of $U^{(\rm ext)}$ due to the change of **r**) is given by the second term on the right-hand side of Eq. (39). Thus the condition of equilibrium, $\nabla U_{\rm G} = 0$, is just the condition of an extremum of the total potential energy, $U + U^{(\rm ext)} + \text{const}$, of the two interacting systems.

For the simplest (and very frequent) case when the applied force is independent of the particle's position, the Gibbs potential energy (39) is just[25]

$$U_{\rm G}(\mathbf{r}) \equiv U(\mathbf{r}) - \mathbf{F}^{(\rm ext)} \cdot \mathbf{r} + \text{const}. \qquad (1.41)$$

As the simplest example, consider a 1D deformation of the usual elastic spring providing the returning force $(-\kappa x)$, where x is the deviation from its equilibrium. As follows from Eq. (22), its potential energy is $U = \kappa x^2/2 + \text{const}$, so its minimum corresponds to $x = 0$. Now let us apply an additional external force F, say independent of x. Then the equilibrium deformation of the spring, $x_0 = F/\kappa$, corresponds to the minimum of not U, but rather of the Gibbs potential energy (41), in our particular case taking the form

$$U_{\rm G} \equiv U - Fx = \frac{\kappa x^2}{2} - Fx. \qquad (1.42)$$

1.5. OK, we've got it – can we go home now?

Sorry, not yet. In many cases, the conservation laws discussed above provide little help, even in systems without dissipation. As a simple example, consider a generalization of the bead-on-the-ring problem shown in Fig. 3, in which the ring is rotated by external forces, with a constant angular velocity ω, about its vertical diameter.[26] In this problem (to which I will repeatedly return below, using it as an

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[24] Unfortunately, in most textbooks, the association of the (unavoidably used) notion of $U_{\rm G}$ with the glorious name of Josiah Willard Gibbs is postponed until a course of statistical mechanics and/or thermodynamics, where $U_{\rm G}$ is a part of the *Gibbs free energy*, in contrast to U, which is a part of the *Helmholtz free energy* – see, e.g., SM Sec. 1.4. I use this notion throughout my series, because the difference between $U_{\rm G}$ and U, and hence that between the Gibbs and Helmholtz free energies, has nothing to do with statistics or thermal motion, and belongs to the whole physics, including not only mechanics but also electrodynamics and quantum mechanics.

[25] Eq. (41) is a particular case of what mathematicians call the *Legendre transformations*.

[26] This is essentially a simplified model of the mechanical control device called the *centrifugal* (or "flyball", or "centrifugal flyball") *governor* – see, e.g., http://en.wikipedia.org/wiki/Centrifugal_governor. (Sometimes the device is called the "Watt's governor", after the famous James Watts who used it in 1788 in one of his first steam

analytical mechanics "testbed"), none of the three conservation laws listed in the last section, holds. In particular, the bead's energy,

$$E = \frac{m}{2}v^2 + mgh, \tag{1.43}$$

is *not* constant, because the external forces rotating the ring may change it. Of course, we still can solve the problem using Newton's laws, but this is even more complex than for the above case of the ring at rest, in particular because the force **N** exerted on the bead by the ring now may have three rather than two Cartesian components, which are not simply related. On the other hand, it is clear that the bead still has just one degree of freedom (say, the angle θ), so its dynamics should not be too complicated.

This case gives us a clue on how situations like this one can be simplified: if we only could exclude the so-called *reaction forces* such as **N**, that take into account *external constraints* imposed on the particle motion, in advance, that should help a lot. Such a constraint exclusion may be provided by analytical mechanics, in particular its Lagrangian formulation, to which we will now proceed.

Of course, the value of the Lagrangian approach goes far beyond simple systems such as the bead on a rotating ring. Indeed, this system has just two externally imposed constrains: the fixed distance of the bead from the center of the ring, and the instant angle of rotation of the ring about its vertical diameter. Now let us consider the motion of a rigid body. It is essentially a system of a very large number, $N >> 1$, of particles ($\sim 10^{23}$ of them if we think about atoms in a 1-cm-scale body). If the only way to analyze its motion would be to write Newton's laws for each of the particles, the situation would be completely hopeless. Fortunately, the number of constraints imposed on its motion is almost similarly huge. (At negligible deformations of the body, the distances between each pair of its particles should be constant.) As a result, the number of actual degrees of freedom of such a body is small (at negligible deformations, just six – see Sec. 4.1), so with the kind help from analytical mechanics, the motion of the body may be, in many important cases, analyzed even without numerical calculations.

One more important motivation for analytical mechanics is given by the dynamics of "non-mechanical" systems, for example, of the electromagnetic field – possibly interacting with charged particles, conducting bodies, etc. In many such systems, the easiest (and sometimes the only practicable) way to find the equations of motion is to derive them from either the Lagrangian or Hamiltonian function of the system. Moreover, the Hamiltonian formulation of the analytical mechanics (to be reviewed in Chapter 10 below) offers a direct pathway to deriving quantum-mechanical Hamiltonian operators of various systems, which are necessary for the analysis of their quantum properties.

1.6. Self-test problems

1.1. A bicycle, ridden with velocity v on wet pavement, has no mudguards on its wheels. How far behind should the following biker ride to avoid being splashed over? Neglect the air resistance effects.

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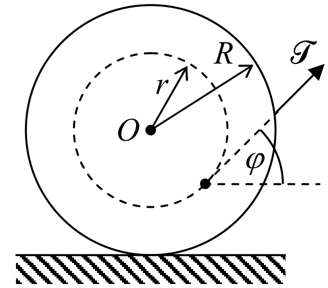
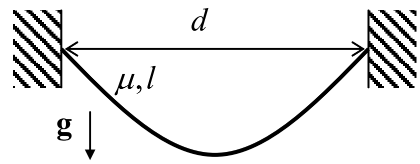
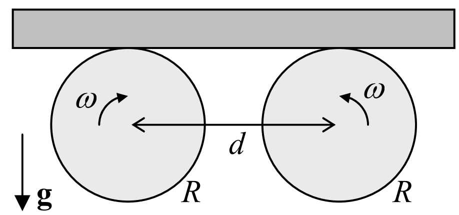
engines, though it had been used in European windmills at least since the early 1600s.) Just as a curiosity: the now-ubiquitous term *cybernetics* was coined by Norbert Wiener in 1948 from the word "governor" (or rather from its Ancient-Greek original κυβερνήτης) exactly in this meaning because the centrifugal governor had been the first well-studied control device.



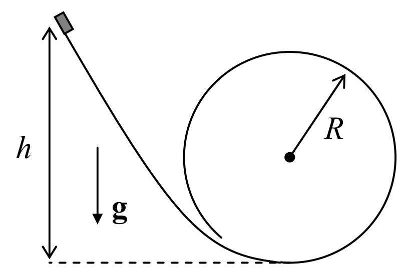
1.2. Two round disks of radius R are firmly connected with a coaxial cylinder of a smaller radius r, and a thread is wound on the resulting spool. The spool is placed on a horizontal surface, and the thread's end is being pooled out at angle φ – see the figure on the right. Assuming that the spool does not slip on the surface, what direction would it roll?



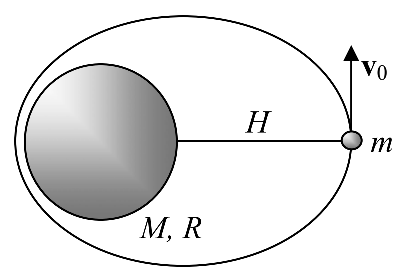
1.3.* Calculate the equilibrium shape of a flexible heavy rope of length l, with a constant mass μ per unit length, if it is hung in a uniform gravity field between two points separated by a horizontal distance d – see the figure on the right.



1.4. A uniform, long, thin bar is placed horizontally on two similar round cylinders rotating toward each other with the same angular velocity ω and displaced by distance d – see the figure on the right. Calculate the laws of relatively slow horizontal motion of the bar within the plane of the drawing, for both possible directions of cylinder rotation, assuming that the kinetic friction force between the slipping surfaces of the bar and each cylinder obeys the simple *Coulomb approximation*[27] $|F| = \mu N$, where N is the normal pressure force between them, and μ is a constant (velocity-independent) coefficient. Formulate the condition of validity of your result.



1.5. A small block slides, without friction, down a smooth slide that ends with a round loop of radius R – see the figure on the right. What smallest initial height h allows the block to make its way around the loop without dropping from the slide if it is launched with negligible initial velocity?

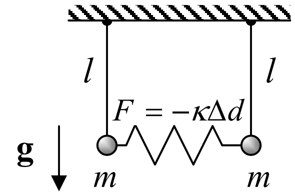


1.6. A satellite of mass m is being launched from height H over the surface of a spherical planet with radius R and mass $M >> m$ – see the figure on the right. Find the range of initial velocities $\mathbf{v}_0$ (normal to the radius) providing closed orbits above the planet's surface.

1.7. Prove that the thin-uniform-disk model of a galaxy allows for small sinusoidal ("harmonic") oscillations of stars inside it, along the direction normal to the disk, and calculate the frequency of these oscillations in terms of Newton's gravitational constant G and the average density ρ of the disk's matter.

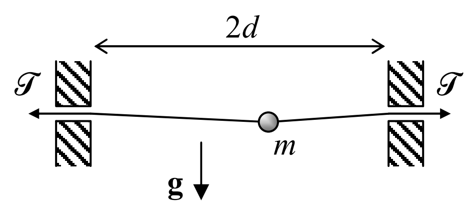
[27] It was suggested in 1785 by the same Charles-Augustin de Coulomb who discovered the famous *Coulomb law* of electrostatics, and hence pioneered the whole quantitative science of electricity – see EM Ch. 1.

1.8. Derive differential equations of motion for small oscillations of two similar pendula coupled with a spring (see the figure on the right), within their common vertical plane. Assume that at the vertical position of both pendula, the spring is not stretched ($\Delta d = 0$).



1.9. One of the popular futuristic concepts of travel is digging a straight railway tunnel through the Earth and letting a train go through it, without initial velocity – driven only by gravity. Calculate the train's travel time through such a tunnel, assuming that the Earth's density ρ is constant, and neglecting the friction and planet-rotation effects.

1.10. A small bead of mass m may slide, without friction, along a light string stretched with force $\mathscr{T} >> mg$, between two points separated by a horizontal distance $2d$ – see the figure on the right. Calculate the frequency of oscillations of the bead about its equilibrium position, within the vertical plane.



1.11. For a rocket accelerating (in free space) due to its working jet motor (and hence spending the jet fuel), calculate the relation between its velocity and the remaining mass.

Hint: For the sake of simplicity, consider the 1D motion.

1.12. Prove the following *virial theorem*:[28] for a set of N particles performing a periodic motion,

$$\overline{T} = -\frac{1}{2}\sum_{k=1}^{N}\overline{\mathbf{F}_k \cdot \mathbf{r}_k} ,$$

where the top bar means averaging over time – in this case over the motion period. What does the virial theorem say about:

(i) a 1D motion of a particle in the confining potential[29] $U(x) = ax^{2s}$, with $a > 0$ and $s > 0$, and
(ii) an orbital motion of a particle in the central potential $U(r) = -C/r$?

Hint: Explore the time derivative of the following scalar function of time: $G(t) \equiv \sum_{k=1}^{N} \mathbf{p}_k \cdot \mathbf{r}_k$.

1.13. As will be discussed in Chapter 8, if a solid body moves through a fluid with a sufficiently high velocity v, the fluid's drag force is approximately proportional to v^2. Use this approximation (introduced by Sir Isaac Newton himself) to find the velocity as a function of time during the body's vertical fall in the air near the Earth's surface.

1.14. A particle of mass m, moving with velocity u, collides head-on with a particle of mass M, initially at rest, increasing its internal energy by ΔE. Calculate the velocities of both particles after the collision, if u is barely sufficient for such an internal energy increase.

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[28] It was first stated by Rudolf Clausius in 1870.

[29] Here and below I am following the (regretful) custom of using the single word "potential" for the potential energy of the particle – just for brevity. This custom is also common in quantum mechanics, but in electrodynamics, these two notions should be clearly distinguished – as they are in the EM part of this series.