Chapter 2. Lagrangian Analytical Mechanics

The goal of this chapter is to describe the Lagrangian formalism of analytical mechanics, which is extremely useful for obtaining the differential equations of motion (*and sometimes their first integrals*) *not only for mechanical systems with holonomic constraints but also for some other dynamic systems.*

2.1. Lagrange equations

In many cases, the constraints imposed on the 3D motion of a system of N particles may be described by N vector (i.e. $3N$ scalar) algebraic equations

$$\mathbf{r}_k = \mathbf{r}_k(q_1, q_2, ..., q_j, ..., q_J, t), \qquad \text{with } 1 \le k \le N, \tag{2.1}$$

where q_j are certain *generalized coordinates* that (together with constraints) completely define the system position. Their number $J \le 3N$ is called the number of the actual *degrees of freedom* of the system. The constraints that allow such a description are called *holonomic*.[1]

For example, for the problem already mentioned in Section 1.5, namely, the bead sliding along a rotating ring (Fig. 1), $J = 1$, because with the constraints imposed by the ring, the bead's position is uniquely determined by just one generalized coordinate – for example, its polar angle θ.

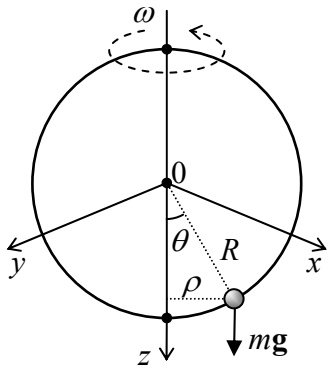


Fig. 2.1. A bead on a rotating ring as an example of a system with just one degree of freedom ($J = 1$).

Indeed, selecting the reference frame as shown in Fig. 1 and using the well-known formulas for the spherical coordinates,[2] we see that in this case, Eq. (1) has the form

$$\mathbf{r} \equiv \{x, y, z\} = \{R\sin\theta\cos\varphi,\ R\sin\theta\sin\varphi,\ R\cos\theta\}, \qquad \text{where } \varphi = \omega t + \text{const}, \tag{2.2}$$

with the last constant depending on the exact selection of the axes x and y and the time origin. Since the angle φ, in this case, is a fixed function of time, and R is a fixed constant, the particle's position in space

[1] Possibly, the simplest counter-example of a *non-holonomic* constraint is a set of inequalities describing the hard walls confining the motion of particles in a closed volume. Non-holonomic constraints are better dealt with by other methods, e.g., by imposing proper boundary conditions on the (otherwise unconstrained) motion.

[2] See, e.g., MA Eq. (10.7).

at any instant t is completely determined by the value of its only generalized coordinate θ. (Note that its dimensionality is different from that of Cartesian coordinates!)

Now returning to the general case of J degrees of freedom, let us consider a set of small *variations* (alternatively called "virtual displacements") δq_j allowed by the constraints. Virtual displacements differ from the actual small displacements (described by *differentials* dq_j proportional to time variation dt) in that δq_j describes not the system's motion as such, but rather its possible variation – see Fig. 1.

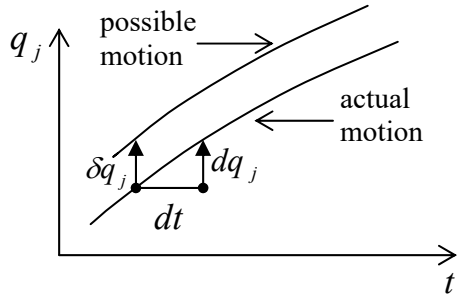


Fig. 2.2. Actual displacement dq_j vs. the virtual one (i.e. variation) δq_j.

Generally, operations with variations are the subject of a special field of mathematics, the calculus of variations.[3] However, the only math background necessary for our current purposes is the understanding that operations with variations are similar to those with the usual differentials, though we need to watch carefully what each variable is a function of. For example, if we consider the variation of the radius vectors (1), at a fixed time t, as functions of independent variations δq_j, we may use the usual formula for the differentiation of a function of several arguments:[4]

$$\delta \mathbf{r}_k = \sum_j \frac{\partial \mathbf{r}_k}{\partial q_j} \delta q_j . \tag{2.3}$$

Now let us break the force acting upon the k^{th} particle into two parts: the frictionless, constraining part $\mathbf{N}_k$ of the reaction force and the remaining part $\mathbf{F}_k$ – including the forces from other sources and possibly the frictional part of the reaction force. Then the 2nd Newton's law for the k^{th} particle of the system may be rewritten as

$$m_k \dot{\mathbf{v}}_k - \mathbf{F}_k = \mathbf{N}_k . \tag{2.4}$$

Since any variation of the motion has to be allowed by the constraints, its $3N$-dimensional vector with N 3D-vector components $\delta \mathbf{r}_k$ has to be perpendicular to the $3N$-dimensional vector of the constraining forces, also having N 3D-vector components $\mathbf{N}_k$. (For example, for the problem shown in Fig. 1, the virtual displacement vector $\delta \mathbf{r}_k$ may be directed only along the ring, while the constraining force $\mathbf{N}$ exerted by the ring, has to be perpendicular to that direction.) This condition may be expressed as

[3] For a concise introduction to the field see, e.g., either I. Gelfand and S. Fomin, *Calculus of Variations*, Dover, 2000, or L. Elsgolc, *Calculus of Variations*, Dover, 2007. An even shorter review may be found in Chapter 17 of Arfken and Weber – see MA Sec. 16. For a more detailed discussion, using many examples from physics, see R. Weinstock, *Calculus of Variations*, Dover, 2007.

[4] See, e.g., MA Eq. (4.2). Also, in all formulas of this section, summations over j are from 1 to J, while those over the particle number k are from 1 to N, so for the sake of brevity, these limits are not explicitly specified.

$$\sum_k \mathbf{N}_k \cdot \delta\mathbf{r}_k = 0, \tag{2.5}$$

where the scalar product of $3N$-dimensional vectors is defined exactly like that of 3D vectors, i.e. as the sum of the products of the corresponding components of the operands. The substitution of Eq. (4) into Eq. (5) results in the so-called *D'Alembert principle*:[5]

D'Alembert principle

$$\boxed{\sum_k (m_k \dot{\mathbf{v}}_k - \mathbf{F}_k) \cdot \delta\mathbf{r}_k = 0.} \tag{2.6}$$

Plugging Eq. (3) into Eq. (6), we get

$$\sum_j \left\{ \sum_k m_k \dot{\mathbf{v}}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} - \mathscr{F}_j \right\} \delta q_j = 0, \tag{2.7}$$

where the scalars $\mathscr{F}_j$, called the *generalized forces*, are defined as follows:[6]

Generalized force

$$\boxed{\mathscr{F}_j \equiv \sum_k \mathbf{F}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j}.} \tag{2.8}$$

Now we may use the standard argument of the calculus of variations: for the left-hand side of Eq. (7) to be zero for an arbitrary selection of independent variations δq_j, the expression in the curly brackets, for every j, should equal zero. This gives us the desired set of $J \le 3N$ equations

$$\sum_k m_k \dot{\mathbf{v}}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} - \mathscr{F}_j = 0; \tag{2.9}$$

what remains is just to recast them in a more convenient form.

First, using the differentiation by parts to calculate the following time derivative:

$$\frac{d}{dt}\left(\mathbf{v}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} \right) = \dot{\mathbf{v}}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} + \mathbf{v}_k \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}_k}{\partial q_j} \right), \tag{2.10}$$

we may notice that the first term on the right-hand side is exactly the scalar product in the first term of Eq. (9).

Second, let us use another key fact of the calculus of variations (which is, essentially, evident from Fig. 3): the differentiation of a variable over time and over the generalized coordinate variation (at a fixed time) are interchangeable operations. As a result, in the second term on the right-hand side of Eq. (10), we may write

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_k}{\partial q_j} \right) = \frac{\partial}{\partial q_j}\left(\frac{d\mathbf{r}_k}{dt} \right) \equiv \frac{\partial \mathbf{v}_k}{\partial q_j}. \tag{2.11}$$

[5] It was spelled out in a 1743 work by Jean le Rond d'Alembert, though the core of this result has been traced to an earlier work by Jacob (Jean) Bernoulli (1667 – 1748) – not to be confused with his son Daniel Bernoulli (1700-1782) who is credited, in particular, for the *Bernoulli equation* for ideal fluids, to be discussed in Sec. 8.4 below.

[6] Note that since the dimensionality of generalized coordinates may be arbitrary, that of generalized forces may also differ from the newton.

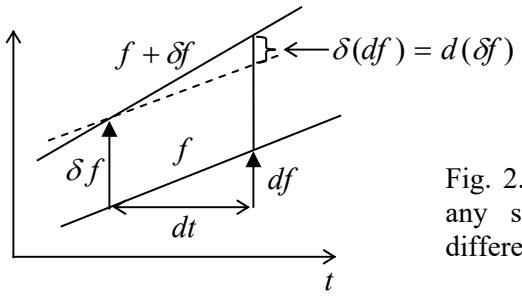


Fig. 2.3. The variation of the differential (of any smooth function *f*) is equal to the differential of its variation.

Finally, let us differentiate Eq. (1) over time:

$$\mathbf{v}_k \equiv \frac{d\mathbf{r}_k}{dt} = \sum_j \frac{\partial \mathbf{r}_k}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}_k}{\partial t}. \tag{2.12}$$

This equation shows that particle velocities $\mathbf{v}_k$ may be considered to be linear functions of the generalized velocities $\dot{q}_j$ *considered as independent variables*, with proportionality coefficients

$$\frac{\partial \mathbf{v}_k}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_k}{\partial q_j}. \tag{2.13}$$

With the account of Eqs. (10), (11), and (13), Eq. (9) turns into

$$\frac{d}{dt}\sum_k m_k \mathbf{v}_k \cdot \frac{\partial \mathbf{v}_k}{\partial \dot{q}_j} - \sum_k m_k \mathbf{v}_k \cdot \frac{\partial \mathbf{v}_k}{\partial q_j} - \mathscr{F}_j = 0. \tag{2.14}$$

This result may be further simplified by making, for the total kinetic energy of the system,

$$T \equiv \sum_k \frac{m_k}{2} v_k^2 = \frac{1}{2}\sum_k m_k \mathbf{v}_k \cdot \mathbf{v}_k, \tag{2.15}$$

the same commitment as for $\mathbf{v}_k$, i.e. considering T a function of not only the generalized coordinates q_j and time t but also of the generalized velocities $\dot{q}_i$ – as variables *independent* of q_j and t. Then we may calculate the partial derivatives of T as

$$\frac{\partial T}{\partial q_j} = \sum_k m_k \mathbf{v}_k \cdot \frac{\partial \mathbf{v}_k}{\partial q_j}, \qquad \frac{\partial T}{\partial \dot{q}_j} = \sum_k m_k \mathbf{v}_k \cdot \frac{\partial \mathbf{v}_k}{\partial \dot{q}_j}, \tag{2.16}$$

and notice that they are exactly the two sums participating in Eq. (14). As a result, we get a system of J *Lagrange equations*,[7]

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - \mathscr{F}_j = 0, \quad \text{for } j = 1, 2, ..., J. \tag{2.17}$$

General Lagrange equations

Their big advantage over the initial Newton's-law equations (4) is that the Lagrange equations do not include the constraining forces $\mathbf{N}_k$, and thus there are only J of them – typically much fewer than $3N$.

[7] They were derived in 1788 by Joseph-Louis Lagrange, who pioneered the whole field of analytical mechanics – not to mention his key contributions to number theory and celestial mechanics.

This is as far as we can go for arbitrary forces. However, if all the forces may be expressed in a form similar to, but somewhat more general than Eq. (1.22): $\mathbf{F}_k = -\nabla_k U(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t)$, where U is the effective potential energy of the system,[8] and ∇_k denotes the spatial differentiation over coordinates of the k^{th} particle, we may recast Eq. (8) into a simpler form:

$$\mathcal{F}_j \equiv \sum_k \mathbf{F}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} = -\sum_k \left(\frac{\partial U}{\partial x_k}\frac{\partial x_k}{\partial q_j} + \frac{\partial U}{\partial y_k}\frac{\partial y_k}{\partial q_j} + \frac{\partial U}{\partial z_i}\frac{\partial z_i}{\partial q_j} \right) \equiv -\frac{\partial U}{\partial q_j}. \tag{2.18}$$

Since we assume that U depends only on particle coordinates (and possibly time), but not velocities: $\partial U / \partial \dot{q}_j = 0$, with the substitution of Eq. (18), the Lagrange equation (17) may be represented in the so-called *canonical form*:

Canonical Lagrange equations

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0,} \tag{2.19a}$$

where L is the *Lagrangian function* (sometimes called just the "Lagrangian"), defined as

Lagrangian function

$$\boxed{L \equiv T - U\ .} \tag{2.19b}$$

(It is crucial to distinguish this function from the mechanical energy (1.26), $E = T + U$.)

Note also that according to Eq. (2.18), for a system under the effect of an additional generalized external force $\mathcal{F}_j(t)$ we have to use, in all these relations, not the internal potential energy $U^{(\text{int})}$ of the system, but its Gibbs potential energy $U \equiv U^{(\text{int})} - \mathcal{F}_j q_j$ – see the discussion in Sec. 1.4.

Using the Lagrangian approach in practice, the reader should always remember, first, that each system has only *one* Lagrange function (19b), but is described by $J \geq 1$ Lagrange equations (19a), with j taking values 1, 2,…, J, and second, that differentiating the function L, we have to consider the generalized velocities as its independent arguments, ignoring the fact they are actually the time derivatives of the generalized coordinates.

2.2. Three simple examples

As the first, simplest example, consider a particle constrained to move along one axis (say, x):

$$T = \frac{m}{2}\dot{x}^2, \qquad U = U(x,t). \tag{2.20}$$

In this case, it is natural to consider x as the (only) generalized coordinate, and $\dot{x}$ as the generalized velocity, so

$$L \equiv T - U = \frac{m}{2}\dot{x}^2 - U(x,t). \tag{2.21}$$

Considering $\dot{x}$ and x as independent variables, we get $\partial L / \partial \dot{x} = m\dot{x}$, and $\partial L / \partial x = -\partial U / \partial x$, so Eq. (19) (the only Lagrange equation in this case of the single degree of freedom!) yields

[8] Note that due to the possible time dependence of U, Eq. (17) does not mean that the forces $\mathbf{F}_k$ have to be conservative – see the next section for more discussion. With this understanding, I will still use for function U the convenient name of "potential energy".

$$\frac{d}{dt}(m\dot{x}) - \left(-\frac{\partial U}{\partial x}\right) = 0, \tag{2.22}$$

evidently the same result as the x-component of the 2nd Newton's law with $F_x = -\partial U/\partial x$. This example is a good sanity check, but it also shows that the Lagrange formalism does not provide too much advantage in this particular case.

Such an advantage is, however, evident in our testbed problem – see Fig. 1. Indeed, taking the polar angle θ for the (only) generalized coordinate, we see that in this case, the kinetic energy depends not only on the generalized velocity but also on the generalized coordinate:[9]

$$\begin{aligned} T &= \frac{m}{2}R^2\left(\dot{\theta}^2 + \omega^2 \sin^2\theta\right), \qquad U = -mgz + \text{const} \equiv -mgR\cos\theta + \text{const}, \\ L &\equiv T - U = \frac{m}{2}R^2\left(\dot{\theta}^2 + \omega^2 \sin^2\theta\right) + mgR\cos\theta + \text{const}. \end{aligned} \tag{2.23}$$

Here it is especially important to remember that at substantiating the Lagrange equation, θ and $\dot{\theta}$ have to be treated as independent arguments of L, so

$$\frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta}, \qquad \frac{\partial L}{\partial \theta} = mR^2\omega^2 \sin\theta\cos\theta - mgR\sin\theta, \tag{2.24}$$

giving us the following equation of motion:

$$\frac{d}{dt}\left(mR^2\dot{\theta}\right) - \left(mR^2\omega^2 \sin\theta\cos\theta - mgR\sin\theta\right) = 0. \tag{2.25}$$

As a sanity check, at $\omega = 0$, Eq. (25) is reduced to the equation (1.18) of the usual pendulum:

$$\ddot{\theta} + \Omega^2 \sin\theta = 0, \quad \text{where } \Omega \equiv \left(\frac{g}{R}\right)^{1/2}. \tag{2.26}$$

We will explore Eq. (25) in more detail later, but please note how simple its derivation was – in comparison with writing the 3D Newton's law and then excluding the reaction force.

Next, though the Lagrangian formalism was derived from Newton's law for mechanical systems, the resulting equations (19) are applicable to other dynamic systems, especially those for which the kinetic and potential energies may be readily expressed via some generalized coordinates. As the simplest example, consider the well-known connection of a capacitor with capacitance C to an inductive coil with self-inductance $\mathscr{L}$[10] (Electrical engineers frequently call it the *LC tank circuit*.)

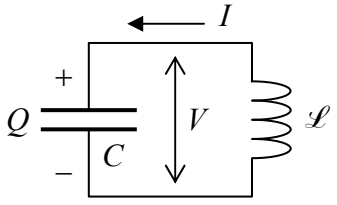


Fig. 2.4. *LC* tank circuit.

[9] The above expression for $T \equiv (m/2)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ may be readily obtained either by the formal differentiation of Eq. (2) over time, or just by noticing that the velocity vector has two perpendicular components: one (of magnitude $R\dot{\theta}$) along the ring, and another one (of magnitude $\omega\rho = \omega R\sin\theta$) normal to the ring's plane.

[10] A fancy font is used here to avoid any chance of confusion between the inductance and the Lagrange function.

As the reader (hopefully :-) knows from their undergraduate studies, at relatively low frequencies we may use the so-called lumped-circuit approximation, in which the total energy of this system is the sum of two components, the electric energy E_e localized inside the capacitor, and the magnetic energy E_m localized inside the inductance coil:

$$E_e = \frac{Q^2}{2C}, \qquad E_m = \frac{\mathscr{L} I^2}{2}. \tag{2.27}$$

Since the electric current I through the coil and the electric charge Q on the capacitor are related by the charge continuity equation $dQ/dt = I$ (evident from Fig. 4), it is natural to declare Q the generalized coordinate of the system, and the current, its generalized velocity. With this choice, the electrostatic energy $E_e(Q)$ may be treated as the potential energy U of the system, and the magnetic energy $E_m(I)$, as its kinetic energy T. With this attribution, we get

$$\frac{\partial T}{\partial \dot{q}} \equiv \frac{\partial E_m}{\partial I} = \mathscr{L} I \equiv \mathscr{L} \dot{Q}, \qquad \frac{\partial T}{\partial q} \equiv \frac{\partial E_m}{\partial Q} = 0, \qquad \frac{\partial U}{\partial q} \equiv \frac{\partial E_e}{\partial Q} = \frac{Q}{C}, \tag{2.28}$$

so the Lagrange equation (19) becomes

$$\frac{d}{dt}\left(\mathscr{L}\dot{Q}\right) - \left(-\frac{Q}{C}\right) = 0, \qquad \text{i.e. } \ddot{Q} + \frac{1}{\mathscr{L}C} Q = 0. \tag{2.29}$$

Note, however, that the above choice of the generalized coordinate and velocity is not unique. Instead, one can use, as the generalized coordinate, the magnetic flux Φ through the inductive coil, related to the common voltage V across the circuit (Fig. 4) by Faraday's induction law $V = -d\Phi/dt$. With this choice, $(-V)$ becomes the generalized velocity, $E_m = \Phi^2/2\mathscr{L}$ should be understood as the *potential* energy, and $E_e = CV^2/2$ treated as the *kinetic* energy. For this choice, the resulting Lagrange equation of motion is equivalent to Eq. (29). If both parameters of the circuit, $\mathscr{L}$ and C, are constant in time, Eq. (29) describes sinusoidal oscillations with the frequency

$$\omega_0 = \frac{1}{\left(\mathscr{L}C\right)^{1/2}}. \tag{2.30}$$

This is of course a well-known result, which may be derived in a more standard way – by equating the voltage drops across the capacitor ($V = Q/C$) and the inductor ($V = -\mathscr{L}dI/dt \equiv -\mathscr{L}d^2Q/dt^2$). However, the Lagrangian approach is much more convenient for more complex systems – for example, for the general description of the electromagnetic field and its interaction with charged particles.[11]

2.3. Hamiltonian function and energy

The canonical form (19) of the Lagrange equation has been derived using Eq. (18), which is formally similar to Eq. (1.22) for a potential force. Does this mean that the system described by Eq. (19) always conserves energy? Not necessarily, because the "potential energy" U that participates in Eq. (18), may depend not only on the generalized coordinates but on time as well. Let us start the analysis of this issue with the introduction of two new (and very important!) notions: the *generalized momentum* corresponding to each generalized coordinate q_j,

[11] See, e.g., EM Secs. 9.7 and 9.8.

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j}, \tag{2.31}$$

Generalized momentum

and the *Hamiltonian function*[12]

$$H \equiv \sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L \equiv \sum_j p_j \dot{q}_j - L\,. \tag{2.32}$$

Hamiltonian function: definition

To see whether the Hamiltonian function is conserved during the motion, let us differentiate both sides of its definition (32) over time:

$$\frac{dH}{dt} = \sum_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right)\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j \right] - \frac{dL}{dt}. \tag{2.33}$$

If we want to make use of the Lagrange equation (19), the last derivative has to be calculated considering L as a function of independent arguments q_j, $\dot{q}_j$, and t, so

$$\frac{dL}{dt} = \sum_j \left(\frac{\partial L}{\partial q_j}\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j \right) + \frac{\partial L}{\partial t}, \tag{2.34}$$

where the last term is the derivative of L as an *explicit* function of time. We see that the last term in the square brackets of Eq. (33) immediately cancels with the last term in the parentheses of Eq. (34). Moreover, using the Lagrange equation (19a) for the first term in the square brackets of Eq. (33), we see that it cancels with the first term in the parentheses of Eq. (34). As a result, we arrive at a very simple and important result:

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}. \tag{2.35}$$

Hamiltonian function: time evolution

The most important corollary of this formula is that if the Lagrangian function does not depend on time explicitly ($\partial L / \partial t = 0$), the Hamiltonian function is an integral of motion:

$$H = \text{const}. \tag{2.36}$$

Let us see how this works, using the first two examples discussed in the previous section. For a 1D particle, the definition (31) of the generalized momentum yields

$$p_x \equiv \frac{\partial L}{\partial v} = mv\,, \tag{2.37}$$

so it coincides with the usual linear momentum – or rather with its x-component. According to Eq. (32), the Hamiltonian function for this case (with just one degree of freedom) is

$$H \equiv p_x v - L = p_x \frac{p_x}{m} - \left(\frac{m}{2}\dot{x}^2 - U \right) = \frac{p_x^2}{2m} + U\,, \tag{2.38}$$

[12] It is named after Sir William Rowan Hamilton, who developed his approach to analytical mechanics in 1833, on the basis of the Lagrangian mechanics. This function is sometimes called just the "Hamiltonian", but it is advisable to use the full term "Hamiltonian function" in classical mechanics, to distinguish it from the *Hamiltonian operator* used in quantum mechanics, whose abbreviation to *Hamiltonian* is extremely common. (The relation of these two notions will be discussed in Sec. 10.1 below.)

i.e. coincides with the particle's mechanical energy $E = T + U$. Since the Lagrangian does not depend on time explicitly, both H and E are conserved.

However, it is not always that simple! Indeed, let us return again to our testbed problem (Fig. 1). In this case, the generalized momentum corresponding to the generalized coordinate θ is

$$p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta}, \tag{2.39}$$

and Eq. (32) yields:

$$\begin{aligned} H \equiv p_\theta\dot{\theta} - L &= mR^2\dot{\theta}^2 - \left[\frac{m}{2}R^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) + mgR\cos\theta\right] + \text{const} \\ &\equiv \frac{m}{2}R^2\left(\dot{\theta}^2 - \omega^2\sin^2\theta\right) - mgR\cos\theta + \text{const}. \end{aligned} \tag{2.40}$$

This means that (as soon as $\omega \neq 0$), the Hamiltonian function *differs* from the mechanical energy

$$E \equiv T + U = \frac{m}{2}R^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) - mgR\cos\theta + \text{const}. \tag{2.41}$$

The difference, $E - H = mR^2\omega^2\sin^2\theta$ (besides an inconsequential constant), may change at the bead's motion along the ring, so although H *is* an integral of motion (since $\partial L/\partial t = 0$), the energy is generally *not* conserved.

In this context, let us find out when these two functions, E and H, do coincide. In mathematics, there is a notion of a *homogeneous function* $f(x_1, x_2,\ldots)$ *of degree* λ, defined in the following way: for an arbitrary constant a,

$$f(ax_1, ax_2,...) = a^\lambda f(x_1, x_2,...). \tag{2.42}$$

Such functions obey the following *Euler theorem*:[13]

$$\sum_j \frac{\partial f}{\partial x_j}x_j = \lambda f, \tag{2.43}$$

which may be simply proved by differentiating both parts of Eq. (42) over a and then setting this parameter to the particular value $a = 1$. Now, consider the case when the kinetic energy is a quadratic form of all generalized velocities $\dot{q}_j$:

$$T = \sum_{j,j'} t_{jj'}(q_1, q_2,...,t)\,\dot{q}_j\dot{q}_{j'}, \tag{2.44}$$

with no other terms. It is evident that such T satisfies the definition (42) of a homogeneous function of the velocities with $\lambda = 2$,[14] so the Euler theorem (43) gives

$$\sum_j \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j = 2T. \tag{2.45}$$

[13] This is just one of many theorems bearing the name of their author – the genius mathematician Leonhard Euler (1707-1783).

[14] Such functions are called *quadratic-homogeneous*.

But since U is independent of the generalized velocities, $\partial L/\partial \dot{q}_j = \partial T/\partial \dot{q}_j$, and the left-hand side of Eq. (45) is exactly the first term in the definition (32) of the Hamiltonian function, so in this case

$$H = 2T - L = 2T - (T - U) = T + U = E. \tag{2.46}$$

So, for a system with a kinetic energy of the type (44), for example, a free particle with T considered as a function of its Cartesian velocities,

$$T = \frac{m}{2}\left(v_x^2 + v_y^2 + v_z^2\right), \tag{2.47}$$

the notions of the Hamiltonian function and mechanical energy are identical. Indeed, some textbooks, very regrettably, do not distinguish these notions at all! However, as we have seen from our bead-on-the-rotating-ring example, these variables do not always coincide. For that problem, the kinetic energy, in addition to the term proportional to $\dot{\theta}^2$, has another, velocity-independent term – see the first of Eqs. (23) – and hence is *not* a quadratic-homogeneous function of the angular velocity, giving $E \neq H$.

Thus, Eq. (36) expresses a new conservation law, generally different from that of mechanical energy conservation.

2.4. Other conservation laws

Looking at the Lagrange equation (19), we immediately see that if $L \equiv T - U$ is independent of some generalized coordinate q_j, $\partial L/\partial q_j = 0$,[15] then the corresponding generalized momentum is an integral of motion:[16]

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j} = \text{const}. \tag{2.48}$$

For example, for a 1D particle with the Lagrangian (21), the momentum p_x is conserved if the potential energy is constant (and hence the x-component of force is zero) – of course. As a less obvious example, let us consider a 2D motion of a particle in the field of central forces. If we use polar coordinates r and φ in the role of generalized coordinates, then the Lagrangian function[17]

$$L \equiv T - U = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - U(r) \tag{2.49}$$

is independent of φ, and hence the corresponding generalized momentum,

$$p_\varphi \equiv \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}, \tag{2.50}$$

[15] Such coordinates are frequently called *cyclic*, because in some cases (like φ in Eq. (49) below) they represent periodic coordinates such as angles. However, this terminology is somewhat misleading, because some "cyclic" coordinates (e.g., x in our first example) have nothing to do with rotation.

[16] This fact may be considered a particular case of a more general mathematical statement called the *Noether theorem* – named after its author, Emmy Nöther, sometimes called the "greatest woman mathematician ever lived". Unfortunately, because of time/space restrictions, for its discussion I have to refer the interested reader elsewhere – for example to Sec. 13.7 in H. Goldstein *et al*., *Classical Mechanics*, 3rd ed. Addison Wesley, 2002.

[17] Note that here $\dot{r}^2$ is the square of the scalar derivative $\dot{r}$, rather than the square of the vector $\dot{\mathbf{r}} = \mathbf{v}$.

is conserved. This is just a particular (2D) case of the angular momentum conservation – see Eq. (1.24). Indeed, for the 2D motion within the [*x, y*] plane, the angular momentum vector,

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} = \begin{vmatrix} \mathbf{n}_x & \mathbf{n}_y & \mathbf{n}_z \\ x & y & z \\ m\dot{x} & m\dot{y} & m\dot{z} \end{vmatrix}, \tag{2.51}$$

has only one component different from zero, namely the component normal to the motion plane:

$$L_z = x(m\dot{y}) - y(m\dot{x}). \tag{2.52}$$

Differentiating the well-known relations between the polar and Cartesian coordinates,

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \tag{2.53}$$

over time, and plugging the result into Eq. (52), we see that

$$L_z = mr^2\dot{\varphi} \equiv p_\varphi. \tag{2.54}$$

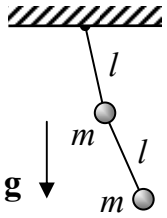
Thus the Lagrangian formalism provides a powerful way of searching for non-evident integrals of motion. On the other hand, if such a conserved quantity is obvious or known *a priori*, it is helpful for the selection of the most appropriate generalized coordinates, giving the simplest Lagrange equations. For example, in the last problem, if we knew in advance that p_φ had to be conserved, this could provide sufficient motivation for using the angle φ as one of the generalized coordinates.

2.5. Exercise problems

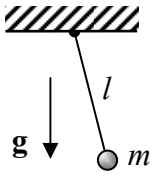
In each of Problems 1-11, for the given system:

(i) introduce a convenient set of generalized coordinates q_j,
(ii) write down the Lagrangian L as a function of $q_j, \dot{q}_j$, and (if appropriate) time,
(iii) write down the Lagrange equation(s) of motion,
(iv) calculate the Hamiltonian function H; find out whether it is conserved,
(v) calculate the mechanical energy E; is $E = H$?; is the energy conserved?
(vi) any other evident integrals of motion?

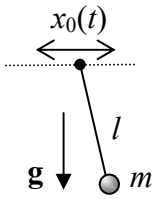
2.1. A double pendulum – see the figure on the right. Consider only the motion within the vertical plane containing the suspension point.



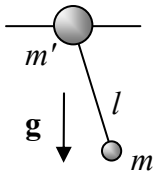
2.2. A stretchable pendulum (i.e. a massive particle hung on an elastic cord that exerts force $F = -\kappa(l - l_0)$, where κ and l_0 are positive constants), also confined to the vertical plane:



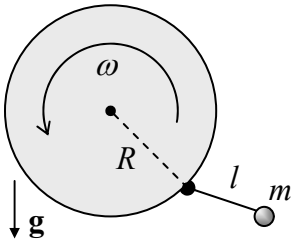
2.3. A fixed-length pendulum hanging from a point whose motion law $x_0(t)$ in the horizontal direction is fixed. (No vertical plane constraint here.)



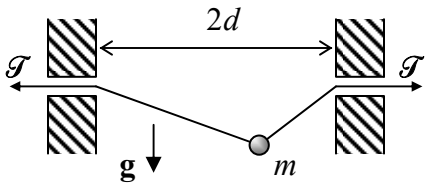
2.4. A pendulum of mass m, hung on another point mass m' that may slide, without friction, along a straight horizontal rail – see the figure on the right. The motion is confined to the vertical plane that contains the rail.



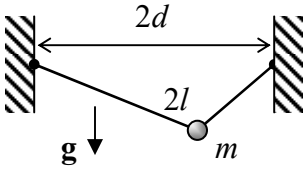
2.5. A point-mass pendulum of length l, attached to the rim of a disk of radius R, which is rotated in a vertical plane with a constant angular velocity ω – see the figure on the right. (Consider only the motion within the disk's plane.)



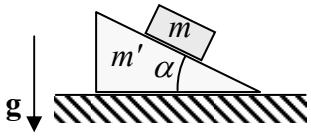
2.6. A bead of mass m, sliding without friction along a light string with a fixed tension $\mathcal{T}$, hung between two horizontally displaced supports – see the figure on the right. Here, in contrast to the similar Problem 1.10, the tension $\mathcal{T}$ may be comparable with the bead's weight mg, and the motion is *not* restricted to the vertical plane.



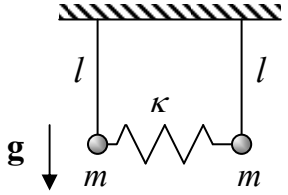
2.7. A bead of mass m, sliding without friction along a light string of a fixed length $2l$, that is hung between two support points displaced horizontally by distance $2d < 2l$ – see the figure on the right. As in the previous problem, the motion is not restricted to the vertical plane.



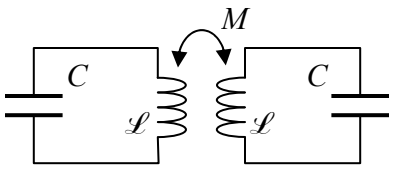
2.8. A block of mass m that can slide, without friction, along the inclined plane surface of a heavy wedge with mass m'. The wedge is free to move, also without friction, along a horizontal surface – see the figure on the right. (Both motions are within the vertical plane containing the steepest slope line.)



2.9. The two-pendula system that was the subject of Problem 1.8 – see the figure on the right.



2.10. A system of two similar, inductively coupled *LC* circuits – see the figure on the right.



2.11.* A small *Josephson junction* – the system consisting of two superconductors (S) weakly coupled by Cooper-pair tunneling through a thin insulating layer (I) that separates them – see the figure on the right.

Hints:

(i) At not very high frequencies (whose quantum $\hbar\omega$ is lower than the binding energy 2Δ of the Cooper pairs), the Josephson effect in a sufficiently small junction may be described by the following coupling energy:

$$U(\varphi) = -E_J \cos\varphi + \text{const},$$

where the constant E_J describes the coupling strength, while the variable φ (called the *Josephson phase difference*) is connected to the voltage V across the junction by the famous *frequency-to-voltage relation*

$$\frac{d\varphi}{dt} = \frac{2e}{\hbar}V,$$

where $e \approx 1.602\times10^{-19}$ C is the fundamental electric charge and $\hbar \approx 1.054\times10^{-34}$ J·s is the Planck constant.[18]

(ii) The junction (as any system of two close conductors) has a substantial electric capacitance C.

[18] More discussion of the Josephson effect and the physical sense of the variable φ may be found, for example, in EM Sec. 6.5 and QM Secs. 1.6 and 2.8, but the given problem may be solved without that additional information.

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