Chapter 4. Rigid Body Motion

This chapter discusses the motion of rigid bodies, with a heavy focus on its most nontrivial part: rotation. Some byproducts of this analysis enable a discussion, at the end of the chapter, of the motion of point particles as observed from non-inertial reference frames.

4.1. Translation and rotation

It is natural to start a discussion of many-particle systems from a (relatively :-) simple limit when the changes of distances $r_{kk'} \equiv |\mathbf{r}_k - \mathbf{r}_{k'}|$ between the particles are negligibly small. Such an abstraction is called the (*absolutely*) *rigid body*; it is a reasonable approximation in many practical problems, including the motion of solid samples. In other words, this model neglects *deformations* – which will be the subject of the next chapters. The rigid-body approximation reduces the number of degrees of freedom of the system of N particles from $3N$ to just six – for example, three Cartesian coordinates of one point (say, 0), and three angles of the system's rotation about three mutually perpendicular axes passing through this point – see Fig. 1.[1]

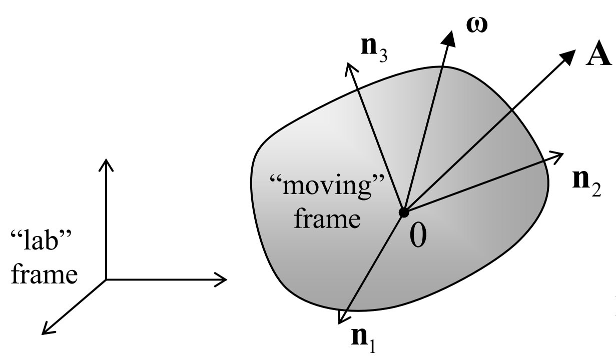


Fig. 4.1. Deriving Eq. (8).

As it follows from the discussion in Secs. 1.1-1.3, any purely *translational motion* of a rigid body, at which the velocity vectors $\mathbf{v}$ of all points are equal, is not more complex than that of a point particle. Indeed, according to Eqs. (1.8) and (1.30), in an inertial reference frame, such a body moves exactly as a point particle upon the effect of the net external force $\mathbf{F}^{(\text{ext})}$. However, the rotation is a bit more tricky.

Let us start by showing that an arbitrary elementary displacement of a rigid body may be always considered as a sum of the translational motion and of what is called a *pure rotation*. For that, consider a "moving" reference frame $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$, firmly bound to the body, and an arbitrary vector $\mathbf{A}$ (Fig. 1). The vector may be represented by its Cartesian components A_j in that moving frame:

$$\mathbf{A} = \sum_{j=1}^{3} A_j \mathbf{n}_j \, . \tag{4.1}$$

[1] An alternative way to arrive at the same number six is to consider three points of the body, which uniquely define its position. If movable independently, the points would have nine degrees of freedom, but since three distances $r_{kk'}$ between them are now fixed, the resulting three constraints reduce the number of degrees of freedom to six.

Let us calculate the time derivative of this vector as observed from a different ("lab") frame, taking into account that if the body rotates relative to this frame, the directions of the unit vectors $\mathbf{n}_j$, as seen from the lab frame, change in time. Hence, in each product contributing to the sum (1), we have to differentiate both operands:

$$\left.\frac{d\mathbf{A}}{dt}\right|_{\text{in lab}} = \sum_{j=1}^{3}\frac{dA_j}{dt}\mathbf{n}_j + \sum_{j=1}^{3} A_j\frac{d\mathbf{n}_j}{dt}. \tag{4.2}$$

On the right-hand side of this equality, the first sum obviously describes the change of vector $\mathbf{A}$ as observed from the moving frame. In the second sum, each of the infinitesimal vectors $d\mathbf{n}_j$ may be represented by its Cartesian components:

$$d\mathbf{n}_j = \sum_{j'=1}^{3} d\varphi_{jj'}\mathbf{n}_{j'}, \tag{4.3}$$

where $d\varphi_{jj'}$ are some dimensionless scalar coefficients. To find out more about them, let us scalar-multiply each side of Eq. (3) by an arbitrary unit vector $\mathbf{n}_{j''}$, and take into account the obvious orthonormality condition:

$$\mathbf{n}_{j'}\cdot\mathbf{n}_{j''} = \delta_{j'j''}, \tag{4.4}$$

where $\delta_{j'j''}$ is the Kronecker delta symbol.[2] As a result, we get

$$d\mathbf{n}_j\cdot\mathbf{n}_{j''} = d\varphi_{jj''}. \tag{4.5}$$

Now let us use Eq. (5) to calculate the first differential of Eq. (4):

$$d\mathbf{n}_{j'}\cdot\mathbf{n}_{j''} + \mathbf{n}_{j'}\cdot d\mathbf{n}_{j''} \equiv d\varphi_{j'j''} + d\varphi_{j''j'} = 0; \qquad \text{in particular,}\quad 2d\mathbf{n}_j\cdot\mathbf{n}_j = 2d\varphi_{jj} = 0. \tag{4.6}$$

These relations, valid for any choice of indices j, j', and j'' of the set $\{1, 2, 3\}$, show that the matrix with elements $d\varphi_{jj'}$ is antisymmetric with respect to the swap of its indices; this means that there are not nine just three non-zero independent coefficients $d\varphi_{jj'}$, all with $j \neq j'$. Hence it is natural to renumber them in a simpler way: $d\varphi_{jj'} = -d\varphi_{j'j} \equiv d\varphi_{j''}$, where the indices j, j', and j'' follow in the "correct" order – either $\{1,2,3\}$, or $\{2,3,1\}$, or $\{3,1,2\}$. It is straightforward to verify (either just by a component-by-component comparison or by using the Levi-Civita permutation symbol[3]) that in this new notation, Eq. (3) may be represented just as a vector product:

$$\boxed{d\mathbf{n}_j = d\boldsymbol{\varphi}\times\mathbf{n}_j,} \tag{4.7}$$

Elementary rotation

where $d\boldsymbol{\varphi}$ is the infinitesimal vector defined by its Cartesian components $d\varphi_j$ in the rotating reference frame $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$.

This relation is the basis of all rotation kinematics. Using it, Eq. (2) may be rewritten as

$$\boxed{\left.\frac{d\mathbf{A}}{dt}\right|_{\text{in lab}} = \left.\frac{d\mathbf{A}}{dt}\right|_{\text{in mov}} + \sum_{j=1}^{3} A_j\frac{d\boldsymbol{\varphi}}{dt}\times\mathbf{n}_j \equiv \left.\frac{d\mathbf{A}}{dt}\right|_{\text{in mov}} + \boldsymbol{\omega}\times\mathbf{A}, \quad \text{where } \boldsymbol{\omega}\equiv\frac{d\boldsymbol{\varphi}}{dt}.} \tag{4.8}$$

Vector's evolution in time

To reveal the physical sense of the vector $\boldsymbol{\omega}$, let us apply Eq. (8) to the particular case when $\mathbf{A}$ is the radius vector $\mathbf{r}$ of a point of the body, and the lab frame is selected in a special way: its origin has the

[2] See, e.g., MA Eq. (13.1).

[3] See, e.g., MA Eq. (13.2). Using this symbol, we may write $d\varphi_{jj'} = -d\varphi_{j'j} \equiv \varepsilon_{jj'j''}d\varphi_{j''}$ for *any* choice of j, j', and j''.

same position and moves with the same velocity as that of the moving frame, in the particular instant under consideration. In this case, the first term on the right-hand side of Eq. (8) is zero, and we get

$$\left.\frac{d\mathbf{r}}{dt}\right|_{\text{in special lab frame}} = \boldsymbol{\omega}\times\mathbf{r}, \tag{4.9}$$

were vector **r** itself is the same in both frames. According to the vector product definition, the particle velocity described by this formula has a direction perpendicular to the vectors $\boldsymbol{\omega}$ and **r** (Fig. 2), and magnitude $\omega r\sin\theta$. As Fig. 2 shows, the last expression may be rewritten as $\omega\rho$, where $\rho = r\sin\theta$ is the distance from the line that is parallel to the vector $\boldsymbol{\omega}$ and passes through point 0. This is of course just the *pure rotation* about that line (called the *instantaneous axis of rotation*), with the angular velocity ω. According to Eqs. (3) and (8), the *angular velocity vector* $\boldsymbol{\omega}$ is defined by the time evolution of the moving frame alone, so it is the same for all points **r**, i.e. for the rigid body as a whole. Note that nothing in our calculations forbids not only the magnitude but also the direction of the vector $\boldsymbol{\omega}$, and thus of the instantaneous axis of rotation, to change in time; hence the name.

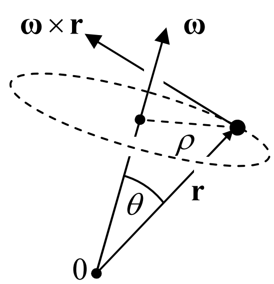


Fig. 4.2. The instantaneous axis and the angular velocity of rotation.

Now let us generalize our result a step further, considering two reference frames that do not rotate versus each other: one ("lab") frame is arbitrary, and another one is selected in the special way described above, so Eq. (9) is valid in it. Since the relative motion of these two reference frames is purely translational, we can use the simple velocity addition rule given by Eq. (1.6) to write

Body point's velocity

$$\boxed{\left.\mathbf{v}\right|_{\text{in lab}} = \left.\mathbf{v}_0\right|_{\text{in lab}} + \left.\mathbf{v}\right|_{\text{in special lab frame}} = \left.\mathbf{v}_0\right|_{\text{in lab}} + \boldsymbol{\omega}\times\mathbf{r},} \tag{4.10}$$

where **r** is the radius vector of a point is measured in the body-bound ("moving") frame 0.

4.2. Inertia tensor

Since the dynamics of each point of a rigid body is strongly constrained by the conditions $r_{kk'}$ = const, this is one of the most important fields of application of the Lagrangian formalism discussed in Chapter 2. For using this approach, the first thing we need to calculate is the kinetic energy of the body in an inertial reference frame. Since it is just the sum of the kinetic energies (1.19) of all its points, we can use Eq. (10) to write:[4]

$$T \equiv \sum\frac{m}{2}\mathbf{v}^2 = \sum\frac{m}{2}\left(\mathbf{v}_0 + \boldsymbol{\omega}\times\mathbf{r}\right)^2 \equiv \sum\frac{m}{2}v_0^2 + \sum m\mathbf{v}_0\cdot(\boldsymbol{\omega}\times\mathbf{r}) + \sum\frac{m}{2}(\boldsymbol{\omega}\times\mathbf{r})^2. \tag{4.11}$$

[4] Actually, all symbols for particle masses, coordinates, and velocities should carry the particle's index, over which the summation is carried out. However, in this section, for the notation simplicity, this index is just implied.

Let us apply to the right-hand side of Eq. (11) two general vector analysis formulas listed in the Math Appendix: the so-called *operand rotation rule* MA Eq. (7.6) to the second term, and MA Eq. (7.7b) to the third term. The result is

$$T = \sum \frac{m}{2} v_0^2 + \sum m\mathbf{r} \cdot (\mathbf{v}_0 \times \boldsymbol{\omega}) + \sum \frac{m}{2} \left[\omega^2 r^2 - (\boldsymbol{\omega} \cdot \mathbf{r})^2 \right]. \tag{4.12}$$

This expression may be further simplified by making a specific choice of the point 0 (from which the radius vectors $\mathbf{r}$ of all particles are measured), namely by using for this point the *center of mass* of the body. As was already mentioned in Sec. 3.4 for the two-point case, the radius vector $\mathbf{R}$ of this point is defined as

$$M\mathbf{R} \equiv \sum m\mathbf{r}, \quad \text{with } M \equiv \sum m, \tag{4.13}$$

so M is the total mass of the body. In the reference frame centered at this point, we have $\mathbf{R} = 0$, so that the second sum in Eq. (12) vanishes, and the kinetic energy is a sum of just two terms:

$$T = T_{\text{tran}} + T_{\text{rot}}, \quad T_{\text{tran}} \equiv \frac{M}{2} V^2, \quad T_{\text{rot}} \equiv \sum \frac{m}{2} \left[\omega^2 r^2 - (\boldsymbol{\omega} \cdot \mathbf{r})^2 \right], \tag{4.14}$$

where $\mathbf{V} \equiv d\mathbf{R}/dt$ is the center-of-mass velocity in our inertial reference frame, and all particle positions $\mathbf{r}$ are measured in the center-of-mass frame. Since the angular velocity vector $\boldsymbol{\omega}$ is common for all points of a rigid body, it is more convenient to rewrite the rotational part of the energy in a form in that the summation over the components of this vector is separated from the summation over the points of the body:

$$\boxed{T_{\text{rot}} = \frac{1}{2} \sum_{j,j'=1}^{3} I_{jj'} \omega_j \omega_{j'},} \tag{4.15}$$

Kinetic energy of rotation

where the 3×3 matrix with elements

$$\boxed{I_{jj'} \equiv \sum m\left(r^2 \delta_{jj'} - r_j r_{j'} \right)} \tag{4.16}$$

Inertia tensor

represents, in the selected reference frame, the *inertia tensor* of the body.[5]

Actually, the term "tensor" for the construct described by this matrix has to be justified, because in physics it implies a certain reference-frame-independent notion, whose matrix elements have to obey certain rules at the transfer between reference frames. To show that the matrix (16) indeed describes such a notion, let us calculate another key quantity, the total angular momentum $\mathbf{L}$ of the same body.[6] Summing up the angular momenta of each particle, defined by Eq. (1.31), and then using Eq. (10) again, in our inertial reference frame we get

$$\mathbf{L} \equiv \sum \mathbf{r} \times \mathbf{p} = \sum m\mathbf{r} \times \mathbf{v} = \sum m\mathbf{r} \times \left(\mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r} \right) \equiv \sum m\mathbf{r} \times \mathbf{v}_0 + \sum m\mathbf{r} \times \left(\boldsymbol{\omega} \times \mathbf{r} \right). \tag{4.17}$$

We see that the momentum may be represented as a sum of two terms. The first one,

[5] While the ABCs of the rotational dynamics were developed by Leonhard Euler in 1765, an introduction of the inertia tensor's formalism had to wait very long – until the invention of the tensor analysis by Tullio Levi-Civita and Gregorio Ricci-Curbastro in 1900 – soon popularized by its use in Einstein's general relativity.

[6] Hopefully, there is very little chance of confusing the angular momentum $\mathbf{L}$ (a vector) and its Cartesian components L_j (scalars with an index) on one hand, and the Lagrangian function L (a scalar without an index) on the other hand.

$$\mathbf{L}_0 \equiv \sum m\mathbf{r}\times\mathbf{v}_0 = M\mathbf{R}\times\mathbf{v}_0, \qquad (4.18)$$

describes the possible rotation of the center of mass about the inertial frame's origin. This term vanishes if the moving reference frame's origin 0 is positioned at the center of mass (where $\mathbf{R} = 0$). In this case, we are left with only the second term, which describes a pure rotation of the body about its center of mass:

$$\mathbf{L} = \mathbf{L}_{\text{rot}} \equiv \sum m\mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r}). \qquad (4.19)$$

Using one more vector algebra formula, the "bac minis cab" rule,[7] we may rewrite this expression as

$$\mathbf{L} = \sum m\left[\boldsymbol{\omega}r^2 - \mathbf{r}(\mathbf{r}\cdot\boldsymbol{\omega})\right]. \qquad (4.20)$$

Let us spell out an arbitrary Cartesian component of this vector:

$$L_j = \sum m\left[\omega_j r^2 - r_j\sum_{j'=1}^{3} r_{j'}\omega_{j'}\right] \equiv \sum m\sum_{j'=1}^{3}\omega_{j'}\left(r^2\delta_{jj'} - r_j r_{j'}\right). \qquad (4.21)$$

By changing the summation order and comparing the result with Eq. (16), the angular momentum may be conveniently expressed via the same matrix elements $I_{jj'}$ as the rotational kinetic energy:

Angular momentum

$$\boxed{L_j = \sum_{j'=1}^{3} I_{jj'}\omega_{j'}\ .} \qquad (4.22)$$

Since $\mathbf{L}$ and $\boldsymbol{\omega}$ are both legitimate vectors (meaning that they describe physical vectors independent of the reference frame choice), the matrix of elements $I_{jj'}$ that relates them is a legitimate tensor. This fact, and the symmetry of the tensor ($I_{jj'} = I_{j'j}$), evident from its definition (16), allow the tensor to be further simplified. In particular, mathematics tells us that by a certain choice of the coordinate axes' orientations, any symmetric tensor may be reduced to a diagonal form

$$I_{jj'} = I_j\delta_{jj'}, \qquad (4.23)$$

where in our case

Principal moments of inertia

$$\boxed{I_j = \sum m\left(r^2 - r_j^2\right) = \sum m\left(r_{j'}^2 + r_{j''}^2\right) \equiv \sum m\rho_j^2\ ,} \qquad (4.24)$$

ρ_j being the distance of the particle from the j^{th} axis, i.e. the length of the perpendicular dropped from the point to that axis. The axes of such a special coordinate system are called the *principal axes*, while the diagonal elements I_j given by Eq. (24), the *principal moments of inertia* of the body. In such a special reference frame, Eqs. (15) and (22) are reduced to very simple forms:

T_{rot} and **L** in principal-axes frame

$$\boxed{T_{\text{rot}} = \sum_{j=1}^{3}\frac{I_j}{2}\omega_j^2\ ,} \qquad (4.25)$$

$$\boxed{L_j = I_j\omega_j.} \qquad (4.26)$$

Both these results remind the corresponding relations for the translational motion, $T_{\text{tran}} = MV^2/2$ and $\mathbf{P} = M\mathbf{V}$, with the angular velocity $\boldsymbol{\omega}$ replacing the linear velocity $\mathbf{V}$, and the tensor of inertia playing the role of scalar mass M. However, let me emphasize that even in the specially selected reference frame, with

[7] See, e.g., MA Eq. (7.5).

its axes pointing in principal directions, the analogy is incomplete, and rotation is generally more complex than translation, because the measures of inertia, I_j, are generally different for each principal axis.

Let me illustrate the last fact on a simple but instructive system of three similar massive particles fixed in the vertices of an equilateral triangle (Fig. 3).

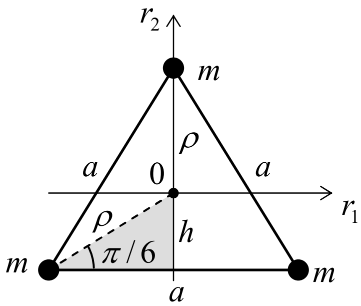


Fig. 4.3. Principal moments of inertia: a simple case study.

Due to the symmetry of the configuration, one of the principal axes has to pass through the center of mass 0 and be normal to the plane of the triangle. For the corresponding principal moment of inertia, Eq. (24) readily yields $I_3 = 3m\rho^2$. If we want to express this result in terms of the triangle's side a, we may notice that due to the system's symmetry, the angle marked in Fig. 3 equals $\pi/6$, and from the shaded right triangle, $a/2 = \rho\cos(\pi/6) \equiv \rho\sqrt{3}/2$, giving $\rho = a/\sqrt{3}$, so, finally, $I_3 = ma^2$.

Let me use this simple case to illustrate the following general *axis shift theorem*, which may be rather useful – especially for more complex systems. For that, let us relate the inertia tensor elements $I_{jj'}$ and $I'_{jj'}$, calculated in two reference frames – one with its origin at the center of mass 0, and another one (0') translated by a certain vector $\mathbf{d}$ (Fig. 4a), so for an arbitrary point, $\mathbf{r}' = \mathbf{r} + \mathbf{d}$. Plugging this relation into Eq. (16), we get

$$\begin{aligned} I'_{jj'} &= \sum m\left[(\mathbf{r}+\mathbf{d})^2\delta_{jj'} - (r_j + d_j)(r_{j'} + d_{j'})\right] \\ &= \sum m\left[(r^2 + 2\mathbf{r}\cdot\mathbf{d} + d^2)\delta_{jj'} - (r_j r_{j'} + r_j d_{j'} + r_{j'} d_j + d_j d_{j'})\right]. \end{aligned} \tag{4.27}$$

Since in the center-of-mass frame, all sums $\sum m r_j$ equal zero, we may use Eq. (16) to finally obtain

$$I'_{jj'} = I_{jj'} + M(\delta_{jj'} d^2 - d_j d_{j'}). \tag{4.28}$$

In particular, this equation shows that if the shift vector $\mathbf{d}$ is perpendicular to one (say, j^{th}) of the principal axes (Fig. 4b), i.e. $d_j = 0$, then Eq. (28) is reduced to a very simple formula:

$$\boxed{I'_j = I_j + Md^2.} \tag{4.29}$$

Principal axis' shift

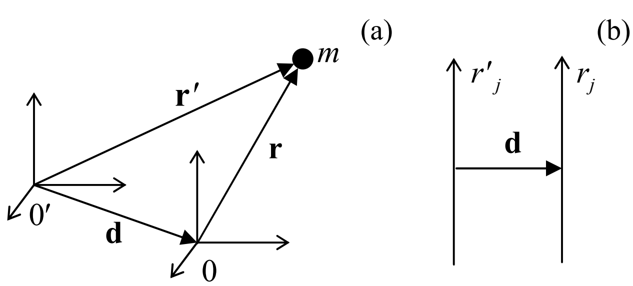


Fig. 4.4. (a) A general coordinate frame's shift from the center of mass, and (b) a shift perpendicular to one of the principal axes.

Now returning to the particular system shown in Fig. 3, let us perform such a shift to the new ("primed") axis passing through the location of one of the particles, still perpendicular to their common plane. Then the contribution of that particular mass to the primed moment of inertia vanishes, and $I'_3 = 2ma^2$. Now, returning to the center of mass and applying Eq. (29), we get $I_3 = I'_3 - M\rho^2 = 2ma^2 - (3m)(a/\sqrt{3})^2 = ma^2$, i.e. the same result as above.

The symmetry situation inside the triangle's plane is somewhat less obvious, so let us start by calculating the moments of inertia for the axes shown vertical and horizontal in Fig. 3. From Eq. (24), we readily get:

$$I_1 = 2mh^2 + m\rho^2 = m\left[2\left(\frac{a}{2\sqrt{3}}\right)^2 + \left(\frac{a}{\sqrt{3}}\right)^2\right] = \frac{ma^2}{2}, \qquad I_2 = 2m\left(\frac{a}{2}\right)^2 = \frac{ma^2}{2}, \tag{4.30}$$

where h is the distance from the center of mass and any side of the triangle: $h = \rho\sin(\pi/6) = \rho/2 = a/2\sqrt{3}$. We see that $I_1 = I_2$, and mathematics tells us that in this case, *any* in-plane axis (passing through the center-of-mass 0) may be considered as principal, and has the same moment of inertia. A rigid body with this property, $I_1 = I_2 \neq I_3$, is called the *symmetric top*. (The last direction is called the *main principal axis* of the system.)

Symmetric top: definition

Despite the symmetric top's name, the situation may be even more symmetric in the so-called *spherical tops*, i.e. highly symmetric systems whose principal moments of inertia are all equal,

Spherical top: definition

$$\boxed{I_1 = I_2 = I_3 \equiv I,} \tag{4.31}$$

Mathematics says that in this case, the moment of inertia for rotation about *any* axis (but still passing through the center of mass) is equal to the same I. Hence Eqs. (25) and (26) are further simplified for any direction of the vector $\boldsymbol{\omega}$:

Spherical top: properties

$$\boxed{T_{\text{rot}} = \frac{I}{2}\omega^2, \qquad \mathbf{L} = I\boldsymbol{\omega},} \tag{4.32}$$

thus making the analogy of rotation and translation complete. (As will be discussed in the next section, this analogy is also complete if the rotation axis is fixed by external constraints.)

Evident examples of a spherical top are a uniform sphere and a uniform spherical shell; its less obvious example is a uniform cube – with masses either concentrated in vertices, or uniformly spread over the faces, or uniformly distributed over the volume. Again, in this case *any* axis passing through the center of mass is a principal one and has the same principal moment of inertia. For a sphere, this is natural; for a cube, rather surprising – but may be confirmed by a direct calculation.

4.3. Fixed-axis rotation

Now we are well equipped for a discussion of the rigid body's rotational dynamics. The general equation of this dynamics is given by Eq. (1.38), which is valid for dynamics of any system of particles – either rigidly connected or not:

$$\dot{\mathbf{L}} = \boldsymbol{\tau}, \tag{4.33}$$

where $\boldsymbol{\tau}$ is the net torque of external forces. Let us start exploring this equation from the simplest case when the axis of rotation, i.e. the direction of vector $\boldsymbol{\omega}$, is fixed by some external constraints. Directing the z-axis along this vector, we have $\omega_x = \omega_y = 0$. According to Eq. (22), in this case, the z-component of the angular momentum,

$$L_z = I_{zz}\omega_z, \tag{4.34}$$

where I_{zz}, though not necessarily one of the principal moments of inertia. still may be calculated using Eq. (24):

$$I_{zz} = \sum m\rho_z^2 = \sum m\left(x^2 + y^2\right), \tag{4.35}$$

with ρ_z being the distance of each particle from the rotation axis z. According to Eq. (15), in this case the rotational kinetic energy is just

$$T_{\text{rot}} = \frac{I_{zz}}{2}\omega_z^2. \tag{4.36}$$

Moreover, it is straightforward to show that if the rotation axis is fixed, Eqs. (34)-(36) are valid even if the axis does not pass through the center of mass – provided that the distances ρ_z are now measured from that axis. (The proof is left for the reader's exercise.)

As a result, we may not care about other components of the vector **L**,[8] and use just one component of Eq. (33),

$$\dot{L}_z = \tau_z, \tag{4.37}$$

because it, when combined with Eq. (34), completely determines the dynamics of rotation:

$$I_{zz}\dot{\omega}_z = \tau_z, \qquad \text{i.e. } I_{zz}\ddot{\theta}_z = \tau_z, \tag{4.38}$$

where θ_z is the angle of rotation about the axis, so $\omega_z = \dot{\theta}$. The scalar relations (34), (36), and (38), describing rotation about a fixed axis, are completely similar to the corresponding formulas of 1D motion of a single particle, with ω_z corresponding to the usual ("linear") velocity, the angular momentum component L_z – to the linear momentum, and I_z – to the particle's mass.

The resulting motion about the axis is also frequently similar to that of a single particle. As a simple example, let us consider what is called the *physical* (or "compound") *pendulum* (Fig. 5) – a rigid body free to rotate about a fixed horizontal axis that does not pass through the center of mass 0, in a uniform gravity field **g**.

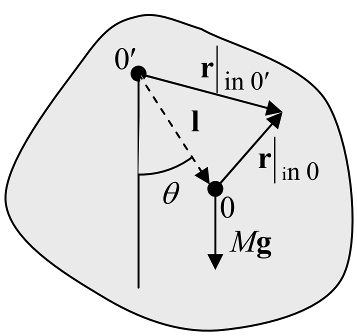


Fig. 4.5. Physical pendulum: a rigid body with a fixed (horizontal) rotation axis 0' that does not pass through the center of mass 0. (The plane of drawing is normal to that axis.)

[8] Note that according to Eq. (22), other Cartesian components of the angular momentum, L_x and L_y, may be different from zero, and even evolve in time. The corresponding torques τ_x and τ_y, which obey Eq. (33), are automatically provided by the external forces that keep the rotation axis fixed.

Let us drop the perpendicular from point 0 to the rotation axis, and call the oppositely directed vector **l** – see the dashed arrow in Fig. 5. Then the torque (relative to the rotation axis 0') of the forces keeping the axis fixed is zero, and the only contribution to the net torque is due to gravity alone:

$$\boldsymbol{\tau}\Big|_{\text{in }0'} \equiv \sum \mathbf{r}\Big|_{\text{in }0'} \times \mathbf{F} = \sum \left(\mathbf{l} + \mathbf{r}\Big|_{\text{in }0}\right) \times m\mathbf{g} = \sum m(\mathbf{l}\times\mathbf{g}) + \sum m\mathbf{r}\Big|_{\text{in }0} \times \mathbf{g} = M\mathbf{l}\times\mathbf{g}. \qquad (4.39)$$

(The last step used the facts that point 0 is the center of mass, so the second term on the right-hand side equals zero, and that the vectors **l** and **g** are the same for all particles of the body.)

This result shows that the torque is directed along the rotation axis, and its (only) component τ_z is equal to $-Mgl\sin\theta$, where θ is the angle between the vectors **l** and **g**, i.e. the angular deviation of the pendulum from the position of equilibrium – see Fig. 5 again. As a result, Eq. (38) takes the form,

$$I'\ddot{\theta} = -Mgl\sin\theta, \qquad (4.40)$$

where I' is the moment of inertia for rotation about the axis 0' rather than about the center of mass. This equation is identical to Eq. (1.18) for the point-mass (sometimes called "mathematical") pendulum, with small-oscillation frequency

Physical pendulum: frequency

$$\boxed{\Omega = \left(\frac{Mgl}{I'}\right)^{1/2} \equiv \left(\frac{g}{l_{\text{ef}}}\right)^{1/2}, \quad \text{with } l_{\text{ef}} \equiv \frac{I'}{Ml}.} \qquad (4.41)$$

As a sanity check, in the simplest case when the linear size of the body is much smaller than the suspension length l, Eq. (35) yields $I' = Ml^2$, i.e. $l_{\text{ef}} = l$, and Eq. (41) reduces to the well-familiar formula $\Omega = (g/l)^{1/2}$ for the point-mass pendulum.

Now let us discuss the situations when a rigid body not only rotates but also moves as a whole. As was mentioned in the introductory chapter, the total linear momentum of the body,

$$\mathbf{P} \equiv \sum m\mathbf{v} = \sum m\dot{\mathbf{r}} = \frac{d}{dt}\sum m\mathbf{r}, \qquad (4.42)$$

satisfies the 2nd Newton's law in the form (1.30). Using the definition (13) of the center of mass, the momentum may be represented as

$$\mathbf{P} = M\dot{\mathbf{R}} = M\mathbf{V}, \qquad (4.43)$$

so Eq. (1.30) may be rewritten as

C.o.m.: law of motion

$$\boxed{M\dot{\mathbf{V}} = \mathbf{F},} \qquad (4.44)$$

where **F** is the vector sum of all external forces. This equation shows that the center of mass of the body moves exactly like a point particle of mass M, under the effect of the net force **F**. In many cases, this fact makes the translational dynamics of a rigid body absolutely similar to that of a point particle.

The situation becomes more complex if some of the forces contributing to the vector sum **F** depend on the rotation of the same body, i.e. if its rotational and translational motions are coupled. Analysis of such coupled motion is rather straightforward if the *direction* of the rotation axis does not change in time, and hence Eqs. (34)-(36) are still valid. Possibly the simplest example is a round cylinder (say, a wheel) rolling on a surface without slippage (Fig. 6). Here the no-slippage condition may be represented as the requirement to the net velocity of the particular wheel's point A that touches the surface to equal zero – in the reference frame bound to the surface. For the simplest case of plane surface (Fig. 6a), this condition may be spelled out using Eq. (10), giving the following relation between the angular velocity ω of the wheel and the linear velocity V of its center:

$$V + r\omega = 0. \tag{4.45}$$

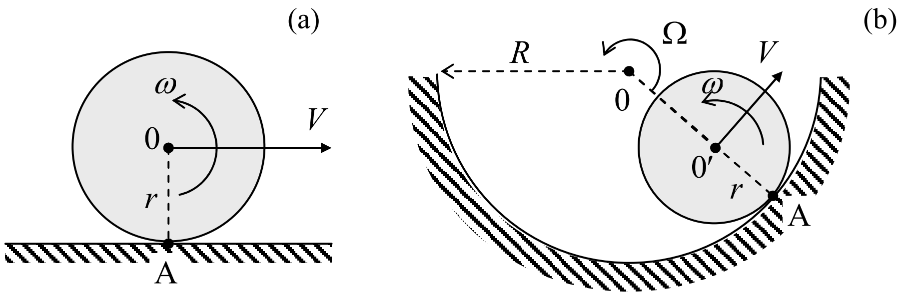


Fig. 4.6. Round cylinder rolling over (a) a plane surface and (b) a concave surface.

Such kinematic relations are essentially holonomic constraints, which reduce the number of degrees of freedom of the system. For example, without the no-slippage condition (45), the wheel on a plane surface has to be considered as a system with two degrees of freedom, making its total kinetic energy (14) a function of two independent generalized velocities, say V and ω:

$$T = T_{\text{tran}} + T_{\text{rot}} = \frac{M}{2}V^2 + \frac{I}{2}\omega^2. \tag{4.46}$$

Using Eq. (45) we may eliminate, for example, the linear velocity and reduce Eq. (46) to

$$T = \frac{M}{2}(\omega r)^2 + \frac{I}{2}\omega^2 \equiv \frac{I_{\text{ef}}}{2}\omega^2, \qquad \text{where } I_{\text{ef}} \equiv I + Mr^2. \tag{4.47}$$

This result may be interpreted as the kinetic energy of pure rotation of the wheel about the instantaneous rotation axis A, with I_{ef} being the moment of inertia about that axis, satisfying Eq. (29).

Kinematic relations are not always as simple as Eq. (45). For example, if a wheel is rolling on a concave surface (Fig. 6b), we need to relate the angular velocities of the wheel's rotation about its axis 0' (say, ω) and that (say, Ω) of its axis' rotation about the center 0 of curvature of the surface. A popular error here is to write $\Omega = -(r/R)\omega$ [WRONG!]. A prudent way to derive the correct relation is to note that Eq. (45) holds for this situation as well, and on the other hand, the same linear velocity of the wheel's center may be expressed as $V = (R - r)\,\Omega$. Combining these formulas, we get the correct relation

$$\Omega = -\frac{r}{R-r}\omega. \tag{4.48}$$

Another famous example of the relation between translational and rotational motion is given by the "sliding-ladder" problem (Fig. 7). Let us analyze it for the simplest case of negligible friction, and the ladder's thickness being small in comparison with its length l.

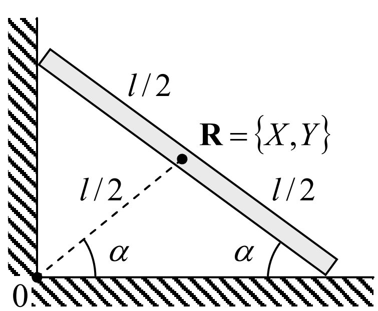


Fig. 4.7. The sliding-ladder problem.

To use the Lagrangian formalism, we may write the kinetic energy of the ladder as the sum (14) of its translational and rotational parts:

$$T = \frac{M}{2}\left(\dot{X}^2 + \dot{Y}^2\right) + \frac{I}{2}\dot{\alpha}^2, \tag{4.49}$$

where X and Y are the Cartesian coordinates of its center of mass in an inertial reference frame, and I is the moment of inertia for rotation about the z-axis passing through the center of mass. (For the uniformly distributed mass, an elementary integration of Eq. (35) yields $I = Ml^2/12$). In the reference frame with the center in the corner 0, both X and Y may be simply expressed via the angle α :

$$X = \frac{l}{2}\cos\alpha, \quad Y = \frac{l}{2}\sin\alpha. \tag{4.50}$$

(The easiest way to obtain these relations is to notice that the dashed line in Fig. 7 has length $l/2$, and the same slope α as the ladder.) Plugging these expressions into Eq. (49), we get

$$T = \frac{I_{\text{ef}}}{2}\dot{\alpha}^2, \qquad I_{\text{ef}} \equiv I + M\left(\frac{l}{2}\right)^2 = \frac{1}{3}Ml^2. \tag{4.51}$$

Since the potential energy of the ladder in the gravity field may be also expressed via the same angle,

$$U = MgY = Mg\frac{l}{2}\sin\alpha, \tag{4.52}$$

α may be conveniently used as the (only) generalized coordinate of the system. Even without writing the Lagrange equation of motion for that coordinate, we may notice that since the Lagrangian function $L \equiv T - U$ does not depend on time explicitly, and the kinetic energy (51) is a quadratic-homogeneous function of the generalized velocity $\dot{\alpha}$, the full mechanical energy,

$$E \equiv T + U = \frac{I_{\text{ef}}}{2}\dot{\alpha}^2 + Mg\frac{l}{2}\sin\alpha = \frac{Mgl}{2}\left(\frac{l\dot{\alpha}^2}{3g} + \sin\alpha\right), \tag{4.53}$$

is conserved, giving us the first integral of motion. Moreover, Eq. (53) shows that the system's energy (and hence dynamics) is identical to that of a physical pendulum with an unstable fixed point $\alpha_1 = \pi/2$, a stable fixed point at $\alpha_2 = -\pi/2$, and frequency

$$\Omega = \left(\frac{3g}{2l}\right)^{1/2} \tag{4.54}$$

of small oscillations near the latter point. (Of course, this fixed point cannot be reached in the simple geometry shown in Fig. 7, where the ladder's fall on the floor would change its equations of motion. Moreover, even before that, the left end of the ladder may detach from the wall. The analysis of this issue is left for the reader's exercise.)

4.4. Free rotation

Now let us proceed to more complex situations when the rotation axis is *not* fixed. A good illustration of the complexity arising in this case comes from the case of a rigid body left alone, i.e. not subjected to external forces and hence with its potential energy U constant. Since in this case, according to Eq. (44), the center of mass (as observed from any inertial reference frame) moves with a constant velocity, we can always use a convenient inertial reference frame with the origin at that point. From the point of view of such a frame, the body's motion is a pure rotation, and $T_{\text{tran}} = 0$. Hence, the system's Lagrangian function is just its rotational energy (15), which is, first, a quadratic-homogeneous function of the components ω_j (which may be taken for generalized velocities), and, second, does not depend on time explicitly. As we know from Chapter 2, in this case the mechanical energy, here equal to T_{rot} alone, is conserved. According to Eq. (15), for the principal-axes components of the vector **ω**, this means

Rotational energy's conservation

$$\boxed{T_{\text{rot}} = \sum_{j=1}^{3} \frac{I_j}{2}\omega_j^2 = \text{const}\,.} \tag{4.55}$$

Next, as Eq. (33) shows, in the absence of external forces, the angular momentum **L** of the body is conserved as well. However, though we can certainly use Eq. (26) to represent this fact as

Angular momentum's conservation

$$\boxed{\mathbf{L} = \sum_{j=1}^{3} I_j \omega_j \mathbf{n}_j = \text{const}\,,} \tag{4.56}$$

where $\mathbf{n}_j$ are the principal axes, this does not mean that all components ω_j are constant, because the principal axes are fixed relative to the rigid body, and hence may rotate with it.

Before exploring these complications, let us briefly mention two conceptually easy, but practically very important cases. The first is a spherical top ($I_1 = I_2 = I_3 = I$). In this case, Eqs. (55) and (56) imply that all components of the vector $\boldsymbol{\omega} = \mathbf{L}/I$, i.e. both the magnitude and the direction of the angular velocity are conserved, for any initial spin. In other words, the body conserves its rotation speed and axis direction, as measured in an inertial frame. The most obvious example is a spherical planet. For example, our Mother Earth, rotating about its axis with angular velocity $\omega = 2\pi/(1\text{ day}) \approx 7.3\times10^{-5}\text{ s}^{-1}$, keeps its axis at a nearly constant angle of 23°27' to the *ecliptic pole*, i.e. to the axis normal to the plane of its motion around the Sun. (In Sec. 6 below, we will discuss some very slow motions of this axis, due to gravity effects.)

Spherical tops are also used in the most accurate gyroscopes, usually with gas-jet or magnetic suspension in vacuum. If done carefully, such systems may have spectacular stability. For example, the gyroscope system of the Gravity Probe B satellite experiment, flown in 2004-2005, was based on quartz spheres – round with a precision of about 10 nm and covered with superconducting thin films (which enabled their magnetic suspension and monitoring). The whole system was stable enough to measure the so-called *geodetic effect* in general relativity (essentially, the space curving by the Earth's mass), resulting in the axis' precession by only 6.6 arc seconds per year, i.e. with an angular velocity of just $\sim10^{-11}\text{s}^{-1}$, with experimental results agreeing with theory with a record ~0.3% accuracy.[9]

The second simple case is that of the symmetric top ($I_1 = I_2 \neq I_3$) with the initial vector **L** aligned with the main principal axis. In this case, $\boldsymbol{\omega} = \mathbf{L}/I_3 = \text{const}$, so the rotation axis is conserved.[10] Such tops, typically in the shape of a *flywheel* (heavy, flat rotor), and supported by *gimbal systems* (also called the "Cardan suspensions") that allow for virtually torque-free rotation about three mutually perpendicular

[9] Still, the main goal of this rather expensive (~$750M) project, an accurate measurement of a more subtle relativistic effect, the so-called *frame-dragging drift* (also called "the Schiff precession"), predicted to be about 0.04 arc seconds per year, has not been achieved.

[10] This is also true for an asymmetric top, i.e. an arbitrary body (with, say, $I_1 < I_2 < I_3$), but in this case the alignment of the vector **L** with the axis $\mathbf{n}_2$ corresponding to the intermediate moment of inertia, is unstable.

axes,[11] are broadly used in more common gyroscopes. Invented by Léon Foucault in the 1850s and made practical later by H. Anschütz-Kaempfe, such gyroscopes have become core parts of automatic guidance systems, for example, in ships, airplanes, missiles, etc. Even if its support wobbles and/or drifts, the suspended gyroscope sustains its direction relative to an inertial reference frame.[12]

However, in the general case with no such special initial alignment, the dynamics of symmetric tops is more complicated. In this case, the vector **L** is still conserved, including its direction, but the vector **ω** is not. Indeed, let us direct the $\mathbf{n}_2$-axis normally to the common plane of the vector **L** and the current instantaneous direction $\mathbf{n}_3$ of the main principal axis (in Fig. 8 below, the plane of the drawing); then, in that particular instant, $L_2 = 0$. Now let us recall that in a symmetric top, the axis $\mathbf{n}_2$ is a principal one. According to Eq. (26) with $j = 2$, the corresponding component ω_2 has to be equal to L_2/I_2, so it is equal to zero. This means that in the particular instant we are considering, the vector **ω** lies in this plane (the common plane of vectors **L** and $\mathbf{n}_3$) as well – see Fig. 8a.

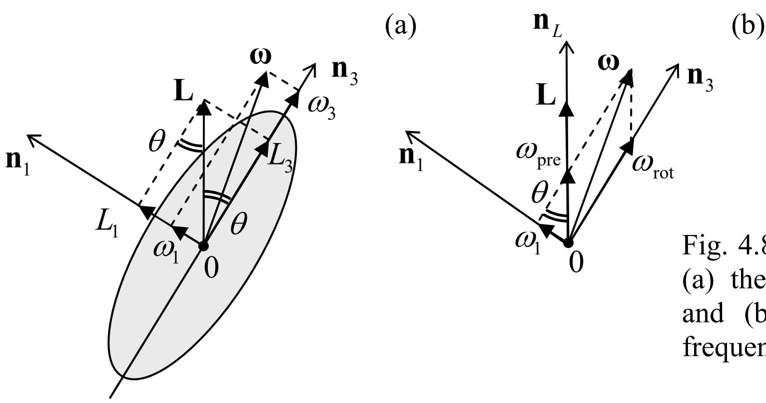


Fig. 4.8. Free rotation of a symmetric top: (a) the general configuration of vectors, and (b) calculating the free precession frequency.

Now consider some point located on the main principal axis $\mathbf{n}_3$, and hence on the plane [$\mathbf{n}_3$, **L**]. Since **ω** is the instantaneous axis of rotation, according to Eq. (9), the point's instantaneous velocity $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$ is directed normally to that plane. This is true for each point of the main axis (besides only one, with $\mathbf{r} = 0$, i.e. the center of mass, which does not move), so the axis as a whole has to move normally to the common plane of the vectors **L**, **ω**, and $\mathbf{n}_3$, while still passing through point 0. Since this conclusion is valid for any moment of time, it means that the vectors **ω** and $\mathbf{n}_3$ rotate about the space-fixed vector **L** together, with some angular velocity ω_{pre}, at each moment staying within one plane. This effect is called the *free* (or "torque-free", or "regular") *precession*, and has to be clearly distinguished it from the completely different effect of the *torque-induced precession*, which will be discussed in the next section.

To calculate ω_{pre}, let us represent the instant vector **ω** as a sum of not its Cartesian components (as in Fig. 8a), but rather of two non-orthogonal vectors directed along $\mathbf{n}_3$ and **L** (Fig. 8b):

$$\boldsymbol{\omega} = \omega_{\text{rot}}\mathbf{n}_3 + \omega_{\text{pre}}\mathbf{n}_L, \qquad \mathbf{n}_L \equiv \frac{\mathbf{L}}{L}. \tag{4.57}$$

---

[11] See, for example, a nice animation available online at http://en.wikipedia.org/wiki/Gimbal.

[12] Currently, optical gyroscopes are becoming more popular for all but the most precise applications. Much more compact but also much less accurate gyroscopes used, for example, in smartphones and tablet computers, are based on the effect of rotation on 2D mechanical oscillators (whose analysis is left for the reader's exercise), and are implemented as micro-electro-mechanical systems (MEMS) – see, e.g., Chapter 22 in V. Kaajakari, *Practical MEMS*, Small Gear Publishing, 2009.

Fig. 8b shows that ω_{rot} has the meaning of the angular velocity of rotation of the body about its main principal axis, while ω_{pre} is the angular velocity of rotation of that axis about the constant direction of the vector **L,** i.e. is exactly the frequency of precession that we are trying to find. Now ω_{pre} may be readily calculated from the comparison of two panels of Fig. 8, by noticing that the same angle θ between the vectors **L** and $\mathbf{n}_3$ participates in two relations:

$$\sin\theta = \frac{L_1}{L} = \frac{\omega_1}{\omega_{\text{pre}}}. \tag{4.58}$$

Since the $\mathbf{n}_1$-axis is a principal one, we may use Eq. (26) for $j = 1$, i.e. $L_1 = I_1\omega_1$, to eliminate ω_1 from Eq. (58), and get a very simple formula

$$\boxed{\omega_{\text{pre}} = \frac{L}{I_1}.} \tag{4.59}$$

Free precession: lab frame

This result shows that the precession *frequency* is constant and independent of the alignment of the vector **L** with the main principal axis $\mathbf{n}_3$, while its *amplitude* (characterized by the angle θ) does depend on the initial alignment, and vanishes if **L** is parallel to $\mathbf{n}_3$.[13] Note also that if all principal moments of inertia are of the same order, ω_{pre} is of the same order as the total angular speed $\omega \equiv |\boldsymbol{\omega}|$ of the rotation.

Now let us briefly discuss the free precession in the general case of an "asymmetric top", i.e. a body with arbitrary $I_1 \neq I_2 \neq I_3$. In this case, the effect is more complex because here not only the *direction* but also the *magnitude* of the instantaneous angular velocity $\boldsymbol{\omega}$ may evolve in time. If we are only interested in the relation between the instantaneous values of ω_j and L_j, i.e. the "trajectories" of the vectors $\boldsymbol{\omega}$ and **L** as observed from the reference frame $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ of the principal axes of the body, rather than in the explicit law of their time evolution, they may be found directly from the conservation laws. (Let me emphasize again that the vector **L**, being constant in an inertial reference frame, generally evolves in the frame rotating with the body.) Indeed, Eq. (55) may be understood as the equation of an ellipsoid in the Cartesian coordinates $\{\omega_1, \omega_2, \omega_3\}$, so for a free body, the vector $\boldsymbol{\omega}$ has to stay on the surface of that ellipsoid.[14] On the other hand, since the reference frame's rotation preserves the *length* of any vector, the *magnitude* (but not the direction!) of the vector **L** is also an integral of motion in the moving frame, and we can write

$$L^2 \equiv \sum_{j=1}^{3} L_j^2 = \sum_{j=1}^{3} I_j^2 \omega_j^2 = \text{const}. \tag{4.60}$$

Hence the trajectory of the vector $\boldsymbol{\omega}$ follows the closed curve formed by the intersection of two ellipsoids, (55) and (60) – the so-called *Poinsot construction*. It is evident that this trajectory is generally "taco-edge-shaped", i.e. more complex than a planar circle, but never very complex either.[15]

The same argument may be repeated for the vector **L**, for whom the first form of Eq. (60) describes a sphere, and Eq. (55), another ellipsoid:

---

[13] For our Earth, free precession's amplitude is so small (corresponding to sub-10-m linear deviations of the symmetry axis from the vector **L** at the surface) that this effect is of the same order as other, more irregular motions of the axis, resulting from turbulent fluid flow effects in the planet's interior and its atmosphere.

[14] It is frequently called the *Poinsot's ellipsoid*, named after Louis Poinsot (1777-1859) who has made several important contributions to rigid body mechanics.

[15] Curiously, the "wobbling" motion along such trajectories was observed not only for macroscopic rigid bodies but also for heavy atomic nuclei – see, e.g., N. Sensharma *et al.*, *Phys. Rev. Lett.* **124**, 052501 (2020).

$$T_{\text{rot}} = \sum_{j=1}^{3} \frac{1}{2I_j} L_j^2 = \text{const} . \tag{4.61}$$

On the other hand, if we are interested in the trajectory of the vector $\boldsymbol{\omega}$ as observed from an *inertial* frame (in which the vector **L** stays still), we may note that the general relation (15) for the same rotational energy T_{rot} may also be rewritten as

$$T_{\text{rot}} = \frac{1}{2}\sum_{j=1}^{3} \omega_j \sum_{j'=1}^{3} I_{jj'}\omega_{j'} . \tag{4.62}$$

But according to the Eq. (22), the second sum on the right-hand side is nothing more than L_j, so

$$T_{\text{rot}} = \frac{1}{2}\sum_{j=1}^{3} \omega_j L_j = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} . \tag{4.63}$$

This equation shows that for a free body (T_{rot} = const, **L** = const), even if the vector $\boldsymbol{\omega}$ changes in time, its endpoint should stay on a plane normal to the angular momentum **L**. Earlier, we have seen that for the particular case of the symmetric top – see Fig. 8b, but for an asymmetric top, the trajectory of the endpoint may not be circular.

If we are interested not only in the trajectory of the vector $\boldsymbol{\omega}$ but also in the law of its evolution in time, it may be calculated using the general Eq. (33) expressed in the principal components ω_j. For that, we have to recall that Eq. (33) is only valid in an inertial reference frame, while the frame $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ may rotate with the body and hence is generally not inertial. We may handle this problem by applying, to the vector **L**, the general kinematic relation (8):

$$\left.\frac{d\mathbf{L}}{dt}\right|_{\text{in lab}} = \left.\frac{d\mathbf{L}}{dt}\right|_{\text{in mov}} + \boldsymbol{\omega}\times\mathbf{L}. \tag{4.64}$$

Combining it with Eq. (33), in the moving frame we get

$$\frac{d\mathbf{L}}{dt} + \boldsymbol{\omega}\times\mathbf{L} = \boldsymbol{\tau} , \tag{4.65}$$

where $\boldsymbol{\tau}$ is the external torque. In particular, for the principal-axis components L_j, related to the components ω_j by Eq. (26), the vector equation (65) is reduced to a set of three scalar *Euler equations*

Euler equations

$$\boxed{I_j\dot{\omega}_j + (I_{j''} - I_{j'})\omega_{j'}\omega_{j''} = \tau_j ,} \tag{4.66}$$

where the set of indices $\{ j, j', j'' \}$ has to follow the usual "right" order – e.g., {1, 2, 3}, etc.[16]

In order to get a feeling of how the Euler equations work, let us return to the particular case of a free symmetric top ($\tau_1 = \tau_2 = \tau_3 = 0$, $I_1 = I_2 \neq I_3$). In this case, $I_1 - I_2 = 0$, so Eq. (66) with $j = 3$ yields ω_3 = const, while the equations for $j = 1$ and $j = 2$ take the following simple form:

$$\dot{\omega}_1 = -\Omega_{\text{pre}}\omega_2 , \qquad \dot{\omega}_2 = \Omega_{\text{pre}}\omega_1 , \tag{4.67}$$

where Ω_{pre} is a constant determined by both the system parameters and the initial conditions:

[16] These equations are of course valid in the simplest case of the fixed rotation axis as well. For example, if $\boldsymbol{\omega} = \mathbf{n}_z\omega$, i.e. $\omega_x = \omega_y = 0$, Eq. (66) is reduced to Eq. (38).

$$\boxed{\Omega_{\text{pre}} \equiv \omega_3 \frac{I_3 - I_1}{I_1}}. \tag{4.68}$$

Free precession: body frame

The system of two equations (67) has a sinusoidal solution with frequency Ω_{pre}, and describes a uniform rotation of the vector $\boldsymbol{\omega}$, with that frequency, about the main axis $\mathbf{n}_3$. This is just another representation of the free precession analyzed above, but this time as observed from the rotating body. Evidently, Ω_{pre} is substantially different from the frequency ω_{pre} (59) of the precession as observed from the lab frame; for example, Ω_{pre} vanishes for the spherical top (with $I_1 = I_2 = I_3$), while ω_{pre}, in this case, is equal to the rotation frequency.[17]

Unfortunately, for the rotation of an asymmetric top (i.e., an arbitrary rigid body) the Euler equations (66) are substantially nonlinear even in the absence of external torque, and may be solved analytically only in just a few cases. One of them is a proof of the already mentioned fact: the free top's rotation about one of its principal axes is stable if the corresponding principal moment of inertia is either the largest or the smallest one of the three. (This proof is easy, and is left for the reader's exercise.)

4.5. Torque-induced precession

The dynamics of rotation becomes even more complex in the presence of external forces. Let us consider the most counter-intuitive effect of *torque-induced precession*, for the simplest case of an axially-symmetric body (which is a particular case of the symmetric top, $I_1 = I_2 \neq I_3$), supported at some point A of its symmetry axis, that does not coincide with the center of mass 0 – see Fig. 9.

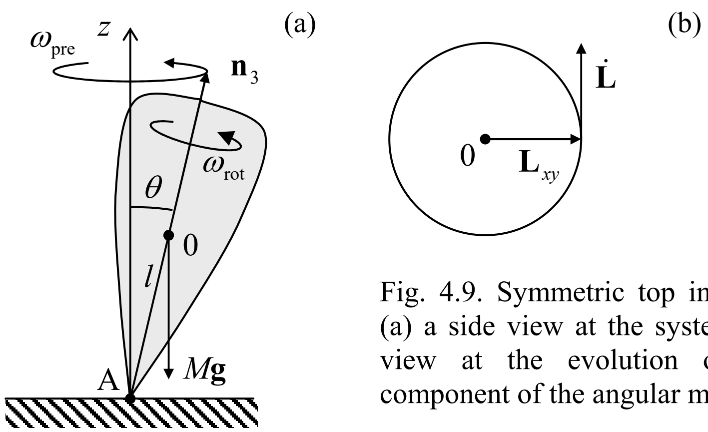


Fig. 4.9. Symmetric top in the gravity field: (a) a side view at the system and (b) the top view at the evolution of the horizontal component of the angular momentum vector.

The uniform gravity field $\mathbf{g}$ creates bulk-distributed forces that, as we know from the analysis of the physical pendulum in Sec. 3, are equivalent to a single force $M\mathbf{g}$ applied in the center of mass – in Fig. 9, point 0. The torque of this force relative to the support point A is

$$\boldsymbol{\tau} = \mathbf{r}_0\big|_{\text{in A}} \times M\mathbf{g} = Ml\mathbf{n}_3 \times \mathbf{g}. \tag{4.69}$$

Hence the general equation (33) of the angular momentum evolution (valid in any inertial frame, for example the one with its origin at point A) becomes

[17] For our Earth with its *equatorial bulge* (see Sec. 6 below), the ratio $(I_3 - I_1)/I_1$ is ~1/300, so that $2\pi/\Omega_{\text{pre}}$ is about 10 months. However, due to the fluid flow effects mentioned above, the observed precession is not very regular.

$$\dot{\mathbf{L}} = Ml\mathbf{n}_3 \times \mathbf{g}. \tag{4.70}$$

Despite the apparent simplicity of this (exact!) equation, its analysis is straightforward only in the limit when the top is spinning about its symmetry axis $\mathbf{n}_3$ with a very high angular velocity ω_{rot}. In this case, we may neglect the contribution to $\mathbf{L}$ due to a relatively small precession velocity ω_{pre} (still to be calculated), and use Eq. (26) to write

$$\mathbf{L} = I_3\boldsymbol{\omega} = I_3\omega_{\text{rot}}\mathbf{n}_3. \tag{4.71}$$

Then Eq. (70) shows that the vector $\dot{\mathbf{L}}$ is perpendicular to both $\mathbf{n}_3$ (and hence $\mathbf{L}$) and $\mathbf{g}$, i.e. lies within a horizontal plane and is perpendicular to the horizontal component $\mathbf{L}_{xy}$ of the vector $\mathbf{L}$ – see Fig. 9b. Since, according to Eq. (70), the magnitude of this vector is constant, $|\dot{\mathbf{L}}| = Mgl\sin\theta$, the vector $\mathbf{L}$ (and hence the body's main axis) rotates about the vertical axis with the following angular velocity:

Torque-induced precession: fast-rotation limit

$$\boxed{\omega_{\text{pre}} = \frac{|\dot{\mathbf{L}}|}{L_{xy}} = \frac{Mgl\sin\theta}{L\sin\theta} \equiv \frac{Mgl}{L} = \frac{Mgl}{I_3\omega_{\text{rot}}}.} \tag{4.72}$$

Thus, rather counter-intuitively, the fast-rotating top does not follow the external, vertical force and, in addition to fast spinning about the symmetry axis $\mathbf{n}_3$, performs a revolution, called the *torque-induced precession*, about the vertical axis.[18] Note that, similarly to the free-precession frequency (59), the torque-induced precession frequency (72) does not depend on the initial (and sustained) angle θ. However, the torque-induced precession frequency is inversely (rather than directly) proportional to ω_{rot}. This fact makes the above simple theory valid in many practical cases. Indeed, Eq. (71) is quantitatively valid if the contribution of the precession into $\mathbf{L}$ is relatively small: $I\omega_{\text{pre}} << I_3\omega_{\text{rot}}$, where I is a certain effective moment of inertia for the precession – to be calculated below. Using Eq. (72), this condition may be rewritten as

$$\omega_{\text{rot}} >> \left(\frac{MglI}{I_3^2}\right)^{1/2}. \tag{4.73}$$

According to Eq. (16), for a body of not too extreme proportions, i.e. with all linear dimensions of the same length scale l, all inertia moments are of the order of Ml^2, so the right-hand side of Eq. (73) is of the order of $(g/l)^{1/2}$, i.e. comparable with the frequency of small oscillations of the same body as the physical pendulum at the absence of its fast rotation.

To develop a quantitative theory that would be valid beyond such approximate treatment, the Euler equations (66) may be used, but are not very convenient. A better approach, suggested by the same L. Euler, is to introduce a set of three independent angles between the principal axes $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ bound to the rigid body, and the axes $\{\mathbf{n}_x, \mathbf{n}_y, \mathbf{n}_z\}$ of an inertial reference frame (Fig. 10), and then express the basic equation (33) of rotation, via these angles. There are several possible options for the definition of such angles; Fig. 10 shows the set of *Euler angles*, most convenient for analyses of fast rotation.[19] As one can see, the first Euler angle, θ, is the usual polar angle measured from the $\mathbf{n}_z$-axis to the $\mathbf{n}_3$-axis. The second one is the azimuthal angle φ, measured from the $\mathbf{n}_x$-axis to the *line of nodes* formed by the intersection of planes $[\mathbf{n}_x, \mathbf{n}_y]$ and $[\mathbf{n}_1, \mathbf{n}_2]$. The last Euler angle, ψ, is measured within the

[18] A semi-quantitative interpretation of this effect is a very useful exercise, highly recommended to the reader.

[19] Of the several choices more convenient in the absence of fast rotation, the most common is the set of so-called *Tait-Brian angles* (called the *yaw*, *pitch*, and *roll*), which are broadly used for aircraft and maritime navigation.

Euler angles

plane [$\mathbf{n}_1$, $\mathbf{n}_2$], from the line of nodes to the $\mathbf{n}_1$-axis. For example, in the simple picture of slow force-induced precession of a symmetric top, that was discussed above, the angle θ is constant, the angle ψ changes rapidly, with the rotation velocity ω_{rot}, while the angle φ evolves with the precession frequency ω_{pre} (72).

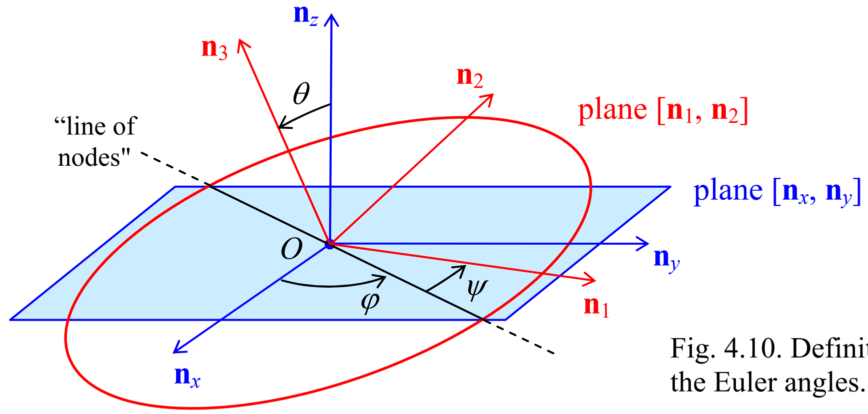


Fig. 4.10. Definition of the Euler angles.

Now we can express the principal-axes components of the instantaneous angular velocity vector, ω_1, ω_2, and ω_3, as measured in the lab reference frame, in terms of the Euler angles. This may be readily done by calculating, from Fig. 10, the contributions of the Euler angles' evolution to the rotation about each principal axis, and then adding them up:

$$\boxed{\begin{aligned}\omega_1 &= \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi,\\ \omega_2 &= \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi,\\ \omega_3 &= \dot{\varphi}\cos\theta + \dot{\psi}.\end{aligned}} \quad (4.74)$$

ω via Euler angles

These relations enable the expression of the kinetic energy of rotation (25) and the angular momentum components (26) via the generalized coordinates θ, φ, and ψ and their time derivatives (i.e. the corresponding generalized velocities), and then using the powerful Lagrangian formalism to derive their equations of motion. This is especially simple to do in the case of symmetric tops (with $I_1 = I_2$), because plugging Eqs. (74) into Eq. (25) we get an expression,

$$T_{\text{rot}} = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\varphi}\cos\theta + \dot{\psi}\right)^2, \quad (4.75)$$

which does not include explicitly either φ or ψ. (This reflects the fact that for a symmetric top we can always select the $\mathbf{n}_1$-axis to coincide with the line of nodes, and hence take $\psi = 0$ at the considered moment of time. Note that this trick does *not* mean we can take $\dot{\psi} = 0$, because the $\mathbf{n}_1$-axis, as observed from an inertial reference frame, moves!) Now we should not forget that at the torque-induced precession, the center of mass moves as well (see, e.g., Fig. 9), so according to Eq. (14), the total kinetic energy of the body is the sum of two terms,

$$T = T_{\text{rot}} + T_{\text{tran}}, \qquad T_{\text{tran}} = \frac{M}{2}V^2 = \frac{M}{2}l^2\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right), \quad (4.76)$$

while its potential energy is just

$$U = Mgl\cos\theta + \text{const}. \quad (4.77)$$

Now we could readily use Eqs. (2.19) to write the Lagrange equations of motion for the Euler angles, but it is simpler to immediately notice that according to Eqs. (75)-(77), the Lagrangian function, $T - U$, does not depend explicitly on the "cyclic" coordinates φ and ψ, so the corresponding generalized momenta (2.31) are conserved:

$$p_{\varphi} \equiv \frac{\partial T}{\partial \dot{\varphi}} = I_{\mathrm{A}} \dot{\varphi} \sin^2 \theta + I_3 (\dot{\varphi} \cos\theta + \dot{\psi}) \cos\theta = \text{const}, \tag{4.78}$$

$$p_{\psi} \equiv \frac{\partial T}{\partial \dot{\psi}} = I_3 (\dot{\varphi} \cos\theta + \dot{\psi}) = \text{const}, \tag{4.79}$$

where $I_{\mathrm{A}} \equiv I_1 + Ml^2$. (According to Eq. (29), I_{A} is just the body's moment of inertia for rotation about a horizontal axis passing through the support point A.) According to the last of Eqs. (74), p_{ψ} is just L_3, i.e. the angular momentum's component along the *precessing* axis $\mathbf{n}_3$. On the other hand, by its very definition (78), p_{φ} is L_z, i.e. the same vector $\mathbf{L}$'s component along the *stationary* axis z. (Actually, we could foresee in advance the conservation of both these components of $\mathbf{L}$ for our system, because the vector (69) of the external torque is perpendicular to both $\mathbf{n}_3$ and $\mathbf{n}_z$.) Using this notation, and solving the simple system of two linear equations (78)-(79) for the angle derivatives, we get

$$\dot{\varphi} = \frac{L_z - L_3 \cos\theta}{I_{\mathrm{A}} \sin^2 \theta}, \qquad \dot{\psi} = \frac{L_3}{I_3} - \frac{L_z - L_3 \cos\theta}{I_{\mathrm{A}} \sin^2 \theta} \cos\theta . \tag{4.80}$$

One more conserved quantity in this problem is the full mechanical energy[20]

$$E \equiv T + U = \frac{I_{\mathrm{A}}}{2} \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta \right) + \frac{I_3}{2} \left(\dot{\varphi} \cos\theta + \dot{\psi} \right)^2 + Mgl \cos\theta. \tag{4.81}$$

Plugging Eqs. (80) into Eq. (81), we get a first-order differential equation for the angle θ, which may be represented in the following physically transparent form:

$$\frac{I_{\mathrm{A}}}{2} \dot{\theta}^2 + U_{\mathrm{ef}}(\theta) = E, \qquad U_{\mathrm{ef}}(\theta) \equiv \frac{(L_z - L_3 \cos\theta)^2}{2 I_{\mathrm{A}} \sin^2 \theta} + \frac{L_3^2}{2 I_3} + Mgl \cos\theta + \text{const}. \tag{4.82}$$

Thus, similarly to the planetary problems considered in Sec. 3.4, the torque-induced precession of a symmetric top has been reduced (without any approximations!) to a 1D problem of the motion of just one of its degrees of freedom, the polar angle θ, in the effective potential $U_{\mathrm{ef}}(\theta)$. According to Eq. (82), very similar to Eq. (3.44) for the planetary problem, this potential is the sum of the actual potential energy U given by Eq. (77), and a contribution from the kinetic energy of motion along two other angles. In the absence of rotation about the axes $\mathbf{n}_z$ and $\mathbf{n}_3$ (i.e., $L_z = L_3 = 0$), Eq. (82) is reduced to the first integral of the equation (40) of motion of a physical pendulum, with $I' = I_{\mathrm{A}}$. If the rotation is present, then (besides the case of very special initial conditions when $\theta(0) = 0$ *and* $L_z = L_3$),[21] the first contribution to $U_{\mathrm{ef}}(\theta)$ diverges at $\theta \rightarrow 0$ and π, so the effective potential energy has a minimum at some non-zero value θ_0 of the polar angle θ – see Fig. 11.

[20] Indeed, since the Lagrangian does not depend on time explicitly, H = const, and since the full kinetic energy T (75)-(76) is a quadratic-homogeneous function of the generalized velocities, we have $E = H$.

[21] In that simple case, the body continues to rotate about the vertical symmetry axis: $\theta(t) = 0$. Note, however, that such motion is stable only if the spinning speed is sufficiently high – see Eq. (85) below.

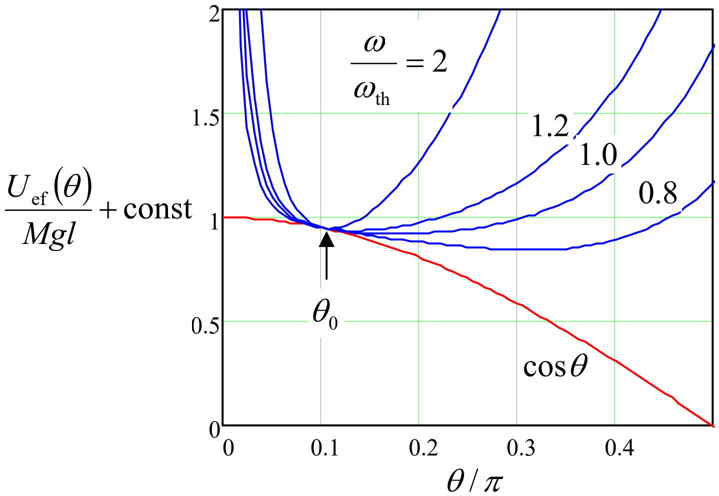


Fig. 4.11. The effective potential energy U_{ef} of the symmetric top, given by Eq. (82), as a function of the polar angle θ, for a particular value (0.95) of the ratio $r \equiv L_z/L_3$ (so that at $\omega_{rot} >> \omega_{th}$, $\theta_0 = \cos^{-1}r \approx 0.1011\pi$), and several values of the ratio ω_{rot}/ω_{th} – see Eq. (85).

If the initial angle $\theta(0)$ is equal to this value θ_0, i.e. if the initial effective energy is equal to its minimum value $U_{ef}(\theta_0)$, the polar angle remains constant through the motion: $\theta(t) = \theta_0$. This corresponds to the pure torque-induced precession whose angular velocity is given by the first of Eqs. (80):

$$\omega_{pre} \equiv \dot{\varphi} = \frac{L_z - L_3\cos\theta_0}{I_A \sin^2\theta_0}. \qquad (4.83)$$

The condition for finding θ_0, $dU_{ef}/d\theta = 0$, is a transcendental algebraic equation that cannot be solved analytically for arbitrary parameters. However, in the high spinning speed limit (73), this is possible. Indeed, in this limit the Mgl-proportional contribution to U_{ef} is small, and we may analyze its effect by successive approximations. In the 0th approximation, i.e. at $Mgl = 0$, the minimum of U_{ef} is evidently achieved at $\cos\theta_0 = L_z/L_3$, turning the precession frequency (83) to zero. In the next, 1st approximation, we may require that at $\theta = \theta_0$, the derivative of the first term of Eq. (82) for U_{ef} over $\cos\theta$, equal to $-L_z(L_z - L_3\cos\theta)/I_A\sin^2\theta$,[22] is canceled with that of the gravity-induced term, equal to Mgl. This immediately yields $\omega_{pre} = (L_z - L_3\cos\theta_0)/I_A\sin^2\theta_0 = Mgl/L_3$, so by identifying ω_{rot} with $\omega_3 \equiv L_3/I_3$ (see Fig. 8), we recover the simple expression (72).

The second important result that may be readily obtained from Eq. (82) is the exact expression for the threshold value of the spinning speed for a vertically rotating top ($\theta = 0$, $L_z = L_3$). Indeed, in the limit $\theta \to 0$ this expression may be readily simplified:

$$U_{ef}(\theta) \approx \text{const} + \left(\frac{L_3^2}{8I_A} - \frac{Mgl}{2}\right)\theta^2. \qquad (4.84)$$

This formula shows that if $\omega_{rot} \equiv L_3/I_3$ is higher than the following threshold value,

$$\boxed{\omega_{th} \equiv 2\left(\frac{MglI_A}{I_3^2}\right)^{1/2},} \qquad (4.85)$$

Threshold rotation speed

[22] Indeed, the derivative of the fraction $1/2I_A\sin^2\theta$, taken at the point $\cos\theta = L_z/L_3$, is multiplied by the numerator, $(L_z - L_3\cos\theta)^2$, which turns to zero at this point.

then the coefficient at θ^2 in Eq. (84) is positive, so U_{ef} has a stable minimum at $\theta_0 = 0$. On the other hand, if ω_3 is decreased below ω_{th}, the fixed point becomes unstable, so the top falls. As the plots in Fig. 11 show, Eq. (85) for the threshold frequency works very well even for non-zero but small values of the precession angle θ_0. Note that if we take $I = I_A$ in the condition (73) of the approximate treatment, it acquires a very simple sense: $\omega_{\text{rot}} >> \omega_{\text{th}}$.

Finally, Eqs. (82) give a natural description of one more phenomenon. If the initial energy is larger than $U_{\text{ef}}(\theta_0)$, the angle θ oscillates between two classical turning points on both sides of the fixed point θ_0 – see Fig. 11 again. The law and frequency of these oscillations may be found exactly as in Sec. 3.3 – see Eqs. (3.27) and (3.28). At $\omega_3 >> \omega_{\text{th}}$, this motion is a fast rotation of the body's symmetry axis $\mathbf{n}_3$ about its average position performing the slow torque-induced precession. Historically, these oscillations are called *nutations*, but their physics is similar to that of the free precession that was analyzed in the previous section, and the order of magnitude of their frequency is given by Eq. (59).

It may be proved that small friction (not taken into account in the above analysis) leads first to a decay of these nutations, then to a slower drift of the precession angle θ_0 to zero, and finally, to a gradual decay of the spinning speed ω_{rot} until it reaches the threshold (85) and the top falls.

4.6. Non-inertial reference frames

Now let us use the results of our analysis of the rotation kinematics in Sec. 1 to complete the discussion of the transfer between two reference frames, which was started in the introductory Chapter 1. As Fig. 12 (which reproduces Fig. 1.2 in a more convenient notation) shows, even if the "moving" frame 0 rotates relative to the "lab" frame 0', the radius vectors observed from these two frames are still related, at any moment of time, by the simple Eq. (1.5). In our new notation:

$$\mathbf{r}' = \mathbf{r}_0 + \mathbf{r}\,. \tag{4.86}$$

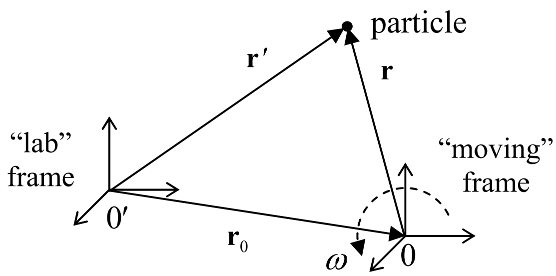


Fig. 4.12. The general case of transfer between two reference frames.

However, as was mentioned in Sec. 1, the general addition rule for velocities is already more complex. To find it, let us differentiate Eq. (86) over time:

$$\frac{d}{dt}\mathbf{r}' = \frac{d}{dt}\mathbf{r}_0 + \frac{d}{dt}\mathbf{r}. \tag{4.87}$$

The left-hand side of this relation is evidently the particle's velocity as measured in the lab frame, and the first term on the right-hand side is the velocity $\mathbf{v}_0$ of point 0, as measured in the same lab frame. The last term is more complex: due to the possible mutual rotation of the frames 0 and 0', that term may not vanish even if the particle does not move relative to the rotating frame 0 – see Fig. 12.

Fortunately, we have already derived the general Eq. (8) to analyze situations exactly like this one. Taking $\mathbf{A} = \mathbf{r}$ in it, we may apply the result to the last term of Eq. (87), to get

Trans-formation of velocity

$$\boxed{\mathbf{v}\big|_{\text{in lab}} = \mathbf{v}_0\big|_{\text{in lab}} + (\mathbf{v} + \boldsymbol{\omega}\times\mathbf{r}),} \tag{4.88}$$

where $\boldsymbol{\omega}$ is the instantaneous angular velocity of an imaginary rigid body connected to the moving reference frame (or we may say, of this frame as such), as measured in the *lab* frame 0', while $\mathbf{v}$ is $d\mathbf{r}/dt$ as measured in the *moving* frame 0. The relation (88), on one hand, is a natural generalization of Eq. (10) for $\mathbf{v} \neq 0$; on the other hand, if $\boldsymbol{\omega} = 0$, it is reduced to simple Eq. (1.8) for the translational motion of the frame 0.

To calculate the particle's acceleration, we may just repeat the same trick: differentiate Eq. (88) over time, and then use Eq. (8) again, now for the vector $\mathbf{A} = \mathbf{v} + \boldsymbol{\omega}\times\mathbf{r}$. The result is

$$\mathbf{a}\big|_{\text{in lab}} \equiv \mathbf{a}_0\big|_{\text{in lab}} + \frac{d}{dt}(\mathbf{v} + \boldsymbol{\omega}\times\mathbf{r}) + \boldsymbol{\omega}\times(\mathbf{v} + \boldsymbol{\omega}\times\mathbf{r}). \tag{4.89}$$

Carrying out the differentiation in the second term, we finally get the goal relation,

Trans-formation of acceleration

$$\boxed{\mathbf{a}\big|_{\text{in lab}} \equiv \mathbf{a}_0\big|_{\text{in lab}} + \mathbf{a} + \dot{\boldsymbol{\omega}}\times\mathbf{r} + 2\boldsymbol{\omega}\times\mathbf{v} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}),} \tag{4.90}$$

where $\mathbf{a}$ is the particle's acceleration as measured in the moving frame. This result is a natural generalization of the simple Eq. (1.9) to the rotating frame case.

Now let the lab frame 0' be inertial; then the 2nd Newton's law for a particle of mass m is

$$m\mathbf{a}\big|_{\text{in lab}} = \mathbf{F}, \tag{4.91}$$

where $\mathbf{F}$ is the vector sum of all forces exerted on the particle. This is simple and clear; however, in many cases it is much more convenient to work in a non-inertial reference frame. For example, when describing most phenomena on the Earth's surface, it is rather inconvenient to use a reference frame bound to the Sun (or to the galactic center, etc.). In order to understand what we should pay for the convenience of using a moving frame, we may combine Eqs. (90) and (91) to write

Inertial "forces"

$$\boxed{m\mathbf{a} = \mathbf{F} - m\mathbf{a}_0\big|_{\text{in lab}} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - 2m\boldsymbol{\omega}\times\mathbf{v} - m\dot{\boldsymbol{\omega}}\times\mathbf{r}.} \tag{4.92}$$

This result means that if we want to use an analog of the 2nd Newton's law in a non-inertial reference frame, we have to add, to the actual net force $\mathbf{F}$ exerted on a particle, four pseudo-force terms, called *inertial forces*, all proportional to the particle's mass. Let us analyze them one by one, always remembering that these are just mathematical terms, not actual physical forces. (In particular, it would be futile to seek a 3rd-Newton's-law counterpart for any inertial force.)

The first term, $-m\mathbf{a}_0|_{\text{in lab}}$, is the only one not related to rotation and is well known from undergraduate mechanics. (Let me hope the reader remembers all these weight-in-the-accelerating-elevator problems.) However, despite its simplicity, this term has more subtle consequences. As an example, let us consider, semi-qualitatively, the motion of a planet, such as our Earth, orbiting a star and also rotating about its own axis – see Fig. 13. The bulk-distributed gravity forces, acting on a planet from its star, are not quite uniform, because they obey the $1/r^2$ gravity law (1.15), and hence are equivalent to a single force applied to a point A slightly offset from the planet's center of mass 0, toward the star. For a spherically symmetric planet, the direction from 0 to A would be exactly aligned with the direction toward the star. However, real planets are not absolutely rigid, so due to the centrifugal "force" (to be discussed momentarily), the rotation about their own axis makes them slightly ellipsoidal – see Fig. 13. (For our Earth, this equatorial bulge is about 10 km.) As a result, the net gravity force is slightly offset from the direction toward the center of mass 0. On the other hand, repeating all the arguments of this section for a body (rather than a point), we may see that, in the reference frame moving with the planet, the inertial force $-M\mathbf{a}_0$ (with the magnitude of the total gravity force, but directed *from* the star) is applied exactly to the center of mass. As a result, this pair of forces creates a torque $\boldsymbol{\tau}$ perpendicular to both the direction toward the star and the vector 0A. (In Fig. 13, the torque vector is perpendicular to the plane of the drawing). If the angle δ between the planet's "polar" axis of rotation and the direction towards the star was fixed, then, as we have seen in the previous section, this torque would induce a slow axis precession about that direction.

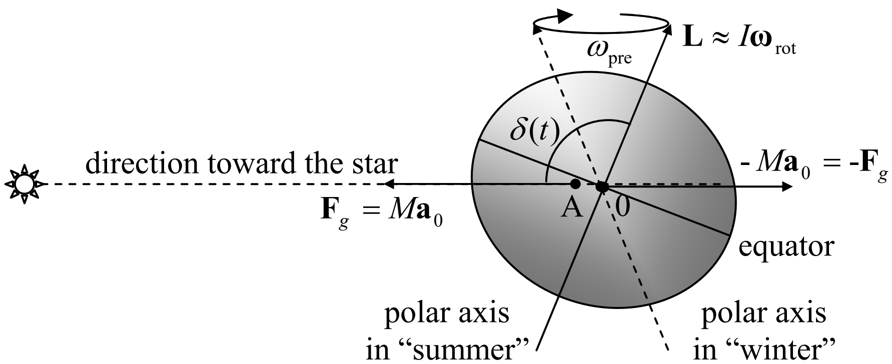


Fig. 4.13. The axial precession of a planet (with the equatorial bulge and the 0A-offset strongly exaggerated).

However, as a result of the orbital motion, the angle δ oscillates in time much faster (once a year) between values $(\pi/2 + \varepsilon)$ and $(\pi/2 - \varepsilon)$, where ε is the axis tilt, i.e. angle between the polar axis (the direction of vectors $\mathbf{L}$ and $\boldsymbol{\omega}_{\text{rot}}$) and the normal to the *ecliptic plane* of the planet's orbit. (For the Earth, $\varepsilon \approx 23.4°$.) A straightforward averaging over these fast oscillations[23] shows that the torque leads to the polar axis' precession about the axis *perpendicular* to the ecliptic plane, keeping ε constant – see Fig. 13. For the Earth, the period $T_{\text{pre}} = 2\pi/\omega_{\text{pre}}$ of this *precession of the equinoxes*, corrected for a substantial effect of the Moon's gravity, is close to 26,000 years.[24]

Returning to Eq. (92), the direction of the second term of its right-hand side,

Centrifugal "force"

$$\boxed{\mathbf{F}_{\text{cf}} \equiv -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}),} \tag{4.93}$$

called the *centrifugal force*, is always perpendicular to, and directed out of the instantaneous rotation axis – see Fig. 14. Indeed, the vector $\boldsymbol{\omega}\times\mathbf{r}$ is perpendicular to both $\boldsymbol{\omega}$ and $\mathbf{r}$ (in Fig. 14, normal to the drawing plane and directed from the reader) and has the magnitude $\omega r\sin\theta = \omega\rho$, where ρ is the distance of the particle from the rotation axis. Hence the outer vector product, with the account of the minus sign, is normal to the rotation axis $\boldsymbol{\omega}$, directed from this axis, and is equal to $\omega^2 r\sin\theta = \omega^2\rho$. The centrifugal "force" is of course just the result of the fact that the centripetal acceleration $\omega^2\rho$, explicit in the inertial reference frame, disappears in the rotating frame. For a typical location of the Earth ($\rho \sim R_{\text{E}} \approx 6\times10^6$ m),

[23] Details of this calculation may be found, e.g., in Sec. 5.8 of the textbook by H. Goldstein *et al.*, *Classical Mechanics*, 3rd ed., Addison Wesley, 2002.
[24] This effect is known from antiquity, apparently discovered by Hipparchus of Rhodes (190-120 BC).

with its angular velocity $\omega_E \approx 10^{-4}$ s^{-1}, the acceleration is rather considerable, of the order of 3 cm/s^2, i.e. ~0.003 g, and is responsible, in particular, for the largest part of the equatorial bulge mentioned above.

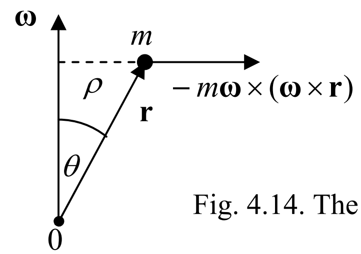


Fig. 4.14. The centrifugal force.

As an example of using the centrifugal force concept, let us return again to our "testbed" problem on the bead sliding along a rotating ring – see Fig. 2.1. In the non-inertial reference frame attached to the ring, we have to add, to the actual forces $m\mathbf{g}$ and $\mathbf{N}$ exerted on the bead, the horizontal centrifugal force[25] directed from the rotation axis, with the magnitude $m\omega^2\rho$. Its component tangential to the ring equals $(m\omega^2\rho)\cos\theta = m\omega^2 R\sin\theta\cos\theta$, and hence the component of Eq. (92) along this direction is $ma = -mg\sin\theta + m\omega^2 R\sin\theta\cos\theta$. With $a = R\ddot{\theta}$, this gives us an equation of motion equivalent to Eq. (2.25), which had been derived in Sec. 2.2 (in the inertial frame) using the Lagrangian formalism.

The third term on the right-hand side of Eq. (92),

$$\boxed{\mathbf{F}_C \equiv -2m\boldsymbol{\omega}\times\mathbf{v},} \qquad (4.94)$$

Coriolis "force"

is the so-called *Coriolis force*,[26] which is different from zero only if the particle moves in a rotating reference frame. Its physical sense may be understood by considering a projectile fired horizontally, say from the North Pole – see Fig. 15.

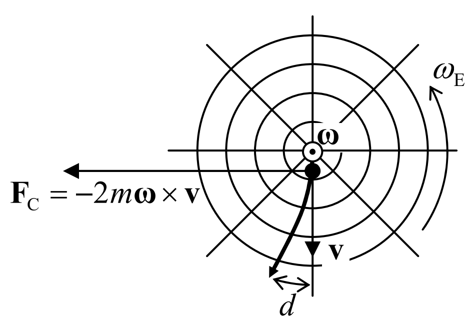


Fig. 4.15. The trajectory of a projectile fired horizontally from the North Pole, from the point of view of an Earth-bound observer looking down. The circles show parallels, while the straight lines mark meridians.

From the point of view of an Earth-based observer, the projectile will be affected by an additional Coriolis force (94), directed westward, with the magnitude $2m\omega_E v$, where $\mathbf{v}$ is the main, southward component of the velocity. This force would cause the westward acceleration $a = 2\omega_E v$, and hence the westward deviation growing with time as $d = at^2/2 = \omega_E vt^2$. (This formula is exact only if d is much smaller than the distance $r = vt$ passed by the projectile.) On the other hand, from the point of

[25] For this problem, all other inertial "forces", besides the Coriolis force (see below) vanish, while the latter force is directed normally to the ring and does not affect the bead's motion along it.

[26] Named after G.-G. de Coriolis (already reverently mentioned in Chapter 1) who described its theory and applications in detail in 1835, though the first semi-quantitative analyses of this effect were given by Giovanni Battista Riccioli and Claude François Dechales already in the mid-1600s, and all basic components of the Coriolis theory may be traced to a 1749 work by Leonard Euler.

view of an inertial-frame observer, the projectile's trajectory in the horizontal plane is a straight line. However, during the flight time t, the Earth's surface slips eastward from under the trajectory by the distance $d = r\varphi = (vt)(\omega_E t) = \omega_E vt^2$, where $\varphi = \omega_E t$ is the azimuthal angle of the Earth's rotation during the flight). Thus, both approaches give the same result – as they should.

Hence, the Coriolis "force" is just a fancy (but frequently very convenient!) way of describing a purely geometric effect pertinent to the rotation, from the point of view of the observer participating in it. This force is responsible, in particular, for the higher right banks of rivers in the Northern hemisphere, regardless of the direction of their flow – see Fig. 16. Despite the smallness of the Coriolis force (for a typical velocity of the water in a river, $v \sim 1$ m/s, it is equivalent to acceleration $a_C \sim 10^{-2}$ cm/s^2 $\sim 10^{-5}\, g$), its multi-century effects may be rather prominent.[27]

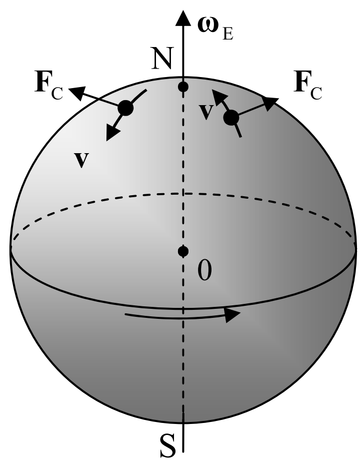


Fig. 4.16. Coriolis forces due to the Earth's rotation, in the Northern hemisphere.

Finally, the last, fourth term of Eq. (92), $-m\dot{\boldsymbol{\omega}} \times \mathbf{r}$, exists only when the rotation frequency changes in time, and may be interpreted as a local-position-specific addition to the first term.

The key relation (92), derived above from Newton's equation (91), may be alternatively obtained from the Lagrangian approach. Indeed, let us use Eq. (88) to represent the kinetic energy of the particle in an inertial "lab" frame in terms of **v** and **r** measured in a *rotating* frame:

$$T = \frac{m}{2}\left[\mathbf{v}_0\big|_{\text{in lab}} + (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r})\right]^2, \tag{4.95}$$

and use this expression to calculate the Lagrangian function. For the relatively simple case of a particle's motion in the field of potential forces, measured from a reference frame that performs a pure rotation (so $\mathbf{v}_0|_{\text{in lab}} = 0$)[28] with a constant angular velocity $\boldsymbol{\omega}$, we get

$$L \equiv T - U = \frac{m}{2}v^2 + m\mathbf{v}\cdot(\boldsymbol{\omega}\times\mathbf{r}) + \frac{m}{2}(\boldsymbol{\omega}\times\mathbf{r})^2 - U \equiv \frac{m}{2}v^2 + m\mathbf{v}\cdot(\boldsymbol{\omega}\times\mathbf{r}) - U_{\text{ef}}, \tag{4.96a}$$

where the effective potential energy,[29]

[27] The same force causes the counterclockwise circulation in the "Nor'easter" storms on the US East Coast, with the radial component of the air velocity directed toward the cyclone's center, due to lower pressure in its middle.
[28] A similar analysis of the cases with $\mathbf{v}_0|_{\text{in lab}} \neq 0$, for example, of a translational relative motion of the reference frames, is left for the reader's exercise.

$$U_{\text{ef}} \equiv U + U_{\text{cf}}, \qquad \text{with } U_{\text{cf}} \equiv -\frac{m}{2}(\boldsymbol{\omega}\times\mathbf{r})^2, \tag{4.96b}$$

is just the sum of the actual potential energy U of the particle and the so-called *centrifugal potential energy*, associated with the centrifugal "force" (93):

$$\mathbf{F}_{\text{cf}} = -\nabla U_{\text{cf}} = -\nabla\left[-\frac{m}{2}(\boldsymbol{\omega}\times\mathbf{r})^2\right] = -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}). \tag{4.97}$$

It is straightforward to verify that the Lagrange equations (2.19), derived from Eqs. (96) considering the Cartesian components of **r** and **v** as generalized coordinates and velocities, coincide with Eq. (92) (with $\mathbf{a}_0|_{\text{in lab}} = 0$, $\dot{\boldsymbol{\omega}} = 0$, and $\mathbf{F} = -\nabla U$).

Now it is very informative to have a look at a by-product of this calculation, the generalized momentum (2.31) corresponding to the particle's coordinate **r** as measured in the rotating reference frame,[30]

Canonical momentum at rotation

$$\boxed{\boldsymbol{p} \equiv \frac{\partial L}{\partial \mathbf{v}} = m(\mathbf{v} + \boldsymbol{\omega}\times\mathbf{r}).} \tag{4.98}$$

According to Eq. (88) with $\mathbf{v}_0|_{\text{in lab}} = 0$, the expression in the parentheses is just $\mathbf{v}|_{\text{in lab}}$. However, from the point of view of the moving frame, i.e. not knowing about the simple physical sense of the vector $\boldsymbol{p}$, we would have a reason to speak about two different linear momenta of the same particle, the so-called *kinetic momentum* $\mathbf{p} = m\mathbf{v}$ and the *canonical momentum* $\boldsymbol{p} = \mathbf{p} + m\boldsymbol{\omega}\times\mathbf{r}$.[31] Let us calculate the Hamiltonian function H defined by Eq. (2.32), and the energy E as functions of the same moving-frame variables:

$$H \equiv \sum_{j=1}^{3}\frac{\partial L}{\partial v_j}v_j - L = \boldsymbol{p}\cdot\mathbf{v} - L = m(\mathbf{v}+\boldsymbol{\omega}\times\mathbf{r})\cdot\mathbf{v} - \left[\frac{m}{2}v^2 + m\mathbf{v}\cdot(\boldsymbol{\omega}\times\mathbf{r}) - U_{\text{ef}}\right] = \frac{mv^2}{2} + U_{\text{ef}}, \tag{4.99}$$

$$E \equiv T + U = \frac{m}{2}v^2 + m\mathbf{v}\cdot(\boldsymbol{\omega}\times\mathbf{r}) + \frac{m}{2}(\boldsymbol{\omega}\times\mathbf{r})^2 + U = \frac{m}{2}v^2 + U_{\text{ef}} + m\mathbf{v}\cdot(\boldsymbol{\omega}\times\mathbf{r}) + m(\boldsymbol{\omega}\times\mathbf{r})^2. \tag{4.100}$$

These expressions clearly show that E and H are *not* equal.[32] In hindsight, this is not surprising, because the kinetic energy (95), expressed in the moving-frame variables, includes a term linear in **v**, and hence

---

[29] For the attentive reader who has noticed the difference between the negative sign in the expression for U_{cf}, and the positive sign before the similar second term in Eq. (3.44): as was already discussed in Chapter 3, it is due to the difference of assumptions. In the planetary problem, even though the angular momentum **L** and hence its component L_z are fixed, the corresponding angular velocity $\dot{\varphi}$ is not. On the opposite, in our current discussion, the angular velocity $\boldsymbol{\omega}$ of the reference frame is assumed to be fixed, i.e. is independent of **r** and **v**.

[30] Here $\partial L/\partial\mathbf{v}$ is just a shorthand for a vector with Cartesian components $\partial L/\partial v_j$. In a more formal language, this is the gradient of the scalar function L in the velocity space.

[31] A very similar situation arises at the motion of a particle with electric charge q in magnetic field $\mathscr{B}$. In that case, the role of the additional term $\boldsymbol{p} - \mathbf{p} = m\boldsymbol{\omega}\times\mathbf{r}$ is played by the product $q\mathscr{A}$, where $\mathscr{A}$ is the vector potential of the field $\mathscr{B} = \nabla\times\mathscr{A}$ – see, e.g., EM Sec. 9.7, and in particular Eqs. (9.183) and (9.192).

[32] Please note the last form of Eq. (99), which shows the physical sense of the Hamiltonian function of a particle in the rotating frame very clearly, as the sum of its kinetic energy (as measured in the moving frame), and the effective potential energy (96b), including that of the centrifugal "force".

is not a quadratic-homogeneous function of this generalized velocity. The difference between these functions may be represented as

$$E - H = m\mathbf{v}\cdot(\boldsymbol{\omega}\times\mathbf{r}) + m(\boldsymbol{\omega}\times\mathbf{r})^2 \equiv m(\mathbf{v}+\boldsymbol{\omega}\times\mathbf{r})\cdot(\boldsymbol{\omega}\times\mathbf{r}) = m\mathbf{v}\big|_{\text{in lab}}\cdot(\boldsymbol{\omega}\times\mathbf{r}). \tag{4.101}$$

Now using the operand rotation rule again, we may transform this expression into a simpler form:[33]

E – H at rotation

$$\boxed{E - H = \boldsymbol{\omega}\cdot\left(\mathbf{r}\times m\mathbf{v}\big|_{\text{in lab}}\right) = \boldsymbol{\omega}\cdot(\mathbf{r}\times\boldsymbol{p}) = \boldsymbol{\omega}\cdot\mathbf{L}\big|_{\text{in lab}}.} \tag{4.102}$$

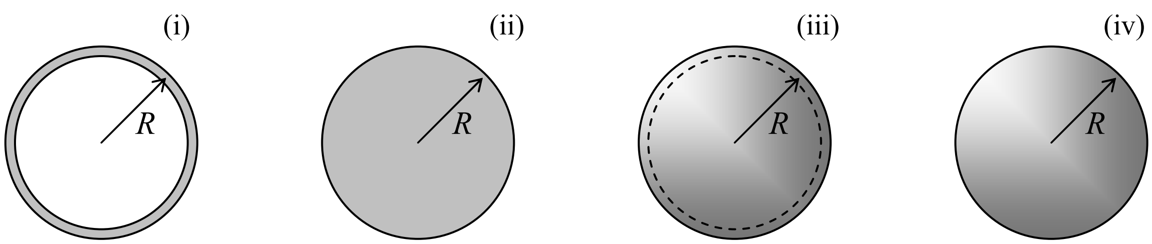
As a sanity check, let us apply this general expression to the particular case of our testbed problem – see Fig. 2.1. In this case, the vector $\boldsymbol{\omega}$ is aligned with the z-axis, so that of all Cartesian components of the vector $\mathbf{L}$, only the component L_z is important for the scalar product in Eq. (102). This component evidently equals $\omega I_z = \omega m\rho^2 = \omega m(R\sin\theta)^2$, so that

$$E - H = m\omega^2 R^2 \sin^2\theta, \tag{4.103}$$

i.e. the same result that follows from the subtraction of Eqs. (2.40) and (2.41).

4.7. Exercise problems

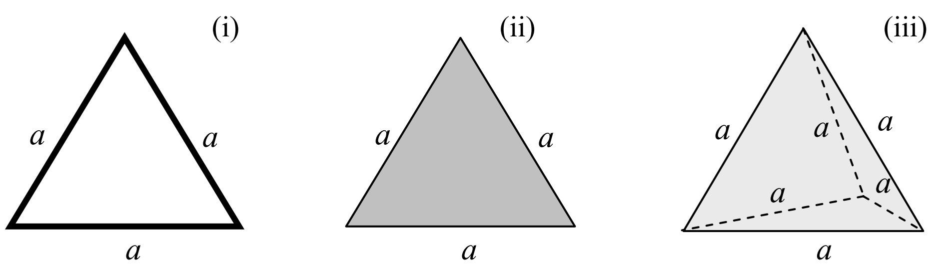
4.1. Calculate the principal moments of inertia for the following uniform rigid bodies:



(i) a thin, planar, round hoop, (ii) a flat round disk, (iii) a thin spherical shell, and (iv) a solid sphere.

Compare the results, assuming that all the bodies have the same radius R and mass M, and give an interpretation of their difference.

4.2. Calculate the principal moments of inertia for the rigid bodies shown in the figure below:



(i) an equilateral triangle made of thin rods with a constant linear mass density μ,
(ii) a thin plate in the shape of an equilateral triangle, with a constant areal mass density σ, and
(iii) a tetrahedron with a constant bulk mass density ρ.

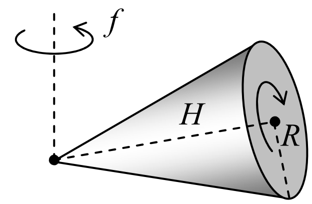
[33] Note that by the definition (1.36), the angular momenta $\mathbf{L}$ of particles merely add up. As a result, the final form of Eq. (102) is valid for an arbitrary system of particles.

Assuming that the total mass of the three bodies is the same, compare the results and give an interpretation of their difference.

4.3. Calculate the principal moments of inertia of a thin uniform plate cut in the form of a right triangle with two $\pi/4$ angles.

4.4. Prove that Eqs. (34)-(36) are valid for the rotation of a rigid body about the fixed z-axis, even if it does not pass through its center of mass.

4.5. Calculate the kinetic energy of a right circular cone with height H, base radius R, and a constant mass density ρ, that rolls over a horizontal surface without slippage, making f turns per second about the vertical axis – see the figure on the right.



4.6. External forces exerted on a rigid body rotating with an angular velocity $\boldsymbol{\omega}$, have zero vector sum but a non-vanishing net torque $\boldsymbol{\tau}$ about its center of mass.

(i) Calculate the work of the forces on the body per unit time, i.e. their instantaneous power.

(ii) Prove that the same result is valid for a body rotating about a fixed axis and the torque's component along this axis.

(iii) Use the last result to prove that at negligible friction, the gear assembly shown in the figure on the right distributes the external torque, applied to its satellite-carrier axis to rotate it about the common axis of two axle shafts, equally to both shafts, even if they rotate with different angular velocities.

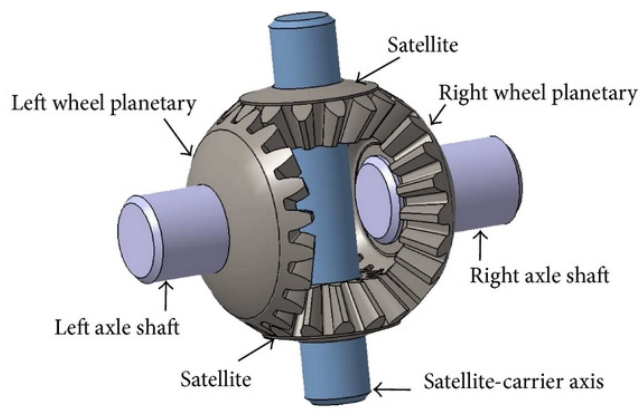
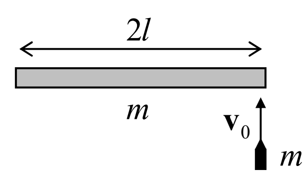
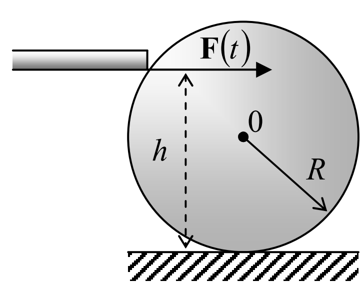


Figure from G. Antoni, *Sci. World J.*, **2014**, 523281 (2014), adapted with permission. Both satellite gears may rotate freely about their common carrier axis.

4.7. The end of a uniform thin rod of length $2l$ and mass m, initially at rest, is hit by a bullet of mass m', flying with a velocity $\mathbf{v}_0$ perpendicular to the rod (see the figure on the right), which gets stuck in it. Use two different approaches to calculate the velocity of the opposite end of the rod right after the collision.



4.8. A ball of radius R, initially at rest on a horizontal surface, is hit with a billiard cue in the horizontal direction, at height h above the table – see the figure on the right. Using the Coulomb approximation for the kinetic friction force between the ball and the surface ($|F_f| = \mu N$), calculate the final linear velocity of the rolling ball as a function of h. Would it matter if the hit point is shifted horizontally (normally to the plane of the drawing)?

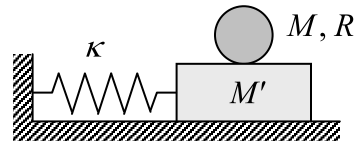


Hint: As in most solid body collision problems, during the short time of the cue hit, all other forces exerted on the ball may be considered negligibly small.

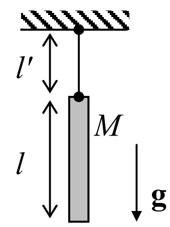
4.9. A round cylinder of radius R and mass M may roll, without slippage, over a horizontal surface. The mass density distribution inside the cylinder is not uniform, so its center of mass is at some distance $l \neq 0$ from its geometrical axis, and the moment of inertia I (for rotation about the axis parallel to the symmetry axis but passing through the center of mass) is different from $MR^2/2$, where M is the cylinder's mass. Derive the equation of motion of the cylinder under the effect of the uniform vertical gravity field, and use it to calculate the frequency of small oscillations of the cylinder near its stable equilibrium position.

4.10. A body may rotate about a fixed horizontal axis – see Fig. 5. Find the frequency of its small oscillations in a uniform gravity field, as a function of the distance l of the axis from the body's center of mass 0, and analyze the result.

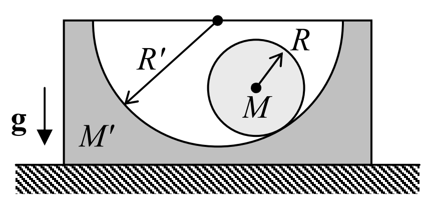
4.11. Calculate the frequency, and sketch the mode of oscillations[34] of a round uniform cylinder of radius R and the mass M, that may roll, without slippage, on a horizontal surface of a block of mass M'. The block, in turn, may move in the same direction, without friction, on an immobile horizontal surface, being connected to it with an elastic spring – see the figure on the right.



4.12. A thin uniform bar of mass M and length l is hung on a light thread of length l' (like a "chime" bell – see the figure on the right). Derive the equations of the system's motion within a vertical plane passing through the suspension point.

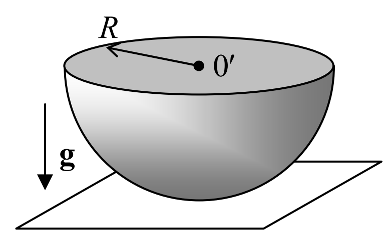


4.13. A uniform round solid cylinder of mass M can roll, without slippage, over a concave round cylindrical surface of a block of mass M', in a uniform gravity field – see the figure on the right. The block can slide without friction on a horizontal surface. Using the Lagrangian formalism,



(i) find the frequency of small oscillations of the system near the equilibrium, and
(ii) sketch the oscillation mode for the particular case $M' = M$, $R' = 2R$.

4.14. A uniform solid hemisphere of radius R and mass M is placed on a horizontal surface – see the figure on the right. Find the frequency of its small oscillations within a vertical plane, for two ultimate cases:

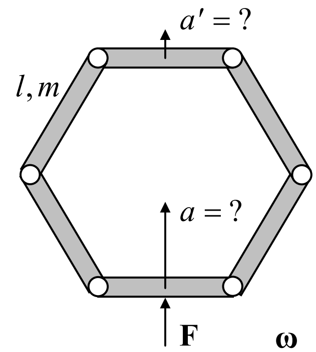


(i) there is *no friction* between the sphere and the horizontal surface;
(ii) the static friction between them is so strong that there is *no slippage*.

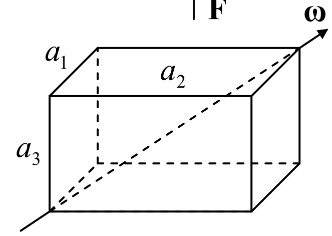
[34] The term *mode* usually refers to the spatial pattern of oscillations; it will be much discussed in later chapters.

4.15. For the "sliding ladder" problem started in Sec. 3 (see Fig. 7), find the critical value α_c of the angle α at that the ladder loses its contact with the vertical wall, assuming that it starts sliding from the vertical position, with a negligible initial velocity.

4.16. Six similar, uniform rods of length l and mass m are connected by light joints so that they may rotate, without friction, versus each other, forming a planar polygon. Initially, the polygon was at rest, and had the correct hexagon shape – see the figure on the right. Suddenly, an external force **F** is applied to the middle of one rod, in the direction of the hexagon's symmetry center. Calculate the accelerations: of the rod to which the force is applied (a), and of the opposite rod (a'), immediately after the application of the force.



4.17. A rectangular cuboid (parallelepiped) with sides a_1, a_2, and a_3, made of a material with a constant mass density ρ, is rotated with a constant angular velocity ω about one of its space diagonals – see the figure on the right. Calculate the torque $\boldsymbol{\tau}$ necessary to sustain this rotation.



4.18. A uniform round ball rolls, without slippage, over a "turntable": a horizontal plane rotated about a vertical axis with a time-independent angular velocity Ω. Derive a self-consistent equation of motion of the ball's center, and discuss its solutions.

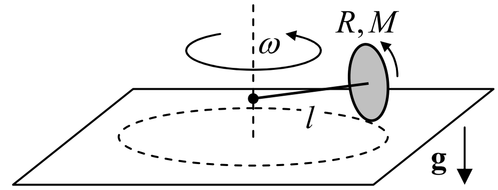
4.19. Calculate the free precession frequency of a uniform thin round disk rotating with an angular velocity ω about a direction very close to its symmetry axis, from the point of view of:

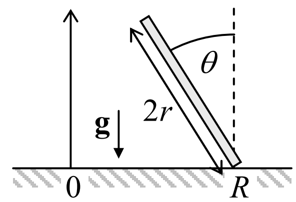
(i) an observer rotating with the disk, and
(ii) a lab-based observer.

4.20. Use the Euler equations to prove the fact mentioned in Sec. 4: free rotation of an arbitrary body ("asymmetric top") about its principal axes with the smallest and largest moments of inertia is stable, while that about the intermediate-I_j axis is not. Illustrate the same fact using the Poinsot construction.

4.21. Give an interpretation of the torque-induced precession, that would explain its direction, by using a simple system exhibiting this effect, as a model.

4.22. One end of a light shaft of length l is firmly attached to the center of a thin uniform solid disk of radius R and mass M, whose plane is perpendicular to the shaft. Another end of the shaft is attached to a vertical axis (see the figure on the right) so that the shaft may rotate about the axis without friction. The disk rolls, without slippage, over a horizontal surface so that the whole system rotates about the vertical axis with a constant angular velocity ω. Calculate the (vertical) supporting force N exerted on the disk by the surface.





4.23. A coin of radius r is rolled over a horizontal surface, without slippage. Due to its tilt θ, it rolls around a circle of radius R – see the figure on the right. Modeling the coin as a very thin round disk, calculate the time period of its motion around the circle.

4.24. Solve the previous problem in the limit when the coin tilt angle θ and the ratio r/R are small, by simpler means, using

(i) an inertial ("lab") reference frame, and
(ii) the non-inertial reference frame moving with the coin's center but not rotating with it.

4.25. A symmetric top on a point support (as shown see, e.g., Fig. 9), rotating around its symmetry axis with a high angular velocity ω_{rot}, is subjected to not only its weight $M\mathbf{g}$ but also an additional force also applied to the top's center of mass, with its vector rotating in the horizontal plane with a constant angular velocity $\omega << \omega_{\text{rot}}$. Derive the system of equations describing the top's motion. Analyze their solution for the simplest case when ω is exactly equal to the frequency (72) of the torque-induced precession in the gravity field alone.

4.26. Analyze the effect of small friction on a fast rotation of a symmetric top around its axis, using a simple model in that the lower end of the body is a right cylinder of radius R.

4.27. An air-filled balloon is placed inside a water-filled container, which moves by inertia in free space, at negligible gravity. Suddenly, force $\mathbf{F}$ is applied to the container, pointing in a certain direction. What direction does the balloon move relative to the container?

4.28. Two planets are in a circular orbit around their common center of mass. Calculate the effective potential energy of a much lighter body (say, a spacecraft) rotating with the same angular velocity, on the line connecting the planets. Sketch the radial dependence of U_{ef} and find out the number of so-called *Lagrange points* in which the potential energy has local maxima. Calculate their position explicitly in the limit when one of the planets is much more massive than the other one.

4.29. Besides the three Lagrange points L_1, L_2, and L_3 discussed in the previous problem, which are located on the line connecting two planets on circular orbits about their mutual center of mass, there are two off-line points L_4 and L_5 – both within the plane of the planets' rotation. Calculate their positions.

4.30. The following simple problem may give additional clarity to the physics of the Coriolis "force". A bead of mass m may slide, without friction, along a straight rod that is rotated within a horizontal plane with a constant angular velocity ω – see the figure on the right. Calculate the bead's acceleration and the force $\mathbf{N}$ exerted on it by the rod, in:

(i) an inertial ("lab") reference frame, and
(ii) the non-inertial reference frame rotating with the rod (but not moving with the bead),

and compare the results.

4.31. Analyze the dynamics of the famous *Foucault pendulum* used for spectacular demonstrations of the Earth's rotation. In particular, calculate the angular velocity of the rotation of its oscillation plane relative to the Earth's surface, at a location with a polar angle ("colatitude") Θ. Assume that the pendulum oscillation amplitude is small enough to neglect nonlinear effects and that its oscillation period is much shorter than 24 hours.

4.32. A small body is dropped down to the surface of Earth from a height $h \ll R_E$, without initial velocity. Calculate the magnitude and direction of its deviation from the vertical, due to the Earth's rotation. Estimate the effect's magnitude for a body dropped from the Empire State Building.

4.33. Calculate the height of solar tides on a large ocean, using the following simplifying assumptions: the tide period (½ of the Earth's day) is much longer than the period of all ocean waves, the Earth (of mass M_E) is a sphere of radius R_E, and its distance r_S from the Sun (of mass M_S) is constant and much larger than R_E.

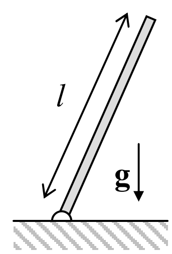
4.34. A satellite is on a circular orbit of radius R, around the Earth. Neglecting the gravity field of the satellite,

(i) write the equations of motion of a small body as observed from the satellite and simplify them for the case when the motion is limited to the satellite's close vicinity;

(ii) use these equations to prove that a body may be placed on an elliptical trajectory around the satellite's center of mass, within its plane of rotation around the Earth. Calculate the ellipse's orientation and eccentricity.

4.35. A non-spherical shape of an artificial satellite may ensure its stable angular orientation relative to the Earth's surface, advantageous for many practical goals. By modeling a satellite as a strongly elongated, axially-symmetric body moving around the Earth on a circular orbit of radius R, find its stable orientation.

4.36. A rigid, straight, uniform rod of length l, with the lower end on a pivot, falls in a uniform gravity field – see the figure on the right. Neglecting friction, calculate the distribution of the bending torque τ along its length, and analyze the result.



Hint: As will be discussed in detail in Sec. 7.5 of the lecture notes, the bending torque's gradient along the rod's length is equal to the rod-normal ("shear") component of the total force between two parts of the rod, mentally separated by its cross-section.

4.37. Let $\mathbf{r}$ be the radius vector of a particle, as measured in a possibly non-inertial but certainly non-rotating reference frame. Taking its Cartesian components for the generalized coordinates, calculate the corresponding generalized momentum $\boldsymbol{p}$ of the particle and its Hamiltonian function H. Compare $\boldsymbol{p}$ with $m\mathbf{v}$, and H with the particle's energy E. Derive the Lagrangian equation of motion in this approach, and compare it with Eq. (92).