Chapter 6. From Oscillations to Waves

In this chapter, the discussion of oscillations is extended to systems with two and more degrees of freedom. This extension naturally leads to another key notion of physics – waves, so far in simple, mostly 1D systems. (In the next chapter, this discussion will be extended to more complex elastic continua.) However, even the limited scope of the models analyzed in this chapter will still enable us to discuss such important general aspects of waves as their dispersion, phase and group velocities, impedance, reflection, and attenuation.

6.1. Two coupled oscillators

Let us discuss oscillations in systems with several degrees of freedom, starting from the simplest case of two linear (harmonic), dissipation-free, 1D oscillators. If the oscillators are independent of each other, the Lagrangian function of their system may be expressed as a sum of two independent terms of the type (5.1):

$$L = L_1 + L_2, \quad L_{1,2} = T_{1,2} - U_{1,2} = \frac{m_{1,2}}{2}\dot{q}_{1,2}^2 - \frac{\kappa_{1,2}}{2}q_{1,2}^2. \tag{6.1}$$

Correspondingly, Eqs. (2.19) for $q_j = q_{1,2}$ yields two independent equations of motion of the oscillators, each one being similar to Eq. (5.2):

$$m_{1,2}\ddot{q}_{1,2} + m_{1,2}\Omega_{1,2}^2 q_{1,2} = 0, \qquad \text{where } \Omega_{1,2}^2 = \frac{\kappa_{1,2}}{m_{1,2}}. \tag{6.2}$$

(In the context of what follows, $\Omega_{1,2}$ are sometimes called the *partial frequencies*.) This means that in this simplest case, an arbitrary motion of the system is just a sum of independent sinusoidal oscillations at two frequencies equal to the partial frequencies (2).

However, as soon as the oscillators are *coupled* (i.e. interact), the full Lagrangian L contains an additional *mixed* term L_{int} depending on both generalized coordinates q_1 and q_2 and/or generalized velocities. As a simple example, consider the system shown in Fig. 1, where two small masses $m_{1,2}$ are constrained to move in only one direction (shown horizontal), and are kept between two stiff walls with three springs.

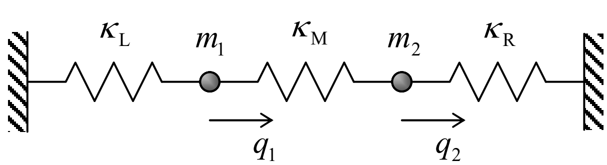

Fig. 6.1. A simple system of two coupled linear oscillators.

In this case, the kinetic energy is still separable, $T = T_1 + T_2$, but the total potential energy, consisting of the elastic energies of three springs, is not:

$$U = \frac{\kappa_L}{2}q_1^2 + \frac{\kappa_M}{2}(q_1 - q_2)^2 + \frac{\kappa_R}{2}q_2^2, \tag{6.3a}$$

where $q_{1,2}$ are the horizontal displacements of the particles from their equilibrium positions. It is convenient to rewrite this expression as

$$U = \frac{\kappa_1}{2}q_1^2 + \frac{\kappa_2}{2}q_2^2 - \kappa q_1 q_2, \qquad \text{where } \kappa_1 \equiv \kappa_L + \kappa_M, \quad \kappa_2 \equiv \kappa_R + \kappa_M, \quad \kappa \equiv \kappa_M. \tag{6.3b}$$

This formula shows that the Lagrangian function $L = T - U$ of this system contains, besides the partial terms (1), a *bilinear* interaction term:

$$L = L_1 + L_2 + L_{\text{int}}, \quad L_{\text{int}} = \kappa q_1 q_2. \tag{6.4}$$

The resulting Lagrange equations of motion are as follows:

Linearly coupled oscillators

$$\boxed{\begin{aligned} m_1\ddot{q}_1 + m_1\Omega_1^2 q_1 &= \kappa q_2, \\ m_2\ddot{q}_2 + m_2\Omega_2^2 q_2 &= \kappa q_1. \end{aligned}} \tag{6.5}$$

Thus the interaction leads to an effective generalized force κq_2 exerted on subsystem 1 by subsystem 2, and the reciprocal effective force κq_1.

Please note two important aspects of this (otherwise rather simple) system of equations. First, in contrast to the *actual* physical interaction forces (such as $F_{12} = -F_{21} = \kappa_M(q_2 - q_1)$ for our system[1]) the *effective* forces on the right-hand sides of Eqs. (5) do not obey the 3rd Newton law. Second, the forces are proportional to the same coefficient κ; this feature is a result of the general bilinear structure (4) of the interaction energy, rather than of any special symmetry.

From our prior discussions, we already know how to solve Eqs. (5), because it is still a system of linear and homogeneous differential equations, so its general solution is a sum of particular solutions of the form similar to Eqs. (5.88),

$$q_1 = c_1 e^{\lambda t}, \qquad q_2 = c_2 e^{\lambda t}, \tag{6.6}$$

with all possible values of λ. These values may be found by plugging Eq. (6) into Eqs. (5), and requiring the resulting system of two linear, homogeneous algebraic equations for the distribution coefficients $c_{1,2}$,

$$\begin{aligned} m_1\lambda^2 c_1 + m_1\Omega_1^2 c_1 &= \kappa c_2, \\ m_2\lambda^2 c_2 + m_2\Omega_2^2 c_2 &= \kappa c_1, \end{aligned} \tag{6.7}$$

to be self-consistent. In our particular case, we get a characteristic equation,

$$\begin{vmatrix} m_1(\lambda^2 + \Omega_1^2) & -\kappa \\ -\kappa & m_2(\lambda^2 + \Omega_2^2) \end{vmatrix} = 0, \tag{6.8}$$

that is quadratic in λ^2, and thus has a simple analytical solution:

$$\begin{aligned} \left(\lambda^2\right)_\pm &= -\frac{1}{2}\left(\Omega_1^2 + \Omega_2^2\right) \mp \left[\frac{1}{4}\left(\Omega_1^2 + \Omega_2^2\right)^2 - \Omega_1^2\Omega_2^2 + \frac{\kappa^2}{m_1 m_2}\right]^{1/2} \\ &\equiv -\frac{1}{2}\left(\Omega_1^2 + \Omega_2^2\right) \mp \left[\frac{1}{4}\left(\Omega_1^2 - \Omega_2^2\right)^2 + \frac{\kappa^2}{m_1 m_2}\right]^{1/2}. \end{aligned} \tag{6.9}$$

[1] Using these expressions, Eqs. (5) may be readily obtained from the Newton laws, but the Lagrangian approach used above will make their generalization, in the next section, more straightforward.

According to Eqs. (2) and (3b), for any positive spring constants, the product $\Omega_1\Omega_2 = (\kappa_L + \kappa_M)(\kappa_R + \kappa_M)/(m_1m_2)^{1/2}$ is always larger than $\kappa/(m_1m_2)^{1/2} = \kappa_M/(m_1m_2)^{1/2}$, so the square root in Eq. (9) is always smaller than $(\Omega_1^2+\Omega_2^2)/2$. As a result, both values of λ^2 are negative, i.e. the general solution to Eq. (5) is a sum of four terms, each proportional to $\exp\{\pm i\omega_\pm t\}$, where both *normal frequencies* (or "natural frequencies", or "eigenfrequencies") $\omega_\pm \equiv i\lambda_\pm$ are real:

Anticrossing: example

$$\boxed{\omega_\pm^2 \equiv -\lambda_\pm^2 = \frac{1}{2}\left(\Omega_1^2+\Omega_2^2\right) \pm \left[\frac{1}{4}\left(\Omega_1^2-\Omega_2^2\right)^2 + \frac{\kappa^2}{m_1m_2}\right]^{1/2}.} \qquad (6.10)$$

A plot of these eigenfrequencies as a function of one of the partial frequencies (say, Ω_1), with the other partial frequency fixed, gives us the famous *anticrossing* (also called the "avoided crossing" or "non-crossing") *diagram* – see Fig. 2. One can see that at weak coupling, the normal frequencies $\omega_\pm$ are close to the partial frequencies $\Omega_{1,2}$ everywhere besides a narrow range near the anticrossing point $\Omega_1 = \Omega_2$. Most remarkably, at passing through this region, ω_+ smoothly "switches" from following Ω_2 to following Ω_1 and vice versa.

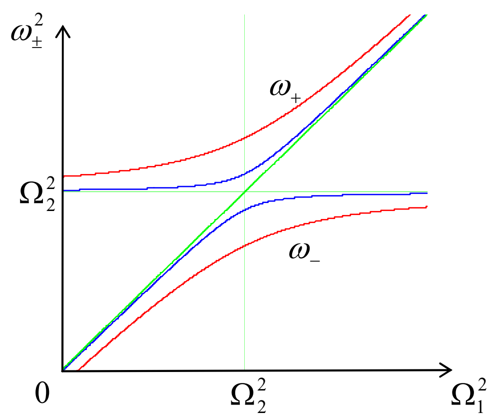

Fig. 6.2. The anticrossing diagram for two values of the normalized coupling strength $\kappa/(m_1m_2)^{1/2}\Omega_2^2$: 0.3 (red lines) and 0.1 (blue lines). In this plot, Ω_1 is assumed to be changed by varying κ_1 rather than m_1, but in the opposite case, the diagram is qualitatively similar.

The reason for this counterintuitive behavior may be found by examining the distribution coefficients $c_{1,2}$ corresponding to each branch of the diagram, which may be obtained by plugging the corresponding value of $\lambda_\pm = -i\omega_\pm$ back into Eqs. (7). For example, at the anticrossing point $\Omega_1 = \Omega_2 \equiv \Omega$, Eq. (10) is reduced to

$$\omega_\pm^2 = \Omega^2 \pm \frac{\kappa}{(m_1m_2)^{1/2}} = \Omega^2\left(1 \pm \frac{\kappa}{(\kappa_1\kappa_2)^{1/2}}\right). \qquad (6.11)$$

Plugging this expression back into any of Eqs. (7), we see that for the two branches of the anticrossing diagram, the distribution coefficient ratio is the same by magnitude but opposite by sign:

$$\left(\frac{c_1}{c_2}\right)_\pm = \mp\left(\frac{m_2}{m_1}\right)^{1/2}, \quad \text{at } \Omega_1 = \Omega_2 . \qquad (6.12)$$

In particular, if the system is symmetric ($m_1 = m_2$, $\kappa_L = \kappa_R$), then at the upper branch, corresponding to $\omega_+ > \omega_-$, we get $c_1 = -c_2$. This means that in this so-called *hard mode*,[2] masses oscillate

[2] In physics, the term "mode" (or "normal mode") is typically used to describe the distribution of a variable in space, at its oscillations with a single frequency. In our current case, when the notion of space is reduced to two oscillator numbers, each mode is fully specified by the corresponding ratio of two distribution coefficients $c_{1,2}$.

in anti-phase: $q_1(t) \equiv -q_2(t)$. The resulting substantial extension/compression of the middle spring (see Fig. 1 again) yields additional returning force which increases the oscillation frequency. On the contrary, at the lower branch, corresponding to ω_-, the particle oscillations are in phase: $c_1 = c_2$, i.e. $q_1(t) \equiv q_2(t)$, so the middle spring is neither stretched nor compressed at all. As a result, in this *soft mode*, the oscillation frequency ω_- is lower than ω_+, and does not depend on κ_M:

$$\omega_-^2 = \Omega^2 - \frac{\kappa}{m} = \frac{\kappa_L}{m} = \frac{\kappa_R}{m} \ . \tag{6.13}$$

Note that for both modes, the oscillations equally engage both particles.

Far from the anticrossing point, the situation is completely different. Indeed, a similar calculation of $c_{1,2}$ shows that on each branch of the diagram, the magnitude of one of the distribution coefficients is much larger than that of its counterpart. Hence, in this limit, any particular mode of oscillations involves virtually only one particle. A slow change of system parameters, bringing it through the anticrossing, leads, first, to a maximal delocalization of each mode at $\Omega_1 = \Omega_2$, and then to a restoration of the localization, but in a different partial degree of freedom.

We could readily carry out similar calculations for the case when the systems are coupled via their velocities, $L_{\text{int}} = m\dot{q}_1\dot{q}_2$, where m is a coupling coefficient – not necessarily a certain physical mass.[3] The results are generally similar to those discussed above, again with the maximum level splitting at $\Omega_1 = \Omega_2 \equiv \Omega$:

$$\omega_\pm^2 = \frac{\Omega^2}{1 \mp |m|/(m_1 m_2)^{1/2}} \approx \Omega^2 \left[1 \pm \frac{|m|}{(m_1 m_2)^{1/2}}\right], \tag{6.14}$$

the last relation being valid for weak coupling. The generalization to the case of simultaneous coordinate and velocity coupling is also straightforward – see the next section.[4]

One more property of weakly coupled oscillators is a periodic slow transfer of energy between them, especially strong at or near the anticrossing point $\Omega_1 = \Omega_2$. Let me leave an analysis of such transfer for the reader's exercise. (Due to the importance of this effect for quantum mechanics, it will be discussed in detail in the QM part of this series.)

6.2. *N* coupled oscillators

The calculations of the previous section may be readily generalized to the case of an arbitrary number (say, *N*) of coupled harmonic oscillators, with an arbitrary type of coupling. It is obvious that in this case Eq. (4) should be replaced with

[3] In mechanics, with $q_{1,2}$ standing for the actual displacements of particles, such coupling is not very natural, but there are many dynamic systems of non-mechanical nature in which such coupling is the most natural one. The simplest example is the system of two *LC* ("tank") circuits, with either capacitive or inductive coupling. Indeed, as was discussed in Sec. 2.2, for such a system, the very notions of the potential and kinetic energies are conditional and interchangeable.

[4] Note that the anticrossing diagram shown in Fig. 2, is even more ubiquitous in quantum mechanics, because, due to the time-oscillatory character of the Schrödinger equation solutions, a weak coupling of any two quantum states leads to qualitatively similar behavior of the eigenfrequencies $\omega_\pm$ of the system, and hence of its *eigenenergies* ("*energy levels*") $E_\pm = \hbar\omega_\pm$ of the system.

$$L = \sum_{j=1}^{N} L_j + \sum_{j,j'=1}^{N} L_{jj'}. \tag{6.15}$$

Moreover, we can generalize the above expression for the mixed terms $L_{jj'}$, taking into account their possible dependence not only on the generalized coordinates but also on the generalized velocities, in a bilinear form similar to Eq. (4). The resulting Lagrangian may be represented in a compact form,

$$L = \sum_{j,j'=1}^{N} \left(\frac{m_{jj'}}{2} \dot{q}_j \dot{q}_{j'} - \frac{\kappa_{jj'}}{2} q_j q_{j'} \right), \tag{6.16}$$

where the off-diagonal terms are index-symmetric: $m_{jj'} = m_{j'j}$, $\kappa_{jj'} = \kappa_{j'j}$, and the factors ½ compensate for the double-counting of each term with $j \neq j'$, at the summation over two independently running indices. One may argue that Eq. (16) is quite general if we still want to keep the equations of motion linear – as they always are if the oscillations are small enough.

Plugging Eq. (16) into the general form (2.19) of the Lagrange equation, we get N equations of motion of the system, one for each value of the index $j' = 1, 2,\ldots, N$:

$$\sum_{j=1}^{N} \left(m_{jj'} \ddot{q}_j + \kappa_{jj'} q_j \right) = 0. \tag{6.17}$$

Just as in the previous section, let us look for a particular solution to this system in the form

$$q_j = c_j e^{\lambda t}. \tag{6.18}$$

As a result, we are getting a system of N linear, homogeneous algebraic equations,

$$\sum_{j=1}^{N} \left(m_{jj'} \lambda^2 + \kappa_{jj'} \right) c_j = 0, \tag{6.19}$$

for the set of N distribution coefficients c_j. The condition that this system is self-consistent is that the determinant of its matrix equals zero:

$$\mathrm{Det}\left(m_{jj'} \lambda^2 + \kappa_{jj'} \right) = 0. \tag{6.20}$$

This characteristic equation is an algebraic equation of degree N for λ^2, and so has N roots $(\lambda^2)_n$. For any Hamiltonian system with stable equilibrium, the matrices $m_{jj'}$ and $\kappa_{jj'}$ ensure that all these roots are real and negative. As a result, the general solution to Eq. (17) is the sum of $2N$ terms proportional to exp $\{\pm i\omega_n t\}$, $n = 1, 2,\ldots, N$, where all N normal frequencies ω_n are real.

Plugging each of these $2N$ values of $\lambda = \pm i\omega_n$ back into a particular set of linear equations (17), one can find the corresponding sets of distribution coefficients $c_{j\pm}$. Generally, the coefficients are complex, but to keep $q_j(t)$ real, the coefficients c_{j+} corresponding to $\lambda = +i\omega_n$, and c_{j-} corresponding to $\lambda = -i\omega_n$ have to be complex-conjugate of each other. Since the sets of the distribution coefficients may be different for each λ_n, they should be marked with two indices, j and n. Thus, at general initial conditions, the time evolution of the j^{th} generalized coordinate may be represented as

$$q_j = \frac{1}{2} \sum_{n=1}^{N} \left(c_{jn} \exp\{+i\omega_n t\} + c_{jn}^* \exp\{-i\omega_n t\} \right) \equiv \mathrm{Re} \sum_{n=1}^{N} c_{jn} \exp\{i\omega_n t\}. \tag{6.21}$$

This formula shows very clearly again the physical sense of the distribution coefficients c_{jn}: a set of these coefficients, with different values of index j but the same mode number n, gives the complex amplitudes of oscillations of the coordinates in this mode, i.e. for the special initial conditions that ensure purely sinusoidal motion of all the system, with frequency ω_n.

The calculation of the normal frequencies and the corresponding modes (distribution coefficient sets) of a particular coupled system with many degrees of freedom from Eqs. (19)–(20) is a task that frequently may be only done numerically.[5] Let us discuss just two particular but very important cases. First, let all the coupling coefficients be small in the following sense: $|m_{jj'}| << m_j \equiv m_{jj}$ and $|\kappa_{jj'}| << \kappa_j \equiv \kappa_{jj}$, for all $j \neq j$, and all partial frequencies $\Omega_j \equiv (\kappa_j/m_j)^{1/2}$ be not too close to each other:

$$\frac{\left|\Omega_j^2 - \Omega_{j'}^2\right|}{\Omega_j^2} >> \frac{\left|\kappa_{jj'}\right|}{\kappa_j}, \frac{\left|m_{jj'}\right|}{m_j}, \quad \text{for all } j \neq j'. \tag{6.22}$$

(Such a situation frequently happens if parameters of the system are "random" in the sense that they do not follow any special, simple rule – for example, the one resulting from some simple symmetry of the system.) Results of the previous section imply that in this case, the coupling does not produce a noticeable change in the oscillation frequencies: $\{\omega_n\} \approx \{\Omega_j\}$. In this situation, oscillations at each eigenfrequency are heavily concentrated in one degree of freedom, i.e. in each set of the distribution coefficients c_{jn} (for a given n), one coefficient's magnitude is much larger than all others.

Now let the conditions (22) be valid for all but one pair of partial frequencies, say Ω_1 and Ω_2, while these two frequencies are so close that the coupling of the corresponding partial oscillators becomes essential. In this case, the approximation $\{\omega_n\} \approx \{\Omega_j\}$ is still valid for all other degrees of freedom, and the corresponding terms may be neglected in Eqs. (19) for j = 1 and 2. As a result, we return to Eqs. (7) (perhaps generalized for velocity coupling) and hence to the anticrossing diagram (Fig. 2) discussed in the previous section. As a result, an extended change of only one partial frequency (say, Ω_1) of a weakly coupled system produces a sequence of frequency anticrossings – see Fig. 3.

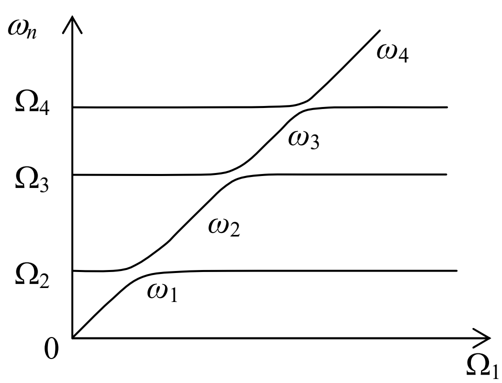

Fig. 6.3. The level anticrossing in a system of N weakly coupled oscillators – schematically.

6.3. 1D waves

The second case when the general results of the last section may be simplified are coupled systems with a considerable degree of symmetry. Perhaps the most important of them are uniform

[5] Fortunately, very effective algorithms have been developed for this *matrix diagonalization* task – see, e.g., references in MA Sec. 16(iii)-(iv). For example, the popular MATLAB software package was initially created exactly for this purpose. ("MAT" in its name stood for "matrix" rather than "mathematics".)

systems that may sustain traveling and standing *waves*. Figure 4a shows a simple example of such a system – a long uniform chain of particles, of mass m, connected with light, elastic springs, pre-stretched with the tension force $\mathscr{T}$ to have equal lengths d. (This system may be understood as a natural generalization of the two-particle system considered in Sec. 1 – cf. Fig. 1.)

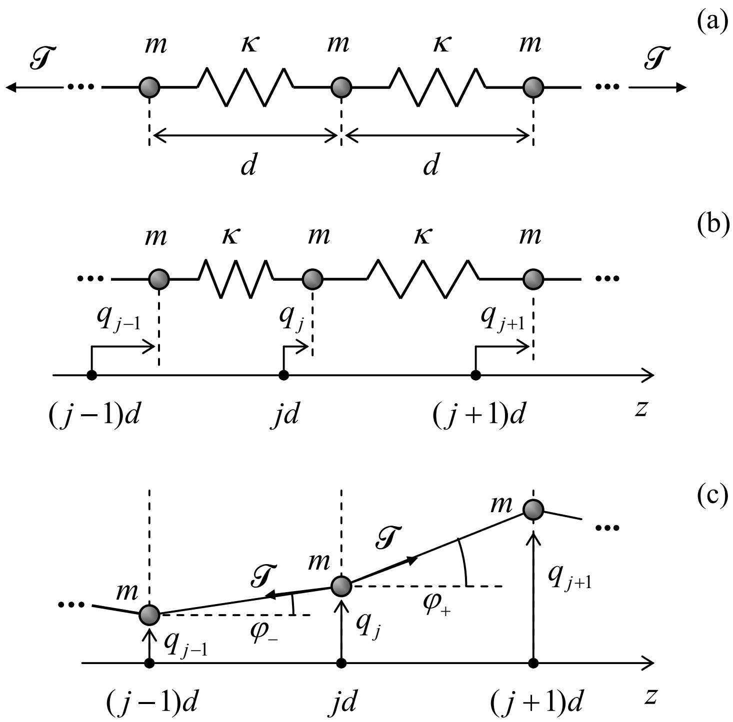

Fig. 6.4. (a) A uniform 1D chain of elastically coupled particles, and their small (b) longitudinal and (c) transverse displacements (much exaggerated for clarity).

The spring's pre-stretch does not affect small *longitudinal* oscillations q_j of the particles about their equilibrium positions $z_j = jd$ (where the integer j numbers the particles sequentially) – see Fig. 4b.[6] Indeed, in the 2$^{\text{nd}}$ Newton law for such a longitudinal motion of the j^{th} particle, the forces $\mathscr{T}$ and $(-\mathscr{T})$ exerted by the springs on the right and the left of it, cancel. However, the elastic additions, $\kappa\Delta q$, to these forces are generally different:

$$m\ddot{q}_j = \kappa(q_{j+1} - q_j) - \kappa(q_j - q_{j-1}). \tag{6.23}$$

On the contrary, for *transverse* oscillations within one plane (Fig. 4c), the net transverse component of the pre-stretch force exerted on the j^{th} particle, $\mathscr{T}_\text{t} = \mathscr{T}(\sin\varphi_+ - \sin\varphi_-)$, where $\varphi_\pm$ are the force direction angles, does not vanish. As a result, direct contributions to this force from small longitudinal oscillations, with $|q_j| << d$, $\mathscr{T}/\kappa$, are negligible. Also, due to the first of these strong conditions, the angles $\varphi_\pm$ are small, and hence may be approximated, respectively, as $\varphi_+ \approx (q_{j+1} - q_j)/d$ and $\varphi_- \approx (q_j - q_{j-1})/d$. Plugging these expressions into a similar approximation, $\mathscr{T}_\text{t} \approx \mathscr{T}(\varphi_+ - \varphi_-)$ for the transverse force, we see that it may be expressed as $\mathscr{T}(q_{j+1} - q_j)/d - \mathscr{T}(q_j - q_{j-1})/d$, i.e. is absolutely similar

[6] Note the need for a clear distinction between *the equilibrium position* z_j of the j^{th} point and its *deviation* q_j from it. Such distinction has to be sustained in the continuous limit (see below), where it is frequently called the *Eulerian description* – named after L. Euler, even though it was introduced to mechanics by J. d'Alembert. In this course, the distinction is emphasized by using different letters – respectively, z and q (in the 3D case, $\mathbf{r}$ and $\mathbf{q}$).

to that in the longitudinal case, just with the replacement $\kappa \to \mathscr{T}/d$. As a result, we may write the equation of motion of the j^{th} particle for these two cases in the same form:

$$m\ddot{q}_j = \kappa_{\text{ef}}(q_{j+1} - q_j) - \kappa_{\text{ef}}(q_j - q_{j-1}), \tag{6.24}$$

where κ_{ef} is the "effective spring constant", equal to κ for the longitudinal oscillations, and to $\mathscr{T}/d$ for the transverse oscillations.[7]

Apart from the (formally) infinite size of the system, Eq. (24) is just a particular case of Eq. (17), and thus its particular solution may be looked for in the form (18), where, in light of our previous experience, we may immediately take $\lambda^2 \equiv -\omega^2$. With this substitution, Eq. (24) gives the following simple form of the general system of equations (19) for the distribution coefficients c_j:

$$\left(-m\omega^2 + 2\kappa_{\text{ef}}\right)c_j - \kappa_{\text{ef}}c_{j+1} - \kappa_{\text{ef}}c_{j-1} = 0. \tag{6.25}$$

Now comes the most important conceptual step toward the wave theory. The *translational symmetry* of Eq. (25), i.e. its invariance with respect to the replacement $j \to j + 1$, allows it to have particular solutions of the following form:

$$c_j = ae^{i\alpha j}, \tag{6.26}$$

where the coefficient α may depend on ω (and system's parameters), but not on the particle number j. Indeed, plugging Eq. (26) into Eq. (25) and canceling the common factor $e^{i\alpha j}$, we see that this *differential* equation is indeed identically satisfied, provided that α obeys the following *algebraic* characteristic equation:

$$\left(-m\omega^2 + 2\kappa_{\text{ef}}\right) - \kappa_{\text{ef}}e^{+i\alpha} - \kappa_{\text{ef}}e^{-i\alpha} = 0. \tag{6.27}$$

The physical sense of the solution (26) becomes clear if we use it and Eq. (18) with $\lambda = \mp i\omega$, to write

$$\boxed{q_j(t) = \text{Re}\left[a\exp\left\{i(kz_j \mp \omega t)\right\}\right] = \text{Re}\left[a\exp\left\{ik(z_j \mp v_{\text{ph}}t)\right\}\right],} \tag{6.28}$$

1D traveling wave

where the *wave number* k is defined as $k \equiv \alpha/d$. Eq. (28) describes a *sinusoidal*[8] *traveling wave* of particle displacements, which propagates, depending on the sign before v_{ph}, to the right or the left along the particle chain, with the so-called *phase velocity*

$$\boxed{v_{\text{ph}} \equiv \frac{\omega}{k}.} \tag{6.29}$$

Phase velocity

Perhaps the most important characteristic of a wave system is the so-called *dispersion relation*, i.e. the relation between the wave's frequency ω and its wave number k – one may say, between the temporal and spatial frequencies of the wave. For our current system, this relation is given by Eq. (27) with $\alpha \equiv kd$. Taking into account that $(2 - e^{+i\alpha} - e^{-i\alpha}) \equiv 2(1 - \cos\alpha) \equiv 4\sin^2(\alpha/2)$, the dispersion relation may be rewritten in a simpler form:

[7] The re-derivation of Eq. (24) from the Lagrangian formalism, with the simultaneous strict proof that the small oscillations in the longitudinal direction and the two mutually perpendicular transverse directions are all independent of each other, is a very good exercise, left for the reader.

[8] In optics and quantum mechanics, such waves are usually called *monochromatic*; I will try to avoid this term until the corresponding parts (EM and QM) of my series.

$$\omega = \pm\omega_{\max} \sin\frac{\alpha}{2} \equiv \pm\omega_{\max} \sin\frac{kd}{2}, \qquad \text{where } \omega_{\max} \equiv 2\left(\frac{\kappa_{\text{ef}}}{m}\right)^{1/2}. \tag{6.30}$$

This result, sketched in Fig. 5, is rather remarkable in several aspects. I will discuss them in some detail, because most of these features are typical for waves of any type (including even the "de Broglie waves", i.e. wavefunctions, in quantum mechanics), propagating in periodic structures.

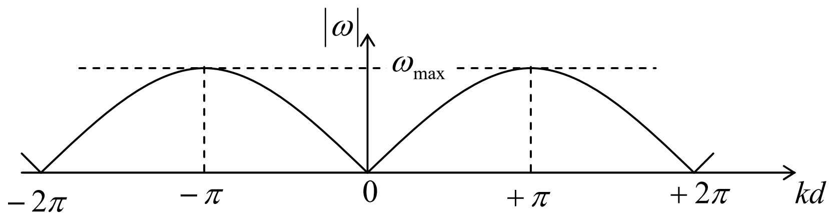

Fig. 6.5. The dispersion relation (30).

First, at low frequencies, $\omega << \omega_{\max}$, the dispersion relation (31) is linear:

$$\omega = \pm vk, \qquad \text{where } v \equiv \left|\frac{d\omega}{dk}\right|_{k=0} = \frac{\omega_{\max} d}{2} = \left(\frac{\kappa_{\text{ef}}}{m}\right)^{1/2} d\,. \tag{6.31}$$

Plugging Eq. (31) into Eq. (29), we see that the constant v plays, in the low-frequency limit, the role of the phase velocity for waves of any frequency. Due to its importance, this *acoustic wave*[9] limit will with be the subject of the special next section.

Second, when the wave frequency *is* comparable with $\omega_{\max}$, the dispersion relation is *not* linear, and the system is *dispersive*. This means that as a wave, whose Fourier spectrum has several essential components with frequencies of the order of $\omega_{\max}$, travels along the structure, its *waveform* (which may be defined as the shape of the line connecting all points $q_j(z)$, at the same time) changes.[10] This effect may be analyzed by representing the general solution of Eq. (24) as the sum (more generally, an integral) of the components (28) with different complex amplitudes a:

1D wave packet

$$\boxed{q_j(t) = \text{Re}\int_{-\infty}^{+\infty} a_k \exp\{i[kz_j - \omega(k)t]\}dk\,.} \tag{6.32}$$

This notation emphasizes the possible dependence of the component wave amplitudes a_k and frequencies ω on the wave number k. While the latter dependence is given by the dispersion relation, in our current case by Eq. (30), the function a_k is determined by the initial conditions. For applications, the case when a_k is substantially different from zero only in a narrow interval, of a width $\Delta k << k_0$ around some central value k_0, is of special importance. The Fourier transform reciprocal to Eq. (32) shows that this is true, in particular, for the so-called *wave packet* – a sinusoidal ("carrier") wave modulated by a spatial *envelope* function of a large width $\Delta z \sim 1/\Delta k >> 1/k_0$ – see, e.g., Fig. 6.

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[9] This term is purely historical. Though the usual sound waves in air, which are the subject of acoustics, belong to this class, the waves we are discussing may have frequencies both well below and well above the human ear's sensitivity range.

[10] The waveform's deformation due to *dispersion* (which we are considering now) should be clearly distinguished from its possible change due to *attenuation* due to energy dissipation – which is *not* taken into account is our current energy-conserving model – cf. Sec. 6 below.

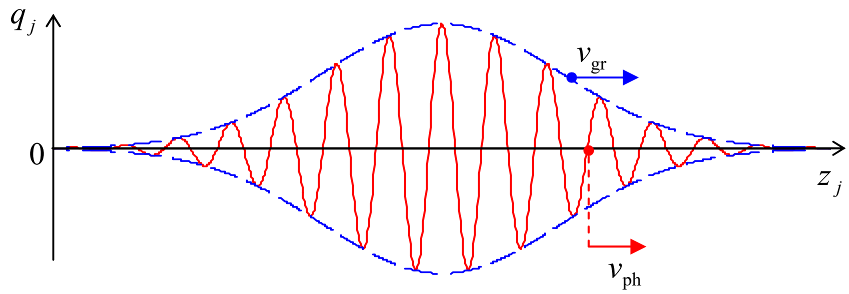

Fig. 6.6. The phase and group velocities of a wave packet.

Using the strong inequality $\Delta k \ll k_0$, the wave packet's propagation may be analyzed by expending the dispersion relation $\omega(k)$ into the Taylor series at point k_0, and, in the first approximation in $\Delta k/ k_0$, restricting the expansion to its first two terms:

$$\omega(k) \approx \omega_0 + \left.\frac{d\omega}{dk}\right|_{k=k_0} \tilde{k}, \qquad \text{where } \omega_0 \equiv \omega(k_0), \text{ and } \tilde{k} \equiv k - k_0. \tag{6.33}$$

In this approximation, Eq. (32) yields

$$\begin{aligned} q_j(t) &\approx \text{Re} \int_{-\infty}^{+\infty} a_k \exp\left\{ i\left[\left(k_0 + \tilde{k}\right) z_j - \left(\omega_0 + \left.\frac{d\omega}{dk}\right|_{k=k_0} \tilde{k} \right) t \right] \right\} dk \\ &\equiv \text{Re}\left[\exp\{ i(k_0 z_j - \omega_0 t)\} \int_{-\infty}^{+\infty} a_k \exp\left\{ i\tilde{k}\left(z_j - \left.\frac{d\omega}{dk}\right|_{k=k_0} t \right) \right\} dk \right]. \end{aligned} \tag{6.34}$$

Comparing the last expression with the initial form of the wave packet,

$$q_j(0) = \text{Re} \int_{-\infty}^{+\infty} a_k e^{ikz_j} dk \equiv \text{Re}\left[\exp\{ik_0 z_j\} \int_{-\infty}^{+\infty} a_k \exp\{i\tilde{k} z_j\} dk \right], \tag{6.35}$$

and taking into account that the phase factors before the integrals in the last forms of Eqs. (34) and (35) do not affect its envelope, we see that in this approximation, the envelope sustains its initial form and propagates along the system with the so-called *group velocity*

$$\boxed{v_{gr} \equiv \left.\frac{d\omega}{dk}\right|_{k=k_0}.} \tag{6.36}$$

Group velocity

Except for the acoustic wave limit (31), this velocity, which characterizes the propagation of the waveform's envelope, is different from the phase velocity (29), which describes the propagation of the carrier wave, e.g., the spatial position of one of its zeros – see the red and blue arrows in Fig. 6.[11]

Next, for our particular dispersion relation (30), the difference between v_{ph} and v_{gr} increases as ω approaches ω_{max}, with the group velocity (36) tending to zero, while the phase velocity stays almost constant. The physics of such a maximum frequency available for the wave propagation may be readily understood by noticing that according to Eq. (30), at $\omega = \omega_{max}$, the wave number k equals $n\pi/d$, where n

[11] Taking into account the next term in the Taylor expansion of the function $\omega(q)$, proportional to $d^2\omega/dq^2$, we would find that the dispersion leads to a gradual change of the envelope's form. Such changes play an important role in quantum mechanics, so they are discussed in detail in the QM part of these lecture notes.

is an odd integer, and hence the phase shift $\alpha \equiv kd$ is an odd multiple of π. Plugging this value into Eq. (28), we see that at $\omega = \omega_{\max}$, the oscillations of two adjacent particles are in anti-phase, for example:

$$q_0(t) = \mathrm{Re}\left[a \exp\{-i\omega t\}\right], \qquad q_1(t) = \mathrm{Re}\left[a \exp\{i\pi - i\omega t\}\right] = -q_0(t). \tag{6.37}$$

It is clear, especially from Fig. 4b for longitudinal oscillations, that at such a phase shift, all the springs are maximally stretched/compressed (just as in the hard mode of the two coupled oscillators analyzed in Sec. 1), so it is natural that this mode has the highest possible frequency.

This fact invites a natural question: what happens with the system if it is agitated at a frequency $\omega > \omega_{\max}$, say by an external force exerted on its boundary? Reviewing the calculations that have led to the dispersion relation (30), we see that they are all valid not only for real but also for any complex values of k. In particular, at $\omega > \omega_{\max}$ it gives

$$k = \frac{(2n-1)\pi}{d} \pm \frac{i}{\Lambda}, \qquad \text{where } n = 1, 2, 3, ..., \quad \Lambda \equiv \frac{d}{2\cosh^{-1}(\omega/\omega_{\max})}. \tag{6.38}$$

Plugging this relation into Eq. (28), we see that the wave's amplitude becomes an exponential function of the particle's position:

$$\left| q_j \right| = \left| a \right| e^{\pm j \,\mathrm{Im}\, kd} \propto \exp\{\pm z_j / \Lambda\}. \tag{6.39}$$

Physically this means that penetrating into the structure, the wave decays exponentially (from the excitation point), dropping by a factor of $e \approx 3$ at the so-called *penetration depth* Λ. (According to Eq. (38), at $\omega \sim \omega_{\max}$ this depth is of the order of the distance d between the adjacent particles, and decreases but rather slowly as the frequency is increased beyond $\omega_{\max}$.) Such a limited penetration is a very common property of waves, including electromagnetic waves penetrating into various plasmas and superconductors, and the quantum-mechanical de Broglie waves penetrating into classically forbidden regions of space. Note that this effect of "wave expulsion" from the medium's bulk does not require any energy dissipation.

Finally, one more fascinating feature of the dispersion relation (30) is its periodicity: if the relation is satisfied with some wave number $k_0(\omega)$, it is also satisfied with any $k_n(\omega) = k_0(\omega) + 2\pi n/d$, where n is an integer. This property is independent of the particular dynamics of the system and is a common property of all systems that are d-periodic in the usual ("direct") space. It has especially important implications for the quantum de Broglie waves in periodic systems – for example, crystals – leading, in particular, to the famous band/gap structure of their energy spectrum.[12]

6.4. Acoustic waves

Now let us return to the limit of low-frequency, dispersion-free *acoustic waves*, with $|\omega| << \omega_0$, propagating with the frequency-independent velocity (31). Such waves are the general property of *any* elastic continuous medium and obey a simple (and very important) partial differential equation. To derive it, let us note that in the acoustic wave limit, $|kd| << 1$,[13] the phase shift $\alpha \equiv kd$ is very close to

[12] For more detail see, e.g., QM Sec. 2.5.

[13] Strictly speaking, per the discussion at the end of the previous section, in this reasoning, k means the distance of the wave number from the closest point $2\pi n/d$ – see Fig. 5 again.

$2\pi n$. This means that the differences $q_{j+1}(t) - q_j(t)$ and $q_j(t) - q_{j-1}(t)$, participating in Eq. (24), are relatively small and may be approximated with $\partial q/\partial j \equiv \partial q/\partial(z/d) \equiv d(\partial q/\partial z)$, with the derivatives taken at middle points between the particles: respectively, $z_+ \equiv (z_{j+1} - z_j)/2$ and $z_- \equiv (z_j - z_{j-1})/2$. Let us now consider z as a continuous argument, and introduce the particle displacement $q(z, t)$ – a continuous function of space and time, satisfying the requirement $q(z_j, t) = q_j(t)$. In this notation, in the limit $kd \to 0$, the sum of the last two terms of Eq. (24) becomes $-\kappa d\,[\partial q/\partial z(z_+) - \partial q/\partial z(z_-)]$, and hence may be approximated as $-\kappa d^2(\partial^2 q/\partial z^2)$, with the second derivative taken at point $(z_+ - z_-)/2 \equiv z_j$, i.e. exactly at the same point as the time derivative. As a result, the whole set of *ordinary* differential equations (24), for different j, is reduced to just *one partial* differential equation

$$m\frac{\partial^2 q}{\partial t^2} - \kappa_{\text{ef}} d^2 \frac{\partial^2 q}{\partial z^2} = 0. \tag{6.40a}$$

Using Eq. (31), we may rewrite this *1D wave equation* in a more general form

1D wave equation

$$\boxed{\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right) q(z,t) = 0.} \tag{6.40b}$$

The most important property of the wave equation (40), which may be verified by an elementary substitution, is that it is satisfied by either of two *traveling wave* solutions (or their linear superposition):

$$q_+(z,t) = f_+(t - z/v), \qquad q_-(z,t) = f_-(t + z/v), \tag{6.41}$$

where $f_\pm$ are *any* smooth functions of one argument. The physical sense of these solutions may be revealed by noticing that the displacements $q_\pm$ do not change at the addition of an arbitrary change Δt to their time argument, provided that it is accompanied by an addition of the proportional addition of $\mp v\Delta t$ to their space argument. This means that with time, the waveforms just move (respectively, to the left or the right), with the constant speed v, retaining their form – see Fig. 7.[14]

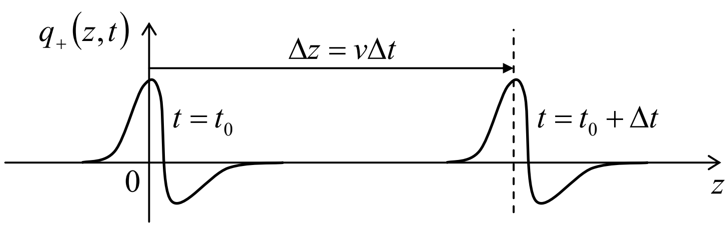

Fig. 6.7. Propagation of a traveling wave in a dispersion-free 1D system.

Returning to the simple model shown in Fig. 4, let me emphasize that the acoustic-wave velocity v is different for the waves of two types: for the longitudinal waves (with $\kappa_{\text{ef}} = \kappa$, see Fig. 4b),

$$v = v_l \equiv \left(\frac{\kappa}{m}\right)^{1/2} d, \tag{6.42}$$

while for the transverse waves (with $\kappa_{\text{ef}} = \mathscr{T}/d$, see Fig. 4c):

[14] From the point of view of Eq. (40), the only requirement to the "smoothness" of the functions $f_\pm$ is to be doubly differentiable. However, we should not forget that in our case the wave equation is only an approximation of the discrete Eq. (24), so according to Eq. (30), the traveling waveform conservation is limited by the acoustic wave limit condition $\omega \ll \omega_{\max}$, which should be fulfilled for all Fourier components of these functions.

$$v = v_t = \left(\frac{\mathscr{T}}{md}\right)^{1/2} d \equiv \left(\frac{\mathscr{T}d}{m}\right)^{1/2} \equiv \left(\frac{\mathscr{T}}{\mu}\right)^{1/2}, \tag{6.43}$$

where the constant $\mu \equiv m/d$ has a simple physical sense of the particle chain's mass per unit length. Evidently, these velocities, in the same system, may be rather different.

The wave equation (40), with its only parameter v, may conceal the fact that any wave-supporting system is characterized by one more key parameter. In our current model (Fig. 4), this parameter may be revealed by calculating the forces $F_\pm(z, t)$ accompanying any of the traveling waves (41) of particle displacements. For example, in the acoustic wave limit $kd \to 0$ we are considering now, the force exerted by the j^{th} particle on its right neighbor may be approximated as

$$F(z_j, t) \equiv \kappa_{\text{ef}} \left[q_j(t) - q_{j+1}(t) \right] \approx -\kappa_{\text{ef}} \left.\frac{\partial q}{\partial z}\right|_{z=z_j} d, \tag{6.44}$$

where, as was discussed above, κ_{ef} is equal to κ for the longitudinal waves, and to $\mathscr{T}/d$ for the transverse waves. But for the traveling waves (41), the partial derivatives $\partial q_\pm/\partial z$ are equal to $\mp \dot{f}_\pm / v$ (where the dot means the differentiation over the full arguments of the functions $f_\pm$), so the corresponding forces are equal to

$$F_\pm = \mp \frac{\kappa_{\text{ef}} d}{v} \dot{f}_\pm, \tag{6.45}$$

i.e. are proportional to the particle's velocities $u = \partial q/\partial t$ in these waves,[15] $u_\pm = \dot{f}_\pm$, for the same z and t. This means that the ratio

$$\frac{F_\pm(z,t)}{u_\pm(z,t)} = -\kappa_{\text{ef}} d \frac{\partial q_\pm / \partial z}{\partial q_\pm / \partial t} = -\kappa_{\text{ef}} d \frac{(\mp \dot{f}_\pm)/v}{\dot{f}_\pm} \equiv \pm \frac{\kappa_{\text{ef}} d}{v}, \tag{6.46}$$

depends only on the wave propagation direction, but is independent of z and t, and also of the propagating waveform. Its magnitude,

Wave impedance

$$\boxed{Z \equiv \left| \frac{F_\pm(z,t)}{u_\pm(z,t)} \right| = \frac{\kappa_{\text{ef}} d}{v} = (\kappa_{\text{ef}} m)^{1/2},} \tag{6.47}$$

characterizing the dynamic "stiffness" of the system for the propagating waves, is called the *wave impedance*.[16] Note that the impedance is determined by the *product* of the system's generic parameters κ_{ef} and m, while the wave velocity (31) is proportional to their *ratio*, so these two parameters are completely independent, and both are important. According to Eq. (47), the wave impedance, just as the wave velocity, is also different for the longitudinal and transverse waves:

$$Z_l = \frac{\kappa d}{v_l} \equiv (\kappa m)^{1/2}, \qquad Z_t = \frac{\mathscr{T}}{v_t} \equiv (\mathscr{T}\mu)^{1/2}. \tag{6.48}$$

[15] Of course, the *particle's* velocity u (which is proportional to the wave amplitude) should not be confused with the *wave's* velocity v (which is independent of this amplitude).

[16] This notion is regretfully missing from many physics (but not engineering!) textbooks.

(Note that the first of these expressions for Z coincides with the one used for a single oscillator in Sec. 5.6. In that case, Z may be also recast in a form similar to Eq. (46), namely, as the ratio of the force and velocity amplitudes at free oscillations.)

One of the wave impedance's key functions is to scale the power carried by a traveling wave:

Traveling wave's power

$$\mathscr{P}_\pm \equiv F_\pm(z,t)u_\pm(z,t) = -\kappa_{\text{ef}} d \frac{\partial q_\pm}{\partial z}\frac{\partial q_\pm}{\partial t} = \pm \frac{\kappa_{\text{ef}} d}{v}\dot{f}_\pm^2 \equiv \pm Z\dot{f}_\pm^2 . \tag{6.49}$$

Two remarks about this important result. First, the sign of $\mathscr{P}$ depends only on the direction of the wave propagation, but not on the waveform. Second, the instant value of the power does not change if we move with the wave in question, i.e. measure $\mathscr{P}$ at points with $z \pm vt = \text{const}$. This is natural because in the Hamiltonian system we are considering, the wave energy is conserved. Hence, the wave impedance Z characterizes the energy *transfer* along the system rather than its *dissipation*.

Another important function of the wave impedance notion becomes clear when we consider waves in nonuniform systems. Indeed, our previous analysis assumed that the 1D system supporting the waves (Fig. 4) is exactly periodic, i.e. macroscopically uniform, and extends all the way from $-\infty$ to $+\infty$. Now let us examine what happens when this is not true. The simplest and very important example of such nonuniform systems is a *sharp interface*, i.e. a point (say, $z = 0$) at which system parameters experience a jump while remaining constant on each side of the interface – see Fig. 8.

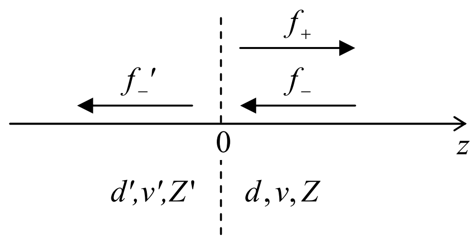

Fig. 6.8. Partial reflection of a wave from a sharp interface.

In this case, the wave equation (40) and its partial solutions (41) are is still valid for $z < 0$ and $z > 0$ – in the former case, with primed parameters. However, the jump of parameters at the interface leads to a partial *reflection* of the incident wave from the interface, so at least on the side of the incidence (in the case shown in Fig. 8, for $z \geq 0$) we need to use two such terms, one describing the incident wave and another one, the reflected wave:

$$q(z,t) = \begin{cases} f_-'(t + z/v'), & \text{for } z \leq 0, \\ f_-(t + z/v) + f_+(t - z/v), & \text{for } z \geq 0. \end{cases} \tag{6.50}$$

To find the relations between the functions f_-, f_+, and f_-' (of which the first one, describing the incident wave, may be considered known), we may use two boundary conditions at $z = 0$. First, the displacement $q_0(t)$ of the particle at the interface has to be the same whether it is considered a part of the left or right sub-system, and it participates in Eqs. (50) for both $z \leq 0$ and $z \geq 0$. This gives us the first boundary condition:

$$f_-'(t) = f_-(t) + f_+(t) . \tag{6.51}$$

On the other hand, the forces exerted on the interface from the left and the right should also have equal magnitude, because the interface may be considered as an object with a vanishing mass, and any nonzero net force would give it an infinite (and hence unphysical) acceleration. Together with Eqs. (45) and (47), this gives us the second boundary condition:

$$Z'\dot{f}_-'(t) = Z\left[\dot{f}_-(t) - \dot{f}_+(t)\right]. \tag{6.52}$$

Integrating both parts of this equation over time, and neglecting the integration constant (which describes a common displacement of all particles rather than their oscillations), we get

$$Z'f_-'(t) = Z\left[f_-(t) - f_+(t)\right]. \tag{6.53}$$

Now solving the system of two linear equations (51) and (53) for $f_+(t)$ and $f_+'(t)$, we see that both these functions are proportional to the incident waveform:

$$f_+(t) = \mathcal{R}\, f_-(t), \qquad f_-'(t) = \mathcal{T}\, f_-(t), \tag{6.54}$$

with the following *reflection* ($\mathcal{R}$) and *transmission* ($\mathcal{T}$) *coefficients*:

Reflection and transmission coefficients

$$\boxed{\mathcal{R} = \frac{Z - Z'}{Z + Z'}, \qquad \mathcal{T} = \frac{2Z}{Z + Z'}.} \tag{6.55}$$

Later in this series, we will see that with the appropriate re-definition of the impedance, these relations are also valid for waves of other physical nature (including the de Broglie waves in quantum mechanics) propagating in 1D continuous structures, and also in continua of higher dimensions, at the normal wave incidence upon the interface.[17] Note that the coefficients $\mathcal{R}$ and $\mathcal{T}$ give the ratios of wave *amplitudes*, rather than their *powers*. Combining Eqs. (49) and (55), we get the following relations for the powers – either at the interface or at the corresponding points of the reflected and transmitted waves:

$$\mathcal{P}_+ = \left(\frac{Z - Z'}{Z + Z'}\right)^2 \mathcal{P}_-, \qquad \mathcal{P}_-' = \frac{4ZZ'}{(Z + Z')^2}\mathcal{P}_-. \tag{6.56}$$

Note that $\mathcal{P}_- + \mathcal{P}_+ = \mathcal{P}_-'$, again reflecting the wave energy conservation.

Perhaps the most important corollary of Eqs. (55)–(56) is that the reflected wave completely vanishes, i.e. the incident wave is completely transmitted through the interface ($\mathcal{P}_+' = \mathcal{P}_+$), if the so-called *impedance matching condition* $Z' = Z$ is satisfied, even if the wave velocities v (32) are different on the left and the right sides of it. On the contrary, the equality of the acoustic velocities in the two continua does *not* guarantee the full transmission of their interface. Again, this is a very general result.

Finally, let us note that for the important particular case of a sinusoidal incident wave:[18]

$$f_-(t) = \mathrm{Re}\left[a\, e^{-i\omega t}\right], \qquad \text{so that } f_+(t) = \mathrm{Re}\left[\mathcal{R} a\, e^{-i\omega t}\right], \tag{6.57}$$

where a is its complex amplitude, the total wave (50) on the right of the interface is

[17] See, e.g. the corresponding parts of this series: QM Sec. 2.3 and EM Sec. 7.3.

[18] In the acoustic wave limit, when the impedances Z and Z', and hence the reflection coefficient $\mathcal{R}$, are real, the factors $\mathcal{R}$ and Z may be taken out from under the Re operators in Eqs. (57)-(59). However, in the current, more general form of these relations, they are also valid for the case of arbitrary frequencies, $\omega \sim \omega_{\max}$, when these factors may be complex.

$$q(z,t) = \mathrm{Re}\left[a\,e^{-i\omega(t+z/v)} + \mathcal{R}\,a\,e^{-i\omega(t-z/v)}\right] \equiv \mathrm{Re}\left[a\left(e^{-ikz} + \mathcal{R}\,e^{+ikz}\right)e^{-i\omega t}\right], \quad \text{for } z \geq 0, \tag{6.58}$$

while according to Eq. (45), the corresponding force distribution is

$$F(z,t) = F_-(z,t) + F_+(z,t) = -Z\dot{f}_-(t - z/v) + Z\dot{f}_+(t - z/v) = \mathrm{Re}\left[i\omega Z a\left(e^{-ikz} - \mathcal{R}\,e^{+ikz}\right)e^{-i\omega t}\right]. \tag{6.59}$$

These expressions will be used in the next section.

6.5. Standing waves

Now let us consider the two limits in which Eqs. (55) predicts a *total wave reflection* ($\mathcal{T} = 0$): $Z'/Z \to \infty$ (when $\mathcal{R} = -1$) and $Z'/Z \to 0$ (when $\mathcal{R} = +1$). According to Eq. (53), the former limit corresponds to $f_-(t) + f_+(t) \equiv q(0, t) = 0$, i.e. to vanishing oscillations at the interface. This means that this particular limit describes a perfectly *rigid boundary*, not allowing the system's end to oscillate at all. In this case, Eqs. (58)-(59) yield

$$q(z,t) = \mathrm{Re}\left[a\left(e^{-ikz} - e^{+ikz}\right)e^{-i\omega t}\right] \equiv -2\,\mathrm{Re}\left[a\,e^{-i\omega t}\right]\sin kz, \tag{6.60}$$

$$F(z,t) = \mathrm{Re}\left[i\,\omega Z a\left(e^{-ikz} + e^{+ikz}\right)e^{-i\omega t}\right] \equiv 2\omega Z\,\mathrm{Re}\left[a\;e^{-i(\omega t - \pi/2)}\right]\cos kz. \tag{6.61}$$

These equalities mean that we may interpret the process on the right of the interface using two mathematically equivalent, but physically different languages: either as the sum of two *traveling waves* (the incident one and the reflected one, propagating in opposite directions), or as a single *standing wave*. Note that in contrast with the traveling wave (Fig. 9a, cf. Fig. 7), in the standing sinusoidal wave (Fig. 9b) all particles oscillate in time with the same phase.

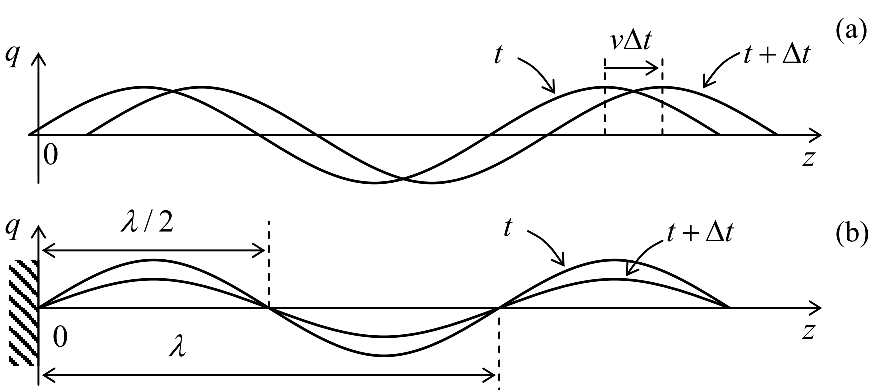

Fig. 6.9. The time evolution of (a) a traveling sinusoidal wave, and (b) a standing sinusoidal wave at a rigid boundary.

Note also that the phase of the force oscillations (61) is shifted, both in space and in time, by $\pi/2$ relative to the particle displacement oscillations. (In particular, at the rigid boundary the force amplitude reaches its maximum.) As a result, the average power flow vanishes, so the *average* energy of the standing wave does not change, though its *instant* energy still oscillates, at each spatial point, between its kinetic and potential components – just as at the usual harmonic oscillations of one particle. A similar standing wave, but with a maximum of the displacement q, and with a zero ("node") of the force F, is formed at the open boundary, with $Z'/Z \to 0$, and hence $\mathcal{R} = +1$.

Now I have to explain why I have used the sinusoidal waveform for the wave reflection analysis. Let us consider a 1D wave system, which obeys Eq. (40), of a finite length l, limited by two rigid walls (located, say, at $z = 0$ and $z = l$), which impose the corresponding boundary conditions,

$$q(0,t) = q(l,t) = 0, \tag{6.62}$$

on its motion. Naturally, a sinusoidal traveling wave, induced in the system, will be reflected from both ends, forming the standing wave patterns of the type (60) near each of them. These two patterns are compatible if l is exactly equal to an integer number (say, n) of $\lambda/2$, where $\lambda \equiv 2\pi/k$ is the wavelength:

$$l = n\frac{\lambda}{2} \equiv n\frac{\pi}{k}. \tag{6.63}$$

This requirement yields the following spectrum of possible wave numbers:

$$k_n = n\frac{\pi}{l}, \tag{6.64}$$

where the list of possible integers n may be limited to non-negative values: n = 1, 2, 3,… (Indeed, negative values give absolutely similar waves (60), while $n = 0$ yields $k_n = 0$, and the corresponding wave vanishes at all points: $\sin(0\cdot z) \equiv 0$.) In the acoustic wave limit we are discussing, Eq. (31), $\omega = \pm vk$, may be used to translate this wave-number spectrum into an equally simple spectrum of possible standing-wave frequencies:[19]

$$\omega_n = vk_n = n\frac{\pi v}{l}, \qquad \text{with } n = 1,2,3,... \tag{6.65}$$

Now let us notice that this spectrum, and the corresponding standing-wave patterns,[20]

$$q^{(n)}(z,t) = 2\,\mathrm{Re}\left[a_n \exp\{-i\omega_n t\}\right] \sin k_n z, \qquad \text{for } 0 \leq z \leq l, \tag{6.66}$$

may be calculated in a different way, by a direct solution of the wave equation (41) with the boundary conditions (62). Indeed, let us look for the general solution of this partial differential equation in the so-called *variable-separated* form[21]

$$q(z,t) = \sum_n Z_n(z) T_n(t), \tag{6.67}$$

where each partial product $Z_n(z)T_n(t)$ is supposed to satisfy the equation on its own. Plugging this partial solution into Eq. (40), and then dividing all its terms by the same product, Z_nT_n, we may rewrite the result as

$$\frac{1}{v^2}\frac{1}{T_n}\frac{d^2T_n}{dt^2} = \frac{1}{Z_n}\frac{d^2Z_n}{dz^2}. \tag{6.68}$$

Here comes the punch line of the variable separation method: since the left-hand side of the equation may depend only on t, while its right-hand side, only on z, Eq. (68) may be valid only if both sides are constant. Denoting this constant as $-k_n^2$, we get two similar ordinary differential equations,[22]

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[19] Again, negative values of ω may be dropped, because they give similar real functions $q(z, t)$.

[20] They describe, in particular, the well-known transverse standing waves on a guitar string.

[21] This *variable separation method* is very general and is discussed in all parts of this series, especially in EM Chapter 2.

[22] The first of them is the 1D form of what is frequently called the *Helmholtz equation*.

$$\frac{d^2 Z_n}{dz^2} + k_n^2 Z_n = 0, \qquad \frac{d^2 T_n}{dt^2} + \omega_n^2 T_n = 0, \qquad \text{where } \omega_n^2 \equiv v^2 k_n^2, \tag{6.69}$$

with well-known (and similar) sinusoidal solutions

$$Z_n = c_n \cos k_n z + s_n \sin k_n z, \qquad T_n = u_n \cos \omega_n t + v_n \sin \omega_n z \equiv \text{Re}\left[a_n \exp\{-i\omega_n t\}\right], \tag{6.70}$$

where c_n, v_n, u_n, and v_n (or, alternatively, $a_n \equiv u_n + iv_n$) are constants. The first of these relations, with all k_n different, may satisfy the boundary conditions only if for all n, $c_n = 0$, and $\sin k_n l = 0$, giving the same wave number spectrum (64) and hence the own frequency spectrum (65), so the general solution (67) of the so-called *boundary problem*, given by Eqs. (40) and (62), takes the form

$$q(z,t) = \text{Re}\sum_n a_n \exp\{-i\omega_n t\} \sin k_n z, \tag{6.71}$$

where the complex amplitudes a_n are determined by the initial conditions.

Hence such sinusoidal standing waves (Fig. 10a) are not just an assumption, but a natural property of the 1D wave equation. It is also easy to verify that the result (71) is valid for the same system with different boundary conditions, though with a modified wave number spectrum. For example, if the rigid boundary condition ($q = 0$) is implemented at $z = 0$, and the so-called *open* boundary condition ($F = 0$, i.e. $\partial q/\partial z = 0$) is imposed at $z = l$, the spectrum becomes

$$k_n = \left(n - \frac{1}{2}\right)\frac{\pi}{l}, \qquad \text{with } n = 1, 2, 3, ..., \tag{6.72}$$

so the lowest standing waves look like Fig. 10b shows.[23]

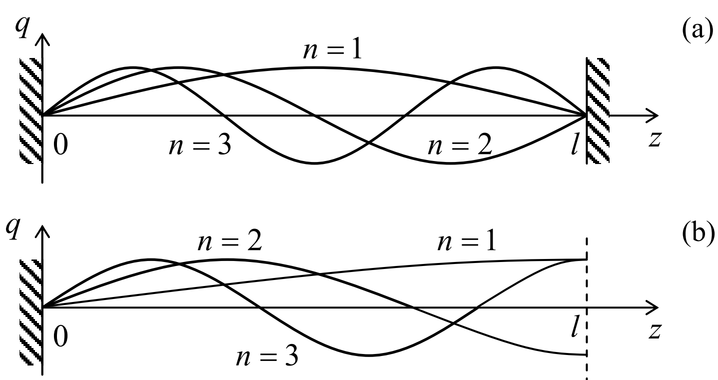

Fig. 6.10. The lowest standing wave modes for the 1D systems with (a) two rigid boundaries, and (b) one rigid and one open boundary.

Note that the difference between the sequential values of k_n is still a constant:

$$k_{n+1} - k_n = \frac{\pi}{l}, \tag{6.73}$$

the same one as for the spectrum (64). This is natural because in both cases the transfer from the n^{th} mode to the $(n + 1)^{\text{th}}$ mode corresponds just to an addition of one more half-wave – see Fig. 10. (This conclusion is valid for any combination of rigid and free boundary conditions.) As was discussed above, for the discrete-particle chain we have started with (Fig. 4), the wave equation (40), and hence the above

[23] The lowest standing wave of the system, with the smallest k_n and ω_n, is usually called its *fundamental mode*.

derivation of Eq. (71), are only valid in the acoustic wave limit, i.e. when the distance d between the particles is much less than the wavelengths $\lambda_n \equiv 2\pi/k_n$ of the mode under analysis. For a chain of length l, this means that the number of particles, $N \sim l/d$, has to be much larger than 1. However, a remarkable property of Eq. (71) is that it remains valid, with the same wave number spectrum (64), not only in the acoustic limit but also for arbitrary $N > 0$. Indeed, since $\sin k_n z \equiv (\exp\{+ik_n z\} - \exp\{-ik_n z\})/2$, each n^{th} term of Eq. (71) may be represented as a sum of two traveling waves with equal but opposite wave vectors. As was discussed in Sec. 3, such a wave is a solution of equation (24) describing the discrete-particle system for *any* k_n, with the only condition that its frequency obeys the general dispersion relation (30), rather than its acoustic limit (65).

Moreover, the expressions for k_n (with appropriate boundary conditions), such as Eq. (64) or Eq. (72), also survive the transition to arbitrary N, because their derivation above was based only on the sinusoidal form of the standing wave. The only new factor arising in the case of arbitrary N is that due to the equidistant property (73) of the wave number spectrum, as soon as n exceeds N, the waveforms (71), at particle locations $z_j = jd$, start to repeat. For example,

$$\sin k_{n+N} z_j = \sin(k_n + N\Delta k) jd = \sin\left(k_n + N\frac{\pi}{d}\right) jd = \sin(k_n z_j + \pi j N) \equiv \pm \sin k_n z_j \, . \tag{6.74}$$

Hence the system has only N different (linearly-independent) modes. But this result is in full compliance with the general conclusion made in Sec. 2, that *any* system of N coupled oscillators has exactly N own frequencies and corresponding oscillation modes. So, our analysis of a particular system shown in Fig. 4, just exemplifies this general conclusion. Fig. 11 below illustrates this result for a particular finite value of N; the curve connecting the points shows exactly the same dispersion relation as was shown in Fig. 5, but now it is just a guide for the eye, because for a system with a finite length l, the wave number spectrum is discrete, and the intermediate values of k and ω do not have an immediate physical sense.[24] Note that the own frequencies of the system are generally not equidistant, while the wave numbers are.

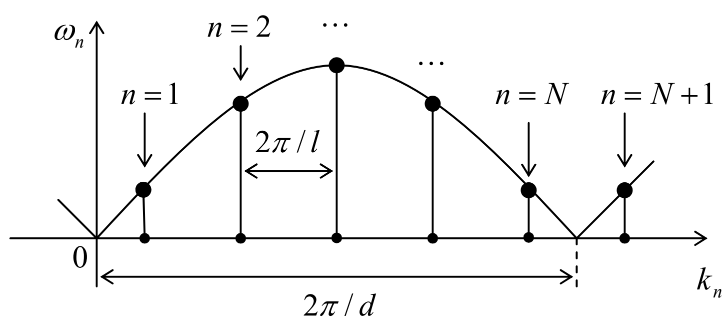

Fig. 6.11. The wave numbers and own frequencies of a chain of a finite number N of particles in a chain with one rigid and one open boundary – schematically.

This insensitivity of the spacing (73) between the adjacent wave numbers to the particular physics of a macroscopically uniform system is a very general fact, common for waves of any nature, and is broadly used for analyses of systems with a very large number of particles (such as human-size crystals, with $N \sim 10^{23}$). For N so large, the effect of the boundary conditions, e.g., the difference

[24] Note that Fig. 11 shows the case of one rigid and one open boundary (see Fig. 10b), where $l = Nd$; for a conceptually simpler system with two rigid boundaries (Fig. 10a) we would need to take $l = (N + 1)d$ because neither of the end points can oscillate.

between the spectra (64) and (72) is negligible, and they may be summarized as the following rule for the number of different standing waves within some interval $\Delta k >> \pi/l$:

$$\Delta N \equiv \frac{\Delta k\big|_{\text{standing}}}{k_{n+1}-k_n} = \frac{l}{\pi}\Delta k\big|_{\text{standing}} . \tag{6.75a}$$

For such analyses, it is frequently more convenient to work with traveling waves rather than standing ones. In this case, we have to take into account that (as was just discussed above) each standing wave (66) may be decomposed into two traveling waves with wave numbers $\pm k_n$, so the interval Δk doubles, and Eq. (75a) becomes[25]

$$\boxed{\Delta N = \frac{l}{2\pi}\Delta k\big|_{\text{traveling}} .} \tag{6.75b}$$

Traveling wave number

Note that this counting rule is valid for waves of just one type. As was discussed above, for the model system we have studied (Fig. 4), there are 3 types of such waves – one longitudinal and two transverse, so if we need to count them all, ΔN should be multiplied by 3.

6.6. Wave decay and attenuation

Now let us discuss the effects of energy dissipation on the 1D waves, on the example of the same uniform system shown in Fig. 4. The simplest description of this effect is the linear drag that may be described, as it was done for a single oscillator in Sec. 5.1, by adding the term $\eta dq_j/dt$, to Eq. (24) for each particle:

$$m\ddot{q}_j + \eta\dot{q}_j - \kappa_{\text{ef}}(q_{j+1}-q_j) + \kappa_{\text{ef}}(q_j - q_{j-1}) = 0 . \tag{6.76}$$

(In a uniform system, the drag coefficient η should be similar for all particles, though it may be different for the longitudinal and transverse oscillations.)

To analyze the dissipation effect on the *standing* waves, we may again use the variable separation method, i.e. look for the solution of Eq. (76) in the form similar to Eq. (67), naturally re-adjusting it for our current discrete case:

$$q(z_j,t) = \sum_n Z_n(z_j)T_n(t) . \tag{6.77}$$

After dividing all terms by $mZ_n(z_j)T_n(t)$ and separating the time-dependent and space-dependent terms, we get

$$\frac{\ddot{T}_n}{T_n} + \frac{\eta}{m}\frac{\dot{T}_n}{T_n} = \frac{\kappa_{\text{ef}}}{m}\left[\frac{Z_n(z_{j+1})}{Z_n(z_j)} + \frac{Z_n(z_{j+1})}{Z_n(z_j)} - 2\right] = \text{const} . \tag{6.78}$$

As we know from the previous section, the resulting equation for the function $Z_n(z_j)$ is satisfied if the variable separation constant is equal to $-\omega_n^2$, where ω_n obeys the dispersion relation (30) for the wave number k_n, properly calculated for the *dissipation-free* system, with the account of the given boundary

[25] Note that this simple, but very important relation is frequently derived using the so-called *Born-Carman boundary condition* $q_0(t) \equiv q_N(t)$, which implies bending the system of interest into a closed loop. For a 1D system with $N >> 1$, such mental exercise may be somehow justified, but for systems of higher dimension, it is hardly physically plausible – and is unnecessary.

conditions – see, e.g. Eqs. (62) and (72). Hence for the function $T_n(t)$, we are getting the following ordinary differential equation:

$$\ddot{T}_n + 2\delta\dot{T}_n + \omega_n^2 T_n = 0, \qquad \text{with } \delta \equiv \frac{\eta}{2m}, \tag{6.79}$$

which is absolutely similar to Eq. (5.6b) for a single linear oscillator, which was studied in Sec. 5.1. As we already know, it has the solution (5.9) describing the free oscillation decay with the relaxation time given by (5.10), $\tau = 1/\delta$, and hence similar for all modes.[26]

Hence, the above analysis of the dissipation effect on *free* standing waves has not brought any surprises, but it gives us a hint of how their *forced* oscillations, induced by some external forces $F_j(t)$ exerted on the particles, may be analyzed. Indeed, representing each of the forces as a sum over the system's modes (spatial harmonics),

$$f(z_j, t) \equiv \frac{F_j(t)}{m} = \sum_n f_n(t) Z_n(z_j), \tag{6.80}$$

and using the variable separation (77), we arrive at the natural generalization of Eq. (79):

$$\ddot{T}_n + 2\delta\dot{T}_n + \omega_n^2 T_n = f_n(t), \tag{6.81}$$

which is identical to Eq. (5.13b) for a single oscillator. This fact enables us to use Eq. (5.27), with $G(\tau) \to G_n(\tau)$, for the calculation of each $T_n(t)$. Now finding the functions $f_n(t)$ from Eq. (80) by the usual reciprocal Fourier transform, and plugging these results into Eq. (77), we get the following generalization of Eq. (5.27):

$$q(z_j, t) = \sum_{j'=1}^{N} \int_0^\infty f(z_{j'}, t-\tau)\, \mathcal{G}(z_j, z_{j'}, \tau)\, d\tau, \quad \text{where } \mathcal{G}(z_j, z_{j'}, \tau) \equiv \sum_n G_n(\tau) Z_n(z_j) Z_n(z_{j'}). \tag{6.82}$$

(Here the mutually orthogonal functions $Z_n(z_j)$ are assumed to be normalized, i.e. the sums of their squares over $j = 1, 2,\ldots, N$ to equal 1.) Such function $\mathcal{G}(z_j, z_{j'}, \tau)$ is called the *spatial-temporal Green's function* of the system – in our current case, of a discrete 1D set of N particles located at points $z_j = jd$. The reader is challenged to spell out this function for at least one of the particular cases discussed above and use it to solve at least one forced-oscillation problem.

Now let us discuss the dissipation effects on the *traveling* waves, where they may take a completely different form of *attenuation*. Let us discuss it on a simple example when one end (located at $z = 0$) of a very long chain ($l \to \infty$) is externally forced to perform sinusoidal oscillations of a certain frequency ω and a fixed amplitude A_0. In this case, it is natural to look for a particular solution to Eq. (76) in a form very different from Eq. (77):

$$q(z_j, t) = \text{Re}\left[c_j e^{-i\omega t}\right], \tag{6.83}$$

---

[26] Even an elementary experience with acoustic guitars shows that for their strings, this conclusion of our theory is not valid: higher modes ("overtones") decay substantially faster, leaving the fundamental mode oscillations for a slower decay. This is a result of another important energy dissipation (i.e. the wave decay) mechanism, not taken into account in Eq. (76) – the radiation of the sound into the guitar's body through the string supports, mostly through the bridge. Such radiation may be described by a proper modification of the boundary conditions (62), in terms of the ratio of the wave impedance (47) of the string and those of the supports.

with time-independent but generally complex amplitudes c_j. As our discussion of a single oscillator in Sec. 5.1 implies, this is not the general, but rather a partial solution, which describes the forced oscillations in the system, to that it settles after some initial transient process. (At non-zero damping, we may be sure that free oscillations fade after a finite time, and thus may be ignored for most purposes.)

Plugging Eq. (83) into Eq. (76), we reduce it to an equation for the amplitudes c_j,

$$\left(-m\omega^2 - i\omega\eta + 2\kappa_{\text{ef}}\right)c_j - \kappa_{\text{ef}}c_{j+1} - \kappa_{\text{ef}}c_{j-1} = 0\,, \qquad (6.84)$$

which is a natural generalization of Eq. (25). As a result, partial solutions of the set of these equations (for $j = 0, 1, 2,\ldots$) may be looked for in the form (26) again, but now, because of the new, imaginary term in Eq. (84), we should be ready to get a complex phase shift α, and hence a complex wave number $k \equiv \alpha/d$.[27] Indeed, the resulting characteristic equation for k,

$$\sin^2\frac{kd}{2} = \frac{\omega^2}{\omega_{\max}^2} + i\frac{2\omega\delta}{\omega_{\max}^2} \qquad (6.85)$$

(where $\omega_{\max}$ is defined by Eq. (30), and the damping coefficient is defined just as in a single oscillator, $\delta \equiv \eta/2m$), does not have a real solution even at $\omega < \omega_{\max}$. Using the well-known expressions for the sine function of a complex argument,[28] Eq. (85) may be readily solved in the most important low-damping limit $\delta << \omega$. In the linear approximation in δ, it does not affect the real part of k, but makes its imaginary part different from zero:

$$k = \pm\frac{2}{d}\left(\sin^{-1}\frac{\omega}{\omega_{\max}} + i\frac{\delta}{\omega_{\max}}\right) \equiv \pm\left(\frac{2}{d}\sin^{-1}\frac{\omega}{\omega_{\max}} + i\frac{\delta}{v}\right), \qquad \text{for } -\pi \le \operatorname{Re}k \le \pi\,, \qquad (6.86)$$

with a periodic extension to other periods – see Fig. 5. Just as was done in Eq. (28), due to two values of the wave number, generally we have to take c_j in the form of not a single wave (26), but of a linear superposition of two partial solutions:

$$c_j = \sum_{\pm} c_{\pm}\exp\left\{\pm i\operatorname{Re}kz_j \mp \frac{\delta}{v}z_j\right\}, \qquad (6.87)$$

where the constants $c_{\pm}$ should be found from the boundary conditions. In our particular case, when $|c_0| = A_0$ and $c_\infty = 0$, only one of these two waves, namely the wave exponentially decaying at its penetration into the system, is different from zero: $|c_+| = A_0$, $c_- = 0$. Hence our solution describes a single wave, with the real amplitude and the oscillation energy decreasing as

$$\boxed{A_j \equiv \left|c_j\right| = A_0\exp\left\{-\frac{\delta}{v}z_j\right\}, \qquad E_j \propto A_j^2 \propto \exp\left\{-\alpha z_j\right\}, \qquad \text{with } \alpha = \frac{2\delta}{v},} \qquad (6.88)$$

Wave attenuation

i.e. with a frequency-independent *attenuation constant* $\alpha = 2\delta/v$,[29] so the spatial scale of wave penetration into a dissipative system is given by $l_d \equiv 1/\alpha$. Certainly, our simple solution (88) is only valid for a system of length $l >> l_d$; otherwise, we would need the second term in the sum (87) to describe the wave reflected from its opposite end.

[27] As a reminder, we have already met such a situation in the absence of damping, but at $\omega > \omega_{\max}$ – see Eq. (38).

[28] See, e.g., MA Eq. (3.5).

[29] I am sorry to use for the attenuation the same letter α as for the phase shift in Eq. (26) and a few of its corollaries, but both notations are traditional.

6.7. Nonlinear and parametric effects

Now let me discuss (because of the lack of time, very briefly, and on a semi-quantitative level), the new nonlinear and parametric phenomena that appear in oscillatory systems with more than one degree of freedom – cf. Secs. 5.4-5.8. One important new effect here is the *mutual phase locking* of (two or more) weakly coupled self-excited oscillators with close frequencies: if the own frequencies of the oscillators are sufficiently close, their oscillation frequencies "stick together" to become exactly equal. Though the dynamics of this process is very close to that of the phase locking of a single oscillator by an external signal, which was discussed in Sec. 5.4, it is rather counter-intuitive in view of the results discussed in Sec. 1, and in particular, the anticrossing diagram shown in Fig. 2. The analysis of the effect using the van der Pol method (which is left for the reader's exercise) shows that the origin of the difference is the oscillators' nonlinearity, which makes the oscillation amplitudes virtually independent of the phase evolution – see Eq. (5.68) and its discussion.

One more new effect is the so-called *non-degenerate parametric excitation*. It may be illustrated on the example of just two coupled oscillators – see Sec. 1 above. Let us assume that the coupling constant κ participating in Eqs. (5) is not constant, but oscillates in time – say with some frequency ω_p. In this case, the forces acting on each oscillator from its counterpart, described by the right-hand sides of Eqs. (5), will be proportional to $\kappa q_{2,1}(1 + \mu \cos\omega_p t)$. Assuming that the oscillations of q_1 and q_2 are close to sinusoidal ones, with certain frequencies $\omega_{1,2}$, we see that the force exerted on each oscillator contains the so-called *combinational frequencies*

$$\omega_p \pm \omega_{2,1}. \tag{6.89}$$

If one of these frequencies is close to the own oscillation frequency of the oscillator, we can expect a substantial parametric interaction between the oscillators (on top of the constant coupling effects discussed in Sec. 1). According to Eq. (89), this may happen in two cases:

Parametric interaction conditions

$$\boxed{\begin{aligned} \omega_p &= \omega_1 + \omega_2, \quad &(6.90a)\\ \omega_p &= \omega_1 - \omega_2. \quad &(6.90b)\end{aligned}}$$

The quantitative analysis (also highly recommended to the reader) shows that in the case (90a), the parameter modulation indeed leads to energy "pumping" into the oscillations.[30] As a result, a sufficiently large μ, at sufficiently small damping coefficients $\delta_{1,2}$ and the effective detuning

$$\xi \equiv \omega_p - (\Omega_1 + \Omega_2), \tag{6.91}$$

may lead to a simultaneous self-excitation of two frequency components $\omega_{1,2}$. These frequencies, while being approximately equal to the corresponding own frequencies $\Omega_{1,2}$ of the system, are related to the *pumping frequency* ω_p by the exact relation (90a), but otherwise are arbitrary, e.g., may be incommensurate (Fig. 12a), thus justifying the term *non-degenerate* parametric excitation.[31] (The parametric excitation of a single oscillator, which was analyzed in Sec. 5.5, is a particular, *degenerate* case of such excitation, with $\omega_1 = \omega_2 = \omega_p/2$.) On the other hand, for the case described by Eq. (90b), the parameter modulation always extracts energy *from* the oscillations, effectively increasing the system's damping.

[30] Hence the common name of ω_p – the pumping frequency.

[31] Note that in some publications, the term *parametric down-conversion* (PDC) is used instead.

Somewhat counterintuitively, this difference between the two cases (90) may be interpreted more simply by using the basic notions of quantum mechanics. Namely, the equality $\omega_p = \omega_1 + \omega_2$ enables a decay of an external photon of energy $\hbar\omega_p$ into two photons of energies $\hbar\omega_1$ and $\hbar\omega_2$ of the oscillators. On the contrary, the complementary relation (90b), meaning that $\omega_1 = \omega_p + \omega_2$, results in a pumping-induced decay of photons of frequency ω_1.

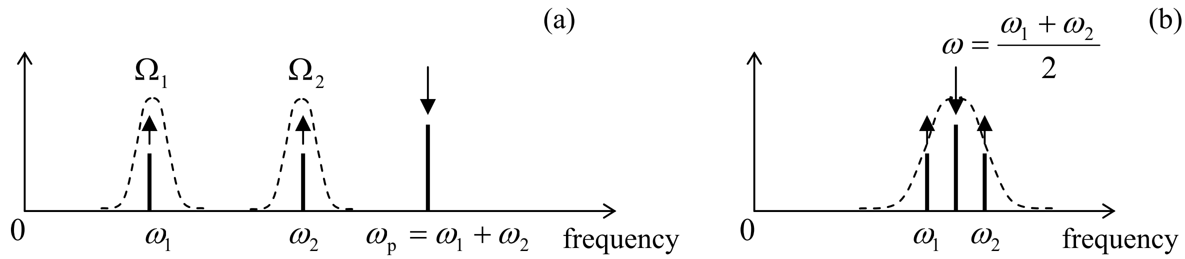

Fig. 6.12. Spectra of oscillations at (a) the non-degenerate parametric excitation, and (b) the four-wave mixing. The arrow directions symbolize the energy flows into and out of the system.

Note that even if the frequencies ω_1 and ω_2 of the parametrically excited oscillations are incommensurate, the oscillations are highly correlated. Indeed, the quantum-mechanical theory of this effect[32] shows that the generated photons are *entangled*. This fact makes the parametric excitation very popular for a broad class of experiments in several currently active fields including quantum computation and encryption, and the Bell inequality/local reality studies.[33]

Proceeding to nonlinear phenomena, let us note, first of all, that the simple reasoning that accompanied Eq. (5.108) in Sec. 5.8, is also valid in the case when oscillations consist of two (or more) sinusoidal components with incommensurate frequencies. Replacing the notation 2ω with ω_p, we see that the non-degenerate parametric excitation of the type (90a) is possible in a system of two coupled oscillators with a quadratic nonlinearity (of the type γq^2), "pumped" by an intensive external signal at frequency $\omega_p \approx \Omega_1 + \Omega_2$. In optics, it is often more convenient to have all three of these frequencies within the same, relatively narrow range. A simple calculation, similar to the one made in Eqs. (5.107)-(5.108), shows that this may be done using the cubic nonlinearity[34] of the type αq^3, which allows a similar parametric energy exchange at the frequency relation shown in Fig. 12b:

Four-wave mixing

$$\boxed{2\omega = \omega_1 + \omega_2, \qquad \text{with } \omega \approx \omega_1 \approx \omega_2 .} \tag{6.92a}$$

This process is often called the *four-wave mixing*, because it may be interpreted quantum-mechanically as the transformation of *two* externally-delivered photons, each with energy $\hbar\omega$, into two other photons of energies $\hbar\omega_1$ and $\hbar\omega_2$. The word "wave" in this term stems from the fact that at optical frequencies, it is hard to couple a sufficient volume of a nonlinear medium with lumped-type resonators. It is much easier to implement the parametric excitation (as well as other nonlinear phenomena such as the higher harmonic generation) of light in *distributed systems* of a linear size much larger than the involved wavelengths. In such systems, the energy transfer from the incoming wave of frequency ω to

[32] Which is, surprisingly, not much more complex than the classical theory – see, e.g., QM Sec.5.5.

[33] See, e.g., QM Secs. 8.5 and 10.3, respectively.

[34] In optics, such nonlinearity is implemented using transparent crystals such as lithium niobate ($LiNbO_3$), with the cubic-nonlinear dependence of the electric polarization on the applied electric field: $\mathscr{P} \propto \mathscr{E} + \alpha\mathscr{E}^3$.

generated waves of frequencies ω_1 and ω_2 is gradually accumulated at their joint propagation along the system. From the analogy between Eq. (85) (describing the evolution of the wave's amplitude *in space*), and the usual equation of the linear oscillator (describing its evolution *in time*), it is clear that this energy transfer accumulation requires not only the frequencies ω but also the wave numbers k be in similar relations. For example, the four-wave mixing requires not only the frequency balance (92a) but also a similar relation

$$2k = k_1 + k_2, \tag{6.92b}$$

to be fulfilled. Since all three frequencies are close, this relation is easy to arrange. Unfortunately, due to the lack of time/space, for more discussion of this very interesting subject, called *nonlinear optics*, I have to refer the reader to special literature.[35]

It may look like a dispersion-free media, with $\omega/k = v =$ const, is the perfect solution for arranging the nonlinear/parametric interaction of waves, because in such media, for example, Eq. (92b) automatically follows from Eq. (92a). However, in such a medium, not only the desirable three parametrically interacting waves but also all their harmonics, have the same velocity. At these conditions, the energy transfer rates between all harmonics are of the same order. Perhaps the most important result of such a multi-harmonic interaction is that intensive incident traveling waves, interacting with a nonlinear medium, may develop sharply non-sinusoidal waveforms, in particular those with an almost instant change of the field at a certain moment. Such *shock waves*, especially the mechanical ones, are of large interest for certain applications – some of them not quite innocent, e.g., the dynamics of explosion in the usual (chemical) and nuclear bombs.[36]

To conclude this chapter, let me note that the above discussion of 1D acoustic waves will be extended, in Sec. 7.7, to elastic 3D media. There we will see that generally, the waves obey a more complex equation than the apparently natural generalization of Eq. (40):

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) q(\mathbf{r}, t) = 0, \tag{6.93}$$

where ∇^2 is the 3D Laplace operator. This fact adds to the complexity of traveling-wave and standing-wave phenomena in higher dimensions. Moreover, in multi-dimensional systems, including such pseudo-1D systems as thin rods and pseudo-2D systems such as thin membranes, even static elastic deformations may be very nontrivial. An introduction to the general theory of small deformations, with a focus on elastic continua, will be the subject of the next chapter.

6.8. Exercise problems

For each of the systems specified in Problems 1-6:

(i) introduce convenient generalized coordinates q_j of the system,

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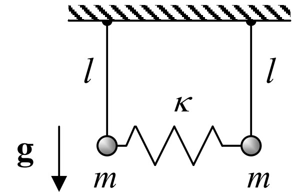
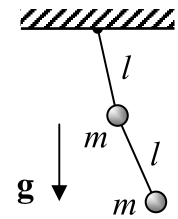
[35] See, e.g., N. Bloembergen, *Nonlinear Optics*, 4^{th} ed., World Scientific, 1996, or a more modern treatment by R. Boyd, *Nonlinear Optics*, 3^{rd} ed., Academic Press, 2008. This field is currently very active. As just a single example, let me mention the recent experiments with parametric amplification of ultrashort (~20-fs) optical pulses to peak power as high as $\sim 5\times10^{12}$ W – see X. Zeng *et al*., *Optics Lett*. **42**, 2014 (2017).

[36] The classical (and perhaps still the best) monograph on the subject is Ya. Zeldovich, *Physics of Shock Waves and High-Temperature Phenomena,* Dover, 2002.

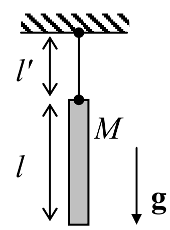
(ii) calculate the frequencies of its small harmonic oscillations near the equilibrium,
(iii) calculate the corresponding distribution coefficients, and
(iv) sketch the oscillation modes.



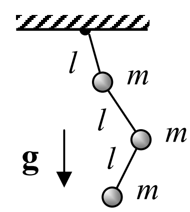
6.1. Two elastically coupled pendula confined to the vertical plane that contains both suspension points, with the parameters shown in the figure on the right (see also Problems 1.8 and 2.9).



6.2. The double pendulum confined to the vertical plane containing the support point (which was the subject of Problem 2.1), with $m' = m$ and $l = l'$ – see the figure on the right.

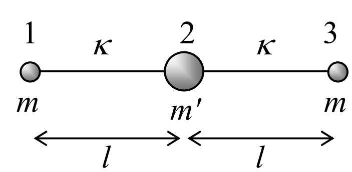


6.3 The chime bell considered in Problem 4.12 (see the figure on the right), for the particular case $l = l'$.

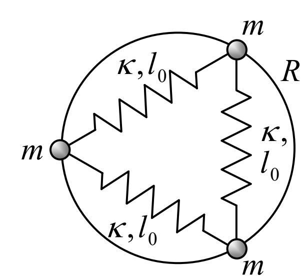


6.4. The triple pendulum shown in the figure on the right, with the motion confined to a vertical plane containing the support point.

Hint: You may use any (e.g., numerical) method to calculate the characteristic equation's roots.



6.5. The symmetric three-particle system shown in the figure on the right, where the connections between the particles not only act as usual elastic springs (giving potential energies $U = \kappa(\Delta l)^2/2$) but also resist bending, giving additional potential energy $U' = \kappa'(l\theta)^2/2$, where θ is the (small) bending angle.[37]

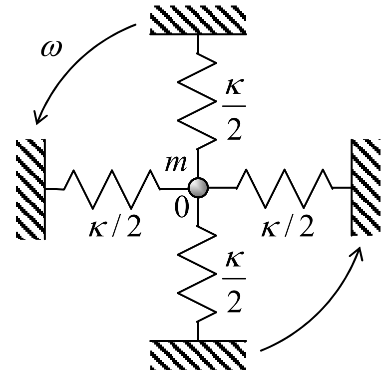


6.6. Three similar beads of mass m, which may slide along a round ring of radius R without friction, connected with similar springs with elastic constants κ and equilibrium lengths l_0 (not necessarily equal to $\sqrt{3}R$) – see the figure on the right.

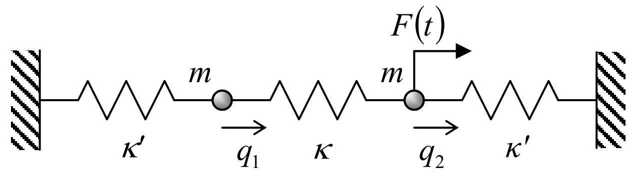
[37] This is a reasonable model for small oscillations of linear molecules such as the now-infamous CO_2.

6.7. On the example of the model considered in Problem 1, explore free oscillations in a system of two similar and weakly coupled linear oscillators.

6.8. A small body is held by four similar elastic springs as shown in the figure on the right. Analyze the effect of rotation of the system as a whole about the axis normal to its plane, on the body's small oscillations within this plane. Assume that the oscillation frequency is much higher than the angular velocity ω of the rotation. Discuss the physical sense of your results and possible ways of using such systems for measurement of the rotation.



6.9. An external longitudinal force $F(t)$ is applied to the right particle of the system shown in Fig. 1, with $\kappa_L = \kappa_R = \kappa'$ and $m_1 = m_2 \equiv m$ (see the figure on the right), and the response $q_1(t)$ of the left particle to this force is being measured.



(i) Calculate the temporal Green's function for this response.

(ii) Use this function to calculate the response to the following force:

$$F(t) = \begin{cases} 0, & \text{for } t < 0, \\ F_0 \sin \omega t, & \text{for } 0 \le t, \end{cases}$$

with constant amplitude F_0 and frequency ω.

6.10. Use the Lagrangian formalism to re-derive Eqs. (24) for both the longitudinal and the transverse oscillations in the system shown in Fig. 4a.

6.11. Calculate the energy (per unit length) of a sinusoidal traveling wave propagating in the 1D system shown in Fig. 4a. Use your result to calculate the average power flow created by the wave, and compare it with Eq. (49) in the acoustic wave limit.

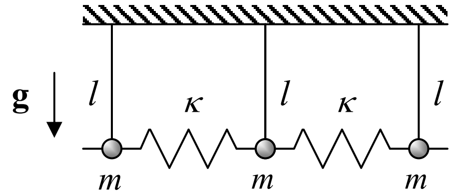
6.12. Calculate spatial distributions of the kinetic and potential energies in a standing sinusoidal 1D acoustic wave and analyze their evolution in time.

6.13. The midpoint of a guitar string of length l has been slowly pulled off sideways by a distance $h \ll l$ from its equilibrium position, and then let go. Neglecting energy dissipation, use two different approaches to calculate the midpoint's displacement as a function of time.

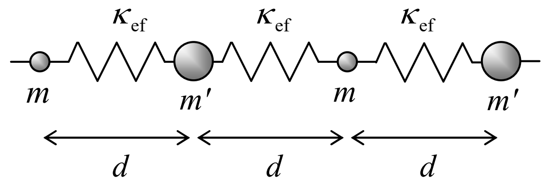
Hint: You may like to use the following series: $\displaystyle\sum_{m=1}^{\infty} \frac{\cos(2m-1)\xi}{(2m-1)^2} = \frac{\pi^2}{8}\left(1 - \frac{\xi}{\pi/2}\right)$, for $0 \le \xi \le \pi$.

6.14. Spell out the spatial-temporal Green's function (82) for waves in a uniform 1D system of N points, with rigid boundary conditions (62). Explore the acoustic limit of your result.

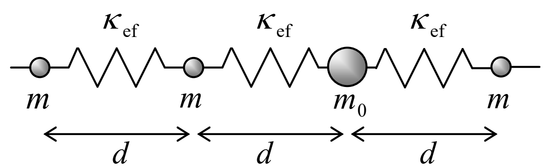
6.15. Calculate the dispersion law $\omega(k)$ and the highest and lowest frequencies of small longitudinal waves in a long chain of similar, spring-coupled pendula – see the figure on the right.



6.16. Calculate and analyze the dispersion relation $\omega(k)$ for small waves in a long chain of elastically coupled particles with alternating masses – see the figure on the right. In particular, discuss the dispersion relation's period Δk, and its evolution at $m' \to m$.



6.17. Analyze the traveling wave's reflection from a "point inhomogeneity": a single particle with a different mass $m_0 \neq m$, in an otherwise uniform 1D chain – see the figure on the right.

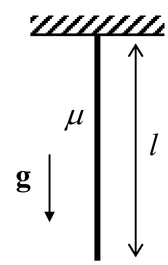


6.18.*

(i) Explore an approximate way to analyze small waves in a continuous 1D system with parameters slowly varying along its length.

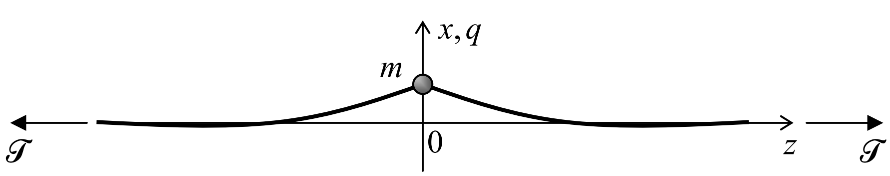
(ii) Apply this method to calculate the frequencies of transverse standing waves on a freely hanging heavy rope of length l, with a constant mass μ per unit length – see the figure on the right.

(iii) For the three lowest standing wave modes, compare the results with those obtained in the solution of Problem 4 for the triple pendulum.



Hint: The reader familiar with the WKB approximation in quantum mechanics (see, e.g., QM Sec. 2.4) is welcome to adapt it for this classical application. Another possible starting point is the van der Pol approximation discussed in Sec. 5.3, which should be translated from the time domain to the space domain.

6.19. A particle of mass m is attached to an infinite string, of mass μ per unit length, stretched with tension $\mathscr{T}$. The particle is confined to move along the x-axis perpendicular to the string (see the figure below), in an additional smooth potential $U(x)$ with a minimum at $x = 0$. Assuming that the waves on the string are excited only by the motion of the particle (rather than any external source), reduce the system of equations describing the system to an ordinary differential equation for small oscillations $x(t)$, and calculate their Q-factor of due to the drag caused by the string.



6.20.* Use the van der Pol method to analyze the mutual phase locking of two weakly coupled self-oscillators with the dissipative nonlinearity, for the cases of:

(i) the direct coordinate coupling described by Eq. (5), and
(ii) a bilinear but otherwise arbitrary coupling of two similar oscillators.

Hint: In Task (ii), describe the coupling by an arbitrary linear operator, and express the result via its Fourier image.

6.21.* Extend Task (ii) of the previous problem to the mutual phase locking of N similar self-oscillators. In particular, explore the in-phase mode's stability for the case of so-called *global coupling* via a single force F contributed equally by all oscillators.

6.22.* Find the condition of non-degenerate parametric excitation in a system of two coupled oscillators described by Eqs. (5), but with time-dependent coupling: $\kappa \to \kappa(1 + \mu \cos\omega_p t)$, with $\omega_p \approx \Omega_1 + \Omega_2$.

Hint: Use the van der Pol method, assuming the modulation depth μ, the static coupling coefficient κ, and the detuning $\xi \equiv \omega_p - (\Omega_1 + \Omega_2)$ are all sufficiently small.

6.23. Show that the cubic nonlinearity of the type αq^3 indeed enables the parametric interaction ("four-wave mixing") of oscillations with incommensurate frequencies related by Eq. (92a).

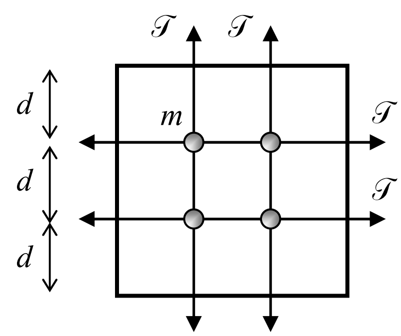
6.24. In the first nonvanishing approximation in small oscillation amplitudes, calculate their effect on the frequencies of the double-pendulum system that was the subject of Problem 1.

6.25. Calculate the velocity of small transverse waves propagating on a thin, planar, elastic membrane, with a constant mass $\mathscr{m}$ per unit area, pre-stretched with force τ per unit width.

6.26. A membrane discussed in the previous problem is stretched on a thin but firm plane frame of area $a \times a$.

(i) Calculate the frequency spectrum of small transverse standing waves in the system; sketch a few lowest wave modes.

(ii) Compare the results with those for a discrete-point analog of this system, with four particles of equal masses m, connected with light flexible strings that are stretched, with equal tensions $\mathscr{T}$, on a similar frame – see the figure on the right.



Hint: The frames do not allow the membrane edges/string ends to deviate from their planes.

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