Chapter 7. Deformations and Elasticity

The objective of this chapter is a discussion of small deformations of 3D continua, with a focus on the elastic properties of solids. The reader will see that such deformations are nontrivial even in the absence of their evolution in time, so several key problems of statics will need to be discussed before proceeding to such dynamic phenomena as elastic waves in infinite media and thin rods.

7.1. Strain

As was already discussed in Chapters 4-6, in a *continuum*, i.e. a system of particles so close to each other that the system discreteness may be neglected, particle displacements **q** may be considered as a continuous function of not only time but also space. In this chapter, we will consider only *small* deviations from the rigid-body approximation discussed in Chapter 4, i.e. small *deformations*. The deformation smallness allows us to consider the displacement vector **q** as a function of the *initial* (pre-deformation) position of the particle, **r**, and time t – just as was done in Chapter 6 for 1D waves.

The first task of the deformation theory is to exclude from consideration the types of motion considered in Chapter 4, namely the body's translation and rotation, unrelated to deformations. This means, first of all, that the variables describing deformations should not depend on the displacement's part that is independent of the position **r** (i.e. is common for the whole media), because that part corresponds to a translational shift rather than to a deformation (Fig. 1a). Moreover, even certain non-uniform displacements do not contribute to deformation. For example, Eq. (4.9) (with $d\mathbf{r}$ replaced with $d\mathbf{q}$ to comply with our current notation) shows that a small displacement of the type

$$d\mathbf{q}|_{\text{rotation}} = d\boldsymbol{\varphi}\times\mathbf{r}, \tag{7.1}$$

where $d\boldsymbol{\varphi} = \boldsymbol{\omega}dt$ is an infinitesimal vector common for the whole continuum, corresponds to its elementary rotation of the body about the direction of that vector, and has nothing to do with its deformation (Fig. 1b).

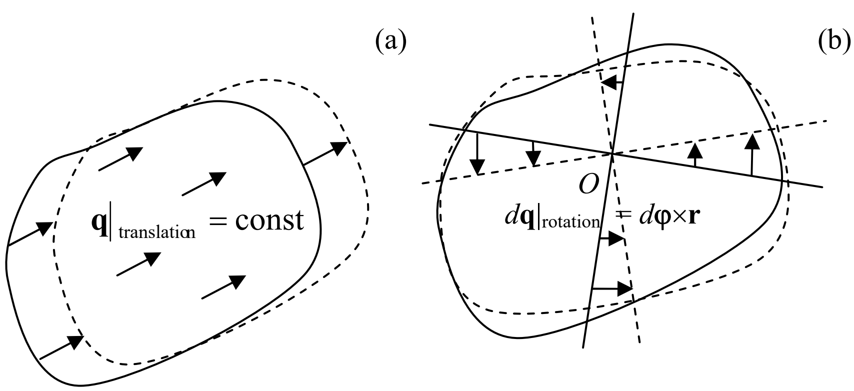

Fig. 7.1. Two types of displacement vector distributions that are unrelated to deformation: (a) translation and (b) rotation.

This is why to develop an adequate quantitative characterization of deformation, so far for fixed t, we should start with finding suitable functions of the spatial distribution of displacements, $\mathbf{q}(\mathbf{r})$, that exist only due to deformations. One such measure is the change of the distance $dl \equiv |d\mathbf{r}|$ between two close points:

$$(dl)^2\Big|_{\text{after deformation}} - (dl)^2\Big|_{\text{before deformation}} = \sum_{j=1}^{3}(dr_j + dq_j)^2 - \sum_{j=1}^{3}(dr_j)^2 \ , \qquad (7.2)$$

where dq_j is the j^{th} Cartesian component of the difference $d\mathbf{q}$ between the displacements $\mathbf{q}$ of these close points. If the deformation is small in the sense $|d\mathbf{q}| << dl$, we may keep, in this expression, only the terms proportional to the first power of the infinitesimal vector $d\mathbf{q}$:

$$(dl)^2\Big|_{\text{after deformation}} - (dl)^2\Big|_{\text{before deformation}} = \sum_{j=1}^{3}\left[2dr_j dq_j + (dq_j)^2\right] \approx 2\sum_{j=1}^{3} dr_j dq_j \ . \qquad (7.3)$$

Since q_j is a function of three independent scalar arguments r_j, its full differential (at fixed time) may be represented as

$$dq_j = \sum_{j'=1}^{3} \frac{\partial q_j}{\partial r_{j'}} dr_{j'} . \qquad (7.4)$$

The coefficients $\partial q_j/\partial r_{j'}$ may be considered as elements of a tensor providing a linear relation between the vectors d**r** and d**q**.[1] Plugging Eq. (4) into Eq. (2), we get

$$(dl)^2\Big|_{\text{after deformation}} - (dl)^2\Big|_{\text{before deformation}} = 2\sum_{j,j'=1}^{3} \frac{\partial q_j}{\partial r_{j'}} dr_j dr_{j'} \ . \qquad (7.5)$$

The convenience of the tensor $\partial q_j/\partial r_{j'}$ for characterizing deformations is that it automatically excludes the translation displacement (Fig. 1a), which is independent of r_j. Its drawback is that its particular elements are still affected by the rotation of the body – even though the sum (5) is not. Indeed, according to the vector product's definition, Eq. (1) may be represented in Cartesian coordinates as

$$dq_j\Big|_{\text{rotation}} = \left(d\varphi_{j'} r_{j''} - d\varphi_{j''} r_{j'}\right)\varepsilon_{jj'j''} , \qquad (7.6)$$

where $\varepsilon_{jj'j''}$ is the Levi-Civita symbol. Differentiating Eq. (6) over a particular Cartesian coordinate of vector **r**, and taking into account that this partial differentiation (∂) is independent of (and hence may be swapped with) the differentiation (d) over the common rotation angle φ, we get the amounts

$$d\left(\frac{\partial q_j}{\partial r_{j'}}\right)_{\text{rotation}} = -\varepsilon_{jj'j''} d\varphi_{j''} \quad \text{and} \quad d\left(\frac{\partial q_{j'}}{\partial r_j}\right)_{\text{rotation}} = -\varepsilon_{j'jj''} d\varphi_{j''} = \varepsilon_{jj'j''} d\varphi_{j''} , \qquad (7.7)$$

which may differ from 0. However, notice that the *sum* of these two differentials equals zero for any $d\boldsymbol{\varphi}$, which is possible only if[2]

$$\left(\frac{\partial q_{j'}}{\partial r_j} + \frac{\partial q_j}{\partial r_{j'}}\right)_{\text{rotation}} = 0, \qquad \text{for } j \neq j' . \qquad (7.8)$$

This is why it is convenient to rewrite Eq. (5) in a mathematically equivalent form,

[1] Since both $d\mathbf{q}$ and $d\mathbf{r}$ are legitimate physical vectors (whose Cartesian components are properly transformed as the transfer between reference frames), the 3×3 matrix with elements $\partial q_j/\partial r_{j'}$ is indeed a legitimate physical tensor – see the discussion in Sec. 4.2.

[2] As a result, the full sum (5), which includes three partial sums (8), is not affected by rotation – as we already know.

$$(dl)^2\Big|_{\text{after deformation}} - (dl)^2\Big|_{\text{before deformation}} = 2\sum_{j,j'=1}^{3} s_{jj'} dr_j dr_{j'}, \tag{7.9a}$$

where $s_{jj'}$ are the elements of the so-called *symmetrized strain tensor*, defined as

Strain tensor

$$s_{jj'} \equiv \frac{1}{2}\left(\frac{\partial q_j}{\partial r_{j'}} + \frac{\partial q_{j'}}{\partial r_j}\right). \tag{7.9b}$$

(Note that this modification does not affect the diagonal elements $s_{jj} = \partial q_j/\partial r_j$). So, the advantage of the symmetrized tensor (9b) over the initial tensor with elements $\partial q_j/\partial r_{j'}$ is that according to Eq. (8), at pure rotation, all elements of the symmetrized strain tensor vanish.

Now let us discuss the physical meaning of this tensor. As was already mentioned in Sec. 4.2, any symmetric tensor may be diagonalized by an appropriate selection of the reference frame axes. In such principal axes, $s_{jj'} = s_{jj}\delta_{jj'}$, so Eq. (4) takes a simple form:

$$dq_j = \frac{\partial q_j}{\partial r_j} dr_j = s_{jj} dr_j. \tag{7.10}$$

We may use this expression to calculate the change of each side of an elementary cuboid (parallelepiped) with its sides dq_j parallel to the principal axes:

$$dr_j\Big|_{\text{after deformation}} - dr_j\Big|_{\text{before deformation}} \equiv dq_j = s_{jj} dr_j, \tag{7.11}$$

and of the cuboid's volume $dV = dr_1 dr_2 dr_3$:

$$dV\Big|_{\text{after deformation}} - dV\Big|_{\text{before deformation}} = \prod_{j=1}^{3}(dr_j + s_{jj}dr_j) - \prod_{j=1}^{3} dr_j = dV\left[\prod_{j=1}^{3}(1+s_{jj}) - 1\right], \tag{7.12}$$

Since all our analysis is only valid in the linear approximation in small $s_{jj'}$, Eq. (12) is reduced to

$$dV\Big|_{\text{after deformation}} - dV\Big|_{\text{before deformation}} \approx dV \sum_{j=1}^{3} s_{jj} \equiv dV\,\mathrm{Tr}(\mathrm{s}), \tag{7.13}$$

where Tr (*trace*)[3] of any matrix (in particular, any tensor) is the sum of its diagonal elements; in our current case

$$\mathrm{Tr}(\mathrm{s}) \equiv \sum_{j=1}^{3} s_{jj}. \tag{7.14}$$

The tensor theory shows that the trace does not depend on the particular choice of the coordinate axes; so, the diagonal elements of the strain tensor characterize the medium's compression/extension.

Next, what is the meaning of its off-diagonal elements? It may be illustrated by the simplest example of a purely *shear deformation* shown in Fig. 2. (The geometry means to be uniform along the z-axis normal to the plane of the drawing.) In this case, all displacements (assumed small) have just one Cartesian component – in Fig. 2, along the x-axis: $\mathbf{q} = \mathbf{n}_x \alpha y$ (with $\alpha << 1$), so the only nonzero element of the initial strain tensor $\partial q_j/\partial r_{j'}$ is $\partial q_x/\partial y = \alpha$, and the symmetrized tensor (9b) is

[3] The traditional European notation for Tr is Sp (from the German *Spur* meaning "trace" or "track").

$$\mathrm{s} = \begin{pmatrix} 0 & \alpha/2 & 0 \\ \alpha/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (7.15)$$

Evidently, the change of volume, given by Eq. (13), vanishes in this case. Thus, off-diagonal elements of the tensor s characterize shear deformations.

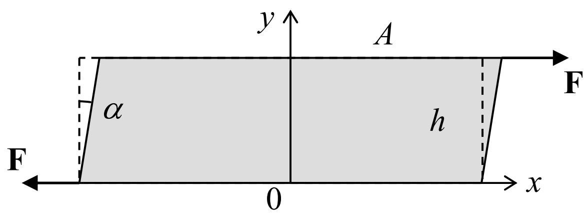

Fig. 7.2. An example of pure shear.

To conclude this section, let me note that Eq. (9) is only valid in Cartesian coordinates. For the solution of some important problems with the axial or spherical symmetry, it is frequently convenient to express six different elements of the symmetric strain tensor in either cylindrical or spherical coordinates via three components of the displacement vector **q** in the same coordinates. A straightforward differentiation of the definitions of these curvilinear coordinates, similar to that used to derive the well-known expressions for spatial derivatives of arbitrary functions,[4] yields, in particular, the following formulas for the diagonal elements of the tensor:

(i) in the cylindrical coordinates:

$$s_{\rho\rho} = \frac{\partial q_\rho}{\partial \rho}, \qquad s_{\varphi\varphi} = \frac{1}{\rho}\left(q_\rho + \frac{\partial q_\varphi}{\partial \varphi} \right), \qquad s_{zz} = \frac{\partial q_z}{\partial z}. \qquad (7.16)$$

(ii) in the spherical coordinates:

$$s_{rr} = \frac{\partial q_r}{\partial r}, \qquad s_{\theta\theta} = \frac{1}{r}\left(q_r + \frac{\partial q_\theta}{\partial \theta} \right), \qquad s_{\varphi\varphi} = \frac{1}{r}\left(q_r + q_\theta \frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta}\frac{\partial q_\varphi}{\partial \varphi} \right). \qquad (7.17)$$

These expressions, which will be used below for the solution of some problems for symmetrical geometries, may be a bit counter-intuitive. Indeed, Eq. (16) shows that even for a purely radial, axially-symmetric deformation, $\mathbf{q} = q(\rho)\mathbf{n}_\rho$, the angular element of the strain tensor does not vanish: $s_{\varphi\varphi} = q/\rho$. (According to Eq. (17), in the spherical coordinates, both angular elements of the tensor exhibit the same property.) Note, however, that this relation describes a simple geometric fact: the change of the lateral distance $\rho d\varphi << \rho$ between two close points at the same distance from the symmetry axis, at a small change of ρ that keeps the angle $d\varphi$ between the directions towards these two points intact.

7.2. Stress

Now let us discuss the forces that cause the strain – or, from a legitimate alternative point of view, are caused by the strain. Internal forces acting inside (i.e. between arbitrarily defined parts of) a

[4] See, e.g., MA Eqs. (10.1)-(10.12).

continuum may be also characterized by a tensor. This *stress tensor,*[5] with elements $\sigma_{jj'}$, relates the Cartesian components of the vector $d\mathbf{F}$ of the force acting on an elementary area dA of an (in most cases, just imagined) interface between two parts of a continuum, to the components of the elementary vector $d\mathbf{A} = \mathbf{n}dA$ normal to the area – see Fig. 3:

Stress tensor

$$dF_j = \sum_{j'=1}^{3} \sigma_{jj'} dA_{j'} . \tag{7.18}$$

The usual sign convention here is to take the *outer* normal $d\mathbf{n}$, i.e. to direct $d\mathbf{A}$ out of "our" part of the continuum, i.e. the part on which the calculated force $d\mathbf{F}$ is exerted – by the complementary part.

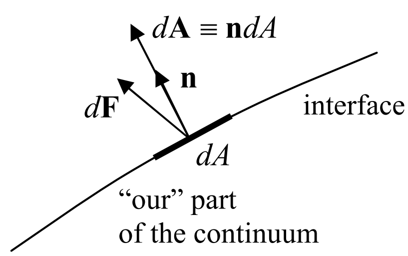

Fig. 7.3. The definition of vectors $d\mathbf{A}$ and $d\mathbf{F}$.

In some cases, the stress tensor's structure is very simple. For example, as will be discussed in detail in the next chapter, static and ideal *fluids* (i.e. liquids and gases) may only provide forces normal to any interface and usually are directed toward "our" part of the body, so

Pressure

$$d\mathbf{F} = -\mathcal{P}d\mathbf{A}, \qquad \text{i.e. } \sigma_{jj'} = -\mathcal{P}\delta_{jj'}, \tag{7.19}$$

where the scalar $\mathcal{P}$ (in most cases positive) is called *pressure*, and generally may depend on both the spatial position and time. This type of stress, with $\mathcal{P} > 0$, is frequently called *hydrostatic compression* – even if it takes place in solids, as it may.

However, in the general case, the stress tensor also has off-diagonal terms, which characterize the shear stress. For example, if the shear strain in Fig. 2 is caused by the shown pair of forces $\pm\mathbf{F}$, they create internal forces $F_x\mathbf{n}_x$, with $F_x > 0$ if we speak about the force acting upon a part of the sample below the imaginary horizontal interface we are discussing. To avoid a horizontal acceleration of each horizontal slice of the sample, the forces should not depend on y, i.e. $F_x = \text{const} = F$. Superficially, it may look that in this case, the only nonzero element of the stress tensor is $dF_x/dA_y = F/A = \text{const}$, so tensor is asymmetric, in contrast to the strain tensor (15) of the same system. Note, however, that the displayed pair of forces $\pm\mathbf{F}$ creates not only the shear stress but also a nonzero rotating torque $\boldsymbol{\tau} = -Fh\mathbf{n}_z = -(dF_x/dA_y)Ah\mathbf{n}_z = -(dF_x/dA_y)V\mathbf{n}_z$, where $V = Ah$ is the sample's volume. So, if we want to perform a static stress experiment, i.e. avoid the sample's rotation, we need to apply some other forces, e.g., a pair of vertical forces creating an equal and opposite torque $\boldsymbol{\tau}' = (dF_y/dA_x)V\mathbf{n}_z$, implying that $dF_y/dA_x = dF_x/dA_y = F/A$. As a result, the stress tensor becomes symmetric, and similar in structure to the symmetrized strain tensor (15):

[5] It is frequently called the *Cauchy stress tensor*, partly to honor Augustin-Louis Cauchy who introduced this notion (and is responsible for the development, mostly in the 1820s, much of the theory described in this chapter), and partly to distinguish it from other possible definitions of the stress tensor, including the *1st* and *2nd Piola-Kirchhoff tensors*. For the small deformations discussed in this course, all these notions coincide.

$$\sigma = \begin{pmatrix} 0 & F/A & 0 \\ F/A & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.20}$$

In many situations, the body may be stressed not only by forces applied to their surfaces but also by some volume-distributed (*bulk*) forces $d\mathbf{F} = \mathbf{f}dV$, whose certain effective *bulk density* $\mathbf{f}$. (The most evident example of such forces is gravity. If its field is uniform as described by Eq. (1.16), then $\mathbf{f} = \rho\mathbf{g}$, where ρ is the mass density.) Let us derive the key formula describing the summation of the interface and bulk forces. For that, consider again an elementary cuboid with sides dr_j parallel to the corresponding coordinate axes $\mathbf{n}_j$ (Fig. 4) – now not necessarily the principal axes of the stress tensor.

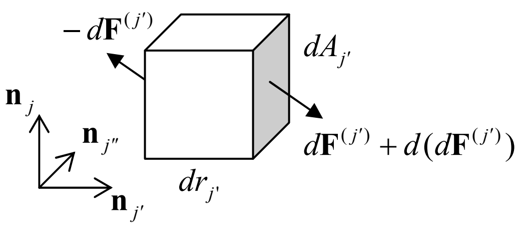

Fig. 7.4. Deriving Eq. (23).

If elements $\sigma_{jj'}$ of the tensor do not depend on position, the force $d\mathbf{F}^{(j')}$ acting on the j'^{th} face of the cuboid is exactly balanced by the equal and opposite force acting on its opposite face, because the vectors $d\mathbf{A}^{(j')}$ at these faces are equal and opposite. However, if $\sigma_{jj'}$ is a function of $\mathbf{r}$, then the net force $d(d\mathbf{F}^{(j')})$ does not vanish. (In this expression, the first differential sign refers to the elementary shift $dr_{j'}$, while the second one, to the elementary area $dA_{j'}$.) Using the expression $\sigma_{jj'}dA_{j'}$ for the j'^{th} contribution to the sum (18), in the first order in $d\mathbf{r}$ the j^{th} components of the vector $d(d\mathbf{F}^{(j')})$ is

$$d(dF_j^{(j')}) = d\left(\sigma_{jj'}dA_{j'}\right) = \frac{\partial\sigma_{jj'}}{\partial r_{j'}}dr_{j'}dA_{j'} \equiv \frac{\partial\sigma_{jj'}}{\partial r_{j'}}dV, \tag{7.21}$$

where the cuboid's volume $dV = dr_{j'}dA_{j'}$ evidently does not depend on the index j'. The addition of these force components for all three pairs of cuboid faces, i.e. the summation of Eqs. (21) over all three values of the upper index j', yields the following relation for the j^{th} Cartesian component of the net force exerted on the cuboid:

$$d(dF_j) = \sum_{j'=1}^{3} d(dF_j^{(j')}) = \sum_{j'=1}^{3} \frac{\partial\sigma_{jj'}}{\partial r_{j'}}dV \ . \tag{7.22}$$

Since any volume may be broken into such infinitesimal cuboids, Eq. (22) shows that the space-varying stress is equivalent to a volume-distributed force $d\mathbf{F}_{\text{ef}} = \mathbf{f}_{\text{ef}}dV$, whose *effective* (not real!) bulk density $\mathbf{f}_{\text{ef}}$ has the following Cartesian components

$$\left(f_{\text{ef}}\right)_j = \sum_{j'=1}^{3} \frac{\partial\sigma_{jj'}}{\partial r_{j'}}, \tag{7.23}$$

Euler-Cauchy principle

so in the presence of genuinely bulk forces $d\mathbf{F} = \mathbf{f}dV$, the densities $\mathbf{f}_{\text{ef}}$ and $\mathbf{f}$ just add up. This is the so-called *Euler-Cauchy stress principle*.

Let us use this addition rule to spell out the 2nd Newton law for a unit volume of a continuum:

$$\rho \frac{\partial^2 \mathbf{q}}{\partial t^2} = \mathbf{f}_{\text{ef}} + \mathbf{f} . \tag{7.24}$$

Using Eq. (23), the j^{th} Cartesian component of Eq. (24) may be represented as

Continuum dynamics: equation

$$\rho \frac{\partial^2 q_j}{\partial t^2} = \sum_{j'=1}^{3} \frac{\partial \sigma_{jj'}}{\partial r_{j'}} + f_j . \tag{7.25}$$

This is the key equation of the continuum's dynamics (and statics), which will be repeatedly used below.

For the solution of some problems, it is also convenient to have a general expression for the work $\delta\mathscr{W}$ of the stress forces at a virtual deformation $\delta\mathbf{q}$ – understood in the same variational sense as the virtual displacements $\delta\mathbf{r}$ in Sec. 2.1. Using the Euler-Cauchy principle (23), for any volume V of a medium not affected by volume-distributed forces, we may write[6]

$$\delta\mathscr{W} = -\int_V \mathbf{f}_{\text{ef}} \cdot \delta\mathbf{q}\, d^3r = -\sum_{j=1}^{3} \int_V (f_{\text{ef}})_j \,\delta q_j d^3r = -\sum_{j,j'=1}^{3} \int_V \frac{\partial \sigma_{jj'}}{\partial r_{j'}} \delta q_j d^3r. \tag{7.26}$$

Let us work out this integral by parts for a volume so large that the deformations δq_j on its surface are negligible. Then, swapping the operations of the variation and the spatial differentiation (just like it was done with the time differentiation in Sec. 2.1), we get

$$\delta\mathscr{W} = \sum_{j,j'=1}^{3} \int_V \sigma_{jj'} \delta \frac{\partial q_j}{\partial r_{j'}} d^3r. \tag{7.27}$$

Assuming that the tensor $\sigma_{jj'}$ is symmetric, we may rewrite this expression as

$$\delta\mathscr{W} = \frac{1}{2} \sum_{j,j'=1}^{3} \int_V \left(\sigma_{jj'} \delta \frac{\partial q_j}{\partial r_{j'}} + \sigma_{j'j} \delta \frac{\partial q_j}{\partial r_{j'}} \right) d^3r. \tag{7.28}$$

Now, swapping indices j and j' in the second expression, we finally get

$$\delta\mathscr{W} = \frac{1}{2} \sum_{j,j'=1}^{3} \int_V \delta \left(\frac{\partial q_j}{\partial r_{j'}} \sigma_{jj'} + \frac{\partial q_{j'}}{\partial r_j} \sigma_{jj'} \right) d^3r = -\sum_{j,j'=1}^{3} \int_V \sigma_{jj'} \delta s_{jj'} d^3r , \tag{7.29}$$

where $s_{jj'}$ are the elements of the strain tensor (9b). It is natural to rewrite this important formula as

Work of stress forces

$$\delta\mathscr{W} = \int_V \delta w(\mathbf{r}) d^3r, \qquad \text{where } \delta w(\mathbf{r}) \equiv \sum_{j,j'=1}^{3} \sigma_{jj'} \delta s_{j'j} , \tag{7.30}$$

and interpret the locally-defined scalar function $\delta w(\mathbf{r})$ as the work of the stress forces per unit volume, at a small variation of the deformation.

As a sanity check, for the pure pressure (19), Eq. (30) is reduced to the obviously correct result $\delta\mathscr{W} = -\mathscr{P}\delta V$, where V is the volume of the "our" part of the continuum.

[6] Here the sign corresponds to the work of the "external" stress force $d\mathbf{F}$ exerted on "our" part of the continuum by its counterpart – see Fig. 3. Note that some texts make the opposite definition of $\delta\mathscr{W}$, leading to its opposite sign.

7.3. Hooke's law

In order to form a complete system of equations describing the continuum's dynamics, one needs to complement Eq. (25) with an appropriate *constitutive equation* describing the relation between the forces described by the stress tensor $\sigma_{jj'}$, and the deformations **q** described (in the small deformation limit) by the strain tensor $s_{jj'}$. This relation depends on the medium, and generally may be rather complicated. Even leaving alone various anisotropic solids (e.g., crystals) and macroscopically-inhomogeneous materials (like ceramics or sand), strain typically depends not only on the current value of stress (possibly in a nonlinear way) but also on the previous history of stress application. Indeed, if strain exceeds a certain *plasticity threshold*, atoms (or nanocrystals) may slip to their new positions and never come back even if the strain is reduced. As a result, deformations become irreversible – see Fig. 5.

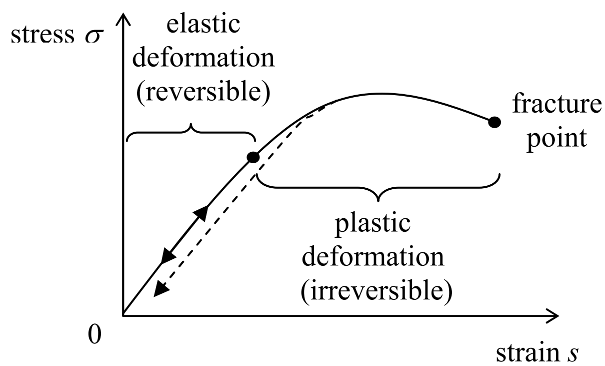

Fig. 7.5. A typical relation between the stress and strain in solids (schematically).

Only below the thresholds of nonlinearity and plasticity (which are typically close to each other), the strain is nearly proportional to stress, i.e. obeys the famous *Hooke's law*.[7] However, even in this *elastic range*, the law is not quite simple, and even for an isotropic medium is described not by one but by two constants, called the *elastic moduli*. The reason for that is that most elastic materials resist the strain accompanied by a volume change (say, the hydrostatic compression) differently from how they resist a shear deformation.

To describe this difference, let us first represent the symmetrized strain tensor (9b) in the following mathematically equivalent form:

$$s_{jj'} = \left(s_{jj'} - \frac{1}{3}\delta_{jj'}\mathrm{Tr}\left(\mathrm{s}\right)\right) + \left(\frac{1}{3}\delta_{jj'}\mathrm{Tr}\left(\mathrm{s}\right)\right). \qquad (7.31)$$

According to Eq. (13), the *traceless tensor* in the first parentheses does not give any contribution to the volume change, e.g., may be used to characterize a purely shear deformation, while the second term describes the hydrostatic compression alone. Hence we may expect that the stress tensor may be represented (again, within the elastic deformation range only!) as

$$\sigma_{jj'} = 2\mu\left(s_{jj'} - \frac{1}{3}\mathrm{Tr}\left(\mathrm{s}\right)\delta_{jj'}\right) + 3K\left(\frac{1}{3}\mathrm{Tr}\left(\mathrm{s}\right)\delta_{jj'}\right), \qquad (7.32)$$

Hooke's law via μ and K

where K and μ are constants. (The inclusion of coefficients 2 and 3 into Eq. (32) is justified by the simplicity of some of its corollaries – see, e.g., Eqs. (36) and (41) below.) Indeed, experiments show that

[7] Named after Robert Hooke (1635-1703), the polymath who was the first to describe the law in its simplest, 1D version.

Hooke's law in this form is followed, at small strain, by all isotropic materials. In accordance with the above discussion, the constant μ (in some texts, denoted as G) is called the *shear modulus*, while the constant K (sometimes denoted B), the *bulk modulus*. The two left columns of Table 1 show the approximate values of these moduli for typical representatives of several major classes of materials.[8]

Table 7.1. Elastic moduli, density, and sound velocities of a few representative materials (approximate values)

Material	K (GPa)	μ (GPa)	E (GPa)	ν	ρ (kg/m^3)	v_l (m/s)	v_t (m/s)
Diamond[(a)]	600	450	1,100	0.20	3,500	1,830	1,200
Hardened steel	170	75	200	0.30	7,800	5,870	3,180
Water[(b)]	2.1	0	0	0.5	1,000	1,480	0
Air[(b)]	0.00010	0	0	0.5	1.2	332	0

[(a)] Averages over crystallographic directions (~10% anisotropy).
[(b)] At the so-called *ambient conditions* (T = 20°C, $\mathcal{P}$ = 1 bar ≡ 10^5 Pa).

To better appreciate these values, let us first discuss the quantitative meaning of K and μ, using two simple examples of elastic deformation. However, in preparation for that, let us first solve the set of nine (or rather six different) linear equations (32) for $s_{jj'}$. This is easy to do, due to the simple structure of these equations: they relate the elements $\sigma_{jj'}$ and $s_{jj'}$ with the same indices, but the tensor's trace effect. This slight complication may be readily overcome by noticing that according to Eq. (32),

$$\mathrm{Tr}\,(\sigma) \equiv \sum_{j=1}^{3} \sigma_{jj} = 3K\,\mathrm{Tr}\,(\mathrm{s}), \qquad \text{so that } \mathrm{Tr}\,(\mathrm{s}) = \frac{1}{3K}\mathrm{Tr}\,(\sigma). \tag{7.33}$$

Plugging this result into Eq. (32) and solving it for $s_{jj'}$, we readily get the reciprocal relation, which may be represented in a similar form:

$$s_{jj'} = \frac{1}{2\mu}\left(\sigma_{jj'} - \frac{1}{3}\mathrm{Tr}(\sigma)\delta_{jj'}\right) + \frac{1}{3K}\left(\frac{1}{3}\mathrm{Tr}(\sigma)\delta_{jj'}\right). \tag{7.34}$$

Now let us apply Hooke's law, in the form of Eqs. (32) or (34), to two simple situations in which the strain and stress tensors may be found without using the full differential equation of the elasticity theory and boundary conditions for them. (That will be the subject of the next section.) The first situation is the hydrostatic compression when the stress tensor is diagonal, and all its diagonal elements are equal – see Eq. (19).[9] For this case, Eq. (34) yields

$$s_{jj'} = -\frac{\mathcal{P}}{3K}\delta_{jj'}, \tag{7.35}$$

[8] Since the strain tensor elements, defined by Eq. (9), are dimensionless, while the strain, defined by Eq. (18), has a dimensionality similar to pressure (of force per unit area), so do the elastic moduli K and μ.
[9] It may be proved that such a situation may be implemented not only in a fluid with pressure $\mathcal{P}$ but also in a solid sample of an *arbitrary* shape, for example by placing it into a compressed fluid.

i.e. regardless of the shear modulus, the strain tensor is also diagonal, with all diagonal elements equal. According to Eqs. (11) and (13), this means that all linear dimensions of the body are reduced by a similar factor, so its shape is preserved, while the volume is reduced by

$$\frac{\Delta V}{V} = \sum_{j=1}^{3} s_{jj} = -\frac{\mathcal{P}}{K}. \tag{7.36}$$

This formula clearly shows the physical sense of the bulk modulus K as the *reciprocal compressibility*. As Table 1 shows, the values of K may be dramatically different for various materials, and even for such "soft stuff" as water, this modulus is actually rather high. For example, even at the bottom of the deepest, 10-km ocean well ($\mathcal{P} \approx 10^3$ bar ≈ 0.1 GPa), the water's density increases by just about 5%. As a result, in most human-scale experiments, water may be treated as an *incompressible fluid* – the approximation that will be widely used in the next chapter. Many solids are even much less compressible – see, for example, the first two rows of Table 1.

Quite naturally, the most compressible media are gases. For a portion of gas, a certain background pressure $\mathcal{P}$ is necessary just for containing it within its volume V, so Eq. (36) is only valid for small increments of pressure, $\Delta\mathcal{P}$:

$$\frac{\Delta V}{V} = -\frac{\Delta\mathcal{P}}{K}. \tag{7.37}$$

Moreover, the compression of gases also depends on thermodynamic conditions. (In contrast, for most condensed media, the temperature effects are very small.) For example, at ambient conditions, most gases are reasonably well described by the equation of state called the *ideal classical gas*:

$$\mathcal{P}V = Nk_B T, \qquad \text{i.e. } \mathcal{P} = \frac{Nk_B T}{V}. \tag{7.38}$$

where N is the number of molecules in volume V, and $k_B \approx 1.38\times10^{-23}$ J/K is the Boltzmann constant.[10] For a small volume change ΔV at a constant temperature T, this equation gives

$$\Delta\mathcal{P}\big|_{T=\text{const}} = -\frac{Nk_B T}{V^2}\Delta V = -\frac{\mathcal{P}}{V}\Delta V, \qquad \text{i.e. } \frac{\Delta V}{V}\Big|_{T=\text{const}} = -\frac{\Delta\mathcal{P}}{P}. \tag{7.39}$$

Comparing this expression with Eq. (36), we get a remarkably simple result for the isothermal compression of gases,

$$K\big|_{T=\text{const}} = \mathcal{P}, \tag{7.40}$$

which means in particular that the bulk modulus listed in Table 1 is actually valid, at the ambient conditions, for almost any gas. Note, however, that the change of thermodynamic conditions (say, from isothermal to adiabatic[11]) may affect the compressibility of the gas..

Now let us consider the second, rather different, fundamental experiment: a purely shear deformation shown in Fig. 2. Since the traces of the matrices (15) and (20), which describe this situation, are equal to 0, for their off-diagonal elements, Eq. (32) gives merely $\sigma_{jj'} = 2\mu s_{jj'}$, so the deformation angle α (see Fig. 2) is just

[10] For the derivation and a detailed discussion of Eq. (37), see, e.g., SM Sec. 3.1.

[11] See, e.g., SM Sec. 1.3.

$$\alpha = \frac{1}{\mu}\frac{F}{A}. \tag{7.41}$$

Note that the angle does not depend on the thickness h of the sample, though of course the maximal linear deformation $q_x = \alpha h$ is proportional to the thickness. Naturally, as Table 1 shows, $\mu = 0$ for all fluids because they do not resist static shear stress.

However, not all situations, even apparently simple ones, involve just either K or μ. Let us consider stretching a long and thin elastic rod of a uniform cross-section of area A – the so-called *tensile stress experiment* shown in Fig. 6.[12]

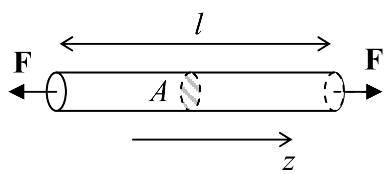

Fig. 7.6. The tensile stress experiment.

Though the deformation of the rod near its clamped ends depends on the exact way forces **F** are applied (we will discuss this issue later on), we may expect that over most of its length, the tension forces are directed virtually along the rod, $d\mathbf{F} = F_z\mathbf{n}_z$, and hence, with the coordinate choice shown in Fig. 6, $\sigma_{xj} = \sigma_{yj} = 0$ for all j, including the diagonal elements σ_{xx} and σ_{yy}. Moreover, due to the open lateral surfaces, on which, evidently, $dF_x = dF_y = 0$, there cannot be an internal stress force of *any* direction, acting on any elementary internal boundary parallel to these surfaces. This means that $\sigma_{zx} = \sigma_{zy} = 0$. So, of all elements of the stress tensor only one, σ_{zz}, is not equal to zero, and for a uniform sample, $\sigma_{zz} = \text{const} = F/A$. For this case, Eq. (34) shows that the strain tensor is also diagonal, but with different diagonal elements:

$$s_{zz} = \left(\frac{1}{9K} + \frac{1}{3\mu}\right)\sigma_{zz}, \tag{7.42}$$

$$s_{xx} = s_{yy} = \left(\frac{1}{9K} - \frac{1}{6\mu}\right)\sigma_{zz}. \tag{7.43}$$

Since tensile stress is most common in engineering practice (including physical experiment design), both combinations of the elastic moduli participating in these two relations have earned their own names. In particular, the constant in Eq. (42) is usually denoted as $1/E$ (but in many texts, as $1/Y$), where E is called *Young's modulus*:[13]

Young's modulus

$$\frac{1}{E} \equiv \frac{1}{9K} + \frac{1}{3\mu}, \qquad \text{i.e. } E \equiv \frac{9K\mu}{3K+\mu}. \tag{7.44}$$

---

[12] Though the analysis of compression in this situation gives similar results, in practical experiments a strong compression of a long sample may lead to the loss of the horizontal stability – the so-called *buckling* – of the rod.

[13] Named after another polymath, Thomas Young (1773-1829) – somewhat unfairly, because his work on elasticity was predated by a theoretical analysis by L. Euler in 1727 and detailed experiments by Giordano Riccati in 1782.

As Fig. 6 shows, in the tensile stress geometry $s_{zz} \equiv \partial q_z/\partial z = \Delta l/l$, so Young's modulus scales the linear relation between the relative extension of the rod and the force applied per unit area:[14]

$$\frac{\Delta l}{l} = \frac{1}{E}\frac{F}{A}. \tag{7.45}$$

The third column of Table 1 above shows the values of this modulus for two well-known solids: diamond (with the highest known value of E of all bulk materials[15]) and the steels (solid solutions of ~10% of carbon in iron) used in construction. Again, for all fluids, Young's modulus equals zero – as it follows from Eq. (44) for $\mu = 0$.

I am confident that most readers of these notes have been familiar with Eq. (42), in the form of Eq. (45), from their undergraduate studies. However, this can hardly be said about its counterpart, Eq. (43), which shows that at the tensile stress, the rod's cross-section dimensions also change. This effect is usually characterized by the following dimensionless *Poisson's ratio*:[16]

$$\nu \equiv -\frac{s_{xx}}{s_{zz}} = -\frac{s_{yy}}{s_{zz}} = -\left(\frac{1}{9K} - \frac{1}{6\mu}\right)\Big/\left(\frac{1}{9K} + \frac{1}{3\mu}\right) \equiv \frac{1}{2}\frac{3K - 2\mu}{3K + \mu}. \tag{7.46}$$

Poisson ratio

According to this formula, for realistic materials with $K > 0$, $\mu \geq 0$, ν may vary from (-1) to (+½), but for the vast majority of materials,[17] its values are between 0 and ½ – see the corresponding column of Table 1. The lower limit of this range is reached in porous materials like cork, whose *lateral dimensions* almost do not change at the tensile stress. Some soft materials such as natural and synthetic rubbers present the opposite case: $\nu \approx$ ½.[18] Since according to Eqs. (13) and (42), the volume change is

$$\frac{\Delta V}{V} = s_{xx} + s_{yy} + s_{zz} = \frac{1}{E}\frac{F}{A}(1 - 2\nu) \equiv (1 - 2\nu)\frac{\Delta l}{l}, \tag{7.47}$$

such materials virtually do not change their *volume* at the tensile stress. The ultimate limit of this trend, $\Delta V/V = 0$, is provided by fluids and gases, because, as it follows from Eq. (46) with $\mu = 0$, their Poisson's ratio ν is exactly ½. However, for most practicable construction materials such as various steels (see Table 1) the relative volume change (47) is as high as ~40% of that of the length.

Due to the tensile stress dominance in practice, the coefficients E and ν are frequently used as a pair of independent elastic moduli, instead of K and μ. Solving Eqs. (44) and (46) for them, we get

$$K = \frac{E}{3(1 - 2\nu)}, \qquad \mu = \frac{E}{2(1 + \nu)}. \tag{7.48}$$

---

[14] According to Eq. (47), E may be thought of as the force (per unit area) that would double the initial sample's length, if only Hooke's law was valid for deformations that large – as it typically isn't.

[15] E is probably somewhat higher (up to 2,000 GPa) in such nanostructures as carbon nanotubes and monatomic sheets (*graphene*), though there is still substantial uncertainty in experimentally measured elastic moduli of these structures – for a review see, e.g., G. Dimitrios *et al*., *Prog. Mater. Sci*. **90**, 75 (2017).

[16] In some older texts, the Poisson's ratio is denoted σ, but its notation as ν dominates modern literature.

[17] The only known exceptions are certain exotic solids with very specific internal microstructure – see, e.g., R. Lakes, *Science* **235**, 1038 (1987) and references therein.

[18] For example, *silicone rubbers* (synthetic polymers broadly used in engineering and physics experiment design) have, depending on their particular composition, synthesis, and thermal curing, ν = 0.47÷0.49, and as a result, combine respectable bulk moduli K = (1.5÷2) GPa with very low Young's moduli: E = (0.0001÷0.05) GPa.

Using these formulas, the two (equivalent) formulations of Hooke's law, expressed by Eqs. (32) and (34), may be rewritten as

Hooke's law via E and ν

$$\sigma_{jj'} = \frac{E}{1+\nu}\left(s_{jj'} + \frac{\nu}{1-2\nu}\text{Tr}(\mathrm{s})\delta_{jj'}\right), \qquad (7.49a)$$

$$s_{jj'} = \frac{1+\nu}{E}\left(\sigma_{jj'} - \frac{\nu}{1+\nu}\text{Tr}(\sigma)\delta_{jj'}\right). \qquad (7.49b)$$

The linear relation between the strain and stress tensor in elastic continua enables one more step in our calculation of the potential energy U due to deformation, which was started at the end of Sec. 2. Indeed, to each infinitesimal part of this strain increase, we may apply Eq. (30), with the elementary work $\delta\mathscr{W}$ of the surface forces increasing the potential energy of "our" part of the body by the equal amount δU. Let us slowly increase the deformation from a completely unstrained state (in which we may take $U = 0$) to a certain strained state, in the absence of bulk forces $\mathbf{f}$, keeping the deformation type, i.e. the relation between the elements of the stress tensor, intact. In this case, all elements of the tensor $\sigma_{jj'}$ are proportional to the same single parameter characterizing the stress (say, the total applied force), and according to Hooke's law, all elements of the tensor $s_{j'j}$ are proportional to that parameter as well. In this case, integration of Eq. (30) through the variation yields the following final value:[19]

Elastic deformation energy

$$U = \int_V u(\mathbf{r})d^3r, \qquad u(\mathbf{r}) = \frac{1}{2}\sum_{j,j'=1}^{3}\sigma_{jj'}s_{j'j}\ . \qquad (7.50)$$

Evidently, this $u(\mathbf{r})$ may be interpreted as the volumic density of the potential energy of the elastic deformation.

7.4. Equilibrium

Now we are fully equipped to discuss the elastic deformation dynamics, but let us start with statics. The static (equilibrium) state may be described by requiring the right-hand side of Eq. (25) to vanish. To find the elastic deformation, we need to plug $\sigma_{jj'}$ from Hooke's law (49a), and then express the elements $s_{jj'}$ via the displacement distribution – see Eq. (9). For a uniform material, the result is[20]

$$\frac{E}{2(1+\nu)}\sum_{j'=1}^{3}\frac{\partial^2 q_j}{\partial r_{j'}^2} + \frac{E}{2(1+\nu)(1-2\nu)}\sum_{j'=1}^{3}\frac{\partial^2 q_{j'}}{\partial r_j \partial r_{j'}} + f_j = 0. \qquad (7.51)$$

Taking into account that the first sum is just the j^{th} component of $\nabla^2\mathbf{q}$, while the second sum is the j^{th} component of $\nabla(\nabla\cdot\mathbf{q})$, we see that all three equations (51) for three Cartesian components (j = 1, 2, and 3) of the deformation vector $\mathbf{q}$, may be conveniently merged into one vector equation

Elastic continuum: equilibrium

$$\frac{E}{2(1+\nu)}\nabla^2\mathbf{q} + \frac{E}{2(1+\nu)(1-2\nu)}\nabla(\nabla\cdot\mathbf{q}) + \mathbf{f} = 0. \qquad (7.52)$$

---

[19] To give additional clarity to the arising factor ½, let me spell out this integration for the simple case of a 1D spring. In this case, Eq. (30) is reduced to $\delta U = \delta\mathscr{W} = F\delta x$, and if the spring's force is elastic, $F = \kappa x$, the integration over x from 0 to its final value yields $U = \kappa x^2/2 \equiv Fx/2$.

[20] As it follows from Eqs. (48), the coefficient before the first sum in Eq. (51) is just the shear modulus μ, while that before the second sum is equal to $(K + \mu/3)$.

For some applications, it is more convenient to recast this equation into a different form, using the well-known vector identity[21] $\nabla^2\mathbf{q} = \nabla(\nabla\cdot\mathbf{q}) - \nabla\times(\nabla\times\mathbf{q})$. The result is

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\nabla(\nabla\cdot\mathbf{q}) - \frac{E}{2(1+\nu)}\nabla\times(\nabla\times\mathbf{q}) + \mathbf{f} = 0. \qquad (7.53)$$

It is interesting that in problems without volume-distributed forces ($\mathbf{f} = 0$), Young's modulus E cancels out. Even more fascinating, in this case, the equation may be re-written in a form not involving Poisson's ratio ν either. Indeed, calculating the divergence of the remaining terms of Eq. (53), taking into account MA Eqs. (9.2) and (11.2), we get a surprisingly simple equation

$$\nabla^2(\nabla\cdot\mathbf{q}) = 0. \qquad (7.54)$$

A natural question here is how the elastic moduli affect the deformation distribution if they do not participate in the differential equation describing it. The answer is different in the following two cases. If what is fixed at the body's boundary are deformations, then the moduli are irrelevant, because the deformation distribution through the body does not depend on them. On the other hand, if the boundary conditions describe fixed stress (or a combination of stress and strain), then the elastic constants creep into the solution via the recalculation of these conditions into the strain. As a simple but representative example, let us calculate the deformation distribution in a (generally, thick) spherical shell under the effect of pressures inside and outside it – see Fig. 7a.

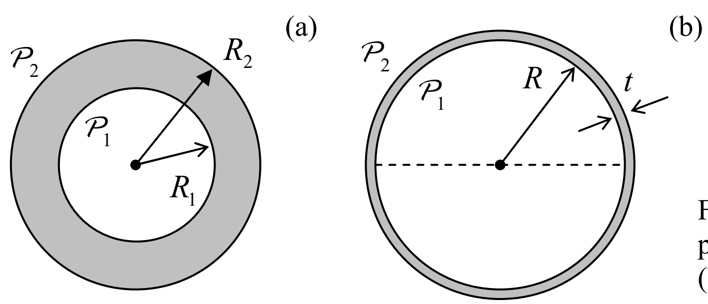

Fig. 7.7. The spherical shell problem: (a) the general case, and (b) the thin shell limit.

Due to the spherical symmetry of the problem, the deformation is obviously spherically symmetric and radial, $\mathbf{q}(\mathbf{r}) = q(r)\mathbf{n}_r$, i.e. is completely described by one scalar function $q(r)$. Since the curl of such a radial vector field is zero,[22] Eq. (53) is reduced to

$$\nabla(\nabla\cdot\mathbf{q}) = 0, \qquad (7.55)$$

This means that the divergence of the function $q(r)$ is constant within the shell. In the spherical coordinates:[23]

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 q\right) = \text{const.} \qquad (7.56)$$

Naming this constant $3a$ (with the numerical factor chosen just for the later notation's convenience), and integrating Eq. (56) over r, we get its solution,

[21] See, e.g., MA Eq. (11.3).
[22] If this is not immediately evident, please have a look at MA Eq. (10.11) with $\mathbf{f} = f_r(r)\mathbf{n}_r$.
[23] See, e.g., MA Eq. (10.10) with $\mathbf{f} = q(r)\mathbf{n}_r$

$$q(r) = ar + \frac{b}{r^2}, \tag{7.57}$$

which also includes another integration constant, b. The constants a and b may be determined from the boundary conditions. Indeed, according to Eq. (19),

$$\sigma_{rr} = \begin{cases} -\mathcal{P}_1, & \text{at } r = R_1, \\ -\mathcal{P}_2, & \text{at } r = R_2. \end{cases} \tag{7.58}$$

In order to relate this stress to strain, let us use Hooke's law, but for that, we first need to calculate the strain tensor components for the deformation distribution (57). Using Eqs. (17), we get

$$s_{rr} = \frac{\partial q}{\partial r} = a - 2\frac{b}{r^3}, \qquad s_{\theta\theta} = s_{\varphi\varphi} = \frac{q}{r} = a + \frac{b}{r^3}, \tag{7.59}$$

so Tr (s) = $3a$. Plugging these relations into Eq. (49a) for σ_{rr}, we obtain

$$\sigma_{rr} = \frac{E}{1+\nu}\left[\left(a - 2\frac{b}{r^3}\right) + \frac{\nu}{1-2\nu}3a\right]. \tag{7.60}$$

Now plugging this relation into Eqs. (58), we get a system of two linear equations for the coefficients a and b. An easy solution to this system yields

$$a = \frac{1-2\nu}{E}\frac{\mathcal{P}_1 R_1^3 - \mathcal{P}_2 R_2^3}{R_2^3 - R_1^3}, \qquad b = \frac{1+\nu}{2E}\frac{(\mathcal{P}_1 - \mathcal{P}_2)R_1^3 R_2^3}{R_2^3 - R_1^3}. \tag{7.61}$$

Formulas (57) and (61) give a complete solution to our problem. (Note that the elastic moduli are back, as was promised.) This solution is rich in physical content and deserves at least some analysis. First of all, note that according to Eq. (48), the coefficient $(1 - 2\nu)/E$ in the expression for a is just $1/3K$, so the first term in Eq. (57) for the net deformation describes the hydrostatic compression. Now note that the second of Eqs. (61) yields $b = 0$ if $R_1 = 0$. Thus for a solid sphere, we have only the hydrostatic compression that was discussed in the previous section. Perhaps less intuitively, making two pressures equal also gives $b = 0$, i.e. the purely hydrostatic compression, for arbitrary $R_2 > R_1$.

However, in the general case, $b \neq 0$, so the second term in the deformation distribution (57), which describes the shear deformation,[24] is also substantial. In particular, let us consider the important thin-shell limit, when $R_2 - R_1 \equiv t \ll R_{1,2} \equiv R$ – see Fig. 7b. In this case, $q(R_1) \approx q(R_2)$ is just the change of the shell radius R, for which Eqs. (57) and (61) (with $R_2^3 - R_1^3 \approx 3R^2 t$) give

$$\Delta R \equiv q(R) \approx aR + \frac{b}{R^2} \approx \frac{(\mathcal{P}_1 - \mathcal{P}_2)R^2}{3t}\left(\frac{1-2\nu}{E} + \frac{1+\nu}{2E}\right) = (\mathcal{P}_1 - \mathcal{P}_2)\frac{R^2}{t}\frac{1-\nu}{2E}. \tag{7.62}$$

Naively, one could think that at least in this limit the problem could be analyzed by elementary means. For example, the total force exerted by the pressure difference $(\mathcal{P}_1 - \mathcal{P}_2)$ on the diametrical cross-section of the shell (see, e.g., the dashed line in Fig. 7b) is $F = \pi R^2(\mathcal{P}_1 - \mathcal{P}_2)$, giving the stress,

[24] Indeed, according to Eq. (48), the material-dependent factor in the second of Eqs. (61) is just $1/4\mu$.

$$\sigma = \frac{F}{A} = \frac{\pi R^2(\mathcal{P}_1 - \mathcal{P}_2)}{2\pi R t} = (\mathcal{P}_1 - \mathcal{P}_2)\frac{R}{2t}, \tag{7.63}$$

directed along the shell's walls. One can check that this simple formula may be indeed obtained, in this limit, from the strict expressions for $\sigma_{\theta\theta}$ and $\sigma_{\varphi\varphi}$, following from the general treatment carried out above. However, if we now tried to continue this approach by using the simple relation (45) to find the small change Rs_{rr} of the sphere's radius, we would arrive at a result with the general structure of Eq. (62), but without the factor $(1 - \nu) < 1$ in the numerator. The reason for this error (which may be as significant as ~30% for typical construction materials – see Table 1) is that Eq. (45), while being valid for *thin rods* of arbitrary cross-section, is invalid for *thin but broad sheets*, and in particular the thin shell in our problem. Indeed, while at the tensile stress, both lateral dimensions of a thin rod may contract freely, in our last problem all dimensions of the shell are under stress – actually, under much more tangential stress than the radial one.[25]

7.5. Rod bending

The general approach to the static deformation analysis, outlined at the beginning of the previous section, may be simplified not only for symmetric geometries but also for uniform thin structures such as *thin plates* (also called "membranes" or "thin sheets") and *thin rods*. Due to the shortage of time, in this course, I will demonstrate typical approaches to such systems only on the example of thin rods. (The theory of thin plates and shells is conceptually similar but mathematically more involved.[26])

Besides the tensile stress analyzed in Sec. 3, the two other major types of rod deformation are *bending* and *torsion*. Let us start from a "local" analysis of bending caused by a pair of equal and opposite external torques $\boldsymbol{\tau} = \pm\mathbf{n}_y\tau_y$ perpendicular to the rod axis z (Fig. 8), assuming that the rod is "quasi-uniform", i.e. that on the scale of this analysis (comparable with the linear scale a of the cross-section) its material parameters and the cross-section A do not change substantially.

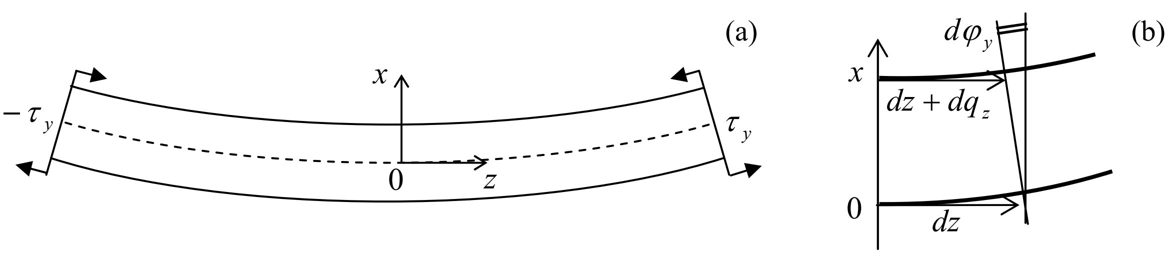

Fig. 7.8. Rod bending, in a local reference frame (specific for each cross-section). The bold arrows show the simplest way to create the two opposite torques τ_y: a couple of opposite forces for each torque.

Just as in the tensile stress experiment (Fig. 6), the components of the stress forces $d\mathbf{F}$, normal to the rod's length, have to equal zero on the surface of the rod. Repeating the arguments made for the tensile stress discussion, we have to conclude that only one diagonal element of the tensor (in Fig. 8, σ_{zz}) may differ from zero:

[25] Strictly speaking, this is only true if the pressure difference is not too small, namely, if $|\mathcal{P}_1 - \mathcal{P}_2| >> \mathcal{P}_{1,2}t/R$.

[26] For its review see, e.g., Secs. 11-15 in L. Landau and E. Lifshitz, *Theory of Elasticity*, 3rd ed., Butterworth-Heinemann, 1986.

$$\sigma_{jj'} = \delta_{jz}\sigma_{zz}. \qquad (7.64)$$

However, in contrast to the tensile stress, at pure static bending, the net force directed along the rod has to vanish:

$$F_z = \int_S \sigma_{zz} d^2 r = 0, \qquad (7.65)$$

where S is the rod's cross-section, so σ_{zz} has to change its sign at some point of the x-axis, selected to lie in the plane of the bent rod. Thus, the bending deformation may be viewed as a combination of a stretch of some layers of the rod (bottom layers in Fig. 8) with compression of other (top) layers.

Since it is hard to make more conclusions about the stress distribution immediately, let us turn over to strain, assuming that the rod's cross-section is virtually constant over the length of our local analysis. From the above representation of bending as a combination of stretching and compression, it is evident that the longitudinal deformation q_z has to vanish along some *neutral line* on the rod's cross-section – in Fig. 8, represented by the dashed line.[27] Selecting the origin of the x-coordinate on this line, and expanding the relative deformation in the Taylor series in x, due to the cross-section smallness we may keep just the first, linear term of the expansion:

$$s_{zz} \equiv \frac{dq_z}{dz} = -\frac{x}{R}. \qquad (7.66)$$

The constant R has the sense of the *curvature radius* of the bent rod. Indeed, on a small segment dz, the cross-section turns by a small angle $d\varphi_y = -dq_z/x$ (Fig. 8b). Using Eq. (66), we get $d\varphi_y = dz/R$, which is the usual definition of the curvature radius R in the differential geometry, for our special choice of the coordinate axes.[28]

Expressions for other elements of the strain tensor are harder to guess (like at the tensile stress, not all of them are equal to zero!), but what we already know about σ_{zz} and s_{zz} is sufficient to start formal calculations. Indeed, plugging Eq. (64) into Hooke's law in the form (49b), and comparing the result for s_{zz} with Eq. (66), we find

$$\sigma_{zz} = -E\frac{x}{R}. \qquad (7.67)$$

From the same Eq. (49b), we could also find the transverse elements of the strain tensor, and conclude that they are related to s_{zz} exactly as at the tensile stress:

$$s_{xx} = s_{yy} = -\nu s_{zz}, \qquad (7.68)$$

and then, integrating these relations along the cross-section of the rod, find the deformation of the cross-section's shape. More important for us, however, is to calculate the relation between the rod's curvature and the net torque acting on a given cross-section S (taking $dA_z > 0$):

$$\tau_y \equiv \int_S (\mathbf{r}\times d\mathbf{F})_y = -\int_S x\sigma_{zz} d^2 r = \frac{E}{R}\int_S x^2 d^2 r \equiv \frac{EI_y}{R}, \qquad (7.69)$$

---

[27] Strictly speaking, that dashed line is the intersection of the *neutral surface* (the continuous set of such neutral lines for all cross-sections of the rod) with the plane of the drawing.

[28] Indeed, for $(dx/dz)^2 << 1$, the general formula MA Eq. (4.3) for the curvature (with the appropriate replacements $f \to x$ and $x \to z$) is reduced to $1/R = d^2x/dz^2 = d(dx/dz)/dz = d(\tan\varphi_y)/dz \approx d\varphi_y/dz$.

where I_y is a geometric constant defined as

$$I_y \equiv \int_S x^2 dxdy. \tag{7.70}$$

Note that this factor, defining the bending rigidity of the rod, grows as fast as a^4 with the linear scale a of the cross-section.[29]

In these expressions, x has to be measured from the neutral line. Let us see where exactly this line passes through the rod's cross-section. Plugging the result (67) into Eq. (65), we get the condition defining the neutral line:

$$\int_S xdxdy = 0. \tag{7.71}$$

This condition allows for a simple interpretation. Imagine a thin sheet of some material, with a constant mass density σ per unit area, cut in the form of the rod's cross-section. If we place a reference frame into its center of mass, then, by its definition,

$$\sigma \int_S \mathbf{r} dxdy = 0. \tag{7.72}$$

Comparing this condition with Eq. (71), we see that one of the neutral lines has to pass through the center of mass of the sheet, which may be called the "center of mass of the cross-section". Using the same analogy, we see that the integral I_y given by Eq. (72) may be interpreted as the moment of inertia of the same imaginary sheet of material, with σ formally equal to 1, for its rotation about the neutral line – cf. Eq. (4.24). This analogy is so convenient that the integral is usually called the *moment of inertia of the cross-section* and denoted similarly – just as has been done above. So, our basic result (69) may be rewritten as

$$\frac{1}{R} = \frac{\tau_y}{EI_y}. \tag{7.73}$$

Rod bending: curvature vs. torque

This relation is only valid if the deformation is small in the sense $R >> a$. Still, since the deviations of the rod from its unstrained shape may accumulate along its length, Eq. (73) may be used for calculations of large "global" deviations of the rod from equilibrium, on a length scale much larger than a. To describe such deformations, Eq. (73) has to be complemented by conditions of the balance of the bending forces and torques. Unfortunately, a general analysis of such deformations requires a bit more differential geometry than I have time for, so I will only discuss this procedure for the simplest case of relatively small transverse deviations $q \equiv q_x$ of an initially horizontal rod from its straight shape that will be used for the z-axis (Fig. 9a), by some forces, possibly including bulk-distributed forces $\mathbf{f} = \mathbf{n}_x f_x(z)$. (Again, the simplest example is a uniform gravity field, for which $f_x = -\rho g = \text{const.}$) Note that in the forthcoming discussion, the reference frame will be global, i.e. common for the whole rod, rather than local (pertaining to each cross-section) as it was in the previous analysis – cf. Fig. 8.

First of all, we may write a static relation for the total vertical force $\mathbf{F} = \mathbf{n}_x F_x(z)$ exerted on the part of the rod to the left of the considered cross-section – located at point z. The differential form of this relation expresses the balance of vertical forces exerted on a small fragment dz of the rod (Fig. 9a), necessary for the absence of its *linear* acceleration: $F_x(z + dz) - F_x(z) + f_x(z)Adz = 0$, giving

[29] In particular, this is the reason why the usual electric wires are made not of a solid copper core, but rather a twisted set of thinner sub-wires, which may slip relative to each other, increasing the wire flexibility.

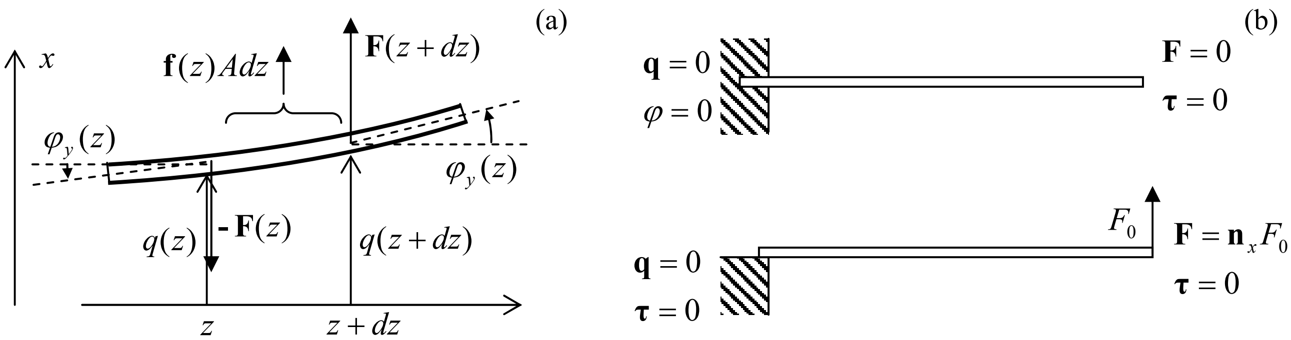

Fig. 7.9. A global picture of rod bending: (a) the forces acting on a small fragment of a rod, and (b) two bending problem examples, each with two typical but different boundary conditions.

$$\frac{dF_x}{dz} = -f_x A, \tag{7.74}$$

where A is the cross-section's area. Note that this vertical component of the internal forces has been neglected in our derivation of Eq. (73), and hence our final results will be valid only if the ratio F_x/A is much smaller than the magnitude of σ_{zz} described by Eq. (67). However, in reality, these are exactly the forces that create the very torque $\boldsymbol{\tau} = \mathbf{n}_y \tau_y$ that in turn causes the bending, and thus have to be taken into account in the analysis of the global picture.

Such an account may be made by writing the balance of the components of the elementary torque exerted on the same rod fragment of length dz, necessary for the absence of its *angular* acceleration: $d\tau_y + F_x dz = 0$, so

$$\frac{d\tau_y}{dz} = -F_x. \tag{7.75}$$

These two equations should be complemented by two geometric relations. The first of them is $d\varphi_y/dz = 1/R$, which has already been discussed above. We may immediately combine it with the basic result (73) of our local analysis, getting:

$$\frac{d\varphi_y}{dz} = \frac{\tau_y}{EI_y}. \tag{7.76}$$

The final equation is the geometric relation evident from Fig. 9a:

$$\frac{dq_x}{dz} = \varphi_y, \tag{7.77}$$

which is (as all expressions of our simple analysis) only valid for small bending angles, $|\varphi_y| << 1$.

The four differential equations (74)-(77) are sufficient for the full solution of the weak-bending problem, if complemented by appropriate boundary conditions. Figure 9b shows the conditions most frequently met in practice. Let us solve, for example, the problem shown on the top panel of Fig. 9b: bending of a rod, "clamped" at one end (say, immersed into a rigid wall), under its own weight. As should be clear from their derivation, Eqs. (74)-(77) are valid for any distribution of parameters A, E, I_y, and ρ over the rod's length, provided that the rod is *quasi-uniform*, i.e. its parameters' changes are so slow that the local relation (76) is still valid at any point. However, just for simplicity, let us consider a uniform rod. The simple structure of Eqs. (74)-(77) allows for their integration one by one, each time using the appropriate boundary conditions. To start, Eq. (74) with $f_x = -\rho g$ = const yields

$$F_x = \rho g A z + \text{const} = \rho g A(z-l), \tag{7.78}$$

where the integration constant has been selected to satisfy the right-end boundary condition: $F_x = 0$ at $z = l$. As a sanity check, at the left wall ($z = 0$), $F_x = -\rho g A l = -mg$, meaning that the whole weight of the rod is exerted on the supporting wall – fine.

Next, by plugging Eq. (78) into Eq. (75) and integrating, we get

$$\tau_y = -\frac{\rho g A}{2}\left(z^2 - 2lz\right) + \text{const} = -\frac{\rho g A}{2}\left(z^2 - 2lz + l^2\right) \equiv -\frac{\rho g A}{2}(z-l)^2, \tag{7.79}$$

where the integration constant's choice ensures the second right-boundary condition: $\tau_y = 0$ at $z = l$ – see Fig. 9b again. Now proceeding in the same fashion to Eq. (76), we get

$$\varphi_y = -\frac{\rho g A}{2EI_y}\frac{(z-l)^3}{3} + \text{const} = -\frac{\rho g A}{6EI_y}\left[(z-l)^3 + l^3\right], \tag{7.80}$$

where the integration constant is selected to satisfy the *clamping condition* at the left end of the rod: $\varphi_y = 0$ at $z = 0$. (Note that this is different from the *support condition* illustrated on the lower panel of Fig. 9b, which allows the *angle* φ_y to be different from zero at $z = 0$, but requires the *torque* to vanish at that point.) Finally, integrating Eq. (77) with φ_y given by Eq. (80), we get the rod's global deformation law,

$$q_x(z) = -\frac{\rho g A}{6EI_y}\left[\frac{(z-l)^4}{4} + l^3 z + \text{const}\right] = -\frac{\rho g A}{6EI_y}\left[\frac{(z-l)^4}{4} + l^3 z - \frac{l^4}{4}\right], \tag{7.81}$$

where the integration constant is selected to satisfy the second left-boundary condition: $q = 0$ at $z = 0$. So, the bending law is sort of complicated even in this very simple problem. It is also remarkable how fast the end's displacement grows with the increase of the rod's length:

$$q_x(l) = -\frac{\rho g A l^4}{8EI_y}. \tag{7.82}$$

To conclude this solution, let us discuss the validity of this result. First, the geometric relation (77) is only valid if $|\varphi_y(l)| << 1$, and hence if $|q_x(l)| << l$. Next, the local formula Eq. (76) is valid if $1/R = \tau(l)/EI_y << 1/a \sim A^{-1/2}$. Using the results (79) and (82), we see that the latter condition is equivalent to $|q_x(l)| << l^2/a$, i.e. is weaker than the former one, because all our analysis has been based on the assumption $l >> a$. Another point of concern may be that the off-diagonal stress element $\sigma_{xz} \sim F_x/A$, which is created by the vertical gravity forces, has been ignored in our local analysis. For that approximation to be valid, this element must be much smaller than the diagonal element $\sigma_{zz} \sim aE/R = a\tau/I_y$ taken into account in that analysis. Using Eqs. (78) and (80), we are getting the following estimates: $\sigma_{xz} \sim \rho g l$, $\sigma_{zz} \sim a\rho g A l^2/I_y \sim a^3 \rho g l^2/I_y$. According to its definition (70), I_y may be crudely estimated as a^4, so we finally get the simple condition $a << l$, which has been assumed from the very beginning of our solution.

7.6. Rod torsion

One more class of analytically solvable elasticity problems is the torsion of quasi-uniform, straight rods by a couple of axially-oriented torques $\boldsymbol{\tau} = \pm\mathbf{n}_z\tau_z$ – see Fig. 10.

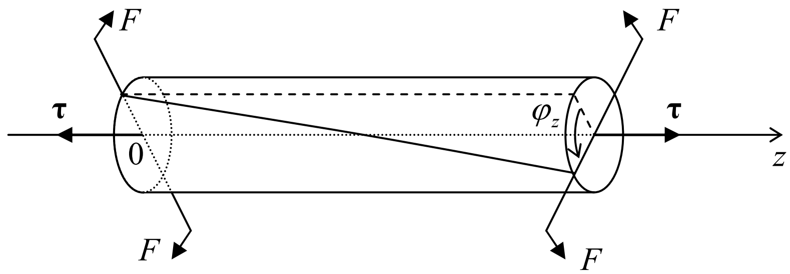

Fig. 7.10. Rod torsion. Just as in Fig. 8, the couples of forces **F** are just vivid representations of the opposite torques $\pm\boldsymbol{\tau}$.

This problem is simpler than the bending in the sense that due to its longitudinal uniformity, $d\varphi_z/dz = \text{const}$, it is sufficient to relate the torque τ_z to the so-called *torsion parameter*

$$\kappa \equiv \frac{d\varphi_z}{dz}. \qquad (7.83)$$

If the deformation is elastic and small (in the sense $\kappa a << 1$, where a is again the characteristic size of the rod's cross-section), κ is proportional to τ_z. Hence our task is to calculate their ratio,

Torsional rigidity: definition

$$C \equiv \frac{\tau_z}{\kappa} \equiv \frac{\tau_z}{d\varphi_z / dz}, \qquad (7.84)$$

called the *torsional rigidity* of the rod.

As the first guess (as we will see below, of a limited validity), one may assume that the torsion does not change either the shape or size of the rod's cross-sections, but leads just to their mutual rotation about a certain central line. Using a reference frame with the origin on that line, this assumption immediately enables the calculation of Cartesian components of the displacement vector $d\mathbf{q}$, by using Eq. (6) with $d\boldsymbol{\varphi} = \mathbf{n}_z d\varphi_z$:

$$dq_x = -yd\varphi_z = -\kappa y dz, \qquad dq_y = xd\varphi_z = \kappa x dz, \qquad dq_z = 0. \qquad (7.85)$$

From here, we can calculate all Cartesian elements (9) of the symmetrized strain tensor:

$$s_{xx} = s_{yy} = s_{zz} = 0, \quad s_{xy} = s_{yx} = 0, \quad s_{xz} = s_{zx} = -\frac{\kappa}{2}y, \quad s_{yz} = s_{zy} = \frac{\kappa}{2}x. \qquad (7.86)$$

The first of these equalities means that the elementary volume does not change, i.e. we are dealing with purely shear deformation. As a result, all nonzero elements of the stress tensor, calculated from Eqs. (32), are proportional to the shear modulus alone:[30]

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0, \qquad \sigma_{xy} = \sigma_{yx} = 0, \qquad \sigma_{xz} = \sigma_{zx} = -\mu\kappa y, \qquad \sigma_{yz} = \sigma_{zy} = \mu\kappa x. \qquad (7.87)$$

[30] Note that for this problem, with a purely shear deformation, using the alternative elastic moduli E and ν would be rather unnatural. If needed, we may always use the second of Eqs. (48): $\mu = E/2(1 + \nu)$.

Now it is straightforward to use this result to calculate the full torque as an integral over the cross-section's area A:

$$\tau_z \equiv \int_A (\mathbf{r}\times d\mathbf{F})_z = \int_A (xdF_y - ydF_x) = \int_A (x\sigma_{yz} - y\sigma_{xz})dxdy. \tag{7.88}$$

Using Eq. (87), we get $\tau_z = \mu\kappa I_z$, i.e.

$$C = \mu I_z, \quad \text{where} \quad I_z \equiv \int_A (x^2+y^2)dxdy\,. \tag{7.89}$$

C at axial symmetry

Again, just as in the case of thin rod bending, we have got an integral, in this case I_z, similar to a moment of inertia, this time for the rotation about the z-axis passing through a certain point of the cross-section. For any axially symmetric cross-section, this has to be its central point. Then, for example, for the practically important case of a uniform round pipe with internal radius R_1 and external radius R_2, Eq. (89) yields

$$C = \mu\, 2\pi \int_{R_1}^{R_2} \rho^3 d\rho = \frac{\pi}{2}\mu\left(R_2^4 - R_1^4\right). \tag{7.90}$$

In particular, for the solid rod of radius R (which may be treated as a pipe with $R_1 = 0$ and $R_2 = R$), this result gives the following torsional rigidity

$$C = \frac{\pi}{2}\mu R^4, \tag{7.91a}$$

while for a hollow pipe of small thickness $t \ll R$, Eq. (90) is reduced to

$$C = 2\pi\mu R^3 t\,. \tag{7.91b}$$

Note that per unit cross-section area A (and hence per unit mass of the rod) the thin pipe's rigidity is twice higher than that of a solid rod:

$$\left.\frac{C}{A}\right|_{\text{thin round pipe}} = \mu R^2 > \left.\frac{C}{A}\right|_{\text{solid round rod}} = \frac{1}{2}\mu R^2\,. \tag{7.92}$$

This fact is one reason for the broad use of thin pipes in engineering and physical experiment design.

However, for rods with axially asymmetric cross-sections, Eq. (89) gives *wrong* results. For example, for a narrow rectangle of area $A = w\times t$ with $t \ll w$, it yields the expression $C = \mu t w^3/12$ [WRONG!], which is even functionally different from the correct result – cf. Eq. (104) below. The reason for this error is that the above analysis does not describe possible bending $q_z(x, y)$ of the rod's cross-section in the direction *along* the rod. (For axially-symmetric rods, such bending is evidently forbidden by the symmetry, so Eq. (89) is valid, and the results (90)-(92) are absolutely correct.)

Let us describe[31] this counter-intuitive effect by taking

$$q_z = \kappa\psi(x,y), \tag{7.93}$$

[31] I would not be terribly shocked if the reader skipped the balance of this section at the first reading. Though the calculation described in it is very elegant, instructive, and typical for the theory of elasticity (and for good physics as a whole!), its results will not be used in other chapters of this course or other parts of this series.

(where ψ is some function to be determined), but still keeping Eq. (87) for two other components of the displacement vector. The addition of ψ does not perturb the equality to zero of the diagonal elements of the strain tensor, as well as of s_{xy} and s_{yx}, but contributes to other off-diagonal elements:

$$s_{xz} = s_{zx} = \frac{\kappa}{2}\left(-y + \frac{\partial \psi}{\partial x}\right), \qquad s_{yz} = s_{zy} = \frac{\kappa}{2}\left(x + \frac{\partial \psi}{\partial y}\right), \tag{7.94}$$

and hence to the corresponding elements of the stress tensor:

$$\sigma_{xz} = \sigma_{zx} = \mu\kappa\left(-y + \frac{\partial \psi}{\partial x}\right), \qquad \sigma_{yz} = \sigma_{zy} = \mu\kappa\left(x + \frac{\partial \psi}{\partial y}\right). \tag{7.95}$$

Now let us find the requirement imposed on the function $\psi(x,y)$ by the fact that the stress force component parallel to the rod's axis,

$$dF_z = \sigma_{zx} dA_x + \sigma_{zy} dA_y = \mu\kappa dA\left[\left(-y + \frac{\partial \psi}{\partial x}\right)\frac{dA_x}{dA} + \left(x + \frac{\partial \psi}{\partial y}\right)\frac{dA_y}{dA}\right], \tag{7.96}$$

has to vanish at the rod's surface(s), i.e. at a cross-section's border. The coordinates $\{x, y\}$ of any point at the border may be considered as unique functions, $x(l)$ and $y(l)$, of the arc l of that line – see Fig. 11.

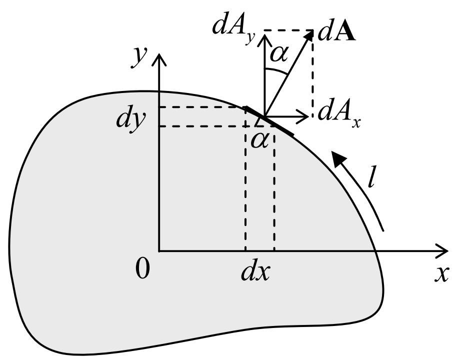

Fig. 7.11. Deriving Eq. (99).

As this sketch shows, the elementary area ratios participating in Eq. (96) may be readily expressed via the derivatives of these functions: $dA_x/dA = \sin\alpha = dy/dl$, $dA_y/dA = \cos\alpha = -dx/dl$, so we may write

$$\left[\left(-y + \frac{\partial \psi}{\partial x}\right)\left(\frac{dy}{dl}\right) + \left(x + \frac{\partial \psi}{\partial y}\right)\left(-\frac{dx}{dl}\right)\right]_{\text{border}} = 0. \tag{7.97}$$

Introducing, instead of ψ, a new function $\chi(x,y)$, defined by its derivatives as

$$\frac{\partial \chi}{\partial x} \equiv \frac{1}{2}\left(-x - \frac{\partial \psi}{\partial y}\right), \qquad \frac{\partial \chi}{\partial y} \equiv \frac{1}{2}\left(-y + \frac{\partial \psi}{\partial x}\right), \tag{7.98}$$

we may rewrite Eq. (97) as

$$2\left(\frac{\partial \chi}{\partial y}\frac{dy}{dl} + \frac{\partial \chi}{\partial x}\frac{dx}{dl}\right)_{\text{border}} \equiv 2\frac{d\chi}{dl}\Big|_{\text{border}} = 0, \tag{7.99}$$

so the function χ has to be constant at each border of the cross-section.

In particular, for a singly-connected cross-section, limited to just one continuous border line (as in Fig. 11), this constant is arbitrary, because according to Eqs. (98), its choice does not affect the longitudinal deformation function $\psi(x,y)$ and hence the deformation as a whole. Now let us use the definition (98) of $\chi(x, y)$ to calculate the 2D Laplace operator of this function:

$$\nabla^2_{x,y}\chi \equiv \frac{\partial^2\chi}{\partial^2 x} + \frac{\partial^2\chi}{\partial^2 y} = \frac{1}{2}\frac{\partial}{\partial x}\left(-x - \frac{\partial\psi}{\partial y}\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(-y + \frac{\partial\psi}{\partial y}\right) \equiv -1. \qquad (7.100)$$

This is a 2D *Poisson equation* (frequently met, for example, in electrostatics), but with a very simple, constant right-hand side. Plugging Eqs. (98) into Eqs. (95), and those into Eq. (88), we may express the torque τ_z, and hence the torsional rigidity C, via the same function:

$$C \equiv \frac{\tau_z}{\kappa} = -2\mu\int_A\left(x\frac{\partial\chi}{\partial x} + y\frac{\partial\chi}{\partial y}\right)dxdy. \qquad (7.101a)$$

C for arbitrary cross-section

Sometimes, it is easier to use this result in either of its two different forms. The first of them may be readily obtained from Eq. (101a) using the integration by parts:

$$C = -2\mu\left(\int dy\int xd\chi + \int dx\int yd\chi\right) = -2\mu\left[\int dy\left(x\chi_{\text{border}} - \int\chi dx\right) + \int dx\left(y\chi_{\text{border}} - \int\chi dy\right)\right]$$
$$= 4\mu\left[\int_A \chi\, dxdy - \chi_{\text{border}}\int_A dxdy\right], \qquad (7.101b)$$

while the proof of one more form,

$$C = 4\mu\int_A\left(\nabla_{x,y}\chi\right)^2 dxdy, \qquad (7.101c)$$

is left for the reader's exercise. Thus, if we need to know the rod's rigidity alone, it is sufficient to calculate the function $\chi(x, y)$ from Eq. (100) with the boundary condition $\chi|_{\text{border}}$ = const, and then plug it into any of Eqs. (101). Only if we are also curious about the longitudinal deformation (93) of the cross-section, we may continue by using Eq. (98) to find the function $\psi(x,y)$ describing this deformation.

Let us see how this recipe works for the two examples discussed above. For the round cross-section of radius R, both the Poisson equation (100) and the boundary condition, χ = const at $x^2 + y^2 = R^2$, are evidently satisfied by the following axially symmetric function:

$$\chi = -\frac{1}{4}(x^2 + y^2) + \text{const}. \qquad (7.102)$$

For this case, Eq. (101a) yields

$$C = 4\mu\int_A\left[\left(-\frac{1}{2}x\right)^2 + \left(-\frac{1}{2}y\right)^2\right]dxdy = \mu\int_A\left(x^2 + y^2\right)d^2r, \qquad (7.103)$$

i.e. the same result (89) that we had for $\psi = 0$. Indeed, plugging Eq. (102) into Eqs. (98), we see that in this case $\partial\psi/\partial x = \partial\psi/\partial y = 0$, so $\psi(x,y)$ = const, i.e. the cross-section is not bent. (As was discussed in Sec. 1, a uniform translation $dq_z = \kappa\psi$ = const does not constitute a deformation.)

Now, turning to a rod with a narrow rectangular cross-section $A = w\times t$ with $t \ll w$, we may use this strong inequality to solve the Poisson equation (100) approximately, neglecting the second derivative of χ along the wider dimension (say, y). The remaining 1D differential equation $d^2\chi/d^2x = -1$, with boundary conditions $\chi|_{x=+t/2} = \chi|_{x=-t/2}$, has an obvious solution: $\chi = -x^2/2 + \text{const}$. Plugging this expression into any form of Eq. (101), we get the following (correct!) result for the torsional rigidity:

$$C = \frac{1}{3}\mu w t^3. \tag{7.104}$$

Now let us have a look at the cross-section bending law (93) for this particular case. Using Eqs. (98), we get

$$\frac{\partial \psi}{\partial y} = -x - 2\frac{\partial \chi}{\partial x} = x, \qquad \frac{\partial \psi}{\partial x} = y + 2\frac{\partial \chi}{\partial y} = y. \tag{7.105}$$

Integrating these differential equations over the cross-section, and taking the integration constant (again, not contributing to the deformation) for zero, we get a beautifully simple result:

$$\psi = xy, \qquad \text{i.e. } q_z = \kappa xy\,. \tag{7.106}$$

It means that the longitudinal deformation of the rod has a "propeller bending" form: while the regions near the opposite corners (on the same diagonal) of the cross-section bend toward one direction of the z-axis, the corners on the other diagonal bend in the opposite direction. (This qualitative conclusion remains valid for rectangular cross-sections with any "aspect ratio" t/w.)

For rods with several surfaces, i.e. with cross-sections limited by several boundaries (say, hollow pipes), finding the function $\chi(x, y)$ requires a bit more care and Eq. (103b) has to be modified because the function may be equal to a different constant at each boundary. Let me leave the calculation of the torsional rigidity for this case for the reader's exercise.

7.7. 3D acoustic waves

Now moving from the statics to dynamics, we may start with Eq. (24), which may be transformed into the vector form exactly as this was done for the static case at the beginning of Sec. 4. Comparing Eqs. (24) and (52), we immediately see that the result may be represented as

Elastic continuum: dynamics

$$\rho\frac{\partial^2 \mathbf{q}}{\partial t^2} = \frac{E}{2(1+\nu)}\nabla^2\mathbf{q} + \frac{E}{2(1+\nu)(1-2\nu)}\nabla(\nabla\cdot\mathbf{q}) + \mathbf{f}(\mathbf{r},t)\,. \tag{7.107}$$

Let us use this general equation for the analysis of the perhaps most important type of time-dependent deformations: *acoustic waves*. First, let us consider the simplest case of a virtually infinite, uniform elastic medium, with no external forces: $\mathbf{f} = 0$. In this case, due to the linearity and homogeneity of the equation of motion, and taking clues from the analysis of the simple 1D model (see Fig. 6.4a) in Secs. 6.3-6.5,[32] we may look for a particular time-dependent solution in the form of a sinusoidal, *linearly polarized*, *plane traveling wave*

Plane, sinusoidal wave

$$\mathbf{q}(\mathbf{r},t) = \text{Re}\left[\mathbf{a}e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\right], \tag{7.108}$$

[32] Note though that Eq. (107) is more complex than the simple wave equation (6.40).

where **a** is the constant complex amplitude of a wave (now a vector!), and **k** is the *wave vector*, whose magnitude is equal to the wave number k. The direction of these two vectors should be clearly distinguished: while **a** determines the wave's *polarization*, i.e. the direction of particle displacements, the vector **k** is directed along the spatial gradient of the full phase of the wave

$$\Psi \equiv \mathbf{k}\cdot\mathbf{r} - \omega t + \arg a\,, \tag{7.109}$$

i.e. along the direction of the wave front propagation.

The importance of the angle between these two vectors may be readily seen from the following simple calculation. Let us point the z-axis of an (inertial) reference frame along the direction of vector **k,** and the x-axis in such direction that the vector **q**, and hence **a**, lie within the $\{x, z\}$ plane. In this case, all variables may change only along the z-axis, i.e. $\nabla = \mathbf{n}_z(\partial/\partial z)$, and the amplitude vector may be represented as the sum of just two Cartesian components:

$$\mathbf{a} = a_x\mathbf{n}_x + a_z\mathbf{n}_z\,. \tag{7.110}$$

Let us first consider a *longitudinal* wave, with the particle motion along the wave direction: $a_x = 0$, $a_z = a$. Then the vector **q** in Eq. (107) describing this wave, has only one (z-) component, so $\nabla\cdot\mathbf{q} = \partial q_z/\partial z$ and $\nabla(\nabla\cdot\mathbf{q}) = \mathbf{n}_z(\partial^2\mathbf{q}/\partial z^2)$, and the Laplace operator gives the same expression: $\nabla^2\mathbf{q} = \mathbf{n}_z(\partial^2\mathbf{q}/\partial z^2)$. As a result, Eq. (107), with $\mathbf{f} = 0$, is reduced to a 1D wave equation

$$\rho\frac{\partial^2 q_z}{\partial t^2} = \left[\frac{E}{2(1+\nu)} + \frac{E}{2(1+\nu)(1-2\nu)}\right]\frac{\partial^2 q_z}{\partial z^2} \equiv \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\frac{\partial^2 q_z}{\partial z^2}\,, \tag{7.111}$$

similar to Eq. (6.40). As we already know from Sec. 6.4, this equation is indeed satisfied with the solution (108), provided that ω and k obey a linear dispersion relation, $\omega = v_l k$, with the following longitudinal wave velocity:

Longitudinal waves: velocity

$$v_l^2 = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho} \equiv \frac{K + (4/3)\mu}{\rho}. \tag{7.112}$$

The last expression allows for a simple interpretation. Let us consider a static experiment, similar to the tensile test experiment shown in Fig. 6, but with a sample much wider than l in both directions perpendicular to the force. Then the lateral contraction is impossible ($s_{xx} = s_{yy} = 0$), and we can calculate the only finite stress element, σ_{zz}, directly from Eq. (34) with Tr (s) = s_{zz}:

$$\sigma_{zz} = 2\mu\left(s_{zz} - \frac{1}{3}s_{zz}\right) + 3K\left(\frac{1}{3}s_{zz}\right) \equiv \left(K + \frac{4}{3}\mu\right)s_{zz}\,. \tag{7.113}$$

We see that the numerator in Eq. (112) is nothing more than the static elastic modulus for such a uniaxial deformation, and it is recalculated into the velocity exactly as the spring constant in the 1D waves considered in Secs. 6.3-6.4 – cf. Eq. (6.42).

Formula (114) becomes especially simple in fluids, where $\mu = 0$, and the wave velocity is described by the well-known expression

Longitudinal waves: velocity in fluids

$$v_l = \left(\frac{K}{\rho}\right)^{1/2}. \tag{7.114}$$

Note, however, that for gases, with their high compressibility and temperature sensitivity, the value of K participating in this formula may differ, at high frequencies, from that given by Eq. (40), because fast compressions/extensions of gas are usually adiabatic rather than isothermal. This difference is noticeable in Table 1, one of whose columns lists the values of v_l for representative materials.

Now let us consider an opposite case of *transverse* waves with $a_x = a$, $a_z = 0$. In such a wave, the displacement vector is perpendicular to $\mathbf{n}_z$, so $\nabla\cdot\mathbf{q} = 0$, and the second term on the right-hand side of Eq. (107) vanishes. On the contrary, the Laplace operator acting on such vector still gives the same non-zero contribution $\nabla^2\mathbf{q} = n_z(\partial^2\mathbf{q}/\partial z^2)$ to Eq. (107), so the equation yields

$$\rho\frac{\partial^2 q_x}{\partial t^2} = \frac{E}{2(1+\nu)}\frac{\partial^2 q_x}{\partial z^2}, \tag{7.115}$$

and we again get the linear dispersion relation, $\omega = v_t k$, but with a different velocity:[33]

Transverse waves: velocity

$$v_t^2 = \frac{E}{2(1+\nu)\rho} = \frac{\mu}{\rho}. \tag{7.116}$$

We see that the speed of the transverse waves depends exclusively on the shear modulus μ of the medium.[34] This is also very natural: in such waves, the particle displacements $\mathbf{q} = \mathbf{n}_x q$ are perpendicular to the elastic forces $d\mathbf{F} = \mathbf{n}_z dF$, so only one element σ_{xz} of the stress tensor is involved. Also, the strain tensor $s_{jj'}$ has no diagonal elements, Tr (s) = 0, so μ is the only elastic modulus actively participating in Hooke's law (32). In particular, fluids cannot carry transverse waves at all (formally, their velocity (116) vanishes), because they do not resist shear deformations. For all other materials, the longitudinal waves are faster than the transverse ones.[35] Indeed, for all known natural materials' Poisson's ratio is positive so the velocity ratio that follows from Eqs. (112) and (116),

$$\frac{v_l}{v_t} = \left(\frac{2-2\nu}{1-2\nu}\right)^{1/2}, \tag{7.117}$$

is above $\sqrt{2} \approx 1.4$. For the most popular construction materials, with $\nu \approx 0.3$, Poisson's ratio is about 2 – see Table 1.

Let me emphasize again that for both the longitudinal and the transverse waves, the dispersion relation between the wave number and frequency is linear: $\omega = vk$. As was already discussed in Chapter 6, in this case of *acoustic waves* (or just "sound"), the phase and group velocities are equal, and waves of more complex form, consisting of several (or many) Fourier components of the type (108), preserve

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[33] Just as in Chapter 6, let me emphasize that the *wave* velocities we are discussing in this section and Sec. 8 below have nothing to do with *particle* velocities $\partial\mathbf{q}/\partial t$. For example, in the transverse wave we are discussing now, v_t is the velocity in the z-direction, while the particles of the medium move across it, in the x-direction. Also, v_l and v_t do not depend on the wave amplitudes, while the particle velocities are proportional to them.

[34] Because of that, one can frequently meet the term *shear waves*. Note also that in contrast to the transverse waves in the simple 1D model analyzed in Chapter 6 (see Fig. 6.4a), those in a 3D continuum do not need a pre-stretch tension $\mathscr{T}$. We will return to the effect of tension in the next section.

[35] Because of this difference between v_l and v_t, in geophysics, the longitudinal waves are known as *P-waves* (with the letter P standing for "primary") because they arrive at the detection site, say from an earthquake, first – before the transverse waves, called the *S-waves*, with S standing for "secondary". (An alternative, also quite logical, decoding of these abbreviations is "pressure waves" and "shear waves".)

their form during propagation. This means that both Eqs. (111) and (115) are satisfied by solutions of the type (6.41):

$$q_{\pm}(z,t) = f_{\pm}\left(t \mp \frac{z}{v}\right), \tag{7.118}$$

where the functions $f_{\pm}$ describe the propagating waveforms. (However, if the initial wave is a mixture, of the type (110), of the longitudinal and transverse components, then these components, propagating with different velocities, will "run from each other".) As one may infer from the analysis of a periodic system model in Chapter 6, the wave dispersion becomes essential at very high (*hypersound*) frequencies where the wave number k becomes close to the reciprocal distance d between the particles of the medium (e.g., atoms or molecules), and hence the approximation of the medium as a continuum, used through this chapter, becomes invalid.

As we already know from Chapter 6, besides the velocity, the waves of each type are characterized by one more important parameter, the wave impedance Z – for acoustic waves frequently called the *acoustic impedance* of the medium. Generalizing Eq. (6.46) to the 3D case, we may define the impedance as the ratio of the force *per unit area* (i.e. the corresponding element of the stress tensor) exerted by the wave, and the particles' velocity. For the longitudinal waves,

$$Z_l \equiv \left|\frac{\sigma_{zz}}{\partial q_z/\partial t}\right| = \left|\frac{\sigma_{zz}}{s_{zz}}\frac{s_{zz}}{\partial q_z/\partial t}\right| = \left|\frac{\sigma_{zz}}{s_{zz}}\frac{\partial q_z/\partial z}{\partial q_z/\partial t}\right|. \tag{7.119}$$

Plugging in Eqs. (108), (112), and (113), we get

Longitudinal waves: impedance

$$Z_l = \left[(K + 4\mu/3)\rho\right]^{1/2}, \tag{7.120}$$

in a clear analogy with the first of Eqs. (6.48). Similarly, for the transverse waves, the appropriately modified definition, $Z_t \equiv |\sigma_{xz}/(\partial q_x/\partial z)|$, yields

Transverse waves: impedance

$$Z_t = (\mu\rho)^{1/2}. \tag{7.121}$$

Just like in the 1D models studied in Chapter 6, one role of the wave impedance is to scale the power $\mathscr{P}$ carried by the wave. For plane 3D waves in infinite media, with their infinite wave front area, it is more appropriate to speak about the *power density*, i.e. power $\wp = d\mathscr{P}/dA$ per unit area of the front, and characterize it by not only its magnitude,

$$\wp = \frac{d\mathbf{F}}{dA}\cdot\frac{\partial \mathbf{q}}{\partial t}, \tag{7.122}$$

but also the direction of the energy propagation, that (for a plane acoustic wave in an isotropic medium) coincides with the direction of the wave vector: $\boldsymbol{\wp} \equiv \wp\mathbf{n}_k$. Using the definition (18) of the stress tensor, the Cartesian components of this *Umov vector*[36] may be expressed as

[36] Named after N. A. Umov, who introduced this concept in 1874 – ten years before a similar notion for electromagnetic waves (see, e.g., EM Sec. 6.4) was suggested by J. Poynting. In a dissipation-free elastic medium, the Umov vector obeys the continuity equation $\partial(\rho v^2/2 + u)/\partial t + \nabla\cdot\boldsymbol{\wp} = 0$, with u given by Eq. (52), which expresses the conservation of the total (kinetic plus potential) energy of the elastic deformation.

$$\mathscr{p}_j = \sum_{j'} \sigma_{jj'} \frac{\partial q_{j'}}{\partial t}. \tag{7.123}$$

Returning to plane waves propagating along axis z, and acting exactly like in Sec. 6.4, for both the longitudinal and transverse waves we again arrive at Eq. (6.49), but for $\mathscr{p}$ rather than $\mathscr{P}$ (due to a different definition of the wave impedance – per unit area rather than per particle chain). For the sinusoidal waves of the type (108), it yields

$$\mathscr{p}_z = \frac{\omega^2 Z}{2} a a^*, \tag{7.124}$$

with Z being the corresponding impedance – either Z_l or Z_t.

Just as in the 1D case, one more important effect, in which the notion of impedance is crucial, is the partial *wave reflection* from at an interface between two media. The two boundary conditions, necessary for the analysis of the reflection, may be obtained from the continuity of the vectors **q** and $d\mathbf{F}$. (The former condition is evident, while the latter one may be obtained by applying the 2nd Newton law to any infinitesimal volume $dV = dAdz$, where the segment dz straddles the interface.) Let us start from the simplest case of the *normal incidence* on a plane interface between two uniform media, each with its own elastic moduli and mass density. Due to the symmetry of the system, it is obvious that the longitudinal/transverse incident wave may only excite similarly polarized reflected and transferred waves. As a result, we may literally repeat the calculations of Sec. 6.4, again arriving at the fundamental relations (6.55) and (6.56), with the replacement of Z and Z' with the corresponding values of either Z_l (120) or Z_t (121). Thus, at the normal incidence, the wave reflection is determined solely by the acoustic *impedances* of the media, while the sound *velocities* are not involved.

The situation, however, becomes more complicated at a nonzero incidence angle $\theta^{(i)}$ (Fig. 12), where the transmitted wave is generally also *refracted*, i.e. propagates under a different angle, $\theta' \neq \theta^{(i)}$, beyond the interface. Moreover, at $\theta^{(i)} \neq 0$ the directions of particle motion (vector **q**) and of the stress forces (vector $d\mathbf{F}$) in the incident wave are neither exactly parallel nor exactly perpendicular to the interface, and thus this wave may serve as an actuator for the reflected and refracted waves of both polarizations – see Fig. 12, drawn for the particular case when the incident wave is transverse. The corresponding four angles, $\theta_t^{(r)}$, $\theta_t^{(r)}$, θ'_l, θ'_t, may be readily related to $\theta_t^{(i)}$ by the "kinematic" condition that the incident wave, as well as the reflected and refracted waves of both types, must have the same spatial distribution along the interface plane, i.e. for the interface particles participating in all five waves. According to Eq. (108), the necessary boundary condition is the equality of the tangential components (in Fig. 12, k_x), of all five wave vectors:

$$k_t \sin\theta_t^{(r)} = k_l \sin\theta_l^{(r)} = k'_l \sin\theta'_l = k'_t \sin\theta'_t = k_x \equiv k_t \sin\theta_t^{(i)}. \tag{7.125}$$

Since the acoustic wave vector magnitudes k, at fixed frequency ω, are inversely proportional to the corresponding wave velocities, we immediately get the following relations:

$$\theta_t^{(r)} = \theta_t^{(i)}, \quad \frac{\sin\theta_l^{(r)}}{v_l} = \frac{\sin\theta'_l}{v'_l} = \frac{\sin\theta'_t}{v'_t} = \frac{\sin\theta_t^{(i)}}{v_t}, \tag{7.126}$$

so generally all four angles are different. (This is of course an analog of the well-known *Snell law* in optics – where, however, only transverse waves are possible.) These relations show that, just like in optics, the direction of a wave propagating into a medium with lower velocity is closer to the normal (in

Fig. 12, to the z-axis). In particular, this means that if $v' > v$, the acoustic waves, at larger angles of incidence, may exhibit the effect of total internal reflection, so well known from optics[37], when the *refracted* wave vanishes. In addition, Eqs. (126) show that in acoustics, the *reflected* longitudinal wave, with velocity $v_l > v_t$, may vanish at sufficiently large angles of the transverse wave incidence.

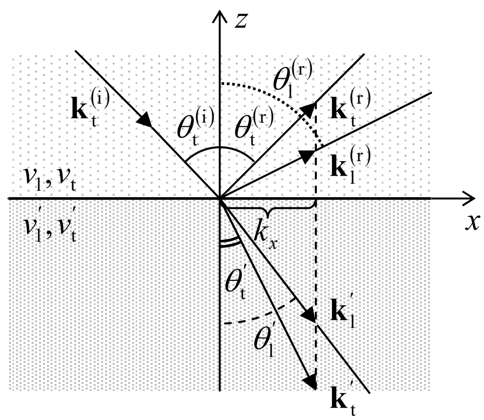

Fig. 7.12. Deriving the "kinematic" conditions (126) of the acoustic wave reflection and refraction (for the case of a transverse incident wave).

All these facts automatically follow from general expressions for amplitudes of the reflected and refracted waves via the amplitude of the incident wave. These relations are straightforward to derive (again, from the continuity of the vectors **q** and $d\mathbf{F}$), but since they are much bulkier than those in the electromagnetic wave theory (where they are called the *Fresnel formulas*[38]), I would not have time/space to spell out and discuss them. Let me only note that, in contrast to the case of normal incidence, these relations involve eight media parameters: the impedances Z, Z', and the velocities v, v' on both sides of the interface, and for both the longitudinal and transverse waves.

There are other interface effects as well. Within certain frequency ranges, interfaces and surfaces of elastic solids may sustain so-called *surface acoustic waves* (SAW), in particular, the *Rayleigh waves* and the *Love waves*.[39] The main feature that distinguishes such waves from their *bulk* (longitudinal and transverse) counterparts discussed above, is that the displacement amplitudes are largest at the interface and decay exponentially into the bulk of both adjacent media, so the waves cannot be plane in the usual sense of being independent of two Cartesian coordinates.

For an analysis of such waves, it is important that in a uniform medium, even non-plane elastic waves may be always separated into independent longitudinal and transverse components. Indeed, it is straightforward (and hence left for the reader) to prove that Eq. (107) may be satisfied by a vector sum $\mathbf{q}(\mathbf{r}, t) = \mathbf{q}_l(\mathbf{r}, t) + \mathbf{q}_t(\mathbf{r}, t)$, with the former component having zero curl ($\nabla\times\mathbf{q}_l = 0$) and propagating with the velocity (112), and the latter component having zero divergence ($\nabla\cdot\mathbf{q}_t = 0$) and propagating with the velocity (116). The plane waves $q_l\mathbf{n}_z$ and $q_t\mathbf{n}_x$ analyzed above certainly fall into these two categories, but in more general waves, there may be no clear association between the longitudinal and transverse components and their polarization.

This is true, in particular, in the Rayleigh waves, where the particle displacement vector **q** may be represented as the sum $\mathbf{q}_l + \mathbf{q}_t$, each of the vectors having more than one Cartesian component. In

[37] See, e.g., EM Sec. 7.5.
[38] Their discussion may be also found in EM Sec. 7.5.
[39] Named, respectively, after Lord Rayleigh (born J. Strutt, 1842-1919) who has theoretically predicted the very existence of surface acoustic waves, and A. Love (1863-1940).

contrast to the bulk waves, the longitudinal and transverse components are coupled via their interaction with the interface, and as a result, propagate with a single velocity v_R. A straightforward analysis of the Rayleigh waves on the *surface* of an elastic solid (i.e. its interface with free space) yields the following equation for v_R:

$$\left(1-\frac{v_R^2}{2v_t^2}\right)^4 = \left(1-\frac{v_R^2}{v_t^2}\right)\left(1-\frac{v_R^2}{v_l^2}\right). \tag{7.127}$$

According to this formula, and Eqs. (112) and (116), for realistic materials with the Poisson index between 0 and ½, the Rayleigh waves are slightly (by 4 to 13%) slower than the bulk transverse waves – and hence substantially slower than the bulk longitudinal waves.

In the simplest case a "1D-plane" Rayleigh wave, independent of one Cartesian coordinate, the net vector **q** has just two Cartesian components (each contributed by $\mathbf{q}_l$ and $\mathbf{q}_t$): one parallel to the propagation direction and hence to the interface, and another one normal to it. As a result, the trajectory of each particle in the wave is an ellipse in the plane normal to the interface. In contrast, the Love waves are purely transverse, with **q** oriented parallel to the interface. However, the interaction of these waves with the interface reduces their velocity v_L in comparison with that (v_t) of the bulk transverse waves, keeping it within the narrow interval between v_t and v_R:

$$v_R < v_L < v_t < v_l\,. \tag{7.128}$$

The practical importance of surface acoustic waves is that their amplitude decays very slowly with distance r from their point-like source: $a \propto 1/r^{1/2}$, while any bulk waves decay much faster, as $a \propto 1/r$. (Indeed, in the latter case the power $\mathscr{P} \propto a^2$, emitted by such source, is distributed over a spherical surface area proportional to r^2, while in the former case all the power goes into a thin surface circle whose length scales as r.) At least two areas of applications of the surface acoustic waves have to be mentioned: geophysics (for earthquake detection and the Earth crust seismology), and electronics (for signal processing, with a focus on frequency filtering). Unfortunately, I cannot dwell on these interesting topics and I have to refer the reader to special literature.[40]

7.8. Elastic waves in restricted geometries

From what was discussed at the end of the last section, it should be pretty clear that generally, the propagation of acoustic waves in elastic bodies of finite size is rather complicated. There is, however, one important limit in which several important simple results may be readily obtained. This is the limit of (relatively) low frequencies, where the corresponding wavelength is much larger than at least one dimension of the system.

Let us consider, for example, various waves that may propagate along thin rods, in this case "thin" meaning that the characteristic size a of the rod's cross-section is much smaller than not only the length of the rod but also the wavelength $\lambda = 2\pi/k$. In this case, there is a considerable range Δz of distances along the rod,

$$a << \Delta z << \lambda\,, \tag{7.129}$$

[40] See, for example, K. Aki and P. Richards, *Quantitative Seismology*, 2nd ed., University Science Books, 2002; and D. Morgan, *Surface Acoustic Waves*, 2nd ed., Academic Press, 2007.

in that we can neglect the material's inertia, and apply the results of our earlier static analyses. For example, for a *longitudinal* wave of stress, which is essentially a wave of periodic tensile extensions and compressions of the rod, within the range (129) we may use the static relation (42):

$$\sigma_{zz} = E s_{zz}. \tag{7.130}$$

In this simple case, it is easier to use the general equation of elastic dynamics not in its vector form (107), but rather in the precursor, Cartesian-component form (25), with $f_j = 0$. For the plane waves of stress, propagating along the z-axis, only one component (with $j' \to z$) of the sum on the right-hand side of that equation is not equal to zero, and it is reduced to

$$\rho \frac{\partial^2 q_j}{\partial^2 t^2} = \frac{\partial \sigma_{jz}}{\partial z}. \tag{7.131}$$

In our current case of longitudinal waves, all components of the stress tensor but σ_{zz} are equal to zero. With σ_{zz} from Eq. (130), and using the definition $s_{zz} = \partial q_z/\partial z$, Eq. (131) is reduced to a simple 1D wave equation,

$$\rho \frac{\partial^2 q_z}{\partial^2 t^2} = E \frac{\partial^2 q_z}{\partial z^2}, \tag{7.132}$$

which shows that the velocity of such *tensile waves* is

$$v = \left(\frac{E}{\rho} \right)^{1/2}. \tag{7.133}$$

Tensile waves: velocity

Comparing this result with Eq. (112), we see that the tensile wave velocity, for any realistic material with a positive Poisson's ratio, is lower than the velocity v_l of longitudinal waves in the bulk of the same material. The reason for this difference is simple: in thin rods, the cross-section is free to oscillate (e.g., shrink in the longitudinal extension phase of the passing wave),[41] so the effective force resisting the longitudinal deformation is smaller than in a border-free space. Since (as it is clearly visible from the wave equation), the scale of the force determines that of v^2, this difference translates into slower waves in rods. Of course, as the wave frequency is increased to $ka \sim 1$, there is a (rather complicated and cross-section-depending) crossover from Eq. (133) to Eq. (112).

Proceeding to *transverse* waves on rods, let us first have a look at long *bending waves* for which the condition (129) is satisfied, so the vector $\mathbf{q} = \mathbf{n}_x q_x$ (with the x-axis being the bending direction – see Fig. 8) is virtually constant in the whole cross-section. In this case, the only element of the stress tensor contributing to the net transverse force F_x is σ_{xz}, so the integral of Eq. (131) over the cross-section is

$$\rho A \frac{\partial^2 q_x}{\partial t^2} = \frac{\partial F_x}{\partial z}, \qquad \text{with } F_x = \int_A \sigma_{xz} d^2 r. \tag{7.134}$$

Now, if Eq. (129) is satisfied, we again may use the static local relations (75)-(77), with all derivatives d/dz duly replaced with their partial form $\partial/\partial z$, to express the force F_x via the bending deformation q_x. Plugging these relations into each other one by one, we arrive at a rather unusual differential equation

[41] For this reason, the tensile waves can be called longitudinal only in a limited sense: while the stress wave is purely longitudinal: $\sigma_{xx} = \sigma_{yy} = 0$, the strain wave is not: $s_{xx} = s_{yy} = -\sigma s_{zz} \neq 0$, i.e. $\mathbf{q}(\mathbf{r}, t) \neq \mathbf{n}_z q_z$.

$$\rho A \frac{\partial^2 q_x}{\partial t^2} = -EI_y \frac{\partial^4 q_x}{\partial z^4}. \tag{7.135}$$

Looking for its solution in the form of a sinusoidal wave (108), we get the following dispersion relation:[42]

Bending waves: dispersion relation

$$\omega^2 = \frac{EI_y}{\rho A} k^4. \tag{7.136}$$

This relation means that the bending waves are not acoustic at *any* frequency, and cannot be characterized by a single velocity that would be valid for all wave numbers k, i.e. for all spatial Fourier components of a waveform. According to our discussion in Sec. 6.3, such *strongly dispersive* systems cannot pass non-sinusoidal waveforms too far without changing their waveform rather considerably.

This situation changes, however, if the rod is pre-stretched with a tension force $\mathscr{T}$ – just as in the discrete 1D model that was analyzed in Sec. 6.3. The calculation of the effect of this force is essentially similar; let us repeat it for the continuous case, for a minute neglecting the bending stress – see Fig. 13.

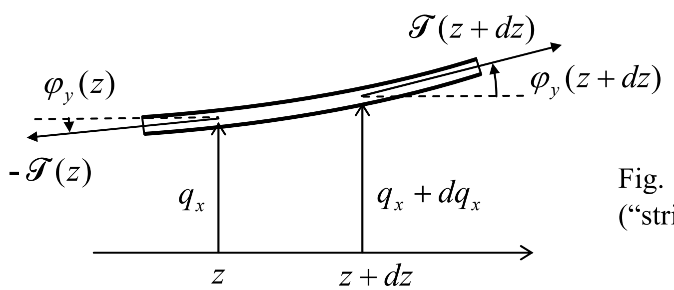

Fig. 7.13. Additional forces in a thin rod ("string"), due to the background tension $\mathscr{T}$.

Still sticking to the limit of small angles φ, the additional vertical component $d\mathscr{T}_x$ of the net force acting on a small rod fragment of length dz is $\mathscr{T}_x(z-dz) - \mathscr{T}_x(z) = \mathscr{T}\varphi_y(z+dz) - \mathscr{T}\varphi_y(z) \approx \mathscr{T}(\partial\varphi_y/\partial z)dz$, so $\partial F_x/\partial z = \mathscr{T}(\partial\varphi_y/\partial z)$. With the geometric relation (77) in its partial-derivative form $\partial q_x/\partial z = \varphi_y$, this additional term becomes $\mathscr{T}(\partial^2 q_x/\partial z^2)$. Now adding it to the right-hand side of Eq. (135), we get the following dispersion relation

$$\omega^2 = \frac{1}{\rho A}\left(EI_y k^4 + \mathscr{T}k^2\right). \tag{7.137}$$

Since the product ρA in the denominator of this expression is just the rod's mass per unit length (which was denoted μ in Chapter 6), at low k (and hence low frequencies), this expression is reduced to the linear dispersion law, with the velocity given by Eq. (6.43):

$$v = \left(\frac{\mathscr{T}}{\rho A}\right)^{1/2}. \tag{7.138}$$

So Eq. (137) describes a smooth crossover from the "guitar-string" acoustic waves to the highly dispersive bending waves (136).

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[42] Note that since the "moment of inertia" I_y, defined by Eq. (70), may depend on the bending direction (unless the cross-section is sufficiently symmetric), the dispersion relation (136) may give different results for different directions of the bending wave polarization.

Now let us consider another type of transverse waves in thin rods – the so-called *torsional waves*, which are essentially the dynamic propagation of the torsional deformation discussed in Sec. 6. The easiest way to describe these waves, again within the limits (129), is to write the equation of rotation of a small segment dz of the rod about the z-axis, passing through the "center of mass" of its cross-section, under the difference of torques $\boldsymbol{\tau} = \mathbf{n}_z \tau_z$ applied on its ends – see Fig. 10:

$$\rho I_z dz \frac{\partial^2 \varphi_z}{\partial t^2} = d\tau_z, \qquad (7.139)$$

where I_z is the "moment of inertia" defined by Eq. (91), which now, after its multiplication by ρdz, i.e. by the mass per unit area, has turned into the genuine moment of inertia of a dz-thick slice of the rod. Dividing both sides of Eq. (139) by dz, and using the static local relation (84), $\tau_z = C\kappa = C(\partial\varphi_z/\partial z)$, we get the following differential equation

$$\rho I_z \frac{\partial^2 \varphi_z}{\partial t^2} = C \frac{\partial^2 \varphi_z}{\partial z^2}. \qquad (7.140)$$

Just as Eqs. (111), (115), and (132), this equation describes an acoustic (dispersion-free) wave, which propagates with the following frequency-independent velocity

$$v = \left(\frac{C}{\rho I_z}\right)^{1/2}. \qquad (7.141)$$

Torsional waves: velocity

As we have seen in Sec. 6, for rods with axially-symmetric cross-sections, the torsional rigidity C is described by the simple relation (89), $C = \mu I_z$, so Eq. (141) is reduced to Eq. (116) for the transverse waves in infinite media. The reason for this similarity is straightforward: in a torsional wave, particles oscillate along small arcs (Fig. 14a), so if the rod's cross-section is round, its surface is stress-free, and does not perturb or modify the motion in any way, and hence does not affect the transverse velocity.

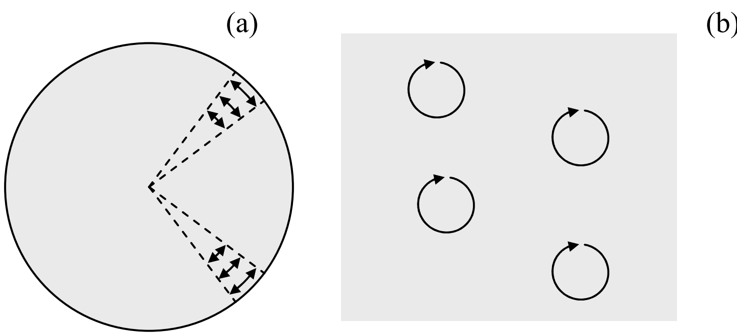


Fig. 7.14. Particle trajectories in two different transverse waves with the same velocity: (a) torsional waves in a thin round rod and (b) circularly polarized waves in an infinite (or very broad) sample.

This fact raises an interesting issue of the relation between the torsional and *circularly polarized* waves. Indeed, in Sec. 7, I have not emphasized enough that Eq. (116) is valid for a transverse wave polarized in any direction perpendicular to the wave vector $\mathbf{k}$ (in our notation, directed along the z-axis). In particular, this means that such waves are doubly degenerate: any isotropic elastic continuum can carry simultaneously two non-interacting transverse waves propagating in the same direction with the same velocity (116), with two mutually perpendicular linear polarizations (directions of the vector $\mathbf{a}$), for example, directed along the x- and y-axes.[43] If both waves are sinusoidal (108), with the same frequency, each point of the medium participates in two simultaneous sinusoidal motions within the [x, y] plane:

$$q_x = \mathrm{Re}\left[a_x e^{i(kz - \omega t)}\right] = A_x \cos\Psi, \qquad q_y = \mathrm{Re}\left[a_y e^{i(kz - \omega t)}\right] = A_y \cos(\Psi + \varphi), \tag{7.142}$$

where $\Psi \equiv kz - \omega t + \varphi_x$, and $\varphi \equiv \varphi_y - \varphi_x$. Basic geometry tells us that the trajectory of such a motion on the [x, y] plane is an ellipse (Fig. 15), so such waves are called *elliptically polarized*. The most important particular cases of such polarization are:

(i) $\varphi = 0$ or π: a *linearly-polarized* wave, with the displacement vector **a** is directed at angle $\theta = \tan^{-1}(A_y/A_x)$ to the x-axis; and

(ii) $\varphi = \pm \pi/2$ and $A_x = A_y$: two possible *circularly-polarized* waves, with the *right* or *left* polarization, respectively.[44]

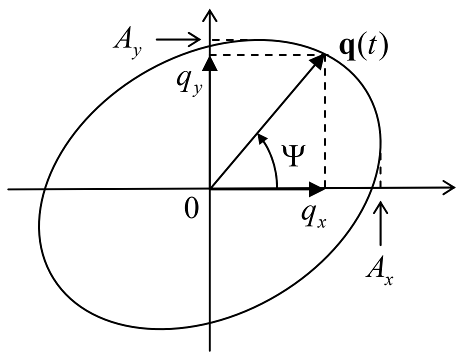


Fig. 7.15. The trajectory of a particle in an elliptically polarized transverse wave, within the plane perpendicular to the direction of wave propagation.

Now comparing the trajectories of particles in the torsional wave in a thin round rod (or pipe) and the circularly polarized wave in a broad sample (Fig. 14), we see that, despite the same wave propagation velocity, these transverse waves are rather different. In the former case (Fig. 14a) each particle moves back and forth along an arc, with the arc's length different for different particles (and vanishing at the rod's center), so the waves are *not* plane. On the other hand, in a circularly polarized wave, all particles move along similar, circular trajectories, so such a wave *is* plane.

To conclude this chapter, let me briefly mention the opposite limit when the size of the body, from whose boundary the waves are completely reflected,[45] is much larger than the wavelength. In this case, the waves propagate almost as in an infinite 3D continuum (which was analyzed in Sec. 7), and the most important new effect is the finite number of wave modes in the body. Repeating the 1D analysis at the end of Sec. 6.5, for each dimension of a 3D cuboid of volume $V = l_1 l_2 l_3$, and taking into account that the numbers k_n in each of the three dimensions are independent, we get the following generalization of

[43] As was discussed in Sec. 6.3, this is also true in the simple 1D model shown in Fig. 6.4a.

[44] The circularly polarized waves play an important role in quantum mechanics, where they may be most naturally quantized, with their elementary excitations (in the case of mechanical waves we are discussing, called *phonons*) having either positive or negative angular momentum $L_z = \pm\hbar$.

[45] For acoustic waves, such a condition is easy to implement. Indeed, from Sec. 7 we already know that the strong inequality of the wave impedances Z is sufficient for such reflection. The numbers in Table 1 show that, for example, the impedance of a longitudinal wave in a typical metal (say, steel) is almost two orders of magnitude higher than that in air, ensuring their virtually full reflection from the surface.

Eq. (6.75) for the number ΔN of different traveling waves with wave vectors within a relatively small volume d^3k of the wave vector space:

$$dN = g\frac{V}{(2\pi)^3}d^3k >> 1, \qquad \text{for } \frac{1}{V} << d^3k << k^3, \tag{7.143}$$

3D density of modes

where $k >>> 1/l_{1,2,3}$ is the center of this volume, and g is the number of different possible wave modes with the same wave vector $\mathbf{k}$. For the mechanical waves analyzed above, with one longitudinal mode, and two transverse modes with different polarizations, $g = 3$.

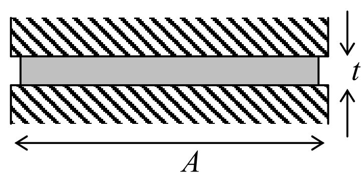
Note that since the derivation of Eqs. (6.75) and (143) does not use other properties of the waves (in particular, their dispersion relations), this mode counting rule is ubiquitous in physics, being valid, in particular, for electromagnetic waves (where $g = 2$) and quantum "de Broglie waves" (i.e. wavefunctions), whose degeneracy factor g is usually determined by the particle's spin.[46]

7.9. Exercise problems

7.1. Derive Eqs. (16).

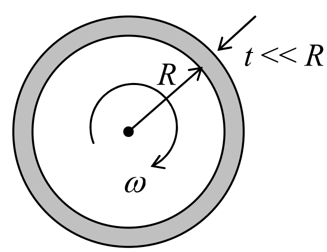
Hint: Besides basic calculus and the definition of the cylindrical coordinates, you may like to use Eq. (4.7) with $d\boldsymbol{\varphi} = (d\varphi)\mathbf{n}_z$.

7.2. A uniform thin sheet of an isotropic elastic material, of thickness t and area $A >> t^2$, is compressed by two plane, parallel, broad, rigid surfaces – see the figure on the right. Assuming that there is no slippage between the sheet and the surfaces, calculate the relative compression $(-\Delta t/t)$ as a function of the compressing force. Compare the result with that for the tensile stress calculated in Sec. 3.



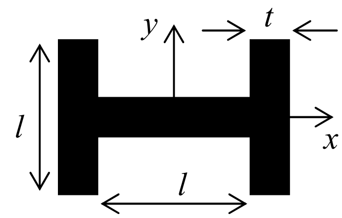
7.3. Two opposite edges of a thin but wide sheet of an isotropic elastic material are clamped in two rigid, plane, parallel walls that are pulled apart with force F, along the sheet's length l. Find the relative extension $\Delta l/l$ of the sheet in the direction of the force and its relative compression $\Delta t/t$ in the perpendicular direction, and compare the results with Eqs. (45)-(46) for the tensile stress and with the solution of the previous problem.

7.4. Calculate the radial extension ΔR of a thin, long, round cylindrical pipe due to its rotation with a constant angular velocity ω about its symmetry axis (see the figure on the right), in terms of the elastic moduli E and ν.

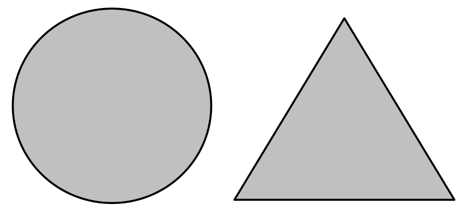


7.5.* A static force $\mathbf{F}$ is exerted on an inner point of a uniform and isotropic elastic body. Calculate the spatial distribution of the deformation created by the force, assuming that far from the point of its application and the points we are interested in, the body's position is kept fixed.

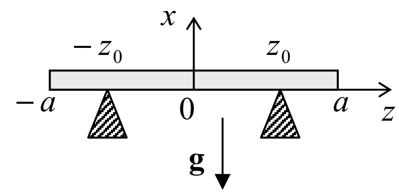
[46] See, e.g., EM Secs. 7.8 and QM Sec. 1.7.



7.6. A long uniform rail with the cross-section shown in the figure on the right is being bent with the same (small) torque twice: first within the xz-plane and then within the yz-plane. Assuming that $t << l$, find the ratio of the bending deformations in these two cases.



7.7. Two thin rods of the same length and mass are made of the same isotropic and elastic material. The cross-section of one of them is a circle, while the other one is an equilateral triangle – see the figure on the right. Which of the rods is stiffer for bending along its length? Quantify the relation. Does the result depend on the bending plane's orientation?

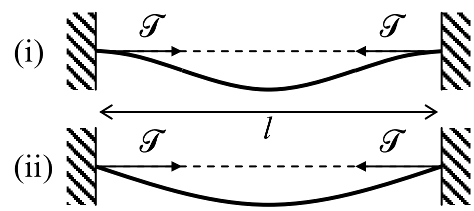


7.8. A thin, uniform, initially straight elastic beam is placed on two point supports at the same height – see the figure on the right. . Calculate the support placements that:

(i) ensure that the beam ends are horizontal, and

(ii) minimize the largest deflection of the beam from the horizontal baseline.

Hint: For Task (ii), an approximate answer (with an accuracy better than 1%) is acceptable.



7.9. Calculate the largest longitudinal compression force $\mathscr{T}$ that may be withstood by a thin, straight, elastic rod without buckling (see the figure on the right) for each of the shown cases:

(i) the rod's ends are clamped, and

(ii) the rod is free to turn about the support points.

7.10. A thin elastic pole with a square cross-section of area $A = a \times a$ is firmly dug into the ground in the vertical position, sticking out by height $h >> a$.

(i) What largest compact mass M may be placed straight on the top of a light pole without stability loss?

(ii) In the absence of such an additional mass, how massive a uniform pole may be to retain its stability?

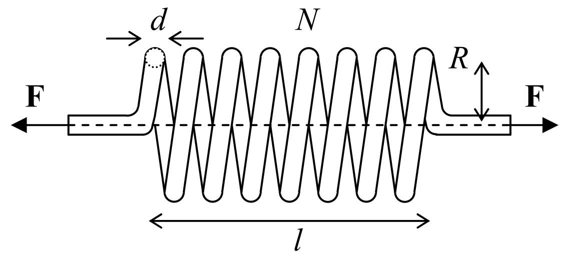
Hint: For Task (ii), you may use the same WKB approximation as in Problem 6.18.

7.11. Calculate the potential energy of a small and slowly changing, but otherwise arbitrary bending deformation of a uniform, initially straight elastic rod. Can the result be used to derive the dispersion relation (136)?

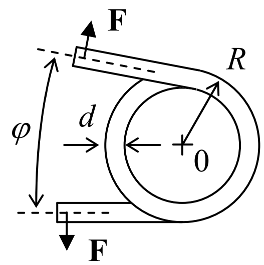
7.12. Calculate the torsional rigidity of a long uniform rod whose cross-section is an ellipse with semi-axes a and b.

7.13. Calculate the potential energy of a small but otherwise arbitrary torsional deformation $\varphi_z(z)$ of a uniform and straight elastic rod.

7.14. Calculate the spring constant $\kappa \equiv dF/dl$ of a coil made of a uniform elastic wire with a circular cross-section of diameter d, wound as a dense round spiral of $N >> 1$ turns of radius $R >> d$ – see the figure on the right.



7.15. The coil studied in the previous problem is now used as what is sometimes called the *torsion spring* – see the figure on the right. Find the corresponding spring constant $d\tau/d\varphi$, where $\boldsymbol{\tau}$ is the torque of the external forces **F** relative to the center of the coil (point 0).



7.16. Use Eqs. (99) and (100) to recast Eq. (101b) for the torsional rigidity C of a thin rod into the form given by Eq. (101c).

7.17.* Generalize Eq. (101b) to the case of rods with more than one cross-section's boundary. Use the result to calculate the torsional rigidity of a thin round pipe, and compare it with Eq. (91).

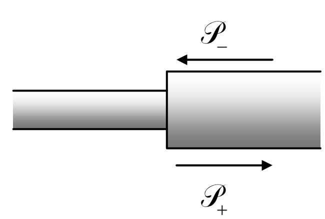
7.18. Prove that in a uniform isotropic medium, an arbitrary (not necessarily plane) elastic wave may be decomposed into a longitudinal wave with $\nabla\times\mathbf{q}_l = 0$ and a transverse wave with $\nabla\cdot\mathbf{q}_t = 0$, and find the equations satisfied by these functions.

7.19.* Use the wave equations derived in the solution of the previous problem and the semi-quantitative description of the Rayleigh surface waves given in Sec. 7 of the lecture notes, to calculate the structure of the waves and to derive Eq. (127).

7.20.* Calculate the modes and frequencies of free radial oscillations of a sphere of radius R, made of a uniform elastic material.

7.21. A long steel wire has a circular cross-section with a 3-mm diameter and is pre-stretched with a constant force of 10 N. Which of the longitudinal and transverse waves with frequency 1 kHz has the largest group velocity in the wire? Accept the following parameters for the steel (see Table 1): E = 170 GPa, $\nu = 0.30$, $\rho = 7.8$ g/cm^3.

7.22. Define and calculate the wave impedances for (i) tensile and (ii) torsional waves in a thin rod, that are appropriate in the long-wave limit. Use the results to calculate the fraction of each wave's power $\mathscr{P}$ reflected from a firm connection of a long rod with a round cross-section to a similar rod with a twice smaller diameter – see the figure on the right.



7.23. Calculate the fundamental frequency of small transverse standing waves on a free uniform thin rod, and the position of displacement nodes in this mode.

Hint: A numerical solution of the final transcendental equation is acceptable.