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Essential Graduate Physics

Lecture Notes and Problems

Exercise Problems
with Model Solutions

Part CM:
Classical Mechanics

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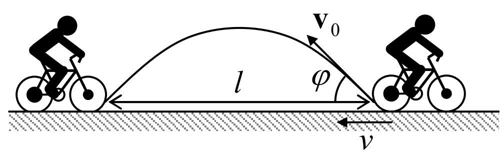
Chapter 1. Review of Fundamentals

Problem 1.1. A bicycle, ridden with velocity v on wet pavement, has no mudguards on its wheels. How far behind should the following biker ride to avoid being splashed over? Neglect the air resistance effects.



Solution: The easiest way to solve this problem is to use the reference frame moving with the bikers. Assuming that their speed is constant, in this reference frame the bike frames are at rest, while the ground moves back with speed v (see the horizontal arrow in the figure on the right), and hence the rim of each wheel moves around its axis with that speed. As a result, in this moving reference frame, the speed of each water drop immediately after its detachment from the tire is the same: $|\mathbf{v}_0| = v$. Since this moving reference frame is inertial, the Newton laws are valid in it, and hence we may use all their corollaries. In particular, after its detachment, each drop follows the well-known parabolic trajectory, and before returning to the initial height passes the distance[1]

$$L = \frac{v^2}{g}\sin 2\varphi, \qquad (*)$$

where φ is the takeoff angle – see the figure above.[2] The distance is largest for drops with $\varphi = \pi/4$:[3]

$$L_{\max} = \frac{v^2}{g}.$$

As the figure above shows, this is the smallest distance to be absolutely safe from splashing, though this expression may be corrected for bike shape details (for example, for certain radius R of the wheel and the bike length l), and for what exactly is meant by the distance between the bikes. For practical values of R and the bike velocities, $v >> (gR)^{1/2} \sim 2$ m/s ~ 5 mph, these corrections are minor because $L_{\max} >> R, l$.

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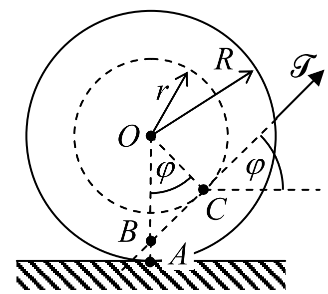
[1] Notice the local (intra-problem) numbering, by asterisks, of the displayed formulas in the solutions, leaving the usual numbers reserved for references to formulas in the lecture notes.

[2] I hope that the reader knows how to derive this formula, but just in case… Since the drop's acceleration during its flight equals g = const and is directed down, placing the reference frame origin at the point of the drop's detachment from the tire and using the traditional orientation of the x- and y-axes, we may spell out Eq. (1.18) of the lecture notes as follows:

$$x(t) = -(v\cos\varphi)\,t, \qquad y(t) = -gt^2/2 + (v\sin\varphi)\,t.$$

Now requiring that the drop returns to the initial height, $y(t) = 0$, for the time of flight we get: $t = (2v/g)\sin\varphi$. Plugging this expression into the formula for $x(t)$, we get $x(t) = -L$, where L is given by Eq. (*).

[3] Note that, curiously enough, relative to the ground, these "most splash-dangerous" drops have the horizontal velocity $(1 - 1/\sqrt{2})\,v \approx +0.29\,v > 0$, i.e. move in the same direction as the bikes, though not that fast.

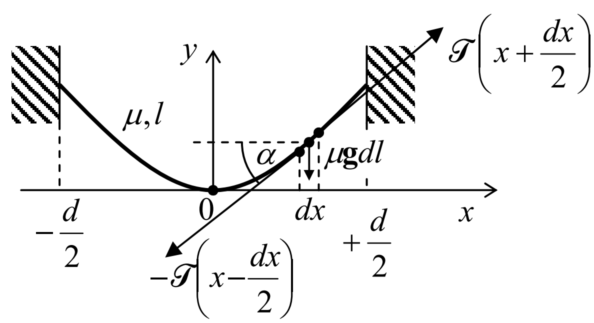


Problem 1.2. Two round disks of radius R are firmly connected with a coaxial cylinder of a smaller radius r, and a thread is wound on the resulting spool. The spool is placed on a horizontal surface, and the thread's end is being pooled out at angle φ – see the figure on the right. Assuming that the spool does not slip on the surface, what direction would it roll?

Solution: The no-slip roll of the spool may be considered as its rotation about the instantaneous axis which coincides with the spool-surface contact line. (In the figure on the right, it is normal to the drawing plane and passes through point A.) Thus the rotation's direction depends on whether the line of the applied force $\mathscr{T}$ passes above or below the axis, i.e. whether point B (where that line crosses the vertical line OA) is located above or below point A. From the right triangle OBC, we get $OB = OC/\cos\varphi \equiv r/\cos\varphi$, while $OA = R$. So, if

$$\frac{r}{\cos\varphi} < R, \qquad \text{i.e. if } \cos\varphi > \frac{r}{R},$$

the spool will roll in the direction of the applied force (in the figure above, to the right), but otherwise, it will roll back. In particular, if the thread is being pulled horizontally ($\varphi = 0$, $\cos\varphi = 1$), the spool will roll to the right, while if it is pulled up ($\varphi = \pi/2$, $\cos\varphi = 0$) it will roll to the left, for any $r < R$.



Problem 1.3.* Calculate the equilibrium shape of a flexible heavy rope of length l, with a constant mass μ per unit length, if it is hung in a uniform gravity field between two points separated by a horizontal distance d – see the figure on the right.

Solution: Let us introduce the Cartesian coordinates as shown in the figure on the right, with the origin at the lowest point of the rope. According to the 2nd Newton law, in equilibrium, the vector sum of the forces acting on each small rope's fragment of a length $dl << l$ should vanish, so for the vector $\mathscr{T}$ of the rope tension force as a function of x, we may write the following vector equality:

$$\mathscr{T}\left(x+\frac{dx}{2}\right) - \mathscr{T}\left(x-\frac{dx}{2}\right) + \mu\mathbf{g}dl = 0. \qquad (*)$$

Here $dx = dl\cos\alpha$ (where α is the rope's slope angle in this particular point, see the figure above) is the horizontal axis' fragment corresponding to dl, so

$$dl = \frac{dx}{\cos\alpha} \equiv \left(1+\tan^2\alpha\right)^{1/2} dx = \left(1+y'^2\right)^{1/2} dx, \qquad \text{where } y' \equiv \frac{dy}{dx} = \tan\alpha.$$

Due to the smallness of dx, we may expand the vector function $\mathscr{T}(x)$ in the Taylor series in dx, and keep only the first (linear) term of the tension difference participating in Eq. (*):

$$\frac{d\mathscr{T}}{dx}dx + \mu\mathbf{g}\left(1+y'^2\right)^{1/2} dx = 0.$$

After the cancellation of $dx \neq 0$, two Cartesian components of this vector equation yield two scalar equations for two unknown scalar functions: $y(x)$ describing the shape of the rope, and $\mathscr{T}(x)$ – the magnitude of its tension:

$$\frac{d\mathscr{T}_x}{dx} \equiv \frac{d}{dx}\left(\mathscr{T}\cos\alpha\right) \equiv \frac{d}{dx}\left[\frac{\mathscr{T}}{\left(1+y'^2\right)^{1/2}}\right] = 0,$$

$$\frac{d\mathscr{T}_y}{dx} \equiv \frac{d}{dx}\left(\mathscr{T}\sin\alpha\right) \equiv \frac{d}{dx}\left[\frac{\mathscr{T}\, y'}{\left(1+y'^2\right)^{1/2}}\right] = \mu g\left(1+y'^2\right)^{1/2}.$$

The first of these equations immediately yields $\mathscr{T}/(1 + y'^2)^{1/2} = \text{const} \equiv \mathscr{T}_0$, where $\mathscr{T}_0$ is the rope's tension at its lowest point (where $y' = 0$). Plugging this relation into the second of the equations, we get the following second-order differential equation for the function we are interested in, $y(x)$:

$$\mathscr{T}_0 y'' = \mu g\left(1+y'^2\right)^{1/2}, \qquad \text{where } y'' \equiv \frac{d^2 y}{dx^2}.$$

It is straightforward to integrate this equation. First, we may represent the second derivative as[4]

$$y'' \equiv \frac{dy'}{dx} = \frac{dy'}{dy}\frac{dy}{dx} \equiv \frac{dy'}{dy}y' = \frac{1}{2}\frac{d\left(y'^2\right)}{dy},$$

so our equation becomes

$$\frac{\mathscr{T}_0}{2}\frac{d\left(y'^2\right)}{dy} = \mu g\left(1+y'^2\right)^{1/2}, \quad \text{or equivalently: } \frac{\mathscr{T}_0}{2}\frac{d\left(1+y'^2\right)}{\left(1+y'^2\right)^{1/2}} = \mu g\, dy.$$

Now we may integrate both parts, getting

$$\mathscr{T}_0\left(1+y'^2\right)^{1/2} = \mu g y + \text{const}.$$

Since we have selected the origin of y at the lowest point of the rope, where $y = 0$ and $y' = 0$, the integration constant also equals $\mathscr{T}_0$, so

$$\mathscr{T}_0\left(1+y'^2\right)^{1/2} = \mu g y + \mathscr{T}_0.$$

Solving this equation for $y' \equiv dy/dx$, and then separating the variables x and y, we get

$$y' = \pm\left\{\left[1+\frac{\mu g}{\mathscr{T}_0}y\right]^2 - 1\right\}^{1/2}, \qquad \frac{dy}{\left\{\left[1+\left(\mu g/\mathscr{T}_0\right)y\right]^2-1\right\}^{1/2}} = \pm dx.$$

It is convenient to integrate both sides of this equation from the lowest point, where $x = 0$ and $y = 0$, to some point $x > 0$, because on this interval, $dy/dx > 0$ (see the figure above), and we may select the positive sign on the right-hand side of the equation. Introducing the dimensionless variable $\xi \equiv 1 + (\mu g/\mathscr{T}_0)y$, so $dy = (\mathscr{T}_0/\mu g)\, d\xi$, we may bring the integral of the left-hand side to a simpler form, getting

[4] This is a very popular transformation, which was already used (for other variables) for the derivation of Eq. (1.20) in the lecture notes, and will be repeatedly used later in this course.

$$\int_{y=0}^{y(x)} \frac{d\xi}{\left(\xi^2 - 1\right)^{1/2}} = \frac{\mu g}{\mathscr{T}_0} x .$$

This integral may be readily worked out using one more substitution: $\xi \equiv \cosh\beta$, so the numerator, $d\xi = \sinh\beta\, d\beta$, and denominator, $(\xi^2 - 1)^{1/2} = (\cosh^2\beta - 1)^{1/2} = \sinh\beta$, are proportional to the same function, $\sinh\beta$, which cancels. As a result, this integral is just $\int d\beta = \beta \equiv \cosh^{-1}\xi \equiv \cosh^{-1} [1 + (\mu g/\mathscr{T}_0)y]$, and we get

$$\cosh^{-1}\left(1 + \frac{\mu g}{\mathscr{T}_0} y\right) = \frac{\mu g x}{\mathscr{T}_0}, \qquad \text{i.e.} \quad y = \frac{\mathscr{T}_0}{\mu g}\left(\cosh\frac{\mu g x}{\mathscr{T}_0} - 1\right). \qquad (**)$$

So, the free-hanging, uniform ropes/chains have the form of the plot of the hyperbolic cosine function.[5] Due to this fact, this curve is sometimes called the *chainette*. (A more popular term for the curve is "catenary", but terms "alysoid" and "funicular" may be also met.) What remains now is to find the constant $\mathscr{T}_0$. This may be done by requiring the sum of all elementary lengths $dl = (1 + y'^2)^{1/2}dx$ of the rope to be equal to its full length l:

$$l \equiv \int_l dl = \int_{-d/2}^{+d/2} \left(1 + y'^2\right)^{1/2} dx = 2\int_0^{d/2} \left(1 + y'^2\right)^{1/2} dx . \qquad (***)$$

From Eq. (**), we get

$$y' = \sinh\frac{\mu g x}{\mathscr{T}_0}, \qquad \text{so that } \left(1 + y'^2\right)^{1/2} = \cosh\frac{\mu g x}{\mathscr{T}_0};$$

with the last equality, the integration in Eq. (***) becomes elementary, giving

$$l = \frac{2\mathscr{T}_0}{\mu g}\sinh\frac{\mu g d}{2\mathscr{T}_0}, \qquad \text{i.e.} \quad \frac{\mu g l}{2\mathscr{T}_0} = \sinh\frac{\mu g d}{2\mathscr{T}_0},$$

or in a convenient dimensionless form:

$$\frac{l}{d}\zeta = \sinh\zeta, \quad \text{where } \zeta \equiv \frac{\mu g d}{2\mathscr{T}_0}. \qquad (****)$$

This is a transcendental equation for ζ (and hence for $\mathscr{T}_0$); since the function $\sinh\zeta$ grows as ζ at $\zeta \to 0$ and faster than that at larger values of its argument, the equation has a single positive root for any $l/d > 1$. Using the well-known asymptotic behaviors of this function for small and large values of its argument, it is straightforward to find that

$$\mathscr{T}_0 \to \mu g l \times \begin{cases} \dfrac{1}{2\sqrt{6}}\left(\dfrac{l}{l-d}\right)^{1/2} \to \infty, & \text{for } l/d \to 1, \\ \dfrac{d/l}{2\ln(l/d)} \to 0, & \text{for } l/d \to \infty. \end{cases}$$

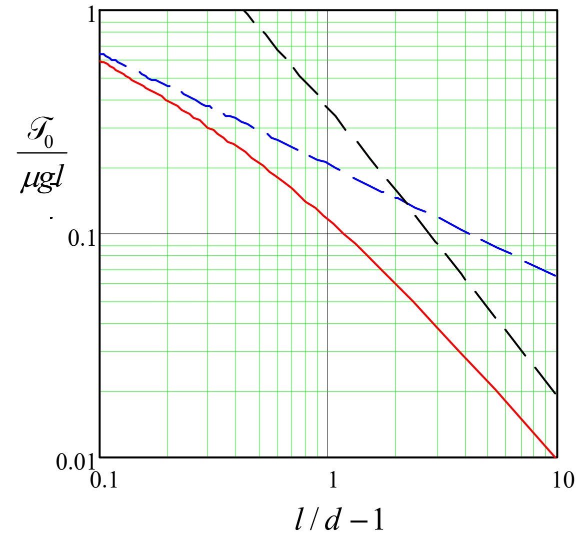
In the former limit, $\mathscr{T}_0$ is much larger than the weight $\mu g l$ of the whole rope, while in the latter limit, is much less than the weight. The dashed lines in the figure below show these two asymptotes together with the quasi-exact (numerically calculated) solution of Eq. (****), shown with the solid red line, on the log-log scale, which is most appropriate in this case.

[5] Additional question: is this solution a good approximation for suspension bridge cables? If not, why?

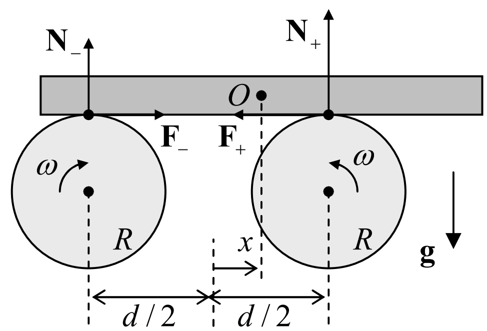
In conclusion, let me note that this problem may be also solved (or rather the differential equation for the function $y(x)$ derived) by the calculus of variations, from the condition that the total potential energy of the rope,

$$U = \int_l \mu g y dl = \mu g \int_{-d/2}^{+d/2} y\left(1 + y'^2\right)^{1/2} dx\,,$$

is minimal at equilibrium, upon the condition of constancy of the rope's length l, i.e. of the integral (***). Though for this particular problem, such a solution is more lengthy, it is highly recommended to the reader as an additional exercise, especially because we will need the calculus of variations several times in this course, starting from the derivation of the Lagrange equations in the next chapter.



Problem 1.4. A uniform, long, thin bar is placed horizontally on two similar round cylinders rotating toward each other with the same angular velocity ω and displaced by distance d – see the figure on the right. Calculate the laws of relatively slow horizontal motions of the bar within the plane of the drawing, for both possible directions of cylinder rotation, assuming that the kinetic friction force between the slipping surfaces of the bar and each cylinder obeys the simple *Coulomb approximation*[6] $|F| = \mu N$, where N is the normal pressure force between them, and μ is a constant (velocity-independent) coefficient. Formulate the condition of validity of your result.



Solution: Let us denote the current horizontal displacement of the bar's center of mass (point O) from the symmetry plane of the cylinder system as x – see the figure above. Then we may write the following two equations for the normal pressure forces $\mathbf{N}_\pm$,

$$N_+ + N_- = Mg,$$
$$N_+\left(\frac{d}{2} - x\right) - N_-\left(\frac{d}{2} + x\right) = 0,$$

where M is the bar's mass. These equations express the balances of, respectively, the vertical forces and their torques, necessary to avoid the vertical linear and the angular accelerations of the bar. (Note that contributions of the friction forces $\mathbf{F}_\pm$ into the torque balance may be ignored only because of the small thickness of the bar.) Solving this simple system of two linear equations, we get

$$N_\pm = Mg\frac{d/2 \pm x}{d}.$$

[6] It was suggested in 1785 by the same C.-A. de Coulomb who discovered the famous *Coulomb law* of electrostatics and hence pioneered the whole quantitative science of electricity.

If the bar motion is relatively slow, $|v| < \omega R$, its surface slips relatively to those of both cylinders, so we can use the Coulomb approximation $|F_\pm| = \mu N_\pm$ for each of the friction forces, and for the total horizontal force, we may write

$$|F| = |F_+ - F_-| = 2\mu Mg \frac{|x|}{d}.$$

The sign of F depends on the direction of the cylinders' rotation. If their top points, on which the bar rests, move toward each other (as shown in the figure above), then the force F_+ is always directed to the left, so taking the shown direction of the displacement x for the positive one, we may write $F_+ = -2\mu Mg(d/2 - x)/d < 0$, while the counterpart force is positive: $F_- = 2\mu Mg(d/2 + x)/d$. As a result,

$$F = F_+ - F_- = -2\mu Mg \frac{x}{d}.$$

In this case, the horizontal component of the 2nd Newton law for the bar reads

$$M\ddot{x} = -2\mu Mg \frac{x}{d}. \qquad (*)$$

This is the well-known equation of 1D motion of a body on an elastic spring with a positive spring constant, $\kappa = 2\mu Mg/d$, and its solutions are sinusoidal oscillations of frequency

$$\omega_0 = \left(\frac{\kappa}{M}\right)^{1/2} \equiv \left(\frac{2\mu g}{d}\right)^{1/2}.$$

Note that this solution is only valid if the displacement amplitude $A \equiv x_{\max}$ is lower than $\omega R/\omega_0$, so the velocity amplitude $\omega_0 A$ is lower than the cylinders' top speed ωR. What happens at larger amplitudes, depends on the static friction coefficient μ_s, more exactly, its relation with the kinetic friction coefficient μ. Though this is beyond the problem's assignment, the reader is encouraged to carry out a semi-quantitative analysis of various cases of such motion.[7]

In the second case, when the cylinders rotate in the direction opposite to that shown in the figure above (i.e. with their top parts moving from each other), both friction forces have opposite directions, and we need to change the sign in the expression for the total horizontal force F. This gives, instead of Eq. (*), the following equation:

$$M\ddot{x} = 2\mu Mg \frac{x}{d}. \qquad (**)$$

Its general solution is a sum of either two exponents or (equivalently) of two hyperbolic functions of time:[8]

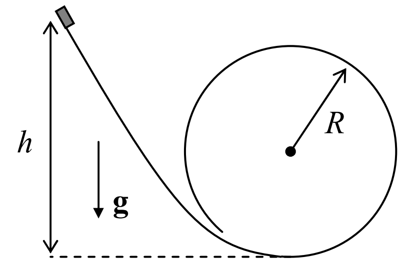
$$x(t) = C_+ e^{\lambda t} + C_- e^{-\lambda t} \equiv C_c \cosh \lambda t + C_s \sinh \lambda t, \qquad \text{with } \lambda = \left(\frac{2\mu g}{d}\right)^{1/2}, \qquad (***)$$

where the constants $C_\pm$ (or alternatively, $C_{c,s}$) are determined by initial conditions – the initial position and velocity of the bar. Note that whatever the conditions are, the displacement x and velocity $v = dx/dt$

[7] See the similar Problem 5.21 below.

[8] This fact may be either verified by its substitution to Eq. (**) or obtained in a regular fashion by looking for the solution in the form $C\exp\{\lambda t\}$, as will be discussed in detail in Sec. 3.2 of the lecture notes.

of the bar will at $t >> 1/\lambda$ grow exponentially. So, at this direction of cylinder rotation, our solution (***) eventually runs out of its validity range $|v| < \omega R$. Again, the reader is encouraged to analyze possible cases of the subsequent motion.



Problem 1.5. A small block slides, without friction, down a smooth slide that ends with a round loop of radius R – see the figure on the right. What smallest initial height h allows the block to make its way around the loop without dropping from the slide if it is launched with negligible initial velocity?

Solution: The most critical point of the motion is evidently the highest point of the round loop, where the block's velocity $\mathbf{v}$ is the lowest, and the block's weight force, $m\mathbf{g}$, is directed exactly along the possible direction of the detachment from the slide's surface. The magnitude of this velocity may be readily calculated from the mechanical energy conservation law written for the initial and critical points:

$$mgh = \frac{mv^2}{2} + 2mgR, \quad \text{giving } v^2 = 2g(h - 2R), \qquad (*)$$

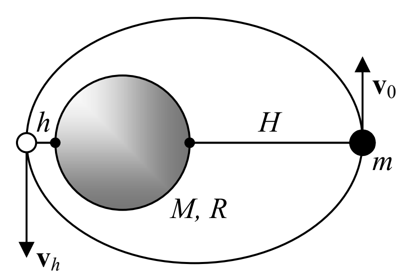
where m is the mass of the block. To avoid the detachment from the slide, this velocity should be so high that the block weight mg could not, alone (without the slide's reaction), provide the necessary centripetal acceleration $a = v^2/R$:

$$mg < m\frac{v^2}{R}.$$

By plugging the last form of Eq. (*) into this condition, it is reduced to a very simple form:

$$h > h_{\min} = \frac{5}{2}R.$$

Note that the result is independent not only of the block's mass m (such independence, due to the weak equivalence principle, is common for all problems where the only substantial force is that of gravity) but also of the gravity acceleration g.



Problem 1.6. A satellite of mass m is being launched from height H over the surface of a spherical planet with radius R and mass $M >> m$ – see the figure on the right. Find the range of initial velocities $\mathbf{v}_0$ (normal to the radius) providing closed orbits above the planet's surface.

Solution: The simplest way to solve this problem is to write the laws of conservation of the angular momentum and the energy for two opposite points of the evidently symmetric orbit (see the figure on the right):

$$mv_0(H+R) = mv_h(h+R), \qquad \frac{m}{2}v_0^2 - G\frac{mM}{H+R} = \frac{m}{2}v_h^2 - G\frac{mM}{h+R}.$$

Solving this system of equations for v_0 and v_h, we get, in particular:

$$v_0^2 = 2GM \frac{h+R}{(H+R)(h+H+2R)}.$$

For the two boundaries of the velocity interval of our interest ($h = 0$ and $h \to \infty$), we get, respectively:

$$\left(v_0^2\right)_{\min} = 2GM \frac{R}{(H+R)(H+2R)}, \qquad \left(v_0^2\right)_{\max} = 2GM \frac{1}{H+R}.$$

For the particular case of the satellite's launch from the planet's surface ($H = 0$), these formulas are reduced to the well-known expressions for the so-called *1st and 2nd space velocities*:[9]

$$v_1 = \left(\frac{GM}{R}\right)^{1/2}, \qquad v_2 = \left(\frac{2GM}{R}\right)^{1/2} = \sqrt{2}\, v_1 \approx 1.41 v_1.$$

For our Earth ($M = M_E \approx 5.97\times10^{24}$ kg, $R = R_E \approx 6.36\times10^{6}$ m), these velocities are close to, respectively, 7.92 and 11.2 km/s.

Problem 1.7. Prove that the thin-uniform-disk model of a galaxy allows for small sinusoidal ("harmonic") oscillations of stars inside it, along the direction normal to the disk, and calculate the frequency of these oscillations in terms of Newton's gravitational constant G and the average density ρ of the disk's matter.

Solution: Let us calculate the net gravitational force $\mathbf{F}$ exerted on the star, of mass m, by the whole galactic disk, within this model. This may be done, for example, by the direct summation of the Newton gravity law (see, e.g., Eq. (1.15) of the lecture notes) for two point masses m and m',

$$\mathbf{F}_{\text{point}} = -G\frac{mm'}{R^3}\mathbf{R}, \qquad \text{where } \mathbf{R} \equiv \mathbf{r} - \mathbf{r}', \qquad (*)$$

over all elementary masses $dm' = \rho(\mathbf{r}')d^3r'$ of the disk:

$$\mathbf{F}(\mathbf{r}) = -Gm\int \frac{\rho(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|^3}(\mathbf{r}-\mathbf{r}')d^3r'.$$

However, even in our simple case of constant density ρ, such integration is a bit cumbersome, because of the vector nature of the integral. It is helpful here (and in many other problems) to use the analogy of the Newton law (*) with the Coulomb law of the electrostatic interaction of two point charges q and q',[10]

$$\mathbf{F}_{\text{point}} = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{R^3}\mathbf{R}. \qquad (**)$$

They are similar with the replacements $q \leftrightarrow m$, $q' \leftrightarrow m'$, and $1/4\pi\varepsilon_0 \leftrightarrow -G$, i.e. $1/\varepsilon_0 \leftrightarrow -4\pi G$. Making these replacements in the well-known Gauss law of electrostatics (which follows from the Coulomb law),[11]

[9] Alternatively, v_2 is called the *escape velocity*.

[10] See, e.g., EM Eq. (1.1).

[11] See, e.g., EM Eq. (1.16), with both sides multiplied by q, so $\mathbf{E} \to q\mathbf{E} = \mathbf{F}$. Note also that the density $\rho(\mathbf{r}')$ has different meanings (and dimensionalities) in Eqs. (*) and (**), but this fact does not affect their analogy.

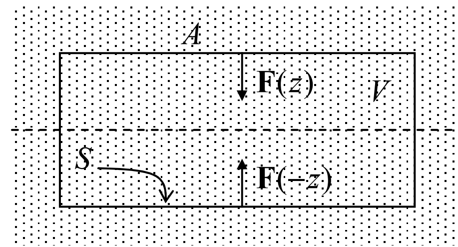
$$\oint_S F_n d^2r = \frac{q}{\varepsilon_0}\int_V \rho(\mathbf{r}')d^3r',$$

we get:

$$\oint_S F_n d^2r = -4\pi Gm\int_V \rho(\mathbf{r}')d^3r'. \qquad (***)$$

Here V is an arbitrary "Gaussian" volume, S is the closed surface limiting the volume, and F_n is the component of the force $\mathbf{F}$ along the outer normal $\mathbf{n}$ to the surface: $F_n \equiv \mathbf{F}\cdot\mathbf{n}$.

For our current problem, it is beneficial to consider the Gaussian volume V in the form of a flat "pillbox", with a thickness $2z$ smaller than that of the galactic disk, and plane "lids" of area A, parallel to the disk's plane – see the figure on the right, where the dashed line indicates the disk's symmetry plane (from which the normal coordinate z will be measured). Taking the pillbox lid area A much smaller than the galactic disk area, we may use the problem's symmetry to argue that the force $\mathbf{F}$ should be:



(i) directed perpendicular to the galactic disk plane, and hence to the pillbox lids: $\mathbf{F} = F_z\mathbf{n}_z$;
(ii) independent of the "horizontal" (in our figure) position: $F_z = F_z(z)$; and
(iii) symmetric relative to the symmetry plane: $F_z(-z) = -F_z(z)$.

With these assumptions, the gravity force flux $\int F_n d^2r$ through the lateral sides of the pillbox vanishes (because on these sides $\mathbf{F}\perp\mathbf{n}$, i.e. $\mathbf{F}\cdot\mathbf{n} = 0$), while the flux through each of two lids is just $F_z(z)A$, so Eq. (***) yields

$$2F(z)A = -4\pi Gm\rho(2zA),$$

giving, finally,

$$F(z) = -\kappa z, \qquad \text{with } \kappa \equiv 4\pi Gm\rho.$$

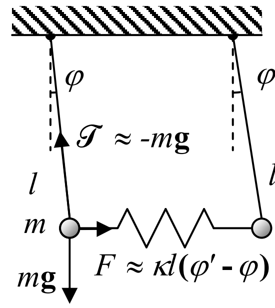
Such an attractive force (trying to return the star to the disk's symmetry plane) proportional to the deviation from the plane, is similar to that provided by a usual elastic spring, and hence causes harmonic oscillations of the star about the symmetry plane, with the frequency

$$\omega = \left(\frac{\kappa}{m}\right)^{1/2} = (4\pi G\rho)^{1/2},$$

independent of the star's mass.

For our galaxy ("Milky Way") in the vicinity of our Sun, the average density ρ is close to 1.4×10^{-20} kg/m^3, and the above formula yields $\omega \approx 3.3\times10^{-15}$ s^{-1}, corresponding to the oscillation period $\mathcal{T} = 2\pi/\omega \approx 60$ million years.[12] The amplitude of our Sun's oscillations (which cannot be calculated from the problem's data, but may be deduced from the experimentally measured Sun's velocity relative to the adjacent stars) is about 2×10^{18} m, i.e. an order of magnitude smaller than the Milky Way disk's thickness ($\sim2\times10^{19}$ m). On the other hand, the amplitude is much larger than the average distance between the stars in our vicinity, $\sim 10^{16}$ m. These two strong relations make this simple model valid for an approximate but very reasonable description of the Sun's motion.

[12] Just for the reader's reference, this oscillation period is much shorter than the period, ~ 240 million years, of the Sun's rotation about the galactic center.



Problem 1.8. Derive differential equations of motion for small oscillations of two similar pendula coupled with a spring (see the figure on the right), within their common vertical plane. Assume that at the vertical position of both pendula, the spring is not stretched ($\Delta d = 0$).

Solution: If the deviations of the pendula from their vertical positions are small: $|\varphi|, |\varphi'| << 1$ (see the figure on the right), then in the linear approximation in φ and φ', the magnitudes of the supporting rods' tensions $\mathscr{T}$ are still mg, and their horizontal components are equal to, respectively, $(-mg\varphi)$ and $(-mg\varphi')$. In the same approximation, the linear displacements of the pendula from the equilibrium (vertical) positions are, respectively, $l\varphi$ and $l\varphi'$, and hence the spring extension Δd is $l(\varphi' - \varphi)$, so the force acting on each pendulum equals $\pm\kappa l(\varphi' - \varphi)$, where κ is the spring constant. As a result, in the linear approximation, the horizontal components of the 2nd Newton law for the two pendula are:

$$m(l\ddot{\varphi}) = \kappa l(\varphi' - \varphi) - mg\varphi,$$
$$m(l\ddot{\varphi}') = -\kappa l(\varphi' - \varphi) - mg\varphi'.$$

The solution of this system of two differential equations will be the subject of Problem 6.1.

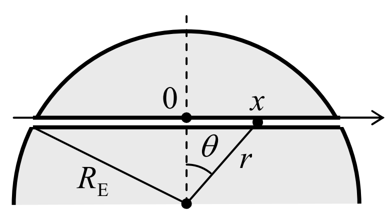
Problem 1.9. One of the popular futuristic concepts of travel is digging a straight railway tunnel through the Earth and letting a train go through it, without initial velocity – driven only by gravity. Calculate the train's travel time through such a tunnel, assuming that the Earth's density ρ is constant, and neglecting the friction and planet rotation effects.

Solution: Let us apply the gravitational analog of the Gauss law, given by Eq. (***) of the model solution of Problem 7,

$$\oint_S F_n d^2r = -4\pi Gm \int_V \rho(\mathbf{r}')d^3r',$$

to a sphere of radius $r \leq R_E$, taking into account that due to the system's symmetry, $\mathbf{F} = \mathbf{n}_r F(r)$, and $F_n = F$. The result shows that the net gravity force felt by the train at distance r from Earth's center is determined only by the planet's mass inside a sphere of this radius,

$$\mathbf{F} = -G\frac{M(r)m}{r^3}\mathbf{r}, \qquad \text{with } M(r) = \rho\frac{4\pi}{3}r^3,$$



where m is the train's mass. With the notation used in the figure on the right, the force's component directed along the tunnel is

$$F_x = F\sin\theta = -G\frac{M(r)m}{r^2}\sin\theta = -\frac{4\pi}{3}Gm\rho r\sin\theta.$$

But the product $r\sin\theta$ is nothing more than the linear displacement x of the train from the middle of the tunnel, so F_x depends on x linearly, similarly to the force of the usual elastic spring with the equilibrium point at $x = 0$:

$$F_x = -\kappa x, \qquad \text{with } \kappa = \frac{4\pi}{3}Gm\rho.$$

The spring constant κ looks even simpler when expressed via the gravity acceleration g on the Earth's surface and its radius R_E. Indeed, by the definition of g,

$$g = \frac{G}{R_E^2} M(R_E) = \frac{G}{R_E^2}\frac{4\pi}{3}R_E^3\rho \equiv \frac{4\pi}{3}G\rho R_E, \qquad \text{so } \kappa = m\frac{g}{R_E}.$$

As a result of this analogy, the equation the of train's motion along the tunnel, $m\ddot{x} = -\kappa x$, is similar to that of the mass on a spring; it describes periodic, sinusoidal oscillations of x in time, with the period

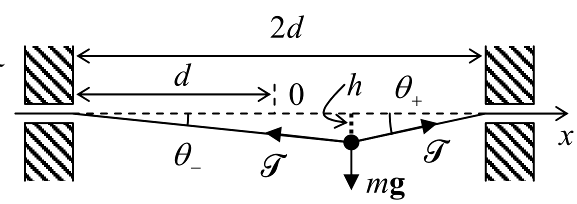
$$\mathcal{T} = \frac{2\pi}{\omega_0}, \qquad \text{where } \omega_0 = \left(\frac{\kappa}{m}\right)^{1/2} = \left(\frac{g}{R_E}\right)^{1/2}.$$

Evidently, the time Δt of a one-way trip of the train through the tunnel, with no initial velocity, is just half of this period:

$$\Delta t = \frac{\mathcal{T}}{2} = \frac{\pi}{\omega_0} = \pi\left(\frac{R_E}{g}\right)^{1/2}.$$

Perhaps the most curious feature of this result is that it is independent of the tunnel's length. The physical reason is that the longer the tunnel, the steeper its average incline toward Earth's center, and hence the larger the train's acceleration. So, if our Earth were uniform, the travel time from any point of its surface to any other point would be the same (about 42 minutes and 13 seconds). In reality, ρ grows toward the Earth's center, so the above result, with ρ being the average density, is accurate only for relatively short tunnels, with length $l \ll R$, while for longer tunnels the travel would be even faster.

Problem 1.10. A small bead of mass m may slide, without friction, along a light string stretched with force $\mathcal{T} \gg mg$, between two points separated by a horizontal distance $2d$ – see the figure on the right. Calculate the frequency of oscillations of the bead about its equilibrium position, within the vertical plane.



Solution: Due to the given condition $\mathcal{T} \gg mg$, the string remains nearly horizontal even under the weight of the bead, so both angles $\theta_\pm$ (see the figure above) are small. As a result, the horizontal motion of the bead is much slower than its vertical oscillations, and the vertical displacement h may be calculated ignoring its dynamics, from the requirement that the sum of the two vertical components of the string tension $\mathcal{T}$, namely $\mathcal{T}\sin\theta_\pm \approx \mathcal{T}\theta_\pm$, counterbalances its weight $m\mathbf{g}$:

$$\mathcal{T}(\theta_- + \theta_+) = mg,$$

plus from the geometric relations evident from the same figure:

$$\theta_- = \frac{h}{d+x}, \qquad \theta_+ = \frac{h}{d-x}, \qquad (*)$$

where x is the horizontal displacement of the bead from its equilibrium position at the center of the string. Solving this simple system of three equations for h and $\theta_\pm$, we get, in particular,

$$\theta_\pm = \frac{mg}{2\mathscr{T} d}(d \pm x).$$

Now we may use this result to calculate the net horizontal component of the forces exerted on the bead:

$$F_x = \mathscr{T} \cos\theta_+ - \mathscr{T} \cos\theta_- \approx \mathscr{T}\left(1-\frac{\theta_+^2}{2}\right) - \mathscr{T}\left(1-\frac{\theta_-^2}{2}\right) = \frac{\mathscr{T}}{2}\left(\frac{mg}{2\mathscr{T} d}\right)^2\left[(d-x)^2-(d+x)^2\right] = -\frac{m^2 g^2}{2\mathscr{T} d}x\,.$$

This force may be represented as $F_x = -\kappa x$, with

$$\kappa = \frac{m^2 g^2}{2\mathscr{T} d} > 0\,,$$

and is always directed toward the equilibrium point $x = 0$, so it is similar to the one provided by the usual elastic spring. Hence the frequency of the bead's oscillations may be found from the well-known formula for the harmonic oscillations of a mass on the spring:

$$\omega = \left(\frac{\kappa}{m}\right)^{1/2} = g\left(\frac{m}{2\mathscr{T} d}\right)^{1/2}. \qquad (**)$$

This result shows, in particular, that $\omega \to 0$ at $\mathscr{T} \to \infty$. This is natural because in this limit, the string becomes strictly horizontal, and the returning horizontal force, which results from the string's slopes, vanishes. Note also that:

– The calculated frequency (**) of the horizontal oscillations of the bead is much smaller than that, $\Omega \sim (2\mathscr{T}/md)^{1/2}$, of its vertical oscillations.[13] This relation confirms the validity of our approach;

– Our result, while being conditioned by the strong inequality $\mathscr{T} >> mg$, is valid for an arbitrary oscillation amplitude $A \equiv x_{\max}$, while it is less (but not necessarily much less!) than d.

Problem 1.11. For a rocket accelerating (in free space) due to its working jet motor (and hence spending the jet fuel), calculate the relation between its velocity and the remaining mass.

Hint: For the sake of simplicity, consider the 1D motion.

Solution: Let us write the law of conservation of the net momentum P of the rocket and a small portion dm of its exhaust gases ejected with the relative velocity u during a small time interval dt, in the so-called *instantaneous rest frame* – an inertial reference frame moving, in the particular instant under consideration, with the same velocity v as the body – in our case, the accelerating rocket:

$$dP \equiv m\,dv + dm\,u = 0\,. \qquad (*)$$

Dividing all terms of this equation by dt, and moving the term proportional to u to the right-hand side, we get the following equation of motion:

$$m\frac{dv}{dt} = -u\frac{dm}{dt}\,.$$

[13] For small, purely vertical oscillations, the formula $\Omega = (2\mathscr{T}/md)^{1/2}$ is exact (prove this!). The coexistence of various oscillations in this system, for an arbitrary ratio $\mathscr{T}/mg$, will be the subject of Problem 3.1.

Its comparison with the usual form of the 2nd Newton law shows that the magnitude of the effective force (in engineering, called *thrust*) of the rocket engine is

$$F_{\text{ef}} = \mu u,$$

where $\mu \equiv -dm/dt > 0$ is the fuel burn rate. For the case when the rate, as well as the exhaust velocity u, are constant in time (meaning, in particular, that $m(t) = m(0) - \mu t$), the resulting equation,

$$\left[m(0) - \mu t\right]\frac{dv}{dt} = \mu u,$$

may be readily integrated to find the velocity and the coordinate of the rocket as functions of time – a useful exercise, highly recommended to the reader.

However, since we are only interested in the relation between the remaining rocket mass and the achieved velocity, we may directly integrate Eq. (*), for an arbitrary $m(t)$:

$$\int \frac{dm}{m} = -\frac{1}{u}\int dv,$$

getting

$$\ln m = -\frac{v}{u} + \text{const.}$$

Now using the initial conditions to find the integration constant, we get the famous *Tsiolkovsky rocket formula*,[14] not conditioned by the constancy of μ:

$$v(t) = v(0) + u \ln \frac{m(0)}{m(t)}.$$

It shows that, a bit counter-intuitively, a rocket may reach velocities much higher than the relative velocity u of the exhaust gases. However, for this, the initial mass of the fuel, contributing to $m(0)$, has to be much larger than that of the ship itself, including the useful payload. This result is the basis for all rocket engineering, notably including multistage designs.

Problem 1.12. Prove the following *virial theorem*:[15] for a set of N particles performing a periodic motion,

$$\overline{T} = -\frac{1}{2}\sum_{k=1}^{N}\overline{\mathbf{F}_k \cdot \mathbf{r}_k},$$

where the top bar means averaging over time – in this case over the motion period. What does the virial theorem say about:

(i) a 1D motion of a particle in the confining potential[16] $U(x) = ax^{2s}$, with $a > 0$ and $s > 0$, and

[14] It was derived, in an implicit form, by W. Moore in 1810, and then re-discovered (and used to discuss rocket motion and space travel) by K. Tsiolkovsky in 1903, just to be re-discovered again by R. Goddard in 1912 and H. Oberth near 1920. Note that the jet motor's idea as such is much older: it may be traced back at least to Hero (a.k.a. Heron) of Alexandria who lived in the 1st century AD.

[15] It was first stated by R. Clausius in 1870. The term *virial* was derived by him from *vis*, the Latin for "force".

(ii) an orbital motion of a particle moving in the central potential $U(r) = -C/r$?

Hint: Explore the time derivative of the following scalar function of time: $G(t) \equiv \sum_{k=1}^{N} \mathbf{p}_k \cdot \mathbf{r}_k$.

Solution: Differentiating the suggested function $G(t)$ by parts,

$$\frac{dG}{dt} \equiv \sum_{k=1}^{N} \dot{\mathbf{p}}_k \cdot \mathbf{r}_k + \sum_{k=1}^{N} \mathbf{p}_k \cdot \dot{\mathbf{r}}_k ,$$

and using Eqs. (1.3), (1.9), and (1.13) of the lecture notes, we get

$$\frac{dG}{dt} = \sum_{k=1}^{N} \mathbf{F}_k \cdot \mathbf{r}_k + \sum_{k=1}^{N} m_k \dot{\mathbf{r}}_k \cdot \dot{\mathbf{r}}_k .$$

The term under the last sum is just twice the kinetic energy (1.19) of the k^{th} particle, so the sum of these terms is twice the total kinetic energy T of the system, and hence

$$\frac{dG}{dt} = \sum_{k=1}^{N} \mathbf{F}_k \cdot \mathbf{r}_k + 2T . \qquad (*)$$

If the system's motion is periodic with a certain time period $\mathcal{T}$, so is the function G: $G(t + \mathcal{T}) = G(t)$, and the time average of its derivative over the period equals zero:[17]

$$\overline{\frac{dG}{dt}} \equiv \frac{1}{\mathcal{T}} \int_{t}^{t+\mathcal{T}} \frac{dG(t')}{dt'} dt' = \frac{1}{\mathcal{T}} \int_{t'=t}^{t'=t+\mathcal{T}} dG(t') = \frac{1}{\mathcal{T}} \left[G(t+\mathcal{T}) - G(t) \right] = 0 ,$$

so the time averaging of Eq. (*) yields

$$0 = \sum_{k=1}^{N} \overline{\mathbf{F}_k \cdot \mathbf{r}_k} + 2\overline{T} ,$$

thus proving the virial theorem.

(i) For a 1D motion of a particle in a time-independent potential $U(x)$, the radius vector $\mathbf{r}$, the velocity $\mathbf{v}$, and the force $\mathbf{F}$ have just one Cartesian component each, with $F_x = -dU/dx$, so the virial theorem is reduced to

$$\overline{T} = \frac{1}{2} \overline{x \frac{dU}{dx}}, \qquad \text{with } T \equiv \frac{m}{2} v^2 \equiv \frac{m}{2} \dot{x}^2 .$$

For the particular case $U(x) = ax^{2s}$,

$$x \frac{dU}{dx} = 2sax^{2s} \equiv 2sU ,$$

so the theorem yields

---

[16] Here and below I am following the (regretful :-) custom of using the single word "potential" for the potential energy of the particle – just for brevity. This custom is also common in quantum mechanics, but in electrodynamics, these two notions should be clearly distinguished – as they are in the EM part of this series.

[17] Actually, this statement (and hence the virial theorem) is asymptotically (i.e. in the limit $\mathcal{T} \to \infty$) valid even if the system is not periodic, but is *stably bound*, meaning that the particles stay together in a limited region of space, and their velocities remain finite.

$$\overline{T} = s\overline{U},$$

for any a and s. (The conditions $a > 0$ and $s > 0$ are necessary to ensure that the particle's motion is periodic, and hence obeys the virial theorem.)

Note that for the most important case of the quadratic confining potential ($s = 1$), this result is reduced to equality of the average values of the kinetic and potential energies – the fact well known from the analysis of the sinusoidal motion of a harmonic oscillator.

(ii) For a particle moving in a central potential $U(r) = -C/r$, the force is directed toward the center:

$$\mathbf{F}(\mathbf{r}) = -\nabla U = -\frac{C}{r^3}\mathbf{r},$$

so the (only) term, $\mathbf{F}\cdot\mathbf{r}$, on the right-hand side of the virial theorem may be expressed as

$$\mathbf{F}\cdot\mathbf{r} = -\frac{C}{r^3}\mathbf{r}\cdot\mathbf{r} = -\frac{C}{r} = U,$$

and the theorem is reduced to a very simple (and important) equality

$$\overline{T} = -\frac{1}{2}\overline{U}.$$

This equality is valid, in particular, for the elliptical orbits of planetary motion, which will be discussed in Chapter 3.

Problem 1.13. As will be discussed in Chapter 8, if a solid body moves through a fluid with a sufficiently high velocity v, the fluid's drag force is approximately proportional to v^2. Use this approximation (introduced by Sir Isaac Newton himself) to find the velocity as a function of time during the body's vertical fall in the air near the Earth's surface.

Solution: According to this approximation and Eq. (1.16) of the lecture notes, the body's motion obeys the 2nd Newton law (1.13) in the form

$$m\frac{dv}{dt} = \alpha v^2 - mg, \tag{*}$$

where $v \equiv dy/dt$, y is the body's height over the Earth's surface, and α is a time-independent drag factor. Chasing the variables v and t to the opposite sides of this differential equation, we reduce it to the form

$$\frac{mdv}{\alpha v^2 - mg} = dt,$$

whose sides may be directly integrated over the corresponding variable:

$$\int \frac{mdv}{\alpha v^2 - mg} = \int dt, \qquad \text{or } \tau \int \frac{d\xi}{1-\xi^2} = -\int dt, \tag{**}$$

where $\xi \equiv v/v_\infty$, with

$$v_\infty \equiv \left(\frac{mg}{\alpha}\right)^{1/2} \text{ and } \tau \equiv \frac{v_\infty}{g} \equiv \left(\frac{m}{g\alpha}\right)^{1/2}.$$

The parameter v_∞ has the sense of the stationary speed of the body's fall, reached at $t >> \tau$,[18] and τ is the natural time scale of our problem.

The right-hand integral in Eq. (**) is elementary, while the left-side one may be readily worked out using the following (easily verifiable) algebraic identity:

$$\frac{1}{1-\xi^2} \equiv \frac{1}{2}\left(\frac{1}{1-\xi} + \frac{1}{1+\xi}\right).$$

As a result of the integration, we get

$$-\frac{\tau}{2}\ln\left(1-\frac{v}{v_\infty}\right) + \frac{\tau}{2}\ln\left(1+\frac{v}{v_\infty}\right) = -t + \text{const}, \quad \text{and hence } \frac{v_\infty + v}{v_\infty - v} = \exp\left\{-2\frac{t}{\tau} + \text{const}\right\}, \quad \text{for } |v| < v_\infty .$$

The constant in the last relation is determined by the initial condition or, equivalently, by the time origin's choice. In particular, if we take $t = 0$ at $v = 0$, then the constant vanishes, so, finally,

$$v(t) = -v_\infty \frac{1-\exp\{-2t/\tau\}}{1+\exp\{-2t/\tau\}} \equiv -v_\infty \tanh\left(\frac{t}{\tau}\right) \to \begin{cases} -v_\infty t/\tau \equiv -gt, & \text{at } t << \tau, \\ -v_\infty, & \text{at } \tau << t. \end{cases}$$

This result describes a natural crossover, on a time interval of the order of τ, from the body's initial downward acceleration with a negligible effect of the air's drag to its fall with the constant velocity $-v_\infty$ determined by the drag.

Note that if our primary interest was in the velocity of the body as a function of its height y rather than time, the equation of motion might be integrated differently, by using the transformation similar to the one employed in Eq. (1.20) of the lecture notes:

$$\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy} = \frac{1}{2}\frac{d(v^2)}{dy}.$$

As a result, Eq. (*) may be rewritten as

$$\frac{m}{\alpha}\frac{1}{2}\frac{d}{dy}\left(\alpha v^2 - mg\right) = \alpha v^2 - mg, \qquad \text{i.e. } \frac{m}{2\alpha}\frac{d\left(mg - \alpha v^2\right)}{mg - \alpha v^2} = dy ,$$

making the integration elementary:

$$\frac{m}{2\alpha}\ln\left|mg - \alpha v^2\right| = y + \text{const}, \quad \text{i.e. } \ln\left|v_\infty^2 - v^2\right| = \frac{2y}{h_0} + \text{const}, \quad \text{where } h_0 \equiv \frac{\alpha}{m} \equiv v_\infty \tau .$$

Let me leave it to the reader to complete this calculation by finding, for example, the final velocity of the body dropped from some height h without initial velocity. (From the last relation, it is clear that the result will depend on the relation between h and h_0.) Other questions this approach enables to answer are how high would the body fly and what would be its velocity upon its return to the Earth's surface if it was projected up from the surface, with a certain velocity v_0. (Working out these answers, be sure to assign the correct drag force sign at the body's way up.)

[18] Unless the body hits the Earth's surface first.

Problem 1.14. A particle of mass m, moving with velocity u, collides head-on with a particle of mass M, initially at rest, increasing its internal energy by ΔE. Calculate the velocities of both particles after the collision, if u is barely sufficient for such an internal energy increase.

Solution: Let us denote the (still to be determined) velocities of the particles after the collision as u' and v, respectively. Then the laws of conservation of the total momentum of the system and of its total energy (including that of the internal excitation) read

$$mu = mu' + Mv, \quad \text{and} \quad \frac{mu^2}{2} = \Delta E + \frac{mu'^2}{2} + \frac{Mv^2}{2}.$$

This system of two algebraic equations for two unknowns, u' and v, may be readily solved by the elimination of one of them (say, u'), using the first equation:

$$u' = u - \frac{M}{m}v, \qquad (*)$$

and then solving the second equation, which becomes a quadratic equation for v. The result (with the physically acceptable positive sign before the square root, giving a reduction of v with the growth of ΔE) is

$$v = \frac{mu}{m+M} + \left[\left(\frac{mu}{m+M}\right)^2 - \frac{2\Delta E}{M}\frac{m}{m+M}\right]^{1/2}.$$

This formula gives a real value for v only if the expression inside the square brackets is non-negative, so the v corresponding to the internal excitation's threshold is given by the first term alone:

$$v_{\mathrm{t}} = \frac{m}{m+M}u. \qquad (**)$$

Now plugging the value of u expressed from this equality into Eq. (*), we immediately see that on this threshold, $u' = v$, regardless of ΔE.

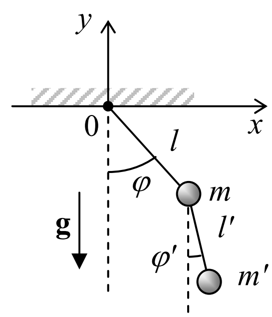
This result acquires a very clear meaning using the notion of the system's center of mass (see, e.g., Sec. 3.4 of the lecture notes): Eq. (**) gives its velocity as well, so at the internal excitation threshold, both final particles move together.[19]

[19] This fact remains valid in the relativity theory as well – see, e.g., EM Sec. 9.3.

Chapter 2. Lagrangian Analytical Mechanics

In each of Problems 1-11, for the given system:

(i) introduce a convenient set of generalized coordinates q_j,
(ii) write down the Lagrangian L as a function of $q_j, \dot{q}_j$, and (if appropriate) time,
(iii) write down the Lagrange equation(s) of motion,
(iv) calculate the Hamiltonian function H; find out whether it is conserved,
(v) calculate the mechanical energy E; is $E = H$?; is the energy conserved?
(vi) any other evident integrals of motion?



Problem 2.1. A double pendulum – see the figure on the right. Consider only the motion within the vertical plane containing the suspension point.

Solution: It is convenient to use the angles φ and φ' of the particle's deviations from their equilibrium (vertical) positions (see the figure on the right) as the generalized coordinates. From this figure, it is straightforward to write the following geometric relations for the Cartesian coordinates of the masses:

$$x = l\sin\varphi, \qquad y = -l\cos\varphi,$$
$$x' = l\sin\varphi + l'\sin\varphi', \qquad y' = -l\cos\varphi - l'\cos\varphi'.$$

Differentiating these relations over time, we get the following expressions for the Cartesian components of particle velocities:

$$\dot{x} = l\dot{\varphi}\cos\varphi, \qquad \dot{y} = l\dot{\varphi}\sin\varphi,$$
$$\dot{x}' = l\dot{\varphi}\cos\varphi + l'\dot{\varphi}'\cos\varphi', \qquad \dot{y}' = l\dot{\varphi}\sin\varphi + l'\dot{\varphi}'\sin\varphi'.$$

Plugging these expressions into the Lagrangian function defined by Eq. (2.19b) of the lecture notes,

$$L \equiv T - U = \left[\frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{m'}{2}\left(\dot{x}'^2 + \dot{y}'^2\right)\right] - \left(mgy + m'gy'\right),$$

we may readily bring the result to the following form:

$$L = \left[\frac{m+m'}{2}l^2\dot{\varphi}^2 + \frac{m'}{2}l'^2\dot{\varphi}'^2 + m'll'\dot{\varphi}\dot{\varphi}'\cos(\varphi - \varphi')\right] + \left[(m+m')gl\cos\varphi + m'gl'\cos\varphi'\right]. \qquad (*)$$

Now plugging this expression into Eq. (2.19a) of the lecture notes, with $q_1 \equiv \varphi$ and $q_2 \equiv \varphi'$, we get the following equations of motion:

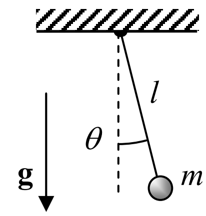
$$\frac{d}{dt}\left[(m+m')l^2\dot{\varphi} + m'll'\dot{\varphi}'\cos(\varphi - \varphi')\right] - \left[-(m+m')gl\sin\varphi - m'll'\dot{\varphi}\dot{\varphi}'\sin(\varphi - \varphi')\right] = 0,$$
$$\frac{d}{dt}\left[m'l'^2\dot{\varphi}' + m'll'\dot{\varphi}\cos(\varphi - \varphi')\right] - \left[-m'gl'\sin\varphi' + m'll'\dot{\varphi}\dot{\varphi}'\sin(\varphi - \varphi')\right] = 0. \qquad (**)$$

Since the kinetic energy of the system, expressed by the first square bracket in Eq. (*), is a quadratic-homogeneous function of the generalized velocities $\dot{\varphi}$ and $\dot{\varphi}'$, the Hamiltonian function H and energy E are equal to each other:

$$H = E = T + U = \left[\frac{m+m'}{2} l^2\dot{\varphi}^2 + \frac{m'}{2} l'^2\dot{\varphi}'^2 + m'll'\dot{\varphi}\dot{\varphi}'\cos(\varphi - \varphi')\right] - \left[(m+m')gl\cos\varphi + m'gl'\cos\varphi'\right],$$

and since $\partial L/\partial t = 0$, they are both conserved at the system's motion.

Problem 2.2. A stretchable pendulum (i.e. a massive particle hung on a spring that exerts force $F = -\kappa(l - l_0)$, where κ and l_0 are positive constants), also confined to the vertical plane.



Solution: The kinetic and gravitational-field potential energies of this system are similar to those of the top pendulum in the previous problem. Taking into account the spring's potential energy, we get

$$L = T - U = \frac{m}{2}\left(\dot{l}^2 + l^2\dot{\theta}^2\right) + mgl\cos\theta - \frac{\kappa}{2}(l - l_1)^2 + \text{const},$$

where $l_1 \equiv l_0 + mg/\kappa$ is the pendulum's length in equilibrium. Using l and θ as the generalized coordinates, from Eq. (2.19a) of the lecture notes, we get the following equations of motion:

$$\frac{d}{dt}\left(m\dot{l}\right) - \left[ml\dot{\theta}^2 + mg\cos\theta - \kappa(l - l_1)\right] = 0,$$

$$\frac{d}{dt}\left(ml^2\dot{\theta}\right) - \left(-mgl\sin\theta\right) = 0.$$

Since the kinetic energy T is a quadratic-homogeneous function of $\dot{l}$ and $\dot{\theta}$, H equals the total energy E:

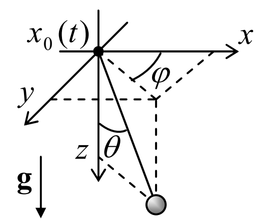
$$H = E = T + U = \frac{m}{2}\left(\dot{l}^2 + l^2\dot{\theta}^2\right) - mgl\cos\theta + \frac{\kappa}{2}(l - l_1)^2 + \text{const},$$

and since $\partial L/\partial t = 0$, both are conserved.

Problem 2.3. A fixed-length pendulum hanging from a point whose motion $x_0(t)$ in the horizontal direction is fixed. (No vertical plane constraint here.)

Solution: As an obvious generalization of the two previous problems, the Lagrangian of this system is

$$L \equiv T - U = \frac{m}{2}\left\{\left[\dot{x} + \dot{x}_0(t)\right]^2 + \dot{y}^2 + \dot{z}^2\right\} + mgz,$$



where x, y, and z are the pendulum coordinates in the (non-inertial) system of the moving support (see the figure on the right). Introducing spherical coordinates shown in the figure:

$$x = l\sin\theta\cos\varphi, \quad y = l\sin\theta\sin\varphi, \quad z = l\cos\theta,$$

we can use the angles θ and φ as the generalized coordinates of the pendulum.[20] In these coordinates,

$$L = T - U = \frac{m}{2}\left[l^2\dot\theta^2 + l^2\dot\varphi^2\sin^2\theta + \dot x_0^2(t) + 2l\dot x_0(t)\left(\dot\theta\cos\theta\cos\varphi - \dot\varphi\sin\theta\sin\varphi\right)\right] + mgl\cos\theta.$$

From here, the Lagrange equations of motion are

$$\ddot\theta - \dot\varphi^2\sin\theta\cos\theta + \frac{\ddot x_0(t)}{l}\cos\theta\cos\varphi + \frac{g}{l}\sin\theta = 0,$$
$$\ddot\varphi\sin^2\theta + 2\dot\theta\dot\varphi\sin\theta\cos\theta - \frac{\ddot x_0(t)}{l}\sin\theta\sin\varphi = 0.$$

We see that the equations depend only on the support point's *acceleration* – as they should because neither a constant displacement of that point nor its motion with a constant velocity can affect the pendulum's motion relative to it.

According to its definition (see Eq. (2.32) of the lecture notes), the Hamiltonian function of the system is

$$H \equiv \frac{\partial L}{\partial\dot\theta}\dot\theta + \frac{\partial L}{\partial\dot\varphi}\dot\varphi - L = \frac{m}{2}\left[l^2\dot\theta^2 + l^2\sin^2\theta\,\dot\varphi^2 - \dot x_0^2(t)\right] - mgl\cos\theta,$$

(where the above constant is taken for zero) while its energy is

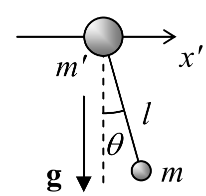
$$E \equiv T + U = \frac{m}{2}\left[l^2\dot\theta^2 + l^2\sin^2\theta\,\dot\varphi^2 + \dot x_0^2(t) + 2l\dot x_0(t)(\dot\theta\cos\theta\cos\varphi - \dot\varphi\sin\theta\sin\varphi)\right] - mgl\cos\theta.$$

We see that $H \neq E$. This is natural, since the kinetic energy of the system,

$$T = \frac{m}{2}\left[l^2\dot\theta^2 + l^2\dot\varphi^2\sin^2\theta + \dot x_0^2(t) + 2l\dot x_0(t)\left(\dot\theta\cos\theta\cos\varphi - \dot\varphi\sin\theta\sin\varphi\right)\right],$$

is *not* a quadratic-homogeneous function of the generalized velocities – because $\dot x_0(t)$ is a fixed function of time and does not qualify as a generalized velocity. Since for an arbitrary function $x_0(t)$, $\partial L/\partial t \neq 0$, neither of H and E is conserved.

Problem 2.4. A pendulum of mass m, hung on another point mass m' that may slide, without friction, along a straight horizontal rail – see the figure on the right. The motion is confined to the vertical plane that contains the rail.



Solution: Convenient generalized coordinates are: the position x' of the mass m' and the angle θ of the pendulum's deviation from equilibrium. Indeed, the Cartesian coordinates of the pendulum may be readily expressed via x' and θ:

$$x = x' + l\sin\theta, \quad y = -l\cos\theta.$$

These relations enable an easy calculation of the Lagrangian function of the generalized coordinates and velocities:

[20] Note that $x_0(t)$ is a fixed (given) function of time rather than a degree of freedom.

$$L \equiv T - U = \frac{m'}{2}\dot{x}'^2 + \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - mgy = \frac{m'}{2}\dot{x}'^2 + \frac{m}{2}\left[\left(\dot{x}' + \dot{\theta} l \cos\theta\right)^2 + \left(\dot{\theta} l \sin\theta\right)^2\right] + mgl\cos\theta$$

$$\equiv \left(\frac{m+m'}{2}\right)\dot{x}'^2 + \frac{m}{2}l^2\dot{\theta}^2 + ml\left(\dot{x}'\dot{\theta} + g\right)\cos\theta,$$

and hence obtaining the following Lagrange equations of motion:

$$\frac{d}{dt}\left[(m+m')\dot{x}' + ml\dot{\theta}\cos\theta\right] = 0,$$
$$\frac{d}{dt}\left[ml^2\dot{\theta} + ml\dot{x}'\cos\theta\right] + ml\left(\dot{x}'\dot{\theta} + g\right)\sin\theta = 0. \qquad (*)$$

Note that the second of these equations includes only one particle's mass, and upon the differentiation of the square bracket, may be much simplified: [21]

$$l\ddot{\theta} + \ddot{x}'\cos\theta + g\sin\theta = 0.$$

Since $\partial L/\partial t = 0$, one integral of motion may be obtained from the conservation of the Hamiltonian function

$$H \equiv \sum_j p_j \dot{q}_j - L = p_{x'}\dot{x}' + p_\theta \dot{\theta} - L,$$

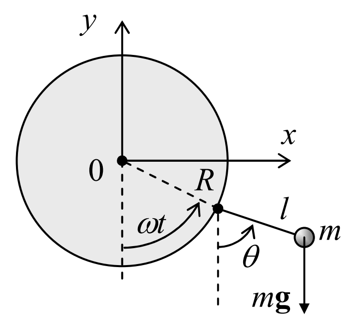
where the generalized momenta $p_{x'}$ and p_θ are given (by definition) by the derivatives of L over the corresponding generalized velocities, i.e. by the square brackets in Eqs. (*). Plugging them into the expression for H, we see that for this particular problem[22]

$$H = \left(\frac{m+m'}{2}\right)\dot{x}'^2 + \frac{m}{2}l^2\dot{\theta}^2 + ml\dot{x}'\dot{\theta}\cos\theta - mgl\cos\theta = T + U \equiv E,$$

i.e. this integral of motion corresponds to the energy conservation law. This is natural for a system moving, without friction, in the potential field of the gravitational force.

The first of Eqs. (*) immediately yields the one more first integral of motion, $p_{x'}$ = const, corresponding to the horizontal component of the total momentum of the system. (Physically, it is conserved because neither of the external forces has a horizontal component.)

Problem 2.5. A point-mass pendulum of length l, attached to the rim of a disk of radius R, which is rotated in a vertical plane with a constant angular velocity ω – see the figure on the right. (Consider only the motion within the disk's plane.)



[21] This equation may be also derived by writing the 2nd Newton law for the particle of mass m in the reference frame connected to the particle of mass m', upon compensating the non-inertiality of the frame by an additional "inertial force" $\mathbf{F}_{in} = -m\mathbf{a}'$ – see Sec. 4.6 of the lecture notes.

[22] The equality $E = H$ might be predicted even without an explicit calculation of H because the kinetic energy T of this system is a quadratic-homogeneous function of the generalized velocities $\dot{x}'$ and $\dot{\theta}$.

Solution: Since the rotation of the disk is fixed, i.e. does not depend on the pendulum's dynamics, the system has just one degree of freedom, and it is convenient to take the angle θ, shown in the figure above, for its (only) generalized coordinate. Indeed, let us express the Cartesian coordinates introduced as shown in the figure, as functions of θ and time:[23]

$$x = R\sin\omega t + l\sin\theta, \qquad y = -R\cos\omega t - l\cos\theta .$$

The (full) differentiation of these expressions over time yields

$$\dot{x} = R\omega\cos\omega t + l\dot{\theta}\cos\theta, \qquad \dot{y} = R\omega\sin\omega t + l\dot{\theta}\sin\theta ,$$

so the kinetic energy of the pendulum is

$$T \equiv \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) = \frac{m}{2}\left[\left(R\omega\cos\omega t + l\dot{\theta}\cos\theta\right)^2 + \left(R\omega\sin\omega t + l\dot{\theta}\sin\theta\right)^2\right]$$
$$\equiv \frac{m}{2}\left[R^2\omega^2 + l^2\dot{\theta}^2 + 2Rl\omega\dot{\theta}\cos(\theta - \omega t)\right],$$

while its potential energy (defined, as usual, to an additive constant) is

$$U = mgy = -mg\left(R\cos\omega t + l\cos\theta\right).$$

Hence the Lagrangian function is

$$L \equiv T - U = \frac{m}{2}\left[R^2\omega^2 + l^2\dot{\theta}^2 + 2Rl\omega\dot{\theta}\cos(\theta - \omega t)\right] + mg\left(R\cos\omega t + l\cos\theta\right),$$

giving the following equation of motion:

$$\frac{d}{dt}p_\theta - \left[-mRl\omega\dot{\theta}\sin(\theta - \omega t) - mgl\sin\theta\right] = 0, \qquad \text{with } p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + mRl\omega\cos(\theta - \omega t).$$

After the explicit differentiation of p_θ, and the division of all terms by ml^2, this equation simplifies:

$$\ddot{\theta} + \frac{R}{l}\omega^2\sin(\theta - \omega t) + \frac{g}{l}\sin\theta = 0 .$$

Note a curious fact: in the absence of gravity, this equation may be rewritten as

$$\ddot{\tilde{\theta}} + \Omega^2\sin\tilde{\theta} = 0, \qquad \text{where } \tilde{\theta} \equiv \theta - \omega t \text{ and } \Omega^2 \equiv \frac{R}{l}\omega^2 ,$$

showing that the rotated pendulum oscillates about the direction toward the disk's center *exactly* as the usual pendulum oscillates about its vertical position – even for arbitrary deviations from the equilibrium, and an arbitrary l/R ratio. This similarity may seem counter-intuitive because the centrifugal inertial "force" $m\omega^2\boldsymbol{\rho}$ (see Sec. 4.6 of the lecture notes) depends on the suspended point's distance ρ from the rotation axis, while the gravity force (driving the usual pendulum's dynamics) does not. The resolution of this paradox may be readily obtained by taking into account that the centrifugal "force" is directed along the vector $\boldsymbol{\rho}$ connecting the rotation center with the suspended mass rather than with the suspension point. (This simple calculation is highly recommended to the reader.)

[23] The angular deviation of the pendulum's suspension point from the vertical line equals ωt + const. To keep equations simple, we may always turn this constant to zero by an appropriate selection of the time origin.

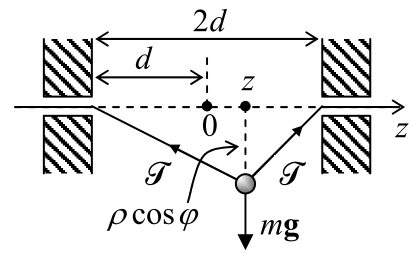
Now returning to an arbitrary g, and taking into account that the term $mR^2\omega^2/2$ does not depend on the pendulum's motion, the Hamiltonian function of the system may be represented as

$$H \equiv p_\theta \dot{\theta} - L = \frac{m}{2} l^2 \dot{\theta}^2 - mg\left(R\cos\omega t + l\cos\theta\right) + \text{const},$$

while its energy is different:

$$E \equiv T + U = \frac{m}{2} l^2 \dot{\theta}^2 - mg\left(R\cos\omega t + l\cos\theta\right) + mRl\omega\dot{\theta}\cos(\theta - \omega t) + \text{const}.$$

Since $\partial L/\partial t \neq 0$ and $E \neq H$, neither of these functions is an integral of motion.



Problem 2.6. A bead of mass m, sliding without friction along a light string with a fixed tension $\mathscr{T}$, hung between two horizontally displaced supports – see the figure on the right. Here, in contrast to the similar Problem 1.10, the tension $\mathscr{T}$ may be comparable with the bead's weight mg, and the motion is *not* restricted to the vertical plane.

Solution: Since the length l of the string between the support points is not fixed, the system has three degrees of freedom. For the generalized coordinates, it is convenient to use the cylindrical coordinates $\{\rho, \varphi, z\}$, where the z-axis passes through the support points, and φ is the angle of rotation of the string/bead plane about this z-axis. In these coordinates,

$$T = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right), \qquad U = -mg\rho\cos\varphi + \mathscr{T}l,$$

where l is the total length of both segments of the string:

$$l = \left[\rho^2 + (z-d)^2\right]^{1/2} + \left[\rho^2 + (z+d)^2\right]^{1/2}.$$

(The second term in the potential energy U accounts for the work necessary to pull out the string to length l, overcoming the fixed tension $\mathscr{T}$.[24]) Using these expressions to compose the Lagrangian function of the system,

$$L = T - U = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right) + mg\rho\cos\varphi - \mathscr{T}\left\{\left[\rho^2 + (z-d)^2\right]^{1/2} + \left[\rho^2 + (z+d)^2\right]^{1/2}\right\},$$

we get the following Lagrange equations of motion:

$$\frac{d}{dt}(m\dot{\rho}) - mg\cos\varphi + \mathscr{T}\left\{\frac{\rho}{\left[\rho^2 + (z-d)^2\right]^{1/2}} + \frac{\rho}{\left[\rho^2 + (z+d)^2\right]^{1/2}}\right\} = 0,$$

$$\frac{d}{dt}\left(m\rho^2\dot{\varphi}\right) + mg\rho\sin\varphi = 0,$$

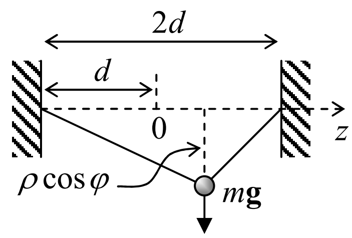
$$\frac{d}{dt}(m\dot{z}) + \mathscr{T}\left\{\frac{z-d}{\left[\rho^2 + (z-d)^2\right]^{1/2}} + \frac{z+d}{\left[\rho^2 + (z+d)^2\right]^{1/2}}\right\} = 0.$$

[24] This net U is essentially the Gibbs potential energy of the bead – see Sec. 1.4 of the lecture notes.

(A partial analysis of these equations will be the subject of Problem 3.1.)

Since $\partial L/\partial t = 0$, and T is a quadratic-homogeneous function of the generalized velocities $\dot{\rho}$, $\dot{\varphi}$, and $\dot{z}$, we may conclude that $E = H$, forming the following first integral of motion:

$$E = H = T + U = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right) - mg\rho\cos\varphi + \mathscr{T}\left\{\left[\rho^2 + (z-d)^2\right]^{1/2} + \left[\rho^2 + (z+d)^2\right]^{1/2}\right\}.$$



Problem 2.7. A bead of mass m, sliding without friction along a light string of a fixed length $2l$, that is hung between two support points displaced horizontally by distance $2d < 2l$ – see the figure on the right. As in the previous problem, the motion is not restricted to the vertical plane.

Solution: Just like in the previous problem, we may use the cylindrical coordinates of the bead, with the z-axis passing through the endpoints of the string, and the angle φ of rotation of the string/bead plane about that axis, measured from the vertical position. In these coordinates, the kinetic and potential energies are also expressed very similarly to those in the previous problem,

$$T = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right), \qquad U = -mg\rho\cos\varphi .$$

However, in our current case, the total length $2l$ of the string is fixed, imposing the following geometric relation between ρ and z:

$$2l = \left[\rho^2 + (d-z)^2\right]^{1/2} + \left[\rho^2 + (d+z)^2\right]^{1/2} = \text{const}, \qquad (*)$$

and hence reducing the number of degrees of freedom from three to two. It is possible to use Eq. (*) for the elimination of either ρ or (better) z from T and U, and hence express the Lagrangian function $L = T - U$ via two remaining generalized coordinates; however, the resulting expressions would be rather bulky.

A better choice of generalized coordinates stems from another form of Eq. (*):[25]

$$\frac{\rho^2}{s^2} + \frac{z^2}{l^2} = 1, \qquad \text{where } s \equiv \left(l^2 - d^2\right)^{1/2} < l . \qquad (**)$$

This formula shows that within the string's plane, the bead moves along an ellipse with semi-major and semi-minor axes equal, respectively, to l and s.[26] It shows also that the only degree of freedom of the bead on that plane may be conveniently described by the angle θ defined as[27]

$$\rho = s\sin\theta, \quad z = l\cos\theta .$$

Plugging these relations into expressions for T and U, we may express L as a function of angles θ and φ (and their time derivatives):

---

[25] It may be readily obtained by squaring both parts of Eq. (*), then isolating the resulting single square root in one part of the resulting equation, squaring the equation again, and collecting/canceling similar terms.

[26] This means that our current problem is equivalent to that of a particle's motion on the surface of a *degenerate ellipsoid* (or "spheroid") – see also EM Sec. 2.4.

[27] In astronomy, the angle θ is sometimes called the *true anomaly*. Note that it coincides with the usual polar angle only in the limit $l/d \to \infty$, i.e. $s \to l$.

$$L = T - U = \frac{m}{2}\left[\left(l^2 \sin^2\theta + s^2 \cos^2\theta\right)\dot{\theta}^2 + s^2 \sin^2\theta\, \dot{\varphi}^2\right] + mgs\sin\theta\cos\varphi .$$

Now it is straightforward to write the Lagrange equations of motion for θ and φ:

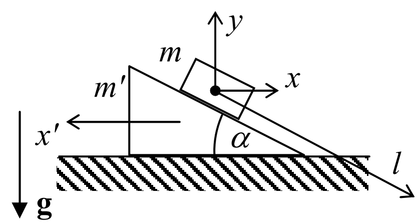
$$\frac{d}{dt}\left[m\left(l^2 \sin^2\theta + s^2 \cos^2\theta\right)\dot{\theta}\right] + md^2\dot{\theta}^2 \sin\theta\cos\theta - ms^2\dot{\varphi}^2 \sin\theta\cos\theta - mgs\cos\theta\cos\varphi = 0,$$

$$\frac{d}{dt}\left(ms^2\dot{\varphi}\sin^2\theta\right) + mgs\sin\theta\sin\varphi = 0.$$

(A partial analysis of these equations will be the subject of Problem 3.2.)

Since $\partial L/\partial t = 0$, and T is a quadratic-homogeneous function of the generalized velocities $\dot{\theta}$ and $\dot{\varphi}$, we may conclude that $E = H$, forming the first integral of motion:

$$E = H = T + U = \frac{m}{2}\left[\left(l^2 \sin^2\theta + s^2 \cos^2\theta\right)\dot{\theta}^2 + s^2 \sin^2\theta\, \dot{\varphi}^2\right] - mgs\sin\theta\cos\varphi .$$



Problem 2.8. A block of mass m that can slide, without friction, along the inclined surface of a heavy wedge (mass m'). The wedge is free to move, also without friction, along a horizontal surface – see the figure on the right. (Both motions are within the vertical plane containing the steepest slope line.)

Solution: From the problem's geometry, the Cartesian coordinates x and y of the block (see the figure above) in a lab frame may be readily related to the horizontal coordinate x' of the wedge and the shift l of the block along the wedge:[28]

$$x = l\cos\alpha - x', \quad y = -l\sin\alpha, \qquad (*)$$

so just two of them (for example, l and x') may be used as the generalized coordinates for this problem. (Other equally convenient choices of two independent generalized coordinates are also possible here.) Using these relations, the Lagrangian function of the system,

$$L \equiv T - U = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{m'}{2}\dot{x}'^2 - mgy ,$$

may be rewritten as

$$L = \frac{m}{2}\dot{l}^2 + \frac{m+m'}{2}\dot{x}'^2 - m\dot{l}\dot{x}'\cos\alpha + mgl\sin\alpha .$$

This formula yields the following Lagrange equations of motion:

$$\frac{d}{dt}\left[m\dot{l} - m\dot{x}'\cos\alpha\right] - mg\sin\alpha = 0 ,$$

$$\frac{d}{dt}\left[(m+m')\dot{x}' - m\dot{l}\cos\alpha\right] = 0 . \qquad (**)$$

[28] At this translational motion, it is not important what exactly points of the block and the wedge we are speaking about, because any different choice would result only in additional inconsequential constants in Eqs. (*).

Since the kinetic energy of the system is a quadratic-homogeneous function of the generalized velocities $\dot{x}'$ and $\dot{l}$, the Hamiltonian function is equal to the total energy E:

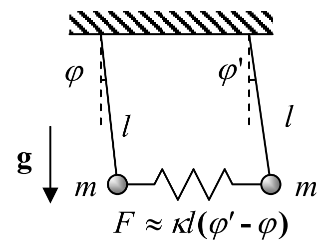
$$H = E = T + U = \frac{m}{2}\dot{l}^2 + \frac{m+m'}{2}\dot{x}'^2 - m\dot{l}\dot{x}'\cos\alpha - mgl\sin\alpha,$$

and since $\partial L/\partial t = 0$, both functions are conserved.

Another integral of motion,

$$(m+m')\dot{x}' - m\dot{l}\cos\alpha = \text{const},$$

immediately follows from Eq. (**). Physically, this is just the horizontal component of the total momentum of this two-body system; it is conserved because neither of the external forces exerted on the system (the gravity forces and the horizontal surface's reaction) has a horizontal component.



Problem 2.9. The two-pendula system that was the subject of Problem 1.8 – see the figure on the right.

Solution: In terms of angular displacements φ and φ' (see the figure on the right), the linear approximation for forces, which may be used for the case of small oscillations, corresponds to the quadratic approximation for the kinetic and potential energies, so we can take

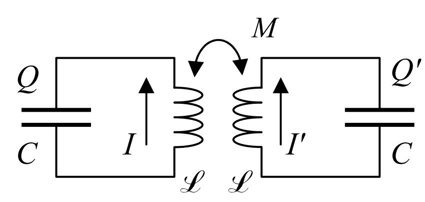
$$L = \frac{m}{2}l^2\left(\dot{\varphi}^2 + \dot{\varphi}'^2\right) - \frac{mgl}{2}\left(\varphi^2 + \varphi'^2\right) - \frac{\kappa}{2}l^2\left(\varphi - \varphi'\right)^2 + \text{const}.$$

This expression yields the Lagrange equations of motion that coincide with the equations derived in the model solution of Problem 1.8 directly from the Newton laws.

Since the kinetic energy is a quadratic function of the angular velocities and the Lagrangian function does not depend on time explicitly, the Hamiltonian function and energy coincide,

$$H = E = \frac{m}{2}l^2\left(\dot{\varphi}^2 + \dot{\varphi}'^2\right) + \frac{mgl}{2}\left(\varphi^2 + \varphi'^2\right) + \frac{\kappa}{2}l^2\left(\varphi - \varphi'\right)^2 + \text{const},$$

and are the first integrals (or rather the same integral) of motion.



Problem 2.10. A system of two similar, inductively coupled LC circuits – see the figure on the right.

Solution: An not very high frequencies, the total magnetic energy of two coupled inductive coils is[29]

$$E_\text{m} = \frac{\mathscr{L}}{2}\left(I^2 + I'^2\right) + MII',$$

where M is the coefficient of mutual inductance of the coils, and $I = \dot{Q}$ and $I' = \dot{Q}'$ are the currents flowing through them – see the figure above. If we use the electric charges Q and Q' of the capacitors as

[29] If this expression is not evident, you may consult, e.g., EM Sec. 5.3, in particular Eq. (5.62).

the generalized coordinates, then E_m should be considered the kinetic energy, while the energy $E_e = Q^2/2C + Q'^2/2C$ of the electric field in the capacitors, the potential energy. As a result, the Lagrangian function of the system is

$$L \equiv T - U = E_e - E_m = \frac{\mathscr{L}}{2}\left(\dot{Q}^2 + \dot{Q}'^2\right) + M\dot{Q}\dot{Q}' - \frac{1}{2C}\left(Q^2 + Q'^2\right),$$

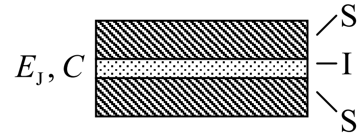
giving the following Lagrange equations of motion,

$$\frac{d}{dt}\left(\mathscr{L}\dot{Q} + M\dot{Q}'\right) + \frac{Q}{C} = 0, \qquad \frac{d}{dt}\left(\mathscr{L}\dot{Q}' + M\dot{Q}\right) + \frac{Q'}{C} = 0,$$

whose structure is similar to those of two coupled mechanical oscillators – cf. Problems 9 and 1.8. As in those cases, both E and H are conserved and equal to each other:

$$H = E = \frac{\mathscr{L}}{2}\left(\dot{Q}^2 + \dot{Q}'^2\right) + M\dot{Q}\dot{Q}' + \frac{1}{2C}\left(Q^2 + Q'^2\right).$$

Problem 2.11.* A small Josephson junction – the system consisting of two superconductors (S), weakly coupled by Cooper-pair tunneling through a thin insulating layer (I) that separates them – see the figure on the right.



Hints:

(i) At not very high frequencies (whose quanta $\hbar\omega$ are lower than the binding energy 2Δ of the Cooper pairs), the Josephson effect in a sufficiently small junction may be well described by the following coupling energy:

$$U(\varphi) = -E_J \cos\varphi + \text{const},$$

where the constant E_J is the coupling strength, while the variable φ (called the *Josephson phase difference*) is connected to the voltage V across the junction by the famous *frequency-to-voltage relation*

$$\omega \equiv \frac{d\varphi}{dt} = \frac{2e}{\hbar}V,$$

where $e \approx 1.602\times10^{-19}$ C is the fundamental charge and $\hbar \approx 1.054\times10^{-34}$ J·s is Plank's constant.[30]

(ii) The junction (as any system of two conductors) has a certain electric capacitance C.

Solution: In order to describe the dynamics of the variable φ, one needs to take into account the (unavoidable, and typically very substantial) capacitance C between the junction electrodes, which provides electric energy

$$E_e = \frac{C}{2}V^2.$$

The Josephson frequency-to-voltage relation cited in the *Hints* shows that if we accept the phase φ as the generalized coordinate, and hence treat $U_J = -E_J \cos\varphi$ + const as the *potential* energy U of the

[30] For more on the Josephson effect and the physical sense of the variable φ, see, e.g., EM Sec. 6.5 and QM Secs. 1.6 and 2.8, but the given problem may be solved without that additional information.

junction, we should associate the voltage V (or rather the fraction $2eV/\hbar$) with the generalized velocity, and hence the E_e, rewritten as

$$E_e = \frac{C}{2}\left(\frac{\hbar}{2e}\right)^2 \dot{\varphi}^2,$$

with the *kinetic* energy T. Plugging the formulas for U and T into the general Lagrange equation of motion,[31] we get

$$\frac{d}{dt}\left[C\left(\frac{\hbar}{2e}\right)^2 \dot{\varphi}\right] + E_J \sin\varphi = 0 .$$

This equation coincides with Eq. (2.26) for a mechanical pendulum, with the small-oscillation frequency

$$\omega_p = \frac{2e}{\hbar}\left(\frac{E_J}{C}\right)^{1/2} .$$

At these oscillations, an excessive part of the superconducting Bose-Einstein condensate of Cooper pairs tunnels through the insulating layer, moving back and forth between the superconducting electrodes, driven by the electric field induced by the resulting electric charge imbalance. Due to a far-reaching analogy between these oscillations and those of the usual collision-free plasma,[32] the frequency ω_p (or its cyclic counterpart $f_p \equiv \omega_p/2\pi$ – typically, of the order of a few GHz) is called the *plasma frequency* of the Josephson junction.

[31] Note that such a simple equation describes only a junction that is not connected to any external circuit. If it is, several fascinating phenomena may arise – see, e.g., the lecture note sections cited in the assignment.

[32] See, e.g., EM Sec. 7.2.

Chapter 3. A Few Simple Problems

Problem 3.1. For the system considered in Problem 2.6 (a bead sliding along a string with a fixed tension $\mathscr{T}$, see the figure on the right), analyze small oscillations of the bead near the equilibrium.

Solution: The Lagrange equations of motion of this system, which were derived in the model solution of Problem 2.6, are

$$\frac{d}{dt}(m\dot{\rho}) - mg\cos\varphi + \mathscr{T}\rho\left\{\frac{1}{\left[\rho^2 + (z-d)^2\right]^{1/2}} + \frac{1}{\left[\rho^2 + (z+d)^2\right]^{1/2}}\right\} = 0,$$

$$\frac{d}{dt}\left(m\rho^2\dot{\varphi}\right) + mg\rho\sin\varphi = 0,$$

$$\frac{d}{dt}(m\dot{z}) + \mathscr{T}\left\{\frac{z-d}{\left[\rho^2 + (z-d)^2\right]^{1/2}} + \frac{z+d}{\left[\rho^2 + (z+d)^2\right]^{1/2}}\right\} = 0,$$

where ρ, φ, and z are the cylindrical coordinates of the bead – see the figure above. As is evident from this sketch, in equilibrium, $\varphi = \varphi_0 \equiv 0$ and $z = z_0 \equiv 0$, while the equilibrium vertical displacement ρ_0 may be readily found from the first equation of motion with $d/dt = 0$, $z = 0$, and $\varphi = 0$:

$$-mg + \frac{2\mathscr{T}\rho_0}{\left(\rho_0^2 + d^2\right)^{1/2}} = 0, \qquad \text{giving } \rho_0 = d\frac{mg/2\mathscr{T}}{\left[1-(mg/2\mathscr{T})^2\right]^{1/2}}. \qquad (*)$$

The last expression (which could be readily obtained from the balance of the vertical components of two string tension forces and the bead's weight in equilibrium as well) duly diverges at $2\mathscr{T} \to mg$, because a lower tension cannot balance the bead's weight even at very large ρ.

Now by linearizing the equations of motion with respect to small deviations $\tilde{\rho} \equiv \rho - \rho_0$, $\tilde{\varphi} \equiv \varphi - \varphi_0 = \varphi$, and $\tilde{z} \equiv z - z_0 = z$ from the equilibrium point, we get

$$m\ddot{\tilde{\rho}} + \frac{2\mathscr{T}d^2}{\left(\rho_0^2 + d^2\right)^{3/2}}\tilde{\rho} = 0,$$

$$m\rho_0^2\ddot{\tilde{\varphi}} + m\rho_0 g\tilde{\varphi} = 0,$$

$$m\ddot{\tilde{z}} + \frac{2\mathscr{T}\rho_0^2}{\left(\rho_0^2 + d^2\right)^{3/2}}\tilde{z} = 0,$$

where Eqs. (*) have been used to simplify the results. These formulas show that small, harmonic oscillations of the generalized coordinates in this particular system are independent, with different frequencies:

$$\omega_\rho^2 = \frac{2\mathscr{T}d^2}{m\left(\rho_0^2 + d^2\right)^{3/2}} \equiv \frac{2\mathscr{T}}{md}\left[1-\left(\frac{mg}{2\mathscr{T}}\right)^2\right]^{3/2},$$

$$\omega_\varphi^2 = \frac{g}{\rho_0} \equiv \frac{2\mathscr{T}}{md}\left[1-\left(\frac{mg}{2\mathscr{T}}\right)^2\right]^{1/2},$$

$$\omega_z^2 = \frac{2\mathscr{T}\rho_0^2}{m\left(\rho_0^2+d^2\right)^{3/2}} \equiv \frac{mg^2}{2\mathscr{T}d}\left[1-\left(\frac{mg}{2\mathscr{T}}\right)^2\right]^{1/2}.$$

In the first form of the result for ω_φ, we may readily recognize the well-known formula for the usual point pendulum. In the already mentioned limit $2\mathscr{T} \to mg$, ω_z also tends to this result – naturally, because in this limit, the string in equilibrium is much longer than $2d$, and the system is close to a spherical pendulum, with equal frequencies of oscillations in two horizontal directions. A bit counter-intuitively, in this limit, the frequency ω_ρ of the bead's vertical oscillations is even lower.

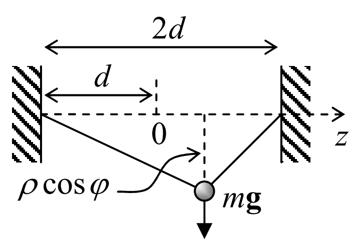
In the opposite limit of a very high tension, $2\mathscr{T}/mg \to \infty$, the frequency of the longitudinal oscillations of the bead (along the string) tends to that calculated, in a simpler way, in the solution of Problem 1.10,

$$\omega_z \to g\left(\frac{m}{2\mathscr{T}d}\right)^{1/2} \to 0,$$

while the frequencies of the two transverse oscillation modes are much higher and coincide:

$$\omega_\rho \to \omega_\varphi \to \left(\frac{2\mathscr{T}}{md}\right)^{1/2} >> \omega_z.$$

This is also natural because in this limit, the gravity force is too small (in comparison with the sting's tension) to affect these oscillation modes. As will be discussed in Chapters 6 and 7 of this course, this hierarchy (two similar transverse modes and one longitudinal mode) is also typical for mechanical waves propagating in any elastic isotropic media.



Problem 3.2. For the system considered in Problem 2.7 (a bead sliding along a string of a fixed length $2l$, see the figure on the right), analyze small oscillations near the equilibrium.

Solution: The equations of motion of this system, derived in the solution of Problem 2.7, were

$$\frac{d}{dt}\left[m\dot{\theta}\left(l^2\sin^2\theta + s^2\cos^2\theta\right)\right] + m\dot{\theta}^2 d^2\sin\theta\cos\theta - ms^2\dot{\varphi}^2\sin\theta\cos\theta - mgs\cos\theta\cos\varphi = 0,$$

$$\frac{d}{dt}\left(ms^2\dot{\varphi}\sin^2\theta\right) + mgs\sin\theta\sin\varphi = 0,$$

where θ is the bead's quasi-polar angle, while φ is the azimuthal angle of rotation of the bead/string plane about the horizontal axis, so the cylindrical coordinates of the bead are $\rho = s\sin\theta$, φ, and $z = l\cos\theta$ – see the figure above. The constant parameter $s \equiv (l^2 - d^2)^{1/2}$ is a measure of the string's "slack".

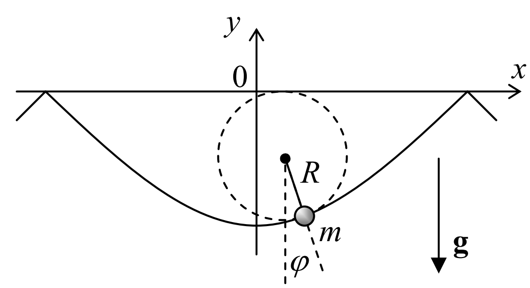
The bead's equilibrium in the middle of the string, in the vertical plane, corresponds to $\theta = \theta_0 \equiv \pi/2$ and $\varphi = \varphi_0 \equiv 0$. Now linearizing the equations of motion with respect to small deviations $\tilde{\theta} \equiv \theta - \theta_0 = \theta - \pi/2$ and $\tilde{\varphi} \equiv \varphi - \varphi_0 = \varphi$ (so $\sin\theta \approx 1$, $\cos\theta \approx \tilde{\theta}$, $\sin\varphi \approx \tilde{\varphi}$, and $\cos\varphi \approx 1$), we get

$$\frac{d}{dt}\left(ml^2\dot{\tilde{\theta}}\right) + mgs\tilde{\theta} = 0, \qquad \frac{d}{dt}\left(ms^2\dot{\tilde{\varphi}}\right) + mgs\tilde{\varphi} = 0.$$

These equations show that small harmonic oscillations of the coordinates θ and φ are independent of each other and have different frequencies:

$$\omega_\theta^2 = \frac{gs}{l^2} \equiv \frac{g\left(l^2 - d^2\right)^{1/2}}{l^2}, \qquad \omega_\varphi^2 = \frac{g}{s} \equiv \frac{g}{\left(l^2 - d^2\right)^{1/2}}, \qquad \text{so } \frac{\omega_\varphi}{\omega_\theta} = \frac{l}{s} \equiv \frac{l}{\left(l^2 - d^2\right)^{1/2}} > 1.$$

Only in the limit $l/d \to \infty$ when a long string's ends are attached to very close points, these frequencies tend to each other and to the well-known frequency $\omega = (g/l)^{1/2}$ of the usual point-mass pendulum. In the opposite limit when the slack s is much smaller than its length l, i.e. the string is stretched almost straight horizontally, the transverse oscillations of the bead are much faster than its longitudinal oscillations: $\omega_\varphi >> \omega_\theta$.



Problem 3.3. A bead is allowed to slide, without friction, along an inverted cycloid in a vertical plane – see the figure on the right. Calculate the frequency of its free oscillations as a function of their amplitude.

Hint: The simplest way to describe a cycloid is to express the Cartesian coordinates of its arbitrary point as functions of some parameter φ.[33] For the inverted cycloid shown in the figure on the right, such *parametric representation* is

$$x = R(\varphi + \sin\varphi), \qquad y = -R(1 + \cos\varphi).$$

Solution: The tangential component of the 2nd Newton's law for the bead is

$$m\frac{dv}{dt} = -mg\frac{dy}{ds}, \qquad (*)$$

where s is the length of the cycloid's part between the bead's position and some fixed point (say, the lowest point) of the curve. Evidently, for a plane curve

$$(ds)^2 = (dx)^2 + (dy)^2.$$

Differentiating the equations given in the *Hint*, we get

$$dx = R(1 + \cos\varphi)d\varphi, \qquad dy = R\sin\varphi d\varphi,$$

$$(ds)^2 = R^2(1 + \cos\varphi)(d\varphi)^2 \equiv 4R^2\cos^2\frac{\varphi}{2}(d\varphi)^2, \quad \text{i.e. } ds = 2R\cos\frac{\varphi}{2}d\varphi \equiv 4Rd\left(\sin\frac{\varphi}{2}\right),$$

[33] This parameter may be understood as the angle of rotation of a circle of the radius R, rolled along a horizontal rail with $y = 0$ (see the dashed lines in the figure above), whose point moves along the cycloid..

so

$$v \equiv \frac{ds}{dt} = 4R\frac{d}{dt}\sin\frac{\varphi}{2}, \quad \text{while} \quad \frac{dy}{ds} = \frac{R\sin\varphi\, d\varphi}{2R\cos(\varphi/2)d\varphi} \equiv \frac{2\sin(\varphi/2)\cos(\varphi/2)}{2\cos(\varphi/2)} \equiv \sin\frac{\varphi}{2}.$$

Plugging these equalities into Eq. (*) and dividing both sides by the product $4mR$, we get

$$\frac{d^2}{dt^2}\sin\frac{\varphi}{2} = -\omega^2\sin\frac{\varphi}{2}, \qquad \text{with } \omega \equiv \frac{1}{2}\left(\frac{g}{R}\right)^{1/2}.$$

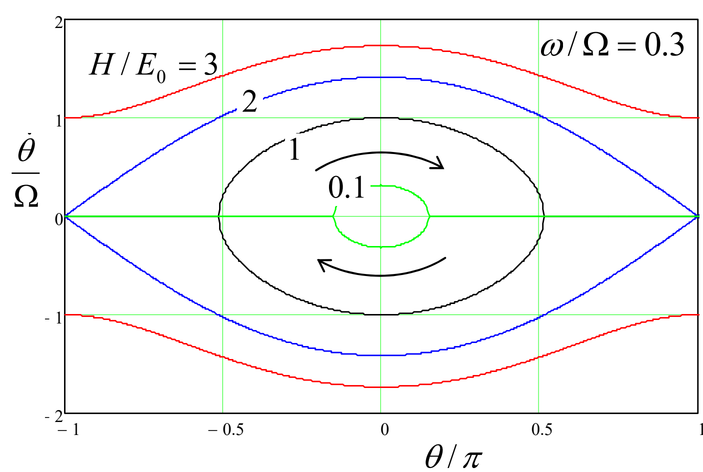
This is the usual equation of motion of a linear ("harmonic") oscillator – see, e.g., Eq. (3.12) of the lecture notes. Hence the variable $\sin(\varphi/2)$ performs sinusoidal oscillations with the amplitude-independent frequency ω.[34] This fact was discovered geometrically in 1659 by C. Huygens (who tried to use it for improving pendulum clocks' accuracy), and is the reason why the cycloid is also called *tautochrone* or *isochrone*.[35]

Problem 3.4. Illustrate the changes of the fixed point set of our testbed system (Fig. 2.1), which was analyzed at the end of Sec. 3.2 of the lecture notes, on the so-called *phase plane* $[\theta, \dot{\theta}]$.

Solution: As was discussed in Secs. 2.3 and 3.1 of the lecture notes, the Hamiltonian function of this system,

$$H = T_{\text{ef}} + U_{\text{ef}}, \quad \text{with } T_{\text{ef}} = \frac{m}{2}R^2\dot{\theta}^2, \quad U_{\text{ef}} = -\frac{m}{2}R^2\omega^2\sin^2\theta - mgR\cos\theta, \qquad (*)$$

is conserved, so on the phase plane $[\theta, \dot{\theta}]$, the point representing the system's state has to move along the line of some constant H (determined by initial conditions). Each of the figures below, calculated numerically from Eq. (*) for a specific value of the ring's rotation speed ω, shows the phase plane with a few such lines corresponding to the specified values of the ratio H/E_0, where $E_0 \equiv mgR$. When examining these plots, remember that the representation point always moves to the right in the upper half-plane, and in the opposite direction in the lower half-plane.



If ω^2 is well below its critical value $\Omega^2 \equiv g/R$ (see, e.g., the figure on the right), the lines near the origin are ellipses, describing harmonic oscillations of the bead about its only stable fixed point $\theta = 0$.[36] As the system's initial energy (and hence H) is increased, the

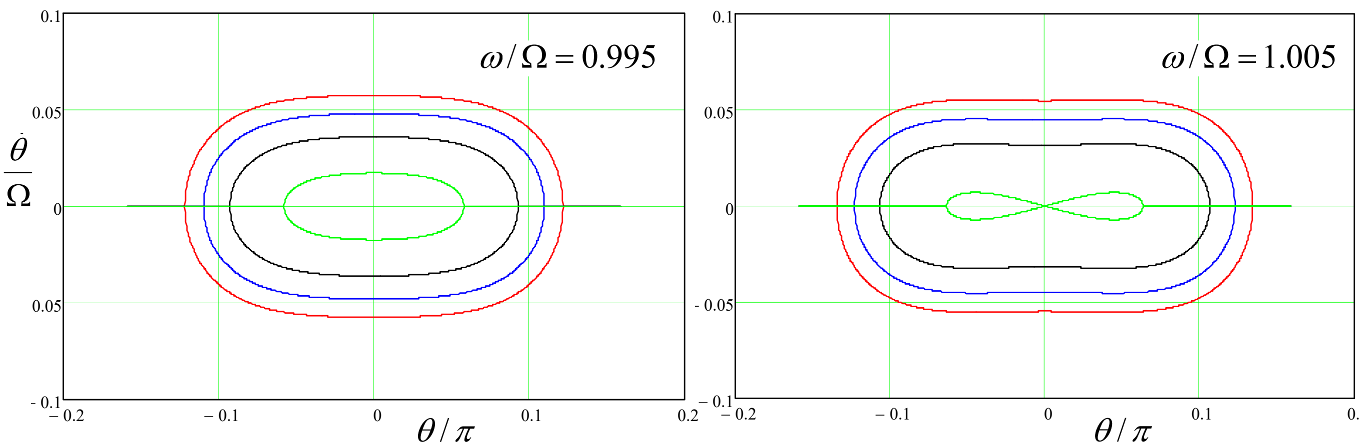
[34] The bead's coordinates s, x, and φ also oscillate with this amplitude-independent frequency, even though at large amplitudes, their waveforms are different from the sinusoidal ones.

[35] From Greek roots *tauto* and *iso* (meaning "same" or "equal") and *chronos* ("time"). The same curve is also called *brachistochrone* (from the Greek root *brachisto*, meaning "fast") because it ensures the shortest time of the particle's slide between two given points under its weight. This fact has played an important role in the development of the calculus of variations.

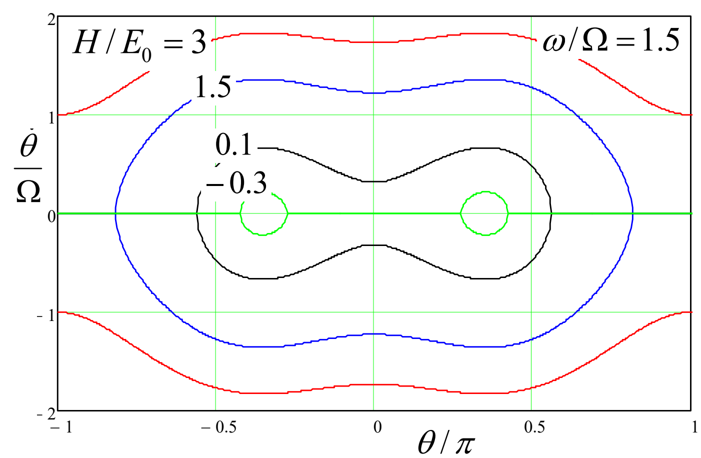
[36] Note that the system is 2π-periodic, so here and below, all statements about the angle θ are correct mod$[2\pi]$.

amplitude of the oscillations grows, and their shape deviates from the sinusoidal one. (This effect, pertaining to the usual pendulum as well, will be discussed in detail in Chapter 5 of this course.) Finally, as the initial energy of the system exceeds $2E_0$, the bead swings over the top point of the ring ($\theta = \pm\pi$), in the direction determined by the initial sign of it velocity.

However, at $\omega^2 \approx \Omega^2$, the near-origin topology of the pattern changes rapidly – compare the two panels of the figure below, plotted for an equidistant series of H, with step $\Delta H = 0.001E_0$. This change (frequently called "bifurcation" – see Chapter 9) illustrates the transition from one stable fixed point $\theta = 0$ at $\omega^2 < \Omega^2$ to two such points $\theta = \pm\cos^{-1}(\Omega^2/\omega^2)$ at $\omega^2 > \Omega^2$ – see Eq. (3.20) and its discussion.



Finally, as the ring's rotation speed ω is further increased, the new stable fixed points separate further, shifting toward their ultimate positions $\pm\pi/2$, now affecting even the global structure of the phase plane trajectories – see the figure on the right, plotted for the same ratio ω/Ω as the function $U_{ef}(\theta)$ in Fig. 3.2(d) of the lecture notes.



As the reader might see in this example, the phase plane is a very convenient tool for a vivid representation of motion in the systems described by nonlinear differential equations. It will be repeatedly used later in this course, especially in Chapters 5 and 9.

<u>Problem 3.5</u>. For a 1D particle of mass m, placed into the potential well $U(q) = \alpha q^{2n}$ (where $\alpha > 0$, and n is a positive integer), calculate the functional dependence of the particle's oscillation period $\mathcal{T}$ on its energy E. Explore the limit $n \to \infty$.

Solution: Let us reduce the integral in Eq. (3.27) of the lecture notes, with $H = E$, $m_{ef} = m$, and $U_{ef} = U(q) = \alpha q^{2n}$,

$$\mathcal{T} = 2\left(\frac{m}{2}\right)^{1/2} \int_B^A \frac{dq}{[E - U(q)]^{1/2}},$$

to a dimensionless form. Using the problem's symmetry with respect to the sign of q (giving $B = -A$), we get

$$\mathcal{T} = 2\left(\frac{m}{2}\right)^{1/2} 2\int_0^A \frac{dq}{\left(E-\alpha q^{2n}\right)^{1/2}} = \left(\frac{8m}{\alpha}\right)^{1/2}\int_0^A \frac{dq}{\left(A^{2n}-q^{2n}\right)^{1/2}} = \left(\frac{8m}{\alpha}\right)^{1/2}\frac{A}{A^n}\int_0^1 \frac{d\xi}{\left(1-\xi^{2n}\right)^{1/2}},$$

where $\xi \equiv q/A$, and the oscillation amplitude A may be found from the classical turning point condition $E - \alpha A^{2n} = 0$:

$$A = \left(E/\alpha\right)^{\frac{1}{2n}}.$$

Combining these two formulas, we get

$$\mathcal{T} \propto A^{1-n} \propto E^{\frac{1}{2n}-\frac{1}{2}}.$$

For the particular case $n = 1$, i.e. for $U(q) = \alpha q^2$, we recover the result (3.29), i.e. the energy-independent period of a harmonic oscillator, while for very steep potential wells with $n \to \infty$, the period decreases with energy as $\sim 1/E^{1/2}$. Note that this analysis did not require working out the dimensionless integral over ξ but if it is needed, it is given in MA Eq. (6.6a).

Problem 3.6. Two small masses m_1 and $m_2 \leq m_1$ may slide without friction over a horizontal surface. They are connected with a spring with an equilibrium length l and an elastic constant κ, and at $t < 0$ are at rest. At $t = 0$, the mass m_1 gets a very short kick with impulse $\mathbf{P} \equiv \int\mathbf{F}(t)dt$ in a direction different from the spring's line. Calculate the largest and smallest magnitude of its velocity at $t > 0$.

Solution: The Lagrangian function of this system has the same general form as for the planetary problems discussed in Sec. 3.4:

$$L \equiv T - U(r) = \frac{m_1}{2}\dot{\mathbf{r}}_1^2 + \frac{m_2}{2}\dot{\mathbf{r}}_2^2 - U(r), \qquad \text{with } r \equiv |\mathbf{r}|, \quad \mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2,$$

where in our current case, $U(r) = \kappa(r - l)^2/2$. Hence, we may use the same approach to its solution, making a transfer from the radius vectors $\mathbf{r}_{1,2}$ of the masses to the radius vector $\mathbf{r}$ of their mutual displacement and the position $\mathbf{R}$ of their center of mass – see Eqs. (3.31)-(3.33) of the lecture notes:

$$\mathbf{R} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}, \quad \text{with } M \equiv m_1 + m_2, \qquad \text{and hence } \mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r}, \quad \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}. \qquad (*)$$

In these new coordinates, the Lagrangian function has the form (3.34)

$$L = \frac{M}{2}\dot{\mathbf{R}}^2 + \frac{m}{2}\dot{\mathbf{r}}^2 - U(r), \quad \text{where } m \equiv \frac{m_1 m_2}{M}.$$

The resulting Lagrange equation for the generalized coordinate $\mathbf{R}$ is very simple: $\ddot{\mathbf{R}} = 0$, just confirming that the center of mass of this system moves as a free point:

$$\dot{\mathbf{R}}(t) \equiv \mathbf{V}(t) = \mathbf{V}(0) = \mathbf{const}, \qquad \mathbf{R}(t) = \mathbf{R}(0) + \mathbf{V}(0)t, \qquad \text{for } t > 0,$$

because after the short kick at $t = 0$, there is no net force exerted on the system. The constant $\mathbf{V}(0)$ in this result may be obtained, for example, by integrating both sides of the 2nd Newton's law (1.30) through the time of the kick:

$$\mathbf{V}(0)=\int \dot{\mathbf{V}}(t)dt=\frac{1}{M}\int \mathbf{F}(t)dt=\frac{\mathbf{P}}{M}.$$

So, in our problem, the reference frame bound to the center of mass of the system (in which $\mathbf{R} \equiv 0$) is inertial, and we may use the Lagrangian function written in it,

$$L=\frac{m}{2}\dot{\mathbf{r}}^2-U(r)\equiv\frac{m}{2}\left(\dot{r}^2+r^2\dot{\varphi}^2\right)-\frac{\kappa}{2}(r-l)^2, \qquad (**)$$

in a regular way. (The last expression uses the polar-coordinate representation of the radius vector $\mathbf{r}$.) In particular, calculating the generalized momenta (2.31) corresponding to the generalized coordinates r and φ,

$$p_r\equiv\frac{\partial L}{\partial \dot{r}}=m\dot{r}, \qquad p_\varphi\equiv\frac{\partial L}{\partial\dot{\varphi}}=mr^2\dot{\varphi}, \qquad (***)$$

we may spell out the Hamiltonian function (2.32) of this system:

$$H\equiv p_r\dot{r}+p_\varphi\dot{\varphi}-L=m\dot{r}^2+mr^2\dot{\varphi}^2-\left[\frac{m}{2}\left(\dot{r}^2+r^2\dot{\varphi}^2\right)-\frac{\kappa}{2}(r-l)^2\right]\equiv\frac{m}{2}\left(\dot{r}^2+r^2\dot{\varphi}^2\right)+\frac{\kappa}{2}(r-l)^2.$$

Since the Lagrangian function (**) does not depend on time explicitly, according to Eq. (2.35), this H is time-independent.[37] Rewriting this first integral of motion in the form

$$H=\frac{m}{2}\dot{\mathbf{r}}^2+\frac{\kappa}{2}(r-l)^2\equiv\frac{m}{2}v^2+\frac{\kappa}{2}(r-l)^2=\text{const},$$

we see that the square $v^2=\mathbf{v}^2$ of the relative velocity $\mathbf{v}\equiv\dot{\mathbf{r}}$ is the largest at the moments when the oscillations $r(t)$ of the distance between the masses pass through the equilibrium point $r = l$, turning the second (potential-energy) part of H to zero. Hence we may write

$$\frac{m}{2}v_{\max}^2=H_{\text{ini}},$$

where H_{ini} is the initial value of H immediately after the kick. Since, according to the problem assignment, at this moment $r = l$ and

$$\mathbf{v}_{\text{ini}}\equiv\dot{\mathbf{r}}_{\text{ini}}\equiv\left(\dot{\mathbf{r}}_1-\dot{\mathbf{r}}_2\right)_{\text{ini}}=\frac{1}{m_1}\int\ddot{\mathbf{r}}(t)dt-0=\frac{1}{m_1}\int\mathbf{F}(t)dt=\frac{\mathbf{P}}{m_1}, \qquad \text{i.e. } v_{\text{ini}}^2=\left(\frac{P}{m_1}\right)^2,$$

so $H_{\text{ini}} = (m/2)(P/m_1)^2$, and hence

$$v_{\max}=\frac{P}{m_1}=v_{\text{ini}}.$$

Now we may return to the lab reference frame, and differentiate Eq. (*) for $\mathbf{r}_1$ over time to calculate the requested velocity of the first mass in that frame:

$$\mathbf{v}_1\equiv\dot{\mathbf{r}}_1=\dot{\mathbf{R}}+\frac{m_2}{M}\dot{\mathbf{r}}=\frac{P}{M}\left(\mathbf{n}_\mathbf{P}+\frac{m_2}{m_1}\mathbf{n}_\text{v}\right), \qquad \text{where } \mathbf{n}_\mathbf{P}\equiv\frac{\mathbf{P}}{P},\ \mathbf{n}_\text{v}\equiv\frac{\mathbf{v}}{v}. \qquad (****)$$

[37] Admittedly, this conclusion is almost evident from the form of Eq. (**). However, since that equality is only valid in a specific (c.o.m.-bound) reference frame, we cannot be too careful. For example, the calculated H is generally different from the full energy E of our system measured in the initial ("lab") reference frame.

Now note that the initial kick induces not only the periodic (but generally not harmonic!) oscillations of the length of the distance vector $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$ about its equilibrium value l, but also its rotation around the center of mass. (Indeed, according to Eq. (***), the generalized momentum p_φ, physically the angular momentum L_z of the system, is also time-independent, and conserves its initial value.) The vectors **v** and hence $\mathbf{n}_\mathbf{v}$ follow this rotation. Since the angular velocity of the rotation is generally incommensurate with the oscillation frequency, at the periodic moments when $v = v_{\max}$, the mutual orientation of the unit-length vectors $\mathbf{n}_\mathbf{P}$ and $\mathbf{n}_\mathbf{v}$ varies; occasionally they become virtually aligned and on other occasions, anti-aligned. At these moments, v_1 takes its extreme values

$$(v_1)_{\max} = \frac{P}{M}\left(1+\frac{m_2}{m_1}\right) \equiv \frac{P}{m_1+m_2}\left(1+\frac{m_2}{m_1}\right) \equiv \frac{P}{m_1},$$

$$(v_1)_{\min} = \frac{P}{M}\left(1-\frac{m_2}{m_1}\right) \equiv \frac{P}{m_1+m_2}\frac{m_1-m_2}{m_1}.$$

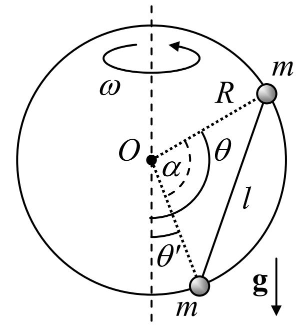
Note that this mass never moves faster than just after the kick, and that if $m_2 = m_1$, there are moments when it virtually stops. The reader is encouraged to prove that the last statement is also true for *any* $m_2 > m_1$. (It is sufficient to analyze Eq. (****), which is valid for this case as well.)

Problem 3.7. Explain why the term $mr^2\dot{\varphi}^2/2$, recast in accordance with Eq. (3.42) of the lecture notes, cannot be merged with $U(r)$ in Eq. (3.38), to form the effective 1D potential energy $U(r) - L_z^2/2mr^2$, with the second term's sign opposite to that given by Eq. (3.44). We have done an apparently similar thing for our testbed bead-on-rotating-ring problem at the very end of Sec. 3.1 – see Eq. (3.6); why the same trick does not work for the planetary problem? Besides a formal explanation, discuss the physics behind this difference.

Solution: Deriving the Lagrange equations in Chapter 2, we have made a commitment to treat the generalized velocities (in the planetary problem, $\dot{r}$ and $\dot{\varphi}$) as variables independent of the generalized coordinates (r and φ). Expressing $mr^2\dot{\varphi}^2/2$ as $L_z^2/2mr^2$, i.e. as a function of r in the Lagrangian function *before* its differentiation, we would violate this commitment and as a result, would get wrong equations of motion. On the contrary, *after* the (correct) equations of motion have been obtained and/or the system's energy has been correctly calculated, we may make whichever substitutions we wish, as we did in Sec. 3.5 of the lecture notes – see Eqs. (3.41)-(3.44).

Concerning the bead-on-a-rotating-ring problem considered in Sec. 3.1, there we indeed moved the kinetic-energy term $(m/2)R^2\omega^2\sin^2\theta$ into the effective potential energy in the Lagrangian function L, i.e. before obtaining the equation of motion. However, that term did not contain any generalized velocity, and hence that operation was legitimate.

The physics behind this difference is that in the planetary problem, the particle is free to move in all three directions (or in two directions, if we take into account the conservation of the direction of the vector **L**) and the further reduction to the 1D motion is performed at conditions where this vector, and hence L_z and L^2, are also conserved – see (3.39), while the instantaneous angular velocity $\dot{\varphi}$ is not fixed, because r in that formula may change. On the contrary, in the bead-on-the-ring problem, the angular velocity ω of the ring's rotation is fixed, i.e. is independent of the system's dynamics.



Problem 3.8. A system of two equal masses *m* on a light rod of a fixed length *l* (frequently called a *dumbbell*) can slide without friction along a vertical ring of radius *R*, rotated about its vertical diameter with a constant angular velocity ω – see the figure on the right. Derive the condition of stability of the lower horizontal position of the dumbbell.

Solution: This problem is a straightforward generalization of our testbed problem (see Fig. 2.1 of the lecture notes) to two masses, so we may readily extend Eqs. (2.23) to write

$$T = \frac{m}{2}R^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) + \frac{m}{2}R^2\left(\dot{\theta}'^2 + \omega^2\sin^2\theta'\right),$$
$$U = -mgR\cos\theta - mgR\cos\theta' + \text{const},$$

where θ and θ' are the polar angles of the mass positions, with the polar axis directed down – see the figure above. Since the difference $\alpha \equiv \theta - \theta'$ between these angles is fixed by the length *l* of the connecting rod,

$$2R\sin\frac{\alpha}{2} = l,$$

i.e. $\theta' = \theta - \alpha$, and $\dot{\theta}' = \dot{\theta}$, the system may be described by just one degree of freedom (say, θ), and its Lagrangian,

$$L \equiv T - U = m\dot{\theta}^2 + \frac{m\omega^2R^2}{2}\left[\sin^2\theta + \sin^2(\theta-\alpha)\right] + mgR\left[\cos\theta + \cos(\theta-\alpha)\right],$$

may be recast into a form similar to Eq. (3.5):

$$L = T_{\text{ef}} - U_{\text{ef}}, \qquad \text{with } T_{\text{ef}} = mR^2\dot{\theta}^2, \quad U_{\text{ef}} = -\frac{m\omega^2R^2}{2}\left[\sin^2\theta + \sin^2(\theta-\alpha)\right] - mgR\left[\cos\theta + \cos(\theta-\alpha)\right],$$

so equilibrium positions of the system correspond to the extrema of the function $U_{\text{ef}}(\theta)$. Calculating its first derivative,

$$\frac{dU_{\text{ef}}}{d\theta} = -m\omega^2R^2\left[\sin\theta\cos\theta + \sin(\theta-\alpha)\cos(\theta-\alpha)\right] + mgR\left[\sin\theta + \sin(\theta-\alpha)\right],$$

we see that for any parameters of the system, the derivative always vanishes at the lower horizontal position[38] of the dumbbell, where $\theta = \alpha/2$, so $\sin(\theta - \alpha) = -\sin(\alpha/2) = -\sin\theta$, while $\cos(\theta - \alpha) = \cos(\alpha/2) = \cos\theta$. (This fact reflects the evident symmetry of the effective potential energy U_{ef} with respect to the angle swap, $\theta \leftrightarrow \theta'$.)

Now we need to establish the condition of this extremum being a minimum, i.e. of the second derivative of function $U_{\text{ef}}(\theta)$ being positive at this point. Rewriting the first derivative in an equivalent form

$$\frac{dU_{\text{ef}}}{d\theta} = -\frac{m\omega^2R^2}{2}\left[\sin 2\theta + \sin 2(\theta-\alpha)\right] + mgR\left[\sin\theta + \sin(\theta-\alpha)\right]$$

[38] It is physically evident (and may be readily verified) that the upper horizontal position (with $\theta = -\alpha/2$, $\theta' = +\alpha/2$), is always an unstable fixed point of the system.

$$\equiv mgR\left\{-\frac{\omega^2}{2\Omega^2}\left[\sin 2\theta+\sin 2(\theta-\alpha)\right]+\left[\sin\theta+\sin(\theta-\alpha)\right]\right\}, \qquad \text{with } \Omega^2\equiv\frac{g}{R},$$

we make the second differentiation elementary:

$$\begin{aligned}\frac{d^2U_{\text{ef}}}{d\theta^2}\Big|_{\theta=\alpha/2} &= mgR\left\{-\frac{\omega^2}{\Omega^2}\left[\cos 2\theta+\cos 2(\theta-\alpha)\right]+\left[\cos\theta+\cos(\theta-\alpha)\right]\right\}_{\theta=\alpha/2} \\ &= 2mgR\left(-\frac{\omega^2}{\Omega^2}\cos\alpha+\cos\frac{\alpha}{2}\right).\end{aligned} \qquad (*)$$

By the definition of the angle α (see the figure above), $0 \le \alpha \le \pi$, so $\cos(\alpha/2)$ is always positive, and Eq. (*) shows that if $\cos\alpha > 0$ (i.e. $\alpha < \pi/2$, i.e. $l/R < \sqrt{2}$), the second derivative is positive (and hence the lower horizontal position of the dumbbell is stable) only if the angular frequency ω of ring's rotation is lower than the following critical value:

$$\omega_{\text{c}}=\Omega\left[\frac{\cos(\alpha/2)}{\cos\alpha}\right]^{1/2}\equiv\left[\frac{g\cos(\alpha/2)}{R\cos\alpha}\right]^{1/2}, \qquad \text{for } \alpha\le\frac{\pi}{2}. \qquad (**)$$

At $\alpha \to 0$ (i.e. at $l/R \to 0$, so the two masses are effectively merged into one), this value tends to Ω, in accordance with the conclusion of our testbed problem analysis at the end of Sec. 3.2 of the lecture notes, providing a good sanity check of our current calculations. As the dumbbell becomes longer, ω_{c} grows because the denominator on the right-hand side of Eq. (**) decreases faster than its numerator. Moreover, if the rod is so long that $l/R > \sqrt{2}$, and hence $\alpha > \pi/2$, so $\cos\alpha < 0$, both terms on the right-hand side of Eq. (*) are positive, and the lower horizontal position is stable at *any* ω.

The reason for this behavior may be most simply interpreted using the notion of the centrifugal "inertial force", which has to be added to real physical forces if we want the 2nd Newton law to be valid in a non-inertial reference frame[39] – in this particular case, in the frame rotating with the ring. If the dumbbell is short, the centrifugal force tries to push it out of the rotation axis, toward one of the nearly-vertical parts of the ring – and at $\omega > \omega_{\text{c}}$, it succeeds, by overcoming gravity forces. However, if l is larger, the rod keeps the masses high on the opposite nearly-vertical sides of the ring even in the absence of rotation, and the centrifugal forces acting on the masses pull the dumbbell in opposite directions, canceling each other at least partly.

Problem 3.9. Analyze the dynamics of the so-called *spherical pendulum* – a point mass hung, in a uniform gravity field **g**, on a light cord of length l, with no motion's confinement to a vertical plane. In particular:

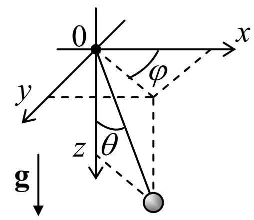
(i) find the integrals of motion and reduce the problem to a 1D one,
(ii) calculate the time period of the possible circular motion around the vertical axis, and
(iii) explore small deviations from the circular motion. (Are the pendulum's orbits closed?)[40]

[39] See Sec. 4.6 below. By the way, my recommendation to the reader is to return to this problem after reading Chapter 4, and re-solve it (or rather re-derive the above expression for the Lagrangian function), working in the rotating reference frame.
[40] Solving this problem is a very good preparation for the analysis of the symmetric top rotation in Sec. 4.5.

Solutions:

(i) This system is a just particular case of that considered in Problem 2.3, with no suspension point's motion, so we may reuse the Lagrangian function calculated in the solution of that problem, by taking $x_0(t) = 0$:

$$L = T - U, \qquad \text{with } T = \frac{ml^2}{2}\left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta\right), \qquad U = -mgl\cos\theta,$$



where the polar angle θ is measured from the z-axis directed down – see the figure on the right. Since in this case, the Lagrangian function does not depend explicitly on time, and the kinetic energy T is a quadratic-homogeneous function of the generalized velocities $\dot{\theta}$ and $\dot{\varphi}$, the total mechanical energy,

$$E = T + U = \frac{ml^2}{2}\left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta\right) - mgl\cos\theta, \qquad (*)$$

does not depend on time, i.e. is an integral of motion. Another integral of motion may be derived from the fact that L does not depend on its azimuthal angle φ, so this variable is cyclic, and the corresponding generalized momentum,

$$p_\varphi \equiv \frac{\partial L}{\partial \dot{\varphi}} = ml^2 \dot{\varphi}\sin^2\theta,$$

is constant. (Since $l\sin\theta$ is the distance $\rho \equiv (x^2 + y^2)^{1/2}$ of the mass from the z-axis, $p_\varphi = m\rho^2\dot{\varphi}$, i.e. it is just the z-component L_z of the angular momentum – see Eq. (3.39) of the lecture notes.) Using this integral to express the azimuthal velocity,

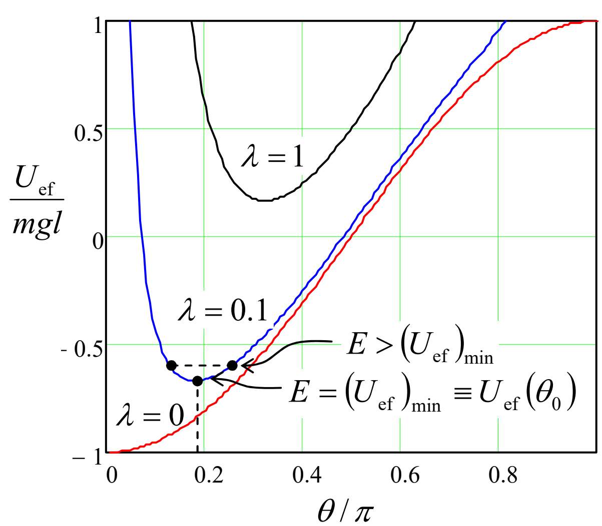
$$\dot{\varphi} = \frac{L_z}{ml^2 \sin^2\theta},$$

and plugging this relation into Eq. (*), we may represent the result in the form

$$E = \frac{ml^2\dot{\theta}^2}{2} + U_{\text{ef}}(\theta), \qquad \text{with } U_{\text{ef}} \equiv \frac{L_z^2}{2ml^2\sin^2\theta} - mgl\cos\theta. \qquad (**)$$

So, the problem has been reduced to an analysis of 1D motion, very similar to how this was done with the planetary problem in Sec. 3.5 of the lecture notes – cf. Eqs. (3.43)-(3.44).

(ii) The figure on the right shows the function $U_{\text{ef}}(\theta)$ on the physically relevant segment $[0, \pi]$, for several values of its only dimensionless parameter $\lambda \equiv L_z^2/m^2l^3g$, which characterizes the initial "spin" of the pendulum. For any value of λ, the function has just one minimum, at the point θ_0 that satisfies the equation $dU_{\text{ef}}/d\theta = 0$; for our case:



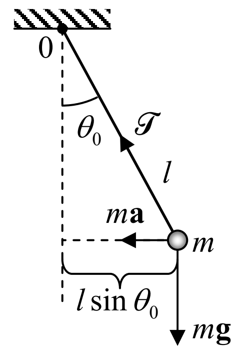
$$-\frac{L_z^2}{ml^2}\frac{\cos\theta_0}{\sin^3\theta_0} + mgl\sin\theta_0 = 0.$$

Instead of solving this equation for θ_0 at a given L_z, it is much easier to use it to express L_z via θ_0,

$$L_z\Big|_{\theta=\theta_0} = \left(\frac{m^2 g l^3 \sin^4\theta_0}{\cos\theta_0}\right)^{1/2}, \qquad (***)$$

and then find the angular velocity $\dot{\varphi}$ corresponding to the point θ_0:

$$\dot{\varphi}_0 \equiv \dot{\varphi}\Big|_{\theta=\theta_0} = \frac{L_z\big|_{\theta=\theta_0}}{ml^2\sin^2\theta_0} = \left(\frac{g}{l\cos\theta_0}\right)^{1/2}. \qquad (****)$$



The physical meaning of this result is very simple (see Fig. 3.5 of the lecture notes and its discussion): if the pendulum's initial energy E has the smallest value $(U_{ef})_{min}$ possible at the given "spin" (L_z), the system has to stay at the fixed point θ_0 – see the lower arrow in the figure above. This corresponds to a circular motion of the pendulum,[41] with a constant polar angle $\theta = \theta_0$. This formula may be also obtained in an easier "undergraduate" way by solving a simple system of two equations for two Cartesian components of the 2nd Newton law for pendulum's motion within a horizontal plane – see the figure on the right:

$$ma \equiv ml\sin\theta_0\dot{\varphi}_0^2 = \mathscr{T}\sin\theta_0, \qquad 0 = mg - \mathscr{T}\cos\theta_0.$$

The time period of this circular motion is

$$\mathcal{T}_0 \equiv \frac{2\pi}{\dot{\varphi}_0} = 2\pi\left(\frac{l\cos\theta_0}{g}\right)^{1/2}.$$

Note that in the limit of small polar angles $\theta_0 \to 0$, when $\cos\theta_0 \to 1$, $\mathcal{T}_0$ coincides with the well-known expression for the period of oscillations of the pendulum in a vertical plane. This is natural because, in the small-angle limit, the sinusoidal oscillations in any two mutually perpendicular vertical planes have the same frequency and are independent.[42] Hence the 2D equation of the pendulum's motion (which, in this limit, is linear), may be satisfied with any linear superposition of such oscillations. In the particular case when the amplitudes of these oscillations are equal, and phases are shifted by $\pi/2$, we come back to the circular motion with the same time period.

(iii) If the energy E of the pendulum is slightly higher than $(U_{ef})_{min} \equiv U_{ef}(\theta_0)$, then according to Eq. (**), the polar angle θ has to oscillate between two points: θ_{max} slightly higher than θ_0, and θ_{min} slightly lower than it – see the horizontal dashed line drawn for $\lambda = 0.1$ in the second figure of this

[41] Sometimes a pendulum performing this particular motion is called the *conical pendulum* because its cord follows the surface of a round cone with a vertical axis.

[42] The easiest way to prove this is to return to the Cartesian coordinates and, in the first nonvanishing approximation in x/l, $y/l \to 0$, reduce the Lagrangian function to the following quadratic form:

$$L \approx \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{mgl}{2}\theta^2 \approx \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{mgl}{2}\left(\frac{x^2}{l^2} + \frac{y^2}{l^2}\right),$$

i.e. to the sum of the Lagrangian functions of two independent harmonic oscillators with the same frequency $\omega_0 = (g/l)^{1/2}$.

solution. The time period of these small oscillations may be calculated by Taylor-expanding the function $U_{\text{ef}}(\theta)$ at the point θ_0, and keeping only the leading (quadratic) term:

$$U_{\text{ef}}(\theta \approx \theta_0) - U_{\text{ef}}(\theta_0) \approx \frac{1}{2}\frac{d^2U_{\text{ef}}}{d\theta^2}\Big|_{\theta=\theta_0}\tilde{\theta}^2, \quad \text{where } \tilde{\theta} \equiv \theta - \theta_0 .$$

This differentiation, followed by the substitution of L_z from Eq. (***), yields

$$\frac{d^2U_{\text{ef}}}{d\theta^2}\Big|_{\theta=\theta_0} = \frac{mgl}{\cos\theta_0}\left(1 + 3\cos^2\theta_0\right).$$

With this replacement, Eq. (**) becomes the standard expression for the energy of a harmonic oscillator with the frequency

$$\omega_\theta = \left[\frac{g}{l\cos\theta_0}\left(1+3\cos^2\theta_0\right)\right]^{1/2}, \qquad \text{so that } \frac{\omega_\theta}{\dot{\varphi}_0} = \left(1+3\cos^2\theta_0\right)^{1/2}.$$

This means that the frequency of small oscillations of the polar angle is generally incommensurate with the angular velocity of the azimuthal angle's rotation – besides certain exact values of θ_0 and hence of the pendulum's energy. Hence, in contrast to the planetary motion in the Coulomb field (3.49), the pendulum's orbits that are different from the circular one are generally open – see Fig. 3.6 of the lecture notes and its discussion. Two exceptions are the limits $\theta_0 \to \pi/2$ (which requires infinite L_z and hence infinite energy) and $\theta_0 \to 0$ (i.e. the limit of very small motions near the equilibrium) when the above formula gives $\omega_\theta = 2\dot{\varphi}_0$. The last relation describes the two increases and two decreases of θ on the general elliptical trajectory of the pendulum, which may be represented as the linear superposition of its oscillations in two mutually perpendicular vertical planes – see the discussion in Part (ii) of this solution.

Problem 3.10. If our planet Earth was suddenly stopped in its orbit around the Sun, how long would it take it to fall on our star? Solve this problem using two different approaches, neglecting the Earth's orbit eccentricity and the Sun's size.

Solution: On one hand, the problem may be solved directly, by integrating the self-evident 1D equation describing the Earth's fall, following from the 2nd Newton's law and his gravity law:

$$M_{\text{E}}\ddot{r} = -\frac{GM_{\text{S}}M_{\text{E}}}{r^2}.$$

After cancellation of M_{E}, this equation may be readily integrated once using the transformation used, in particular, in Eq. (1.20) of the lecture notes:

$$\ddot{r} \equiv \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{dr}\frac{dr}{dt} \equiv \dot{r}\frac{d\dot{r}}{dr} = \frac{d}{dr}\frac{(\dot{r})^2}{2},$$

giving[43]

$$\frac{(\dot{r})^2}{2} = \frac{GM_{\text{S}}}{r} + \text{const}.$$

[43] Actually, this is just the energy conservation law for the purely radial motion in the potential well (3.49) with $\alpha = GM_{\text{S}}M_{\text{E}}$.

The integration constant follows from the implied initial condition $\dot{r} = 0$ at $r = r_0$, where r_0 is the circular orbit's radius, giving

$$\frac{(\dot{r})^2}{2} = GM_S\left(\frac{1}{r} - \frac{1}{r_0}\right), \qquad \text{so} \quad \int_0^{r_0} \frac{dr}{(1/r - 1/r_0)^{1/2}} = (2GM_S)^{1/2} \int_0^{\mathcal{T}_{\text{fall}}} dt\,,$$

where $\mathcal{T}_{\text{fall}}$ is the time we are looking for. The integration of the right-hand side of this equation is elementary, and that of its left-hand side may be carried out, for example, by using the substitution $r \equiv r_0 \sin^2\xi$, so $dr = 2r_0 \sin\xi \cos\xi\, d\xi$ and $(1/r - 1/r_0)^{1/2} = (1/\sin^2\xi - 1)^{1/2}/r_0^{1/2} \equiv \cos\xi / r_0^{1/2}\sin\xi$, so

$$\int_0^{r_0} \frac{dr}{(1/r - 1/r_0)^{1/2}} = r_0^{3/2} \int_0^{\pi/2} 2\sin^2\xi d\xi \equiv r_0^{3/2} \int_0^{\pi/2} (1 - \cos 2\xi) d\xi = \frac{\pi}{2} r_0^{3/2}.$$

So, finally, we have

$$\mathcal{T}_{\text{fall}} = \frac{\pi}{2} \frac{r_0^{3/2}}{(2GM_S)^{1/2}}. \qquad (*)$$

Now we may avoid the need to plug in the experimental values of the three constants in the right-hand part of this result, by comparing it with the period $\mathcal{T}_{\text{rot}}$ of the regular rotation of the Earth around the Sun, i.e. with one year (strictly speaking, the *sidereal year*, close to 365.24 days). For a circular orbit, its radius r_0 and the orbital velocity v_0 are related by the 2nd Newton's law in the form

$$M_E \frac{v_0^2}{r_0} = \frac{GM_S M_E}{r_0^2}, \qquad \text{so that } v_0 = \left(\frac{GM_S}{r_0}\right)^{1/2}, \qquad (**)$$

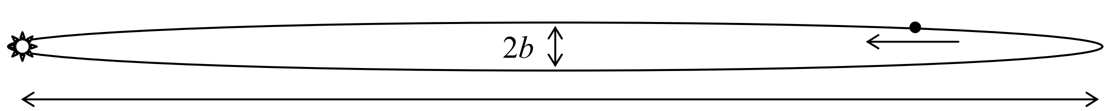
and we may write

$$\mathcal{T}_{\text{rot}} = \frac{2\pi r_0}{v_0} = 2\pi \frac{r_0^{3/2}}{(GM_S)^{1/2}}.$$

Comparing this result with Eq. (*), we get

$$\mathcal{T}_{\text{fall}} = \frac{1}{4\sqrt{2}} \mathcal{T}_{\text{rot}} \approx 64.6 \text{ days}. \qquad (***)$$

Another approach to the same problem is to use the general formulas for elliptic trajectories in the Coulomb-type potential (3.49), which were discussed in Sec. 3.4 of the lecture notes – see especially Fig. 3.7. Indeed, the straight fall of the initially stopped Earth on the Sun may be considered as one-half of an ultimately elongated elliptical orbit (see the figure below), with $b/a \to 0$.[44]



Hence we may apply to it, in particular, the second form of Eq. (3.64b) with $\alpha = GM_S M_E$ and $m = M_E$:

[44] Since, according to Eqs. (3.63), this ratio equals $(1 - e^2)^{1/2}$, the eccentricity of this ellipse tends to 1, so the distance $a(1 - e)$ between the orbit's perihelion and its focus housing the Sun is negligible in comparison with a.

$$\mathcal{T}_{\text{fall}} = \frac{1}{2}\mathcal{T} = \frac{\pi}{2} G M_{\text{S}} M_{\text{E}} \left(\frac{M_{\text{E}}}{2\left|E_{\text{fall}}\right|^3} \right)^{1/2}.$$

The same formula is applicable to the period of the circular motion of the Earth:

$$\mathcal{T}_{\text{rot}} = \pi G M_{\text{S}} M_{\text{E}} \left(\frac{M_{\text{E}}}{2\left|E_{\text{rot}}\right|^3} \right)^{1/2}.$$

A comparison of these two formulas yields

$$\mathcal{T}_{\text{fall}} = \frac{1}{2}\left(\frac{\left|E_{\text{rot}}\right|}{\left|E_{\text{fall}}\right|} \right)^{3/2} \mathcal{T}_{\text{rot}}.$$

What remains is to calculate the energy ratio on the right-hand side of this result. Since E_{fall} is conserved during the fall process, we may calculate it as the potential energy of the Earth on its orbit, because its stoppage kills the kinetic part T_{rot} of its full energy: $E_{\text{fall}} = U_{\text{rot}}$. But as it follows from the virial theorem (see the solution of Problem 1.12),[45] for a circular orbit in the potential (3.49):

$$T_{\text{rot}} = -\frac{1}{2} U_{\text{rot}}, \qquad \text{so } E_{\text{rot}} \equiv T_{\text{rot}} + U_{\text{rot}} = \frac{1}{2} U_{\text{rot}},$$

and we finally get

$$\frac{\left|E_{\text{rot}}\right|}{\left|E_{\text{fall}}\right|} = \frac{1}{2}, \qquad \text{so } \mathcal{T}_{\text{fall}} = \frac{1}{2}\left(\frac{1}{2}\right)^{3/2} \mathcal{T}_{\text{rot}},$$

i.e. the same result as in Eq. (***).

Problem 3.11. The orbits of Mars and Earth around the Sun may be well approximated as coplanar circles[46] with a radii ratio of 3/2. Use this fact, and the Earth's year duration, to calculate the time of travel to Mars spending the least energy on the spacecraft's launch. Neglect the planets' size and the effects of their own gravitational fields.

Solution: As Fig. 3.5 of the lecture notes shows, for a bound motion in an attractive field changing monotonically with distance r, an increase of the $r_{\text{max}}/r_{\text{min}}$ ratio requires additional energy, so for the attractive field of our Sun, expressed by Eq. (3.49), the spacecraft energy minimum corresponds to the elliptic trajectory with $p/(1 - e) = r_{\text{Mars}}$ and $p/(1 + e) = r_{\text{Earth}}$ – see Fig. 3.7 for the nomenclature.[47] From here, we may calculate the required spacecraft orbit's eccentricity:

$$\frac{r_{\text{Mars}}}{r_{\text{Earth}}} = \frac{3}{2} = \frac{1+e}{1-e}, \qquad \text{giving } e = \frac{1}{5}.$$

According to the last form of Eq. (3.64b), the time of the spacecraft's travel between Earth and Mars (evidently, half of the period $\mathcal{T}$ of motion along its elliptic trajectory) is

[45] The same result follows from Eq. (**): $T_{\text{rot}} \equiv M_{\text{E}}v_0^2/2 = GM_{\text{S}}M_{\text{E}}/2r_0 = -U_{\text{rot}}/2$.

[46] Indeed, their eccentricities are close to, respectively, 0.093 and 0.0167.

[47] Evidently, such a trajectory (called the *Hohmann transfer orbit*) may be used for travel only at the optimal mutual position of Earth and Mars on their orbits, which is approximately reached once in several decades.

$$\Delta t = \pi a^{3/2}\left(\frac{1}{GM_\mathrm{S}}\right)^{1/2},$$

while the Earth year's duration[48] is

$$\mathcal{T}_\mathrm{Earth} = 2\pi r_\mathrm{Earth}^{3/2}\left(\frac{1}{GM_\mathrm{S}}\right)^{1/2},$$

where G is the gravity constant and M_S is Sun's mass. We may avoid using these constants by their elimination from the last two relations, getting

$$\Delta t = \frac{\mathcal{T}_\mathrm{Earth}}{2}\left(\frac{a}{r_\mathrm{Earth}}\right)^{3/2}.$$

But according to the first of Eqs. (3.63) (see also Fig. 3.7) of the lecture notes, the ratio a/r_Earth is uniquely defined by the spacecraft orbit's eccentricity:

$$\frac{a}{r_\mathrm{Earth}} = \frac{p/(1-e^2)}{p/(1+e)} \equiv \frac{1}{1-e} = \frac{5}{4},$$

so, finally,

$$\Delta t = \frac{\mathcal{T}_\mathrm{Earth}}{2}\left(\frac{5}{4}\right)^{3/2} \approx 0.700\,\mathcal{T}_\mathrm{Earth} \approx 255\,\mathrm{days}.$$

<u>Problem 3.12</u>. Derive first-order and second-order differential equations for the reciprocal distance $u \equiv 1/r$ as a function of φ, describing the trajectory of a particle's motion in a central potential $U(r)$. Spell out the latter equation for the particular case of the Coulomb potential (3.49) and discuss the result.

Solution: Let us start by rewriting Eq. (3.48) of the lecture notes in the differential form:

$$d\varphi = \pm\frac{L_z dr}{(2m)^{1/2} r^2\left[E - U(r) - L_z^2/2mr^2\right]^{1/2}}, \quad \text{i.e.} \quad \frac{dr}{d\varphi} = \pm r^2\left[E - U(r) - \frac{L_z^2}{2mr^2}\right]^{1/2} \Bigg/ \left(\frac{L_z^2}{2m}\right)^{1/2}. \quad (*)$$

Now from the definition $u \equiv 1/r$, we have $r = 1/u$ and $dr = -du/u^2$. Plugging these relations into Eq. (*) and canceling $(-u^2) \neq 0$, we get

$$\frac{du}{d\varphi} = \mp\left[E - U\left(\frac{1}{u}\right) - \frac{L_z^2}{2m}u^2\right]^{1/2} \Bigg/ \left(\frac{L_z^2}{2m}\right)^{1/2}. \quad (**)$$

This is the required first-order differential equation for u as a function of φ.

In order to get the second-order equation, let us differentiate both parts of Eq. (**) over φ:

[48] With the accuracy of this calculation (which is about a few percent, mostly because of neglecting the Mars orbit's eccentricity $e \sim 0.1$), we may safely ignore the much smaller differences between the genuine Earth's orbit period (called the *sidereal year*) and other astronomical (*tropical*, *anomalistic*, etc.) years, and also various calendar (Julian, Gregorian, etc.) years.

$$\frac{d^2u}{d\varphi^2} = \pm\left[\frac{dU(1/u)}{d\varphi} + 2\frac{L_z^2}{2m}u\frac{du}{d\varphi}\right] \bigg/ 2\left[E - U\left(\frac{1}{u}\right) - \frac{L_z^2}{2m}u^2\right]^{1/2}\left(\frac{L_z^2}{2m}\right)^{1/2}.$$

Now using Eq. (**) to replace the square bracket in the denominator of the last expression with $\mp(L_z^2/2m)^{1/2}du/d\varphi$, representing $dU(1/u)/d\varphi$ as $[dU(1/u)/du]du/d\varphi$ and canceling $du/d\varphi$ in the numerator and denominator, we obtain the requested second-order equation[49]

$$\frac{d^2u}{d\varphi^2} + u = -\frac{m}{L_z^2}\frac{d}{du}U\left(\frac{1}{u}\right). \qquad (***)$$

The very idea of deriving the *second-order* differential equation from the known *first-order* one, i.e. from its first integral, may look awkward because, in dynamics, we usually pursue the opposite way. However, due to the very simple structure of Eq. (***), it may be convenient for some purposes. For example, for the Coulomb field (3.49), $U(r) = -\alpha/r \equiv -\alpha u$, we have $dU/du = -\alpha =$ const, so Eq. (***) is reduced to the well-known linear differential equation of an autonomous harmonic oscillator, with the unit "frequency" and the equilibrium position at $u = u_0 \equiv m\alpha/L_z^2$:

$$\frac{d^2\tilde{u}}{d\varphi^2} + \tilde{u} = 0, \qquad \text{where } \tilde{u} \equiv u - u_0 = u - \frac{m\alpha}{L_z^2}. \qquad (****)$$

In the equilibrium point u_0 we may readily recognize the reciprocal radius $r_0 \equiv 1/u_0 = L_z^2/m\alpha$ given by Eq. (3.52) of the lecture notes for this particular field – see also the discussion following Eq. (3.60) of the notes.

As we already know from the discussion in Sec. 3.4 of the lecture notes, the circular orbit corresponding to the equilibrium of the "harmonic oscillator" (****) requires specific initial conditions, striking the exact balance between the two integrals of motion: L_z and E. If this balance is violated, Eq. (****) describes sinusoidal oscillations of the reciprocal radius u (as a function of the angle φ, not time!), around the value u_0, with a nonvanishing amplitude u_A:

$$u \equiv \frac{1}{r} = u_0 + u_A\cos(\varphi - \varphi_0),$$

where φ_0 is a constant also determined by initial conditions – as well as the amplitude u_A. In this relation we may readily recognize Eq. (3.59) of the lecture notes, describing (depending on the ratio $e = u_A/u_0$) either an elliptic, or the parabolic, or a hyperbolic trajectory of the particle in the Coulomb field.

<u>Problem 3.13</u>. For the motion of a particle in the Coulomb attractive field ($U(r) = -\alpha/r$, with $\alpha > 0$), calculate and sketch the so-called *hodograph*[50] – the trajectory followed by the head of the velocity vector **v**, provided that its tail is kept at the origin.

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[49] It may be also derived from the Lagrange equation corresponding to the Lagrangian function (3.38). Though that way is longer, it still may be recommended to the reader as an exercise.

[50] The use of this notion (in Greek, meaning "path writer") for the characterization of motion may be traced back at least to an 1846 treatise by W. Hamilton. Nowadays, it is most often used in applied fluid mechanics, especially meteorology.

Solution: Perhaps the conceptually simplest way to calculate the hodograph[51] is to start with the well-known relations between the Cartesian and polar coordinates in the plane of the particle's motion:

$$x = r\cos\varphi, \qquad y = r\sin\varphi.$$

Differentiating these relations over time, we may express the Cartesian component of the velocity vector via the time derivatives of the polar coordinates:

$$v_x \equiv \dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi, \qquad v_y \equiv \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi. \qquad (*)$$

For the motion in a central field, the polar angle's speed may be found from Eq. (3.39) of the lecture notes, which expresses the conservation of the angular momentum L_z:

$$\dot{\varphi} = \frac{c}{r^2}, \qquad \text{with } c \equiv \frac{L_z}{m}. \qquad (**)$$

In order to calculate $\dot{r}$, in the particular case of the Coulomb attractive field, we may use Eq. (3.59) of the lecture notes. With the easiest choice of the polar angle's origin, accepted in Fig. 3.7 of the lecture notes, we have

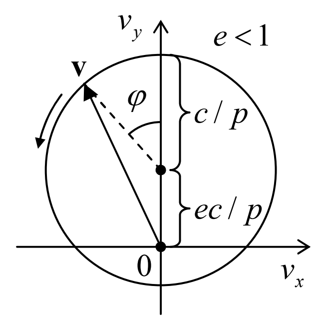
$$r = \frac{p}{1 + e\cos\varphi}, \qquad (***)$$

where the parameters p and e are given by Eqs. (3.60). Differentiating Eq. (***) over time, and then using Eq. (**) and Eq. (***) again, we get

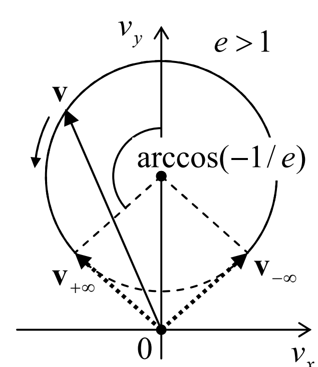
$$\dot{r} = \frac{pe\sin\varphi}{(1 + e\cos\varphi)^2}\dot{\varphi} = \frac{pe\sin\varphi}{(1 + e\cos\varphi)^2}\frac{c}{r^2} = \frac{ec}{p}\sin\varphi.$$

Now plugging this formula and Eq. (**) into Eqs. (*), and then using Eq. (***) to eliminate r from the right-hand side of the result, we get surprisingly simple expressions:

$$v_x = -\frac{c}{p}\sin\varphi, \qquad v_y = \frac{ec}{p} + \frac{c}{p}\cos\varphi.$$



These relations show that the hodograph is an exact circle (see the top figure on the right), even if the particle's trajectory (***) in the direct space is strongly elongated (elliptic) or even open (parabolic or hyperbolic). The only qualitative effect of the trajectory type on the hodograph is that for a closed, elliptic orbit, with an eccentricity e below 1, the circle center's offset from the origin, $ec/p \equiv e(L_z/mp)$, is smaller than the circle's radius, $c/p \equiv L_z/mp$, so the velocity vector periodically sweeps all directions, as shown in the figure above.



In the opposite case of a hyperbolic trajectory, i.e. of $e > 1$, the circle's offset exceeds its radius. In this case, according to Eq. (***), $\cos\varphi$ has to be larger than $(-1/e)$, so the velocity vector sweeps (just once!) only a certain sector of the circle, limited by the vectors $\mathbf{v}_{\pm\infty}$ tangential to it – see the solid arc

[51] There is at least one way to do this without using the Cartesian coordinates; finding it is a good additional exercise left for the reader.

in the bottom figure. (These limiting vectors represent the velocity of a free particle well before and well after the time of its scattering by the attracting center.)

The parabolic trajectory, with $e = 1$, i.e. with the hodograph's circle just touching the origin, represents the borderline case, with all angles swept by the velocity vector, but only once, and with $\mathbf{v}_{\pm\infty} = 0$.

Problem 3.14. Prove that for an arbitrary motion of a particle of mass m in the Coulomb field $U = -\alpha/r$, the vector $\mathbf{A} \equiv \mathbf{p}\times\mathbf{L} - m\alpha\mathbf{n}_r$ (where $\mathbf{n}_r \equiv \mathbf{r}/r$) is conserved.[52] After that:

(i) spell out the scalar product $\mathbf{r}\cdot\mathbf{A}$ and use the result for an alternative derivation of Eq. (3.59), and for a geometric interpretation of the vector $\mathbf{A}$;

(ii) spell out $(\mathbf{A} - \mathbf{p}\times\mathbf{L})^2$ and use the result for an alternative derivation of the hodograph diagram discussed in the previous problem.

Solution: Let us calculate the time derivative of the term $\mathbf{p}\times\mathbf{L}$. As was repeatedly discussed in the lecture notes, at the motion in any central field $U(r)$, the angular momentum vector $\mathbf{L}$ is constant, so

$$\frac{d}{d}(\mathbf{p}\times\mathbf{L}) = \dot{\mathbf{p}}\times\mathbf{L} \equiv \dot{\mathbf{p}}\times(\mathbf{r}\times\mathbf{p}).$$

Also, in our case of a potential central force, the 2nd Newton's law gives $\dot{\mathbf{p}} = \mathbf{F}$, with $\mathbf{F} = -\nabla U(r) = F(r)\mathbf{n}_r$, where $F(r) \equiv -dU/dr$. As a result, we may write:

$$\frac{d}{d}(\mathbf{p}\times\mathbf{L}) = F(r)\mathbf{n}_r\times(\mathbf{r}\times\mathbf{p}) \equiv mF(r)\,\mathbf{n}_r\times(\mathbf{r}\times\dot{\mathbf{r}}).$$

Now let us apply to the involved double vector product the well-known "bac minus cab" rule:[53]

$$\frac{d}{dt}(\mathbf{p}\times\mathbf{L}) = mF(r)[\mathbf{r}(\mathbf{n}_r\cdot\dot{\mathbf{r}}) - \dot{\mathbf{r}}(\mathbf{n}_r\cdot\mathbf{r})] \equiv mF(r)[\mathbf{r}(\mathbf{n}_r\cdot\dot{\mathbf{r}}) - \dot{\mathbf{r}}\,r].$$

In the last expression, the scalar product $\mathbf{n}_r\cdot\dot{\mathbf{r}}$ is just the component of the vector $\dot{\mathbf{r}}$, which is directed along the vector $\mathbf{r}$, i.e. $\dot{r}$, so

$$\frac{d}{d}(\mathbf{p}\times\mathbf{L}) = mF(r)(\mathbf{r}\dot{r} - \dot{\mathbf{r}}\,r) \equiv -mF(r)r^2\left(\frac{\dot{\mathbf{r}}\,r - \mathbf{r}\,\dot{r}}{r^2}\right) = -mF(r)r^2\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) \equiv -mF(r)r^2\,\dot{\mathbf{n}}_r.$$

This equality is valid for any central force; for the Coulomb field, in that $F(r) = -\alpha/r^2$, it yields

$$\frac{d}{d}(\mathbf{p}\times\mathbf{L}) = m\alpha\dot{\mathbf{n}}_r \qquad \text{so} \quad \frac{d}{d}(\mathbf{p}\times\mathbf{L} - m\alpha\mathbf{n}_r) \equiv \frac{d\mathbf{A}}{dt} = 0,$$

i.e. the vector $\mathbf{A}$ is indeed conserved. Since at the orbital motion, the angular momentum $\mathbf{L}$ is normal to the orbit's plane, the vector product $\mathbf{p}\times\mathbf{L}$ and hence the vector $\mathbf{A}$ as a whole lie in that plane.

(i) Let us consider the scalar product

$$\mathbf{r}\cdot\mathbf{A} \equiv \mathbf{r}\cdot(\mathbf{p}\times\mathbf{L}) - m\alpha\,r.$$

[52] This fact, first proved in 1710 by Jacob Hermann, was repeatedly rediscovered during the next two centuries. As a result, the most common name of $\mathbf{A}$ is, rather unfairly, the *Runge-Lenz vector*.

[53] See, e.g., MA Eq. (7.5).

Using the operand rotation rule of vector algebra,[54] the first term may be transformed as

$$\mathbf{r}\cdot(\mathbf{p}\times\mathbf{L})=\mathbf{L}\cdot(\mathbf{r}\times\mathbf{p})\equiv\mathbf{L}\cdot\mathbf{L}=L^2=L_z^2,$$

so

$$\mathbf{r}\cdot\mathbf{A}\equiv rA\cos\varphi=L_z^2-m\alpha r\,,$$

where φ is the angle between the vectors **A** and **r**. Rewriting the last equality as

$$r=\frac{L_z^2}{m\alpha+A\cos\varphi},$$

we see that this is just another form of Eq. (3.59), with the usual choice of the polar angle's reference:

$$r=\frac{p}{1+e\cos\varphi},\quad \text{with } p=\frac{L_z^2}{m\alpha}\text{ and } e=\frac{A}{m\alpha}. \qquad (*)$$

Now looking at the plot of this equation in Fig. 3.7 of the lecture notes, we see that the constant vector **A** is directed toward the orbit's perihelion. Its length may be readily expressed in the terms used in Sec. 3.4, by comparison of the last formula for e with Eq. (3.60):

$$A^2=2mEL_z^2+m^2\alpha^2.$$

(ii) Since, by definition, $\mathbf{A}-\mathbf{p}\times\mathbf{L}\equiv-m\alpha\mathbf{n}_r$, we get

$$(\mathbf{A}-\mathbf{p}\times\mathbf{L})(\mathbf{A}-\mathbf{p}\times\mathbf{L})=m^2\alpha^2\mathbf{n}_r^2\equiv m^2\alpha^2$$

After the parentheses' multiplication, this equality becomes

$$A^2+(\mathbf{p}\times\mathbf{L})^2-2\mathbf{A}\cdot(\mathbf{p}\times\mathbf{L})=m^2\alpha^2. \qquad (**)$$

Since in the planetary problem, the vectors **p** and **L** are normal to each other, the second term is just p^2L^2.[55] Let us rewrite the resulting equality in the Cartesian coordinates, orienting the axes as in Fig. 3.7 of the lecture notes, so $\mathbf{A}=A\mathbf{n}_x$, $\mathbf{L}=L_z\mathbf{n}_z$, $\mathbf{p}=p_x\mathbf{n}_x+p_y\mathbf{n}_y$, and $p^2=p_x^2+p_y^2$. In this case

$$\mathbf{A}\cdot(\mathbf{p}\times\mathbf{L})=A\mathbf{n}_x\cdot\begin{vmatrix}\mathbf{n}_x & \mathbf{n}_y & \mathbf{n}_z\\ p_x & p_y & 0\\ 0 & 0 & L_z\end{vmatrix}=AL_z\mathbf{n}_x\cdot(\mathbf{n}_xp_y-\mathbf{n}_yp_x)=AL_zp_y,$$

and Eq. (**) becomes

$$A^2+L_z^2(p_x^2+p_y^2)-2AL_zp_y=m^2\alpha^2,\qquad \text{i.e. } v_x^2+\left(v_y-\frac{A}{mL_z}\right)^2=\left(\frac{\alpha}{L_z}\right)^2,$$

where $\mathbf{v}=\mathbf{p}/m$ is the particle's velocity. With the account of Eqs. (*), this is the same circular hodograph as was derived and discussed in the previous problem, with the radius $\alpha/L_z=L_z/mp$ and the y-offset $A/mL_z=\alpha e/L_z=e(L_z/mp)$.

The vector **A** is also convenient for the solution of some other problems of classical and quantum mechanics, which are, unfortunately, beyond this series' framework.

[54] See, e.g., MA Eq. (7.6).

[55] In this expression, p is the momentum's magnitude rather than the parameter given by the first of Eqs. (3.60).

Problem 3.15. For a particle moving in the following central potential:

$$U(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2},$$

(i) for positive α and β, and all possible ranges of energy E, calculate the orbit $r(\varphi)$;

(ii) prove that in the limit $\beta \to 0$, for energy $E < 0$, the orbit may be represented as a slowly rotating ellipse;

(iii) express the angular velocity of this slow rotation via the parameters α and β, the particle's mass m, its energy E, and the angular momentum L_z.

Solutions:

(i) For this field, the effective 1D potential (3.44), $U_{ef}(r) \equiv U(r) + L_z^2/2mr^2$, may be represented in the same form:

$$U_{ef}(r) = -\frac{\alpha}{r} + \frac{L_{ef}^{\ 2}}{2mr^2},$$

as for the Coulomb potential (with $\beta = 0$), the only difference being that the effective angular momentum L_{ef} is now different from the actual momentum L_z:

$$L_{ef}^2 \equiv L_z^2 + 2m\beta.$$

This is why the trajectory integral (3.48),

$$\varphi(r) = \frac{L_z}{m}\int \frac{dr}{r^2\left\{\frac{2}{m}\left[E - U_{ef}(r)\right]\right\}^{1/2}},$$

may be worked out exactly as was done in the lecture notes for the Coulomb case, giving

$$r = \frac{p}{1 + e\cos\left[\left(L_{ef}/L_z\right)\varphi\right]}, \qquad p = \frac{L_{ef}^2}{m\alpha}, \qquad e = \left(1 + \frac{2EL_{ef}^2}{m\alpha^2}\right)^{1/2}. \qquad (*)$$

The most important difference between this trajectory and the elliptical Kepler orbits is that at $\beta \neq 0$, the ratio L_{ef}/L_z is not an integer, so r is not a 2π-periodic function of φ, i.e. the trajectory is not closed even within the range of energies (corresponding to $0 < e < 1$) in that r has a finite upper bound – see Fig. 3.6 in the lecture notes.

(ii) The angle $\Delta\varphi$ of the orbit's turn per one period of oscillation of the distance r, i.e. difference of φ between two sequential perihelia (points with $r = r_{min}$), may be found as the difference between the adjacent values of φ at which the cosine function in the denominator of the first Eq. (*) becomes equal to +1. This condition yields

$$\Delta\varphi = 2\pi\frac{L_z}{L_{ef}} = \frac{2\pi}{\left(1 + 2m\beta/L_z^2\right)^{1/2}}.$$

At $\beta \to 0$, this angle is very close to 2π:

$$\delta\varphi \equiv \Delta\varphi - 2\pi \approx -2\pi\frac{m\beta}{L_z^{\ 2}} \to 0,$$

so the trajectory may be indeed considered as a Kepler ellipse slowly rotating with the following angular velocity:

$$\Omega \equiv \frac{|\delta\varphi|}{\mathcal{T}} \to 2\pi \frac{m\beta}{\mathcal{T}L_z^2},$$

where $\mathcal{T}$ is the elliptic orbit's period.

(iii) Since the small parameter β is already taken into account in the last formula explicitly (in the numerator of the right-hand side), the period $\mathcal{T}$ in the denominator may be evaluated using the second of Eqs. (3.64b), which corresponds to $\beta = 0$, so the final result is

$$\Omega \to \frac{\beta}{\alpha L_z^2}\left(8m|E|^3\right)^{1/2}, \quad \text{for } \beta \to 0.$$

(The condition of the quantitative validity of this result is $\Omega\mathcal{T} << 1$.)

Problem 3.16. A star system contains a much lighter planet and an even much smaller mass of dust. Assuming that the attractive gravitational potential of the dust is spherically symmetric and proportional to the square of the distance from the star,[56] calculate the slow precession it gives to a circular orbit of the planet.

Solution: According to the assignment, we may represent the potential energy of the planet with mass m in the form

$$U = -\frac{GmM}{r} + \frac{m\Omega^2 r^2}{2},$$

where $\Omega^2 > 0$ is a small constant,[57] so the effective potential energy (3.44) is

$$U_{\text{ef}}(r) = -\frac{GmM}{r} + \frac{m\Omega^2 r^2}{2} + \frac{L_z^2}{2mr^2},$$

and hence has the following derivatives:

$$\frac{dU_{\text{ef}}(r)}{dr} = \frac{GmM}{r^2} + m\Omega^2 r - \frac{L_z^2}{mr^3}, \qquad \frac{d^2U_{\text{ef}}(r)}{dr^2} = -2\frac{GmM}{r^3} + m\Omega^2 + \frac{3L_z^2}{mr^4}.$$

As was discussed in Sec. 3.4 of the lecture notes, these derivatives are sufficient to calculate the angular velocity ω_φ of the planet on its circular orbit, and the frequency ω_r of small oscillations of its distance r from the star, due to a small perturbation of the orbit – in our case, due to the dust's potential with $\Omega << \omega_\varphi$. Since we need these results only to calculate the small angular velocity of the orbit's precession,[58]

[56] As may be readily shown from the gravitation version of the Gauss law (see, e.g., the model solution of Problem 1.7), this approximation is exact if in the space between the star and the planet's orbit, the dust density is constant.

[57] According to the solution of Problem 1.6, the Ω so defined is the frequency of the slow oscillations the planet would have moving under the effect of the dust alone.

[58] If this expression is not clear, please revisit Fig. 3.6 and its discussion in the lecture notes (and possibly also the model solution of Problem 10).

$$\omega_{\text{pre}} = \tilde{\omega}_r - \tilde{\omega}_\varphi << \omega_\varphi,$$

where the tilde signs denote small variations, it does not matter whether this analysis is carried out assuming the constancy of the orbit's radius or of the angular momentum L_z at this perturbation. The second option gives a bit more physical insight, so let us assume that L_z is independent of Ω.

Let us start with finding the radius $\bar{r}$ of the circular orbit, requiring that

$$\frac{dU_{\text{ef}}}{dr}(\bar{r}) \equiv \frac{GmM}{\bar{r}^2} + m\Omega^2\bar{r} - \frac{L_z^2}{m\bar{r}^3} = 0. \qquad (*)$$

Mostly as a sanity check (but also to have all basic formulas handy), at Ω = 0, this equation yields

$$\frac{GmM}{r_0^2} = \frac{L_z^2}{mr_0^3}, \quad \text{i.e. } r_0 = \frac{L_z^2}{GMm^2}, \qquad (**)$$

where r_0 is the radius of the unperturbed orbit. Since, according to Eq. (3.39) of the lecture notes, $L_z \equiv mr^2\dot{\varphi}$, this result immediately gives the unperturbed frequency (3.53) of the planet's rotation:

$$\omega_0 \equiv \bar{\dot{\varphi}} = \frac{L_z}{mr_0^2} = (GM)^2\left(\frac{m}{L_z}\right)^3. \qquad (***)$$

At small Ω ≠ 0, the planet's radius, at the same L_z, is only slightly different from r_0: $\bar{r} \equiv r_0 + \tilde{r}$, with $|\tilde{r}| << r_0$, so Eq. (*) becomes

$$\frac{GmM}{(r_0+\tilde{r})^2} + m\Omega^2(r_0+\tilde{r}) - \frac{L_z^2}{m(r_0+\tilde{r})^3} = 0.$$

Expanding all terms on the left-hand side of this equation to the Taylor series in small $\Omega^2 << \omega_0^2$ and $\tilde{r} \propto \Omega^2$, and keeping only the terms of two leading orders in these small parameters, we get

$$\frac{GmM}{r_0^2}\left(1 - 2\frac{\tilde{r}}{r_0}\right) + m\Omega^2 r_0 - \frac{L_z^2}{mr_0^3}\left(1 - 3\frac{\tilde{r}}{r_0}\right) = 0.$$

With the account of Eq. (**), this equation for $\tilde{r}$ becomes linear and yields a very simple result:

$$\frac{\tilde{r}}{r_0} = -\frac{\Omega^2}{\left[(GM)^2(m/L_z)^3\right]^2} \equiv -\frac{\Omega^2}{\omega_0^2}. \qquad (****)$$

So, the gravitational attraction of the planet to the dust results (at fixed L_z) in a small shrinkage of its orbit – and hence to a proportional increase of its rotation frequency:

$$\omega_\varphi \equiv \omega_0 + \tilde{\omega}_\varphi = \frac{L_z}{m(r_0+\tilde{r})^2} \approx \frac{L_z}{mr_0^2}\left(1 - 2\frac{\tilde{r}}{r_0}\right), \qquad \text{so that } \frac{\tilde{\omega}_\varphi}{\omega_0} = -2\frac{\tilde{r}}{r_0} = 2\frac{\Omega^2}{\omega_0^2} << 1.$$

What remains is to calculate the frequency ω_r of infinitesimal radial oscillations $\tilde{\tilde{r}} \equiv r(t) - \bar{r}$ about this average radius, in the same (linear) approximation in Ω^2. This may be done, for example, by Taylor-expanding Eq. (3.44) for the planet's energy

$$E = \frac{m\dot{r}^2}{2} + U_{\text{ef}}(r)$$

in small $\tilde{\tilde{r}}$, keeping only the leading terms of this expansion:

$$E \approx \frac{m\dot{\tilde{\tilde{r}}}^2}{2} + U_{\text{ef}}(\bar{r}) + \frac{1}{2}\frac{d^2U_{\text{ef}}}{dr^2}\Big|_{r=\bar{r}}\tilde{\tilde{r}}^2, \qquad \text{for } \tilde{\tilde{r}}^2 << \bar{r}^2,$$

and noticing that besides the background energy $U_{\text{ef}}(\bar{r})$ of the average orbit, this is the usual expression for the energy of a 1D harmonic oscillator with frequency ω_r, where[59]

$$\omega_r^2 = \frac{1}{m}\frac{d^2U_{\text{ef}}}{dr^2}\Big|_{r=\bar{r}\equiv r_0+\tilde{r}} = \frac{1}{m}\left[-2\frac{GmM}{(r_0+\tilde{r})^3} + m\Omega^2 + \frac{3L_z^2}{m(r_0+\tilde{r})^4}\right]$$

$$\approx -2\frac{GM}{r_0^3}\left(1-3\frac{\tilde{r}}{r_0}\right) + \Omega^2 + \frac{3L_z^2}{m^2r_0^4}\left(1-4\frac{\tilde{r}}{r_0}\right) \equiv \omega_0^2 + \Omega^2 - 6\frac{L_z^2}{m^2r_0^4}\frac{\tilde{r}}{r_0} = \omega_0^2 + 7\Omega^2.$$

Since all our analysis is only valid for $\Omega^2 << \omega_0^2$, we may rewrite this result as

$$\tilde{\omega}_r \equiv \omega_r - \omega_0 = \left(\omega_0^2 + 7\Omega^2\right)^{1/2} - \omega_0 \approx \frac{7}{2}\frac{\Omega^2}{\omega_0},$$

so, finally, the requested orbit precession frequency is

$$\omega_{\text{pre}} = \tilde{\omega}_r - \tilde{\omega}_\varphi = \frac{7}{2}\frac{\Omega^2}{\omega_0} - 2\frac{\Omega^2}{\omega_0} \equiv \frac{3}{2}\frac{\Omega^2}{\omega_0} << \omega_0 .$$

Problem 3.17. A particle is moving in the field of an attractive central force with the potential

$$U(r) = -\frac{\alpha}{r^n}, \qquad \text{where } \alpha n > 0 .$$

For what values of n, the circular orbits are stable?

Solution: According to the discussion in Sec. 3.5 of the lecture notes, a circular orbit with radius r_0 corresponds to an extremum of the effective potential energy $U_{\text{ef}}(r)$ defined by Eq. (3.44), and has the orbital stability[60] if this extremum is a minimum – see Fig. 3.5. For our current case,

$$U_{\text{ef}}(r) = -\frac{\alpha}{r^n} + \frac{L_z^2}{2mr^2},$$

so these conditions take the following form:

$$\frac{dU_{\text{ef}}}{dr}\Big|_{r=r_0} \equiv \frac{\alpha n}{r_0^{n+1}} - \frac{L_z^2}{mr_0^3} = 0, \tag{*}$$

$$\frac{d^2U_{\text{ef}}}{dr^2}\Big|_{r=r_0} \equiv -\frac{\alpha n(n+1)}{r_0^{n+2}} + \frac{3L_z^2}{mr_0^4} > 0. \tag{**}$$

Solving Eq. (*) for r_0, and plugging the result into Eq. (**), we may reduce the latter condition to the following form:

[59] The last two steps of this calculation make use of Eqs. (**)-(****).
[60] Note that here the term "orbital" is used in the sense discussed in Sec. 3.2 of the lecture notes.

$$\left.\frac{d^2U_{\text{ef}}}{dr^2}\right|_{r=r_0} = \frac{L_z^2}{mr_0^4}(2-n) > 0 .$$

Since the fraction before the parentheses is always positive, this condition is equivalent to

$$n < 2 .$$

This means that stable circular orbits about an attracting center exist both in potentials with growing $|U(r)|$ (with $n < 0$ and $\alpha < 0$), such as the 3D oscillator's potential $U(r) = m\omega_0^2 r^2/2$ ($n = -2 < 2$), and also in potentials with decreasing $|U(r)|$, but only if the potential's magnitude drops with distance slower than the second term of the effective potential energy U_{ef}. (The most important example is the attractive Coulomb potential (3.49), with $\alpha > 0$ and $n = 1 < 2$.)

Problem 3.18. Determine the condition for a particle of mass m, moving under the effect of a central attractive force

$$\mathbf{F} = -\alpha \frac{\mathbf{r}}{r^3} \exp\left\{-\frac{r}{R}\right\},$$

where α and R are positive constants, to have a stable circular orbit.

Solution: As was discussed in Sec. 3.5 of the lecture notes, the radius r_0 of a circular orbit is the position of the minimum of the effective potential $U_{\text{ef}}(r)$ given by Eq. (3.44), i.e. is the solution of the following equation:

$$\left.\frac{dU_{\text{ef}}}{dr}\right|_{r=r_0} \equiv \left[\frac{dU(r)}{dr} - \frac{L_z^2}{mr^3}\right]_{r=r_0} = 0 . \qquad (*)$$

Since for a central force $\mathbf{F} = F(r)\mathbf{n}_r$, the scalar function $F(r)$ may be represented as $-dU(r)/dr$, in our case we may write

$$\frac{dU_{\text{ef}}}{dr} = -F - \frac{L_z^2}{mr^3} = \frac{\alpha}{r^2}\exp\left\{-\frac{r}{R}\right\} - \frac{L_z^2}{mr^3}, \qquad (**)$$

so the condition (*) gives the following transcendental equation for r_0:

$$\alpha r_0 \exp\left\{-\frac{r_0}{R}\right\} = \frac{L_z^2}{m}. \qquad (***)$$

Before analyzing this equation, let us use Eq. (**) to calculate the second derivative of the effective potential at point r_0, which determines the circular orbit's stability – see Fig. 3.5 of the lecture notes and its discussion:

$$\frac{d^2U_{\text{ef}}}{dr^2} = \left(-\frac{2}{r^3} - \frac{1}{r^2R}\right)\alpha\exp\left\{-\frac{r}{R}\right\} + \frac{3L_z^2}{mr^4} \equiv \frac{1}{r^4}\left[-\left(2+\frac{r}{R}\right)\alpha r\exp\left\{-\frac{r}{R}\right\} + \frac{3L_z^2}{m}\right],$$

so at $r = r_0$, using Eq. (***), we get

$$\left.\frac{d^2U_{\text{ef}}}{dr^2}\right|_{r=r_0} = \frac{1}{r_0^4}\left[-\left(2+\frac{r_0}{R}\right)\alpha r_0\exp\left\{-\frac{r_0}{R}\right\} + \frac{3L_z^2}{m}\right] = \frac{1}{r_0^4}\left[-\left(2+\frac{r_0}{R}\right)\frac{L_z^2}{m} + \frac{3L_z^2}{m}\right] \equiv \frac{1}{r_0^4}\frac{L_z^2}{m}\left(1-\frac{r_0}{R}\right).$$

Since r_0^4, L_z^2, and m are all positive, the second derivative is positive (and hence the circular orbit is stable) if $r_0 < R$.

Now returning to Eq. (***), let us notice that its left-hand side is a smooth, positive function of $r_0 > 0$, turning to zero at $r_0 = 0$ and $r_0 = \infty$, and having one maximum, at $r_0 = R$, equal to $\alpha R/e$. Hence, Eq. (***) has two roots, one of them with $r_0 < R$, corresponding to a stable circular orbit, if

$$\frac{\alpha R}{e} > \frac{L_z^2}{m}, \qquad \text{i.e. if } \; R > \frac{L_z^2 e}{\alpha m}.$$

This condition is always satisfied for the attractive Coulomb potential (3.49), which corresponds to our current case in the limit $R \to \infty$.

Problem 3.19. A particle of mass m, with an angular momentum L_z, moves in the field of an attractive central force with a distance-independent magnitude F. If the particle's energy E is slightly higher than the value $E_{\min}$ corresponding to its circular orbit, what is the time period of its radial oscillations? Compare the period with that of the circular orbit at $E = E_{\min}$.

Solution: Any force $\mathbf{F}(\mathbf{r})$ depending only on the particle's radius vector $\mathbf{r}$ equals $-\nabla U(\mathbf{r})$, where $U(\mathbf{r})$ is the corresponding potential energy. According to MA Eq. (10.8), for an attractive central force $\mathbf{F}(\mathbf{r}) = -F(r)\,\mathbf{n}_r$, we have $U = U(r)$ with $F(r) = dU/dr$. In our particular case, $F(r) = \text{const}$, so $U(r) = Fr + \text{const}$. By ignoring the inconsequential constant, from Eq. (3.44) of the lecture notes, we get the following effective potential energy:

$$U_{\text{ef}}(r) = Fr + \frac{L_z^2}{2mr^2}.$$

According to the discussion in Sec. 3.5, the circular orbit corresponds to the minimum of this function, so its radius r_0 and the corresponding energy $E_{\min}$ are easy to calculate:

$$\left.\frac{dU_{\text{ef}}}{dr}\right|_{r=r_0} \equiv F - \frac{L_z^2}{mr_0^3} = 0, \qquad \text{so } \; r_0 = \left(\frac{L_z^2}{mF}\right)^{1/3} \text{ and } \; E_{\min} \equiv \frac{L_z^2}{2mr_0^2} + U(r_0) = \frac{3}{2}\left(\frac{F^2 L_z^2}{m}\right)^{1/3}.$$

Alternatively, this expression for r_0 may be found from the combination of the 2nd Newton law, $mv_0^2/r_0 = F$, and the expression $L_z = mv_0 r_0$ for the angular momentum, valid for a circular orbit. The solution of this system of two equations gives us not only the above expression for r_0 but also the velocity v_0 of the circular motion and hence its time period:

$$v_0 = \frac{L_z}{mr_0} = \left(\frac{FL_z}{m^2}\right)^{1/3}, \qquad \mathcal{T}_0 = \frac{2\pi r_0}{v_0} = 2\pi\left(\frac{mL_z}{F^2}\right)^{1/3}. \qquad (*)$$

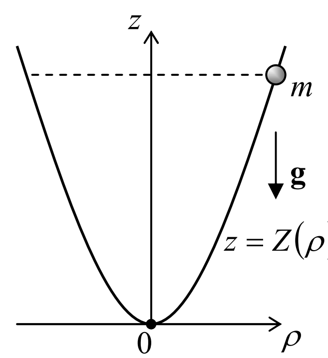
Now if E is only slightly larger than $E_{\min}$, the particle's radius r oscillates around the value r_0 with a small amplitude – see Fig. 3.5 of the lecture notes and its discussion. In order to find the period of these oscillations, we may use the Taylor expansion of the function $U_{\text{ef}}(r)$ at the point r_0, and keep, besides the constant $U_{\text{ef}}(r_0)$, only the leading, quadratic term of this expansion:

$$U_{\text{ef}}(r \approx r_0) \approx U_{\text{ef}}(r_0) + \frac{\kappa}{2}\tilde{r}^2, \quad \text{where } \tilde{r} \equiv r - r_0, \quad \text{and } \kappa \equiv \left.\frac{d^2 U_{\text{ef}}}{dr^2}\right|_{r=r_0} = \frac{3L_z^2}{mr_0^4} = 3\left(\frac{mF^4}{L_z^2}\right)^{1/3}.$$

Within this approximation, Eq. (3.43) of the lecture notes becomes just the well-known expression for the energy of a 1D particle of mass m driven by an elastic spring with the spring constant κ and the equilibrium at point r_0. Such a system performs harmonic oscillations with frequency $\omega = (\kappa/m)^{1/2}$, i.e. with the time period

$$\mathcal{T}_r = \frac{2\pi}{\omega} = 2\pi\left(\frac{m}{\kappa}\right)^{1/2} = \frac{2\pi}{\sqrt{3}}\left(\frac{mL_z}{F^2}\right)^{1/3}.$$

Comparing this result with Eq. (*), we see that the period of radial oscillations is by a factor of $\sqrt{3} \approx 1.732$ shorter than that of the particle's angular rotation (which, in the limit $E \to E_{\min}$, does not differ from the period $\mathcal{T}_0$ of the circular orbital motion). Since this factor is an irrational number, i.e. the periods $\mathcal{T}_r$ and $\mathcal{T}_0$ are incommensurate, the orbit is not closed – see Fig. 3.6 and its discussion.



Problem 3.20. A particle may move without friction, in the uniform gravity field $\mathbf{g} = -g\mathbf{n}_z$, over an axially-symmetric surface that is described, in the cylindrical coordinates $\{\rho, \varphi, z\}$, by a smooth function $Z(\rho)$ – see the figure on the right. Derive the condition of stability of circular orbits of the particle around the symmetry axis z, with respect to small perturbations. For the cases when the condition is fulfilled, find out whether the weakly perturbed orbits are open or closed. Spell out your results for the following particular cases:

(i) a conical surface with $Z = \alpha\rho$,
(ii) a paraboloid with $Z = \kappa\rho^2/2$, and
(iii) a spherical surface with $Z^2 + \rho^2 = R^2$, for $\rho < R$.

Solution: In the cylindrical coordinates, the kinetic energy of a particle of mass m is

$$T \equiv \frac{m}{2}\left(\dot{r}_1^2 + \dot{r}_2^2 + \dot{r}_3^2\right) = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right),$$

so if it is confined to move over a surface with $z = Z(\rho)$, then

$$T = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{Z}^2\right) = \frac{m}{2}\left[\left(1 + Z'^2\right)\dot{\rho}^2 + \rho^2\dot{\varphi}^2\right],$$

where $Z' \equiv dZ/d\rho$. Since neither T nor the potential energy $U = mgZ(\rho)$ depend on the azimuthal angle φ explicitly, this angle is a cyclic generalized coordinate, so the corresponding generalized momentum,

$$L_z \equiv \frac{\partial L}{\partial \dot{\varphi}} = m\rho^2\dot{\varphi},$$

is an integral of motion, whose value is determined by initial conditions. Expressing $\dot{\varphi}$ from this relation and plugging it into T, for the total mechanical energy $E = T + U$ of the point we get

$$E = \frac{m_{\text{ef}}(\rho)}{2}\dot{\rho}^2 + U_{\text{ef}}(\rho), \quad \text{with } m_{\text{ef}}(\rho) = m\left(1 + Z'^2\right) \text{ and } U_{\text{ef}}(\rho) = \frac{L_z^2}{2m\rho^2} + mgZ(\rho). \qquad (*)$$

This expression is generally (though not in detail) similar to Eqs. (3.43)-(3.44) of the lecture notes for the motion in a central potential field,[61] and may be used similarly. Namely, for any given L_z,

[61] See also the solutions of Problems 9 and 17-19.

the circular orbit radius ρ_0 corresponds to the minimum of the function $U_{\text{ef}}(\rho)$, which may be found from the usual condition $dU_{\text{ef}}/d\rho = 0$. For our function (*),

$$\frac{dU_{\text{ef}}}{d\rho} = -\frac{L_z^2}{m\rho^3} + mgZ'(\rho),$$

so ρ_0 satisfies the following equation:

$$\frac{L_z^2}{m^2 g} = \rho_0^3 Z'(\rho_0). \qquad (**)$$

Since the effective mass participating in Eq. (*) is always positive, the condition of radial stability of the circular orbit is $d^2U_{\text{ef}}/d\rho^2 > 0$ at $\rho = \rho_0$; since for our system

$$\frac{d^2U_{\text{ef}}}{d\rho^2} = \frac{3L_z^2}{m\rho^4} + mgZ''(\rho), ,$$

the stability condition is

$$\frac{3L_z^2}{m^2 g} + \rho_0^4 Z''(\rho_0) > 0,$$

where $Z'' \equiv d^2Z/d\rho^2$. By using Eq. (**), this condition may be represented in a simpler form:

$$3Z'(\rho_0) + \rho_0 Z''(\rho_0) > 0. \qquad (***)$$

If a stable orbit is slightly perturbed, i.e. the particle's energy is slightly higher than the minimum of U_{ef}, its radius has a small time-dependent part

$$\tilde{\rho}(t) \equiv \rho(t) - \rho_0, \qquad \text{with } |\tilde{\rho}| << \rho_0.$$

To describe its time evolution, we may expand both parts of Eq. (*) into the Taylor series in small $\tilde{\rho}(t)$, and leave only the first nonvanishing terms:

$$\tilde{E} \equiv E - U_{\text{ef}}(\rho_0) \approx \frac{m_{\text{ef}}(\rho_0)}{2}\dot{\tilde{\rho}}^2 + \frac{1}{2}\frac{d^2U_{\text{ef}}}{d\rho^2}(\rho_0)\tilde{\rho}^2.$$

Comparing this result with Eq. (3.10) of the lecture notes, we see that $\tilde{\rho}(t)$ performs harmonic oscillations with frequency (3.15); according to the above formulas, in our case, this frequency is

$$\omega_\rho = \left[\frac{g}{1+Z'^2}\left(\frac{3Z'}{\rho_0} + Z''\right)\right]^{1/2},$$

where both derivatives of the function $Z(\rho)$ have to be evaluated at point ρ_0.

As was discussed in Sec. 3.4 of the lecture notes, the corresponding orbit $\rho(\varphi)$ is closed if this frequency is commensurate with the angular velocity of the particle's motion around the symmetry axis z, i.e. with $\omega_\varphi \equiv \dot{\varphi} = L_z/m\rho_0^2$. Using Eq. (**), this frequency may be represented as

$$\omega_\varphi = \left(\frac{gZ'}{\rho_0}\right)^{1/2},$$

so for the frequency ratio, we get a formula that does not include g explicitly:

$$\frac{\omega_\rho}{\omega_\varphi} = \left[\frac{1}{1+Z'^2}\left(3+\frac{\rho_0 Z''}{Z'}\right)\right]^{1/2}. \qquad (****)$$

Naturally, this expression gives a real value only if the expression in parentheses is positive, i.e. if the stability condition (***) is fulfilled.

Now we are ready to discuss the given particular cases:

(i) For $Z = \alpha\rho$, i.e. $Z' = \alpha$ and $Z'' = 0$, Eq. (***) reads

$$3\alpha > 0,$$

i.e. circular orbits exist and are stable only on a conical surface with its apex down. In this case, Eq. (****) is reduced to

$$\frac{\omega_\rho}{\omega_\varphi} = \left(\frac{3}{1+\alpha^2}\right)^{1/2},$$

so the orbits are closed only at certain exact values of the slope – for example, when $\alpha = \sqrt{2}$ (giving $\omega_\rho = \omega_\varphi$), $\alpha = 1/\sqrt{2}$ (giving $\omega_\rho = 2\omega_\varphi$), etc.

(ii) For $Z = \kappa\rho^2/2$, i.e. $Z' = \kappa\rho$ and $Z'' = \kappa$, Eq. (***) yields

$$4\kappa\rho_0 > 0,$$

so orbits in a round-parabolic cup exist and are stable only if its bottom looks down – very naturally, and similarly to example (i). In this case ($\kappa > 0$), Eq. (****) yields

$$\frac{\omega_\rho}{\omega_\varphi} = \frac{2}{\left(1+\kappa^2\rho_0^2\right)^{1/2}},$$

so for any orbit very close to the origin, $\omega_\rho/\omega_\varphi \to 2$, i.e. it is virtually closed.

This fact should not be too surprising, because if the motion in such a quadratic potential does not involve the z-component of the kinetic energy (as it happens asymptotically at $Z'^2 \to 0$), it is a linear superposition of two independent harmonic oscillations along two other mutually normal Cartesian coordinates x and y, with the same frequency $(g\kappa)^{1/2}$. The particle's trajectory corresponding to a superposition of such oscillations with arbitrary amplitudes and phases is an ellipse centered at the origin, so the particle's distance ρ from it oscillates twice each rotation period.

(iii) For $Z = \pm(R^2 - \rho^2)^{1/2}$, with $Z' = \mp\rho/(R^2 - \rho^2)^{1/2}$ and $Z'' = \mp R^2/(R^2 - \rho^2)^{3/2}$, Eq. (**) shows that circular orbits may exist only on the lower hemisphere, described by the lower sign in the above formulas. For such orbits, the stability condition (***) takes the form

$$\frac{3\rho_0}{\left(R^2-\rho_0^2\right)^{1/2}} + \frac{\rho_0 R^2}{\left(R^2-\rho_0^2\right)^{3/2}} > 0,$$

so it is fulfilled for any $\rho_0 < R$, while the frequency ratio (****) is

$$\frac{\omega_\rho}{\omega_\varphi} = \left(4 - 3\frac{\rho_0^2}{R^2}\right)^{1/2} \equiv \left(1+3\cos^2\theta_0\right)^{1/2}, \quad \text{where } \theta \equiv \sin^{-1}\frac{\rho}{R}.$$

This result[62] shows that the ratio is commensurate not only at certain special exact values of ρ_0 (such as $(7/12)^{1/2}R$, giving $\omega_\rho/\omega_\varphi = 3/2$), but also asymptotically in the limits $\rho_0 \to R$ (when $\omega_\rho/\omega_\varphi \to 1$) and $\rho_0/R \to 0$ (when $\omega_\rho/\omega_\varphi \to 2$). According to Eq. (**), the former of these limits, where $Z'(\rho_0) \to \infty$, corresponds to very high values of the angular momentum and hence of the kinetic energy of the particle, while the latter limit, on the contrary, corresponds to very low kinetic energies when the particle moves at the very bottom of the surface, where it may be well approximated with a paraboloid:

$$Z \approx \frac{1}{2} Z''\Big|_{\rho=0} \rho^2 = \frac{\rho^2}{2R} \equiv \frac{x^2+y^2}{2R},$$

similar to the previous case (ii) with $\kappa = 1/R$.

Problem 3.21. The gravitational potential (i.e. the gravitational energy of a unit probe mass) of our Milky Way galaxy, averaged over interstellar distances, is reasonably well approximated by the following axially symmetric function:

$$\phi(r,z) = \frac{V^2}{2}\ln\left(r^2 + \alpha z^2\right),$$

where r is the distance from the galaxy's symmetry axis and z is the distance from its central plane, while V and $\alpha > 0$ are constants.[63] Prove that circular orbits of stars in this gravity field are stable, and calculate the frequencies of their small oscillations near such orbits, in the r- and z-directions.

Solution: In the "vertical" (z) direction, the given potential evidently has a minimum at $z = 0$, so any circular orbit may only lie within this central plane. Thus, despite the generally 3D character of motion in this field, for a particle (say, a star) moving in the plane, we may use the 2D effective potential energy given by Eq. (3.44) of the lecture notes:

$$U_{\text{ef}}(r) = m\phi(r,0) + \frac{L_z^2}{2mr^2} = mV^2 \ln r + \frac{L_z^2}{2mr^2} + \text{const},$$

where m is the particle's mass. According to the discussion in Sec. 3.4 of the lecture notes (see, in particular, Fig. 3.5), the orbit's radius r_0, for a given L_z (which is a constant of motion, determined by initial conditions), may be calculated from the requirement $\partial U_{\text{ef}}/\partial r = 0$ at $r = r_0$, giving

$$\frac{mV^2}{r_0} - \frac{L_z^2}{mr_0^3} = 0, \qquad \text{i.e. } r_0 = \frac{L_z}{mV}.$$

The angular velocity of this motion is

$$\omega_0 = \frac{v_0}{r_0} = \frac{L_z}{mr_0^2} = \frac{mV^2}{L_z}.$$

For small deviations $\tilde{r} \equiv r - r_0$ from this orbit in the radial direction, we may expand the effective potential energy in the Taylor series, truncating it after the second nonvanishing term:

[62] Actually, it was already derived in the solution of Problem 9, whose subject was the spherical pendulum, i.e. the same system as our current case (iii), but which was solved using spherical rather than cylindrical coordinates.
[63] Just for the reader's reference, these constants are close to, respectively, 2.2×10^5 m/s and 6.

$$U_{\text{ef}}(r) \approx U_{\text{ef}}(r_0) + \frac{\kappa_r}{2}\tilde{r}^2, \quad \text{where } \kappa_r \equiv \frac{d^2 U_{\text{ef}}}{dr^2}\Big|_{r=r_0} = -\frac{mV}{r_0^2} + \frac{3L_z^2}{mr_0^4} \equiv 2m\left(\frac{mV^2}{L_z}\right)^2.$$

Since $\kappa_r > 0$, the circular orbits are stable with respect to small radial deviations, which obey the standard equation of a linear ("harmonic") oscillator with frequency

$$\omega_r = \left(\frac{\kappa_r}{m}\right)^{1/2} = \sqrt{2}\frac{mV^2}{L_z} \equiv \sqrt{2}\omega_0.$$

To analyze small deviations z from the central plane (with $|z| << r$), we need to restore the z-dependence of the potential energy, but since the deviations take place at $r = r_0$,[64] we may expand only the actual potential energy:

$$U(r_0, z) \equiv m\phi(r_0, z) \approx U(r_0, 0) + \frac{\kappa_z}{2}z^2, \quad \text{where } \kappa_z \equiv \frac{\partial^2 U(r_0, z)}{\partial z^2}\Big|_{z=0} = \alpha\frac{mV^2}{r_0^2} \equiv \alpha\left(\frac{mV^2}{L_z}\right)^2.$$

Since $\kappa_r > 0$, the circular orbits are stable with respect to small z-deviations as well, and these deviations also oscillate sinusoidally, with the frequency

$$\omega_z = \left(\frac{\kappa_z}{m}\right)^{1/2} = \alpha^{1/2}\frac{mV^2}{L_z} \equiv \alpha^{1/2}\omega_0,$$

different from both ω_0 and ω_r. (For our Sun, with $r_0 \approx 2.5\times10^{20}$ m, the corresponding periods are, respectively, 87, 225, and 160 million years.)

<u>Problem 3.22</u>. For particle scattering by a repulsive Coulomb field, calculate the minimum approach distance $r_{\min}$ and the velocity $v_{\min}$ at that point, and analyze their dependence on the impact parameter b (see Fig. 3.9 of the lecture notes) and on the initial velocity v_∞ of the particle.

Solution: Due to the energy conservation during the motion, we may write $E(r = \infty) = E(r = r_{\min})$. The first of these values equals $mv_\infty^2/2$, while, according to Fig. 3.5 of the lecture notes, the second one equals $U_{\text{ef}}(r_{\min})$, where $U_{\text{ef}}(r)$ is the effective potential energy defined by Eq. (3.44), so the energy conservation condition takes the form

$$\frac{mv_\infty^2}{2} = U(r_{\min}) + \frac{L_z^2}{2mr_{\min}^2}.$$

With the account of Eq. (3.49) for the Coulomb potential (with the opposite sign for our case of repulsion, i.e. $U(r) = \alpha/r$), and the relation $L_z = mv_\infty b$ (see Eq. (3.66) of the lecture notes), the condition becomes

$$\frac{mv_\infty^2}{2} = \frac{\alpha}{r_{\min}} + \frac{mv_\infty^2 b^2}{2r_{\min}^2}.$$

Solving this quadratic equation for $r_{\min}$, we get the following expression for its physically meaningful root $r_{\min} > 0$:

---

[64] Such independence of small z- and r-deviations results from the fact that for the given potential, at $z = 0$, $\partial^2\phi/\partial r\partial z = 0$.

$$r_{\min} = \frac{\alpha}{mv_\infty^2} + \left[\left(\frac{\alpha}{mv_\infty^2}\right)^2 + b^2\right]^{1/2},$$

which shows that (for the repulsive interaction only!), $r_{\min}$ is always larger than b, gradually approaching it as the initial velocity v_∞ of the particle is increased.

The easiest way to calculate $v_{\min}$ is to equate the expression $L_z = mv_{\min}r_{\min}$ (evident from looking at Fig. 3.9) for the nearest approach point, to the expression $L_z = mv_\infty b$ already used above. This equality yields the following result,

$$v_{\min} = v_\infty \frac{b}{r_{\min}} = v_\infty \frac{b}{\left[b^2 + \left(\alpha / mv_\infty^2\right)^2\right]^{1/2} + \alpha / mv_\infty^2},$$

showing that $v_{\min}$ is always lower than v_∞, approaching this value only at $b \to \infty$, and vanishing (as it should) in the opposite case of a purely head-on collision with $b = 0$.

Problem 3.23. A particle is launched from afar, with an impact parameter b, toward an attracting center creating the potential

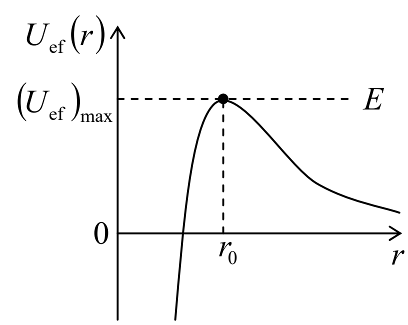
$$U(r) = -\frac{\alpha}{r^n}, \quad \text{with } n > 2 \text{ and } \alpha > 0.$$

(i) For the case when the initial kinetic energy E of the particle is barely sufficient for escaping its capture by this attracting center, express the minimum approach distance via b and n.

(ii) Calculate the capture's total cross-section and explore its limit at $n \to 2$.

Solutions:

(i) Though, formally, the full mechanical energy of the particle in this problem is conserved, its infinitely close approach to point $r = 0$ would lead to an infinite growth of its velocity, and to an inevitable energy loss by this or that inelastic mechanism, and hence to particle's capture by the attracting center. Hence, the given escape condition means that E is smaller than, but virtually equal to the maximum of the effective energy of the radial motion,



$$U_{ef}(r) \equiv U(r) + \frac{L_z^2}{2mr^2} = -\frac{\alpha}{r^n} + \frac{L_z^2}{2mr^2},$$

– see the sketch in the figure on the right. With the account of Eq. (3.66) of the lecture notes, this formula turns into

$$U_{ef}(r) = -\frac{\alpha}{r^n} + \frac{Eb^2}{r^2}. \qquad (*)$$

Now r_0 may be found from the equation $dU_{ef}/dr = 0$. Differentiating Eq. (*), we find that

$$r_0 = \left(\frac{n\alpha}{2Eb^2}\right)^{\frac{1}{n-2}}, \qquad \left(U_{ef}\right)_{\max} \equiv U_{ef}(r_0) = \left(\frac{2Eb^2}{n\alpha}\right)^{\frac{2}{n-2}} Eb^2 \frac{n-2}{n}. \qquad (**)$$

From here, the condition $E = (U_{ef})_{\max}$ becomes

$$\left(\frac{2Eb^2}{n\alpha}\right)^{\frac{2}{n-2}} b^2 \frac{n}{n-2} = 1. \qquad (***)$$

Solving this equation for E, and plugging the result into the first of Eqs. (**), we finally get

$$r_0 = b\left(\frac{n-2}{n}\right)^{1/2},$$

so for any $n > 2$, r_0 is always less than b – a natural result for an attracting center.

(ii) The total cross-section of the capture may be found as

$$\sigma = \pi b_{\min}^2,$$

where $b_{\min}$ is the smallest value of the impact parameter that allows the particle to escape (at the given energy). For $n > 2$, $b_{\min}$ may be readily found from Eq. (***):

$$\sigma = \pi \frac{n}{(n-2)^{(n-2)/n}}\left(\frac{\alpha}{2E}\right)^{2/n}, \quad \text{for } n > 2.$$

In order to explore the limit $n \to 2$, one may use the following mathematical fact:[65]

$$\lim_{n\to 2}\left[(n-2)^{(n-2)}\right] \equiv \lim_{\varepsilon\to 0}\left[\varepsilon^{\varepsilon}\right] = 1,$$

giving

$$\sigma \to \pi\frac{\alpha}{E}, \qquad \text{for } n \to 2. \qquad (****)$$

The last result may be also obtained in a simpler way. At $n = 2$, the effective potential is

$$U_{\text{ef}}(r) = -\frac{\alpha}{r^2} + \frac{Eb^2}{r^2} \equiv \frac{1}{r^2}\left(Eb^2 - \alpha\right).$$

Evidently, the capture is avoided if the effective potential is positive, $Eb^2 > \alpha$, i.e. if $\pi b^2 > \pi\alpha/E$, thus giving Eq. (****) again.

A sanity check: the larger the particle's energy, the smaller σ, i.e. the smaller the chance of its capture at random b.

Problem 3.24. A small body with an initial velocity v_∞ approaches an atmosphere-free planet of mass M and radius R.

(i) Find the condition on the impact parameter b for the body to hit the planet's surface.
(ii) If the body barely avoids the collision, what is its scattering angle?

Solutions:

[65] If you have any doubt in this fact, consider the function $\ln(\varepsilon^{\varepsilon}) \equiv \varepsilon \ln\varepsilon$. At $\varepsilon \to 0$, the logarithm is a much slower function of its argument than the linear factor before it, so $\ln(\varepsilon^{\varepsilon}) \to 0$, and $\varepsilon^{\varepsilon} \equiv \exp\{\ln(\varepsilon^{\varepsilon})\} \to \exp\{0\} = 1$.

(i) The requested condition is $b < b_{max}$, where b_{max} is the impact parameter's value corresponding to the equality $r_{min} = R$, where r_{min} is the lowest point of the small body's orbit. To find r_{min}, we may use the conservation of the body's angular momentum L_z and its mechanical energy E – which may be expressed by Eqs. (3.43)-(3.44) of the lecture notes:

$$E = \frac{m\dot{r}^2}{2} + U_{\text{ef}}(r), \qquad U_{\text{ef}}(r) = U(r) + \frac{L_z^2}{2mr^2}.$$

Applying the energy conservation law to the lowest point of the orbit, $r = r_{min} = R$ (where $\dot{r} = 0$), and the initial point $r \to \infty$ (where $E = mv_\infty^2/2$), we get

$$U_{\text{ef}}(R) = \frac{mv_\infty^2}{2}, \quad \text{i.e.} \quad U(R) + \frac{L_z^2}{2mR^2} = \frac{mv_\infty^2}{2}. \qquad (*)$$

In order to spell out the potential energy, we may use the fact (which was discussed in the model solution of Problem 1.9) that a spherically-symmetric sphere produces, outside it, the gravity field similar to that of a particle of the same mass, located in the sphere's center, so $U(R) = GMm/R$. Using also Eq. (3.66) in the form $L_z = bmv_\infty$, we may rewrite Eq. (*), for $b = b_{max}$, as

$$-\frac{GMm}{R} + \frac{b_{\max}^2 mv_\infty^2}{2R^2} = \frac{mv_\infty^2}{2}.$$

From this equation, we readily get

$$b_{\max} = R\left(1 + \frac{2GM}{Rv_\infty^2}\right)^{1/2}. \qquad (**)$$

For very large initial velocities, $v_\infty^2 >> GM/R$, b_{max} tends to the planet's radius – as it obviously should.

(ii) Plugging Eq. (**) into Eq. (3.69b) of the lecture notes with $\alpha = GMm$ and $E = mv_\infty^2/2$, we get

$$\tan\frac{|\theta|}{2} = \frac{GM}{v_\infty^2 b_{\max}} = \frac{GM/v_\infty^2 R}{\left[1 + (2GM/v_\infty^2 R)\right]^{1/2}}.$$

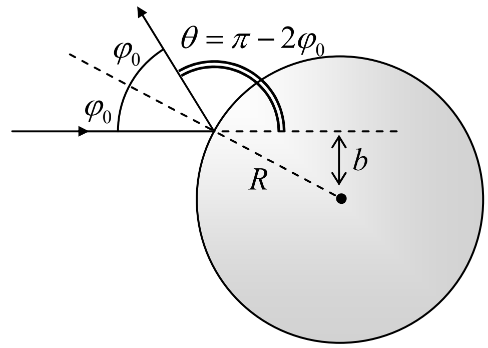
This result shows that while for a very high initial velocity, the scattering angle is very small, as the velocity is decreased well below GM/R (that is just the 1st space velocity for this planet – see the solution of Problem 1.6), the right-hand side of this relation, and hence $\tan(|\theta|/2)$, tends to infinity, so the argument $|\theta|/2$ tends to $\pi/2$, and hence the scattering angle $|\theta|$ tends to π – the perfect *backscattering*.

The fact that the scattering angle may be arbitrary (within the $[0, \pi]$ interval) even for a planet of a substantial radius, is widely used for spacecraft trajectory manipulation and enables very complex interplanetary missions (such as the famous Voyager program[66]), with almost no additional rocket fuel burns.

Problem 3.25. Calculate the differential and total cross-sections of the classical elastic scattering of small particles by a hard sphere of radius R.

[66] See, e.g., http://voyager.jpl.nasa.gov.

Solution: From the simple geometry of such scattering, shown in the figure on the right (whose plane is selected to contain both the initial trajectory of the particle and the center of the sphere), we immediately see that the impact parameter b and the incidence angle φ_0 are related as



$$b = R\sin\varphi_0 = R\sin\frac{\pi-\theta}{2} \equiv R\cos\frac{\theta}{2},$$

so Eq. (3.72) of the lecture notes for the differential cross-section yields

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = R^2\frac{\cos(\theta/2)}{\sin\theta}\left|\frac{d\cos(\theta/2)}{d\theta}\right| = \frac{R^2}{4}, \quad \text{for} \left|\frac{\theta}{2}\right| \leq \frac{\pi}{2}, \quad \text{i.e. for any } 0 \leq \theta \leq \pi .$$

Integrating this simple result over the full body angle 4π, we get the total cross-section of scattering:

$$\sigma \equiv \int_{4\pi}\frac{d\sigma}{d\Omega}d\Omega = \frac{R^2}{4}\int_{4\pi}d\Omega = \frac{R^2}{4}4\pi \equiv \pi R^2 .$$

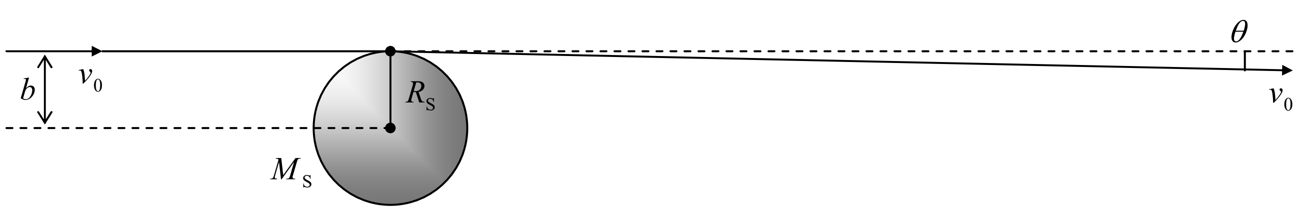
So, in classical mechanics, the total cross-section of a hard sphere is just the area of its geometric cross-section – the fact that could be readily predicted in advance, even without calculating $d\sigma/d\Omega$.

Problem 3.26. The most famous confirmation[67] of Einstein's general relativity theory has come from the observation, by A. Eddington and his associates, of star light's deflection by the Sun, during the May 1919 solar eclipse. Considering light photons as classical particles propagating with the speed of light, $v_0 \to c \approx 3.00\times10^8$m/s, and using the astronomic data for the Sun's mass ($M_S \approx 1.99\times10^{30}$kg) and its radius ($R_S \approx 6.96\times10^8$m), calculate the non-relativistic mechanics' prediction for the angular deflection of the light rays grazing the Sun's surface.

Solution: Given the effect's smallness (confirmed *a posteriori* by the result – see below), we may use Eq. (3.69b) of the lecture notes in the limit $|\theta| << 1$, getting

$$|\theta| = \frac{|\alpha|}{Eb},$$

and also take $b = R_S$ – see the figure below. (Note that this result may be recast just as $|\theta| = U(R_S)/E$.)



[67] It was not the first confirmation, though. The first one came four years earlier from A. Einstein himself, who showed that his theory might explain the difference between the rate of Mercury orbit's precession, known from earlier observations, and the non-relativistic theory of this effect.

With the values $\alpha = -GM_Sm$ (resulting from the comparison of cf. Eq. (1.15) and (3.49) of the lecture notes) and $E = mv_0^2/2$ (where m is the assumed photon's mass), we get an expression independent on m:

$$\left|\theta\right| = \frac{2GM_S}{R_S v_0^2}. \qquad (*)$$

Note that this expression might be also readily obtained without the theory of scattering discussed in Sec. 3.5 of the lecture notes, but just from elementary arguments. Indeed, as the figure above shows, we may calculate $|\theta| << 1$ as the ratio $|v_\perp|/v_0$, where $v_\perp$ is the transverse velocity component acquired by the photon due to the normal component $F_\perp$ of the Sun's attraction force, as described by Eq. (1.15):[68]

$$\left|v_\perp\right| = \frac{\left|p_\perp\right|}{m} = \frac{1}{m}\int_{-\infty}^{+\infty}\left|F_\perp\right|dt = \frac{1}{m}\int_{-\infty}^{+\infty}\frac{GM_S m r_\perp}{r^3}dt = GM_S R_S \int_{-\infty}^{+\infty}\frac{dt}{\left(R_S^2 + v_0^2 t^2\right)^{3/2}} = \frac{2GM_S}{R_S v_0}\int_0^{+\infty}\frac{d\xi}{\left(1+\xi^2\right)^{3/2}} = \frac{2GM_S}{R_S v_0},$$

thus returning us to Eq. (*).

With the astronomic values given in the assignment, and the gravitational constant $G \approx 6.67\times 10^{11}$ SI units, Eq. (*) yields

$$\left|\theta\right| \approx 4.25\times 10^{-6},$$

so the deflection angle is indeed very small (less than one arcsecond!), thus justifying the assumptions made at its derivation. However, the same smallness makes the experimental observation of this effect, and the distinction between the calculated classical value of θ and the (twice larger) value predicted by the general relativity, rather difficult. Indeed, the 1919 Eddington's observation results had been only twice as far from the classical value as from the relativistic value, and the issue was not finally settled until the 1960s.

<u>Problem 3.27</u>. Generalize the expression for the small angle of scattering, obtained in the solution of the previous problem, to a spherically symmetric but otherwise arbitrary potential $U(r)$. Use the result to calculate the differential cross-section of small-angle scattering by the potential $U = C/r^n$, with an integer $n > 0$.

Hint: You may like to use the following table integral: $\displaystyle\int_1^\infty \frac{d\xi}{\xi^{n+1}\left(\xi^2-1\right)^{1/2}} = \pi^{1/2}\frac{\Gamma(n/2+1/2)}{n\Gamma(n/2)}$.

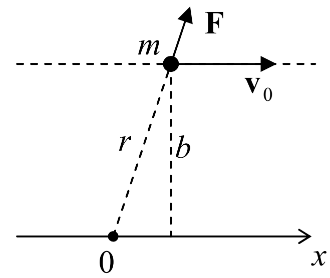
Solution: Acting just as at the second approach to the solution of the previous problem, but with the force $\mathbf{F} = -\nabla U(r) \equiv -\mathbf{n}_r dU/dr$, we may write:

$$v_\perp = \frac{1}{m}\int_{-\infty}^{+\infty}F_\perp dt \approx \frac{1}{m}\int_{-\infty}^{+\infty}\left(-\frac{dU}{dr}\right)\frac{b}{r}dx\frac{dt}{dx} \approx -\frac{b}{mv_0}\int_{-\infty}^{+\infty}\frac{dU}{dr}\frac{dx}{r}, \qquad (*)$$

where x is the direction along the initial straight trajectory of the particle. The approximate signs indicate the steps at that, just in the solution of the previous problem, we are neglecting its deviation from this trajectory during the scattering event (see the figure below) and the change of its speed. In the same approximation, we may take

[68] For the involved table integral, see, e.g., MA Eq. (6.2b) or (6.5b).

$$r^2 = x^2 + b^2, \qquad \text{so } rdr = xdx, \quad \text{i.e. } \frac{dx}{r} = \frac{dr}{x} = \frac{dr}{\left(r^2 - b^2\right)^{1/2}},$$



and since the function under the last integral in Eq. (*) is an even function of r, for the small scattering angle θ we get the following result,[69] valid in the first approximation in θ:

$$\theta \approx \frac{v_\perp}{v_0} = -\frac{b}{mv_0^2} 2\int_b^\infty \frac{dU}{dr} \frac{dr}{\left(r^2 - b^2\right)^{1/2}} \equiv -\frac{b}{E}\int_b^\infty \frac{dU}{dr} \frac{dr}{\left(r^2 - b^2\right)^{1/2}}.$$

In particular, for the potential $U = C/r^n$, we may use the integral provided in the *Hint* to get

$$\theta = \frac{b}{E}\int_b^\infty \frac{Cn}{r^{n+1}} \frac{dr}{\left(r^2 - b^2\right)^{1/2}} \equiv \frac{Cn}{Eb^n}\int_1^\infty \frac{d\xi}{\xi^{n+1}\left(\xi^2 - 1\right)^{1/2}} = \pi^{1/2} \frac{\Gamma(n/2 + 1/2)}{\Gamma(n/2)} \frac{C}{Eb^n}. \qquad (**)$$

As a sanity check, since $\Gamma(1) = 1$ and $\Gamma(½) = \pi^{1/2}$,[70] for the gravitational field, i.e. for $C = -GMm$ and $n = 1$, this formula is reduced to that of the previous problem, with $M_S = M$ and $R_S = b$:

$$\theta = -\frac{GMm}{Eb} \equiv -\frac{U(b)}{E}.$$

To calculate the differential cross-section of such small-angle scattering, we may use a simplified version of Eq. (3.72), valid for $|\theta| \ll 1$:

$$\frac{d\sigma}{d\Omega} = b\left|\frac{db}{\theta\, d\theta}\right|.$$

Combined with Eq. (**) solved for b, it yields the result

$$\frac{d\sigma}{d\Omega} = \frac{1}{n}\left|\frac{\pi^{1/2}\Gamma(n/2 + 1/2)}{\Gamma(n/2)} \frac{C}{E}\right|^{2/n} \theta^{-2-2/n}. \qquad (***)$$

By using Eq. (**) again, it may be represented in a very simple form:

$$\frac{d\sigma}{d\Omega} = \frac{b^2}{n\theta^2}.$$

However, in typical particle-scattering experiments, the impact parameter b cannot be explicitly measured, and the initial form (***) of the result is more relevant.

For the Coulomb field (3.49) with $n = 1$ and $C = -\alpha$, it yields the small-angle limit of the Rutherford formula (3.73):

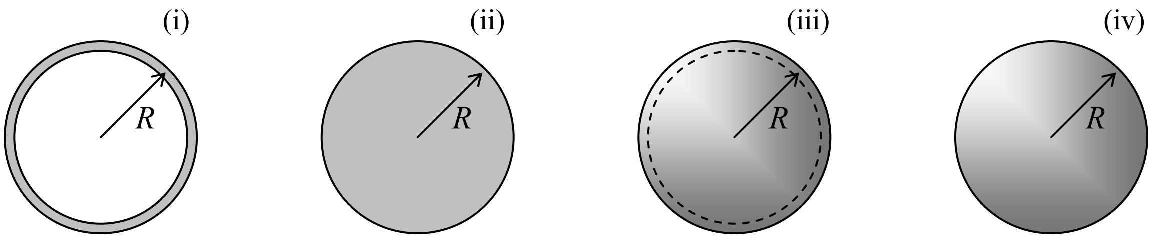
$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{E}\right)^2 \theta^{-4}.$$

On the other hand, for the fields with $n \gg 1$, the differential cross-section approaches the n-independent functional form $d\sigma/d\Omega \propto \theta^{-2}$.

[69] This result may be also obtained from Eq. (3.68) in the first approximation in $U(r)$ – the additional exercise highly recommended to the reader.
[70] See, e.g., MA Eqs. (6.7c) and (6.7e).

Chapter 4. Rigid Body Motion

Problem 4.1. Calculate the principal moments of inertia for the following uniform rigid bodies:



(i) a thin, planar, round hoop, (ii) a flat round disk, (iii) a thin spherical shell, and (iv) a solid sphere.

Compare the results, assuming that all the bodies have the same radius R and mass M, and give an interpretation of their difference.

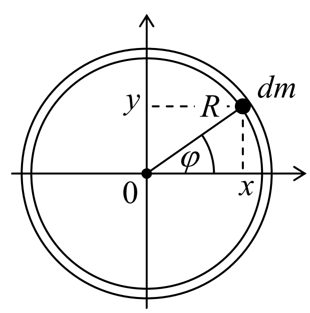
Solutions:

(i) Due to the axial symmetry of the hoop, one of its principal axes of inertia has to coincide with the (say, z-) axis normal to the body's plane. The calculation of the corresponding moment of inertia is very easy, because all points of the hoop are at the same distance $\rho = R$ from the axis, so

$$I_z \equiv \sum m\rho^2 = R^2 \sum m = MR^2.$$

(Here and below, as in Chapter 4 of the lecture notes, the implied-index summation is over all elementary masses of the body, so $\Sigma m = M$.)

For the rotation about any of two other principal axes, which lie in the loop's plane and pass through its center 0, we need an (easy) integration. For example (see the figure on the right),



$$I_x \equiv \sum my^2 = \int y^2 dm = \int_0^{2\pi} (R\sin\varphi)^2 \frac{dm}{d\varphi} d\varphi = R^2 \frac{dm}{d\varphi} \int_0^{2\pi} \sin^2\varphi \, d\varphi = R^2 \frac{dm}{d\varphi}\pi,$$

where, for a uniform hoop, the mass per unit angle ($dm/d\varphi$) is a constant that may be readily related to its total mass M :

$$M \equiv \sum m = \int dm = \int_0^{2\pi} \frac{dm}{d\varphi} d\varphi = \frac{dm}{d\varphi} 2\pi,$$

so $dm/d\varphi = M/2\pi$ and the above result for I_x (and hence for I_y as well) may be expressed as

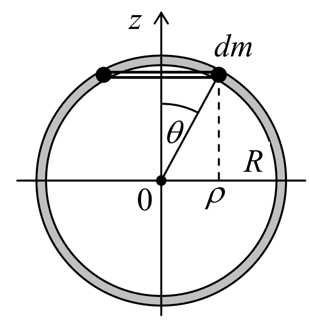
$$I_x = I_y = \frac{1}{2} MR^2.$$

A way to make this and most following calculations more compact is to calculate the ratio I/M from the very beginning, noticing that the functions under the corresponding integrals differ only by ρ^2, where ρ is the distance of the elementary mass dm from the rotation axis. For example, for the hoop:

$$\frac{I_x}{M} = \frac{\int \rho^2 dm}{\int dm} = \frac{\int y^2 dm}{\int dm} = R^2 \int_0^{2\pi} \sin^2 \varphi \, d\varphi \Big/ \int_0^{2\pi} d\varphi = \frac{1}{2} R^2 .$$

(ii) With the solutions of Task (i) on hand, the simplest way to make calculations for a flat disk is to reuse them by representing the disk as a set of many thin hoops of radius r, thickness $dr << r$, area $dA = 2\pi r dr$, mass $dm = (dm/dA)dA = (dm/dA)2\pi r dr$, and the principal moments of inertia $dI_z = r^2 dm = r^2(dm/dA)2\pi r dr$ and $dI_x = dI_y = (1/2)r^2 dm = (1/2)r^2(dm/dA)2\pi r dr$. For a uniform disk, with dm/dA = const, the corresponding integrations yield[71]

$$\frac{I_z}{M} = \frac{\int dI_z}{\int dm} = \frac{\int_0^R r^3 dr}{\int_0^R r dr} = \frac{1}{2} R^2, \qquad \frac{I_x}{M} = \frac{I_y}{M} = \frac{\int dI_x}{\int dm} = \frac{1}{2} \frac{\int_0^R r^3 dr}{\int_0^R r dr} = \frac{1}{4} R^2 .$$



(iii) Due to the spherical symmetry of the spherical shell, all its three principal moments of inertia are equal: $I_x = I_y = I_z = I$, and it is sufficient to calculate one of them, for example, I_z. For that, we may represent the shell as a set of elementary thin hoops (in the plane normal to the z-axis), each with the radius $\rho = R\sin\theta$ (see the figure on the right), the shell's surface area $dA = 2\pi\rho R d\theta = 2\pi R \sin\theta \, R d\theta$, the mass $dm = (dm/dA)dA = (dm/dA)2\pi R\sin\theta \, R d\theta$, and the moment of inertia $dI_z = \rho^2 dm = (R\sin\theta)^2 2\pi R \sin\theta \, R d\theta$. As a result, for a uniform shell, with dm/dA = const, we may write[72]

$$\frac{I}{M} = \frac{I_z}{M} = \frac{\int dI_z}{\int dm} = R^2 \frac{\int_0^\pi \sin^3 \theta \, d\theta}{\int_0^\pi \sin\theta \, d\theta} = R^2 \frac{4/3}{2} = \frac{2}{3} R^2 .$$

(iv) For a solid uniform sphere, $I_x = I_y = I_z = I$, and we can reuse the first of the results of Task (ii), representing the sphere as a set of elementary flat, uniform disks of the radius $\rho = R\sin\theta$ (see the figure above again), the mass $dm = (dm/dV)\pi\rho^2 dz = (dm/dV)\pi(R\sin\theta)^2 d(R\cos\theta)$, and the moment of inertia $dI_z = (1/2)\rho^2 dm = (1/2)(R\sin\theta\,)^2(dm/dV)\pi(R\sin\theta\,)^2 d(R\cos\theta)$. As a result, for a uniform sphere with density dm/dV = const, we get[73]

[71] Note that the results for I_x and I_y in Tasks (i) and (ii) might be also obtained by combining the axial symmetry condition $I_x = I_y$ with the equality $I_x + I_y = I_z$ satisfied by any planar distribution of masses. (Indeed, for all of them $z = 0$, so Eq. (4.24) of the lecture notes yields $I_x = \Sigma m y^2$, $I_y = \Sigma m x^2$, and $I_z = \Sigma m(x^2 + y^2) = I_x + I_y$.)

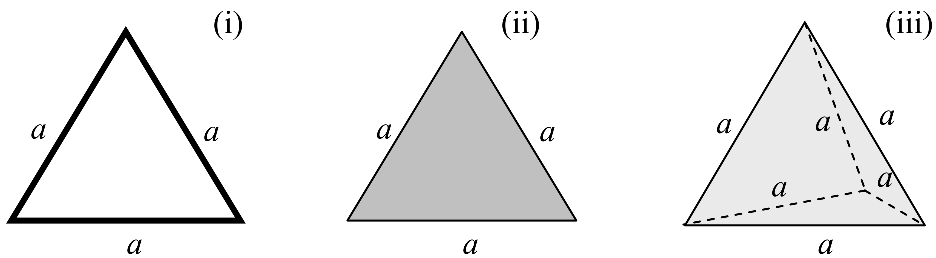
[72] These two (and the following two) integrals over θ may be readily worked out by the following variable substitution $\xi \equiv \cos\theta$, so $\sin\theta d\theta = -d\xi$, $\cos^2\theta \equiv 1 - \sin^2\theta = 1 - \xi^2$, and $\sin^3\theta d\theta = -(1 - \xi^2)d\xi$..

[73] Note that the same final answer may be obtained, perhaps even easier, by reusing the result of Task (iii): representing the solid sphere as a set of spherical shells with the radius r, the mass $dm = (dm/dV)dA = 4\pi(dm/dV)r^2 dr$, and the principal moments of inertia $dI_x = dI_y = dI_z = (2/3)r^2 dm = (2/3)r^2 4\pi(dm/dV)r^2 dr \equiv (8\pi/3)(dm/dV)r^4 dr$, and then integrating the two last results over r, from 0 to R.

$$\frac{I}{M}=\frac{I_z}{M}=\frac{\int dI_z}{\int dm}=\frac{\int(1/2)(R\sin\theta)^4 d(R\cos\theta)}{\int(R\sin\theta)^2 d(R\cos\theta)}=\frac{\int_0^{\pi}\sin^5\theta\, d\theta}{\int_0^{\pi}\sin^3\theta\, d\theta}=\frac{R^2}{2}\frac{16/15}{4/3}=\frac{2}{5}R^2.$$

These results show that the increasing dimensionality of a uniform body, from the round hoop to the round disk and from the spherical shell to the solid sphere, makes the moment of inertia (for the same M and R) smaller because the elementary masses dm are, on average, closer to the rotation axis.

Problem 4.2. Calculate the principal moments of inertia for the rigid bodies shown in the figure below:



(i) an equilateral triangle made of thin rods with a constant linear mass density μ,
(ii) a thin plate in the shape of an equilateral triangle, with a constant areal mass density σ, and
(iii) a tetrahedron with a constant bulk mass density ρ.

Assuming that the total mass of all three bodies is the same, compare the results and give an interpretation of their difference.

Solutions:

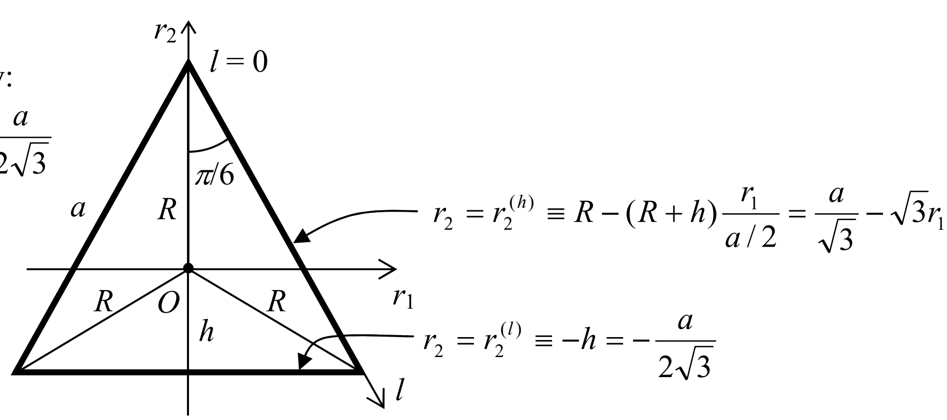
(i) Due to the symmetry discussed in Sec. 4.2 of the lecture notes, the moments of inertia for the rotation about any in-plane axis passing through the center mass are equal, and we can calculate just one of them, e.g., for the rotation about the r_2-axis – see the figure below:

$$I_1=I_2=\mu\int_{-a/2}^{+a/2} r_1^2 dr_1+2\mu\int_0^a r_1^2 dl,$$

where l is the coordinate along the triangle's side.

From trigonometry:

$$R=\frac{a}{\sqrt{3}},\quad h=\frac{a}{2\sqrt{3}}$$



The first integral is elementary, while in the second one, $r_1 = l\sin(\pi/6) = l/2$. As a result, we get

$$I_1 = I_2 = 2\mu \int_0^{a/2} r_1^2 dr_1 + 2\mu \int_0^{a} (l/2)^2 dl = \frac{\mu a^3}{4} = \frac{Ma^2}{12},$$

where $M = 3\mu a$ is the total mass of the triangle.

The third principal moment of inertia (for the rotation about the axis normal to the triangle's plane) is different from $I_{1,2}$, and may be calculated, for example, as the tripled contribution from the rod shown horizontal in the figure above:

$$I_3 = 3\mu \int_{-a/2}^{+a/2} \left(r_1^2 + h^2\right) dr_1 = \frac{\mu a^3}{2} = \frac{Ma^2}{6}.$$

(ii) Due to the similar symmetry, for the uniform plate we may write

$$I_1 = I_2 = \sigma \int r_1^2 d^2 r = 2\sigma \int_0^{a/2} dr_1 \int_{r_2^{(l)}}^{r_2^{(h)}} dr_2\, r_1^2 .$$

Using the lower and higher limits of integration over r_2 for $r_1 \geq 0$, marked in the figure above, we get

$$I_1 = I_2 = 2\sigma \int_0^{a/2} dr_1 \int_{-a/2\sqrt{3}}^{a/\sqrt{3}-\sqrt{3}r_1} dr_2\, r_1^2 = \frac{\sqrt{3}}{96}\sigma a^4 = \frac{Ma^2}{24},$$

where M is the full mass of the triangle,

$$M = \sigma A = \sigma \frac{1}{2}(R + h)a = \frac{\sqrt{3}}{4}\sigma a^2.$$

Similarly,

$$I_3 = \sigma \int \left(r_1^2 + r_2^2\right) d^2 r = 2\sigma \int_0^{a/2} dr_1 \int_{r_2^{(l)}}^{r_2^{(h)}} dr_2 r_1^2 + 2\sigma \int_0^{a/2} dr_1 \int_{r_2^{(l)}}^{r_2^{(h)}} dr_2 r_2^2 .$$

The first integral is just the I_1 calculated above, while the second integral is $I_2 = I_1$. Hence,

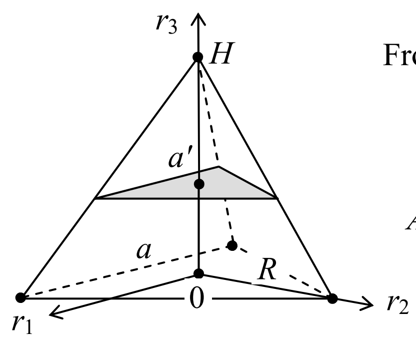
$$I_3 = 2I_1 = \frac{\sqrt{3}}{48}\sigma a^4 = \frac{Ma^2}{12}.$$

Note that for each of the three bodies based on the equilateral triangle (the one solved in Sec. 4.2 of the lecture notes, and Tasks (i) and (ii) of this problem), the ratio $I_3/I_1 = I_3/I_2$ is the same. In hindsight, this is natural, because each of the bodies considered in parts (i) and (ii) may be represented as a system of many troikas of particles equally separated from the center of mass – like in Fig. 4.3 of the lecture notes. Note also that the results of Tasks (i) and (ii) satisfy the more general relation $I_1 + I_2 = I_3$ that is common for all thin planar bodies whose plane is normal to the r_3-axis – see a footnote in the previous problem.

(iii) A little bit counter-intuitively, this task is the easiest one because due to its symmetry, all principal moments of inertia are equal (the tetrahedron is the "spherical top") and we may calculate just one of them, e.g., that for rotation about the vertical axis:

$$I_1 = I_2 = I_3 \equiv I = \rho\int \left(r_1^2 + r_2^2\right) d^3r \equiv \rho\int_0^H dr_3 \frac{I_3(r_3)}{\sigma},$$

where $I_3(r_3)$ is the moment of inertia of the equilateral triangle that is the horizontal cross-section of the tetrahedron at height $r_3 \leq H$ over the base plane (see the figure below), with the side $a' = a - r_3/H = a - (3/2)^{1/2}r_3$.



From trigonometry:

$R = a/\sqrt{3}$,

$H = a(2/3)^{1/2}$,

$A_{\text{base}} = (\sqrt{3}/4)\, a^2$

Using the result of Task (ii), we get

$$I = \rho\int_0^H dr_3 \frac{\sqrt{3}}{48}\left(a - \sqrt{\frac{3}{2}}r_3\right)^4 = \frac{\sqrt{2}}{240}\rho a^5 = \frac{Ma^2}{20},$$

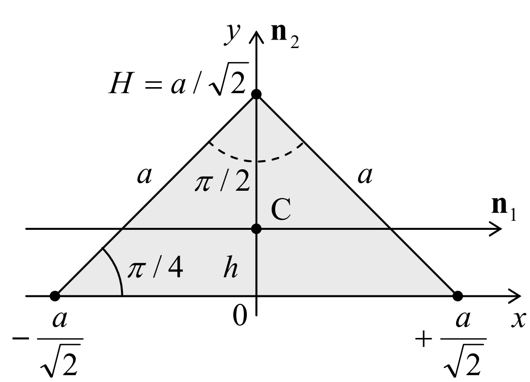
where M is the total mass of the tetrahedron as a triangular pyramid:

$$M = \frac{\rho}{3} A_{\text{base}} H = \frac{\sqrt{2}}{12}\rho a^3 .$$

Comparing the results of Tasks (i)-(iii), we see that with the growth of the body's dimensionality, its rotational inertia drops (at a fixed total mass M), because its points, on average, are closer to the rotation axes.

<u>Problem 4.3</u>. Calculate the principal moments of inertia of a thin uniform plate cut in the form of a right triangle with two $\pi/4$ angles.

Solution: For any flat thin body, one of its principal axes (say, $\mathbf{n}_3$) is normal to its plane, passing through its center of mass C. Due to the evident symmetry of the plate, two other axes are directed as shown in the figure on the right, also passing through point C, so let us start by finding the c.o.m.'s position – namely, its distance h from the long side of the triangle. For the given geometry, the y-component of Eq. (4.13) of the lecture notes reads



$$h = \frac{1}{M}\frac{dM}{dA}\int_0^H \left(x_{\max} - x_{\min}\right) y dy = \frac{1}{a^2/2} 2 \int_0^{a/\sqrt{2}} \left(\frac{a}{\sqrt{2}} - y\right) y dy = \frac{a}{3\sqrt{2}}.$$

Now we may readily calculate I_3 by using the axis shift theorem (4.29) relating it to I_3', the moment of inertia for rotation about the parallel axis passing through the origin 0, which evidently has the same I_3/M ratio as the square of side a:

$$I_3 = I_3' - Mh^2 = M\frac{a^2}{6} - M\left(\frac{a}{3\sqrt{2}}\right)^2 = \frac{Ma^2}{9}.$$

The moment I_2 is even easier to calculate, because for a uniform square plate with side a, with its $I_1 = I_2 = Ma^2/12$, any axis passing through the c.o.m., including the $\mathbf{n}_2$-axis in the figure above, may be considered principal. Discarding the half of the square below the x-axis evidently does not change the I/M ratio, so

$$I_2 = \frac{Ma^2}{12}.$$

Finally, the moment I_1 may be readily calculated by a direct (lengthy but elementary) integration:

$$I_1 = \frac{dM}{dA}\int_{-a/\sqrt{2}}^{+a\sqrt{2}} dx \int_0^{H-x} dy\,(y-h)^2 = \frac{M}{a^2/2}\,2\int_0^{a/\sqrt{2}} dx \int_0^{a/\sqrt{2}-x} dy\left(y - \frac{a}{3\sqrt{2}}\right)^2 = \frac{Ma^2}{36}.$$

As a sanity check: since

$$\frac{Ma^2}{36} + \frac{Ma^2}{12} \equiv \frac{Ma^2}{9},$$

our results satisfy the relation $I_x + I_y = I_z$ valid for any thin body confined to the [x, y] plane – see the solutions of the two previous problems.

Problem 4.4. Prove that Eqs. (4.34)-(4.36) of the lecture notes are valid for the rotation of a rigid body about the fixed z-axis, even if it does not pass through its center of mass.

Solution: For rotation about a fixed z-axis, we may take

$$\boldsymbol{\omega} = \mathbf{n}_z \omega_z,$$

Selecting the coordinate origin somewhere on the rotation axis, let us represent the radius vector of an arbitrary point of the rotating body as

$$\mathbf{r} \equiv \mathbf{n}_x x + \mathbf{n}_y y + \mathbf{n}_z z = \mathbf{n}_\rho \rho + z\mathbf{n}_z,$$

where ρ is the length of vector $\boldsymbol{\rho} \equiv \mathbf{n}_x x + \mathbf{n}_y y$, i.e. the distance of the point $\mathbf{r}$ from the rotation axis, and $\mathbf{n}_\rho$ is the unit vector in the direction of the vector $\boldsymbol{\rho}$, by construction normal to $\mathbf{n}_z$, so $\mathbf{n}_\rho \cdot \mathbf{n}_z = 0$. Then Eq. (4.9) of the lecture notes becomes

$$\mathbf{v} = \mathbf{n}_z \omega_z \times (\mathbf{n}_\rho \rho + z\mathbf{n}_z) = \omega_z \rho\, \mathbf{n}_z \times \mathbf{n}_\rho.$$

Plugging these expressions into the general formula for the total angular moment of the body, we get

$$\mathbf{L} \equiv \sum \mathbf{r} \times m\mathbf{v} = \sum m\omega_z \rho (\mathbf{n}_\rho \rho + z\mathbf{n}_z) \times (\mathbf{n}_z \times \mathbf{n}_\rho) \equiv \omega_z \sum m\rho[\rho\, \mathbf{n}_\rho \times (\mathbf{n}_z \times \mathbf{n}_\rho) + z\,\mathbf{n}_z \times (\mathbf{n}_z \times \mathbf{n}_\rho)].$$

Now applying the "bac minus cab" rule[74] to each of the double vector products in the last expression, for the z-component of the momentum we get

$$L_z \equiv \omega_z \sum m\rho\left(\rho\mathbf{n}_z - z\mathbf{n}_\rho\right)_z = \omega_z \sum m\rho^2 \equiv \omega_z \sum m\left(x^2 + y^2\right),$$

i.e. exactly the combination of Eqs. (4.34) and (4.35).[75]

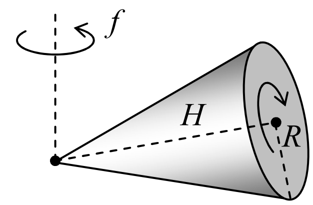
Similarly, the general expression for the kinetic energy of the body's rotation becomes

$$T \equiv \sum \frac{m}{2} v^2 = \sum \frac{m}{2}\left(\omega_z \rho \mathbf{n}_z \times \mathbf{n}_\rho\right)^2 \equiv \omega_z^2 \sum \frac{m}{2}\rho^2\left(\mathbf{n}_z \times \mathbf{n}_\rho\right)^2 .$$

Since, by definition, the unit vectors $\mathbf{n}_z$ and $\mathbf{n}_\rho$ are normal to each other, their vector product is also a unit vector, so the last parentheses squared are equal to 1, and we get the following result:

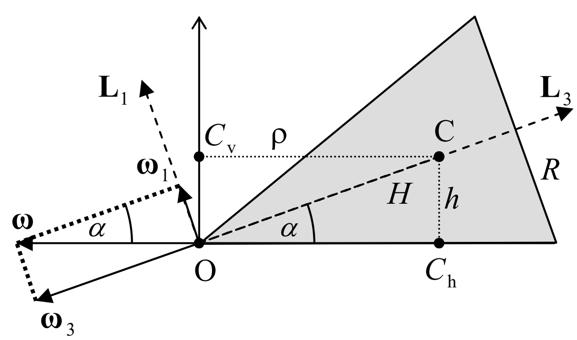
$$T \equiv \omega_z^2 \sum \frac{m}{2}\rho^2 = \omega_z^2 \sum \frac{m}{2}\left(x^2 + y^2\right),$$

which is clearly equivalent to the combination of Eqs. (4.35) and (4.36) of the lecture notes.



Problem 4.5. Calculate the kinetic energy of a right circular cone with height H, base radius R, and a constant mass density ρ, that rolls over a horizontal surface without slippage, making f turns per second about the vertical axis – see the figure on the right.

Solution: The key to the solution of this problem is the realization that the instantaneous axis of the cone's rotation is the line of its contact with the supporting surface. (Indeed, all points of this straight line have zero instantaneous velocity, and according to Eq. (4.9) of the lecture notes, this is only possible on the line with $\boldsymbol{\omega}\times\mathbf{r} = 0$, i.e. on the line of the vector $\boldsymbol{\omega}$ itself.) This means that on the cone's vertical cross-section containing this line (shown in the figure on the right), the vector $\boldsymbol{\omega}$ is horizontal. (Its direction to the left corresponds to the rotation direction shown in the figure of the assignment.) This means that $\boldsymbol{\omega}$ may be represented as the sum of two mutually perpendicular components directed along the principal moments of inertia of the cone, with the lengths



$$\omega_1 = \omega \sin\alpha \quad \text{and} \quad \omega_3 = \omega\cos\alpha ,$$

where α is half of the cone's aperture: $\tan\alpha = R/H$. Hence the kinetic energy of the cone may be calculated from Eq. (4.14) of the lecture notes as

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[74] See, e.g., MA Eq. (7.5).

[75] Note that for an arbitrary body and an arbitrary selection of the origin on the rotation axis, the sum $\Sigma m\rho z$ does not necessarily vanish, so the angular momentum vector generally has other components than L_z – see also Problem 13 and its solution.

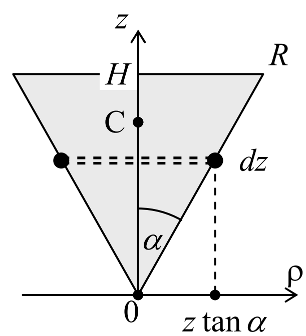
$$T = \frac{M}{2}V^2 + \frac{I_1}{2}(\omega \sin\alpha)^2 + \frac{I_3}{2}(\omega\cos\alpha)^2, \qquad (*)$$

where V is the linear velocity of the cone's center of mass C, and I_1 and I_3 are its principal moments of inertia about that point. Both V and ω may be expressed via the constants provided in the assignment by writing the condition that the c.o.m.'s horizontal and vertical coordinates C_h and C_v (see the figure above) have zero instantaneous velocities:[76]

$$V = \Omega \text{ρ} \equiv \Omega C \cos\alpha, \qquad V = \omega h \equiv \omega C \sin\alpha ,$$

where $\Omega \equiv 2\pi f$, while $C \equiv$ OC is the distance of the cone's center of mass from its vertex. Note that the angular velocity ω may be obtained from these two relations even without calculating C:

$$\omega = \Omega \cot\alpha \equiv \Omega \frac{H}{R}.$$



What remains is to express the constants M, C, I_1, and I_3 via the given dimensions R and H of the cone and its mass density ρ. For that, let us redraw the axial cross-section of the cone as shown in the figure on the right. The mass of a thin horizontal layer of the cone, with thickness dz and radius ρ = $z \tan\alpha$, is evidently

$$dm = \pi\rho \text{ρ}^2 dz \equiv \pi\rho (z\tan\alpha)^2 dz ,$$

while its moments of inertia for the rotation about its center (not about the c.o.m. of the full cone!) have been calculated in the solution of Problem 1 (ii). In our current notation,

$$dI_1 = \frac{1}{4}(z\tan\alpha)^2 dm \equiv \frac{\pi}{4}\rho (z\tan\alpha)^4 dz, \qquad dI_3 = \frac{1}{2}(z\tan\alpha)^2 dm \equiv \frac{\pi}{2}\rho(z\tan\alpha)^4 dz .$$

Integrating the relation for dm over the cone's height, we get the well-known result for its mass:

$$M = \int_{z=0}^{z=H} dm = \pi\rho\tan^2\alpha \int_0^H z^2 dz = \frac{\pi}{3}\rho \tan^2\alpha H^3 \equiv \frac{\pi}{3}\rho R^2 H .$$

A similar integration with the additional weight z yields the c.o.m.'s position – see Eq. (3.32) of the lecture notes:

$$C = \frac{1}{M}\int_{z=0}^{z=H} z\,dm = \frac{\pi\rho\tan^2\alpha}{M}\int_0^H z^3 dz = \frac{\pi\rho\tan^2\alpha}{M}\frac{H^4}{4} \equiv \frac{3}{4}H.$$

To calculate the total I_3 (for the rotation about the cone's symmetry axis), we may just integrate dI_3 over the z-axis:

$$I_3 = \int_{z=0}^{z=H} dI_3 = \frac{\pi}{2}\rho\tan^4\alpha \int_0^H z^4 dz = \frac{\pi}{10}\rho\tan^4\alpha\, H^5 \equiv \frac{\pi}{10}\rho R^4 H \equiv \frac{3}{10}MR^2 .$$

[76] Let me hope that the font difference between the letters ρ (denoting the distance from the vertical axis) and ρ (the mass density) is sufficient to avoid confusion.

However, when integrating dI_1, we have to remember that in Eq. (*), we need the moment of inertia for the rotation about the c.o.m. point C. According to Eq. (4.29), this means that each dI_1 quoted above needs to be complemented with the axis-transfer term $(z - C)^2 dm \equiv \pi\rho\,(z\tan\alpha)^2(z - 3H/4)^2 dz$:

$$I_1 = \pi\rho\int_0^H \left[\frac{1}{4}(z\tan\alpha)^4 + (z\tan\alpha)^2\left(z - \frac{3}{4}H\right)^2\right]dz = \pi\rho H^5\left(\frac{1}{20}\tan^4\alpha + \frac{1}{80}\tan^2\alpha\right)$$
$$\equiv \frac{\pi}{20}\rho R^2 H\left(R^2 + \frac{1}{4}H^2\right) \equiv \frac{3}{20}M\left(R^2 + \frac{1}{4}H^2\right).$$

Plugging all these results into Eq. (*), we finally get

$$T = \frac{3}{40}MH^2\left(1 + 5\cos^2\alpha\right)\Omega^2 \equiv \frac{\pi}{40}\rho\,\frac{6H^2 + R^2}{H^2 + R^2}R^2H^3\Omega^2.$$

4.6. External forces exerted on some rigid body rotating with an angular velocity $\boldsymbol{\omega}$, have zero vector sum but a non-vanishing net torque $\boldsymbol{\tau}$ about its center of mass.

(i) Calculate the work of the forces on the body per unit time, i.e. their instantaneous power.

(ii) Prove that the same result is valid for a body rotating about a fixed axis and the torque's component along this axis.

(iii) Use the last result to prove that at negligible friction, the gear assembly shown in the figure on the right distributes the external torque, applied to its satellite-carrier axis to rotate it about the common axis of two axle shafts, equally to both shafts, even if they rotate with different angular velocities.

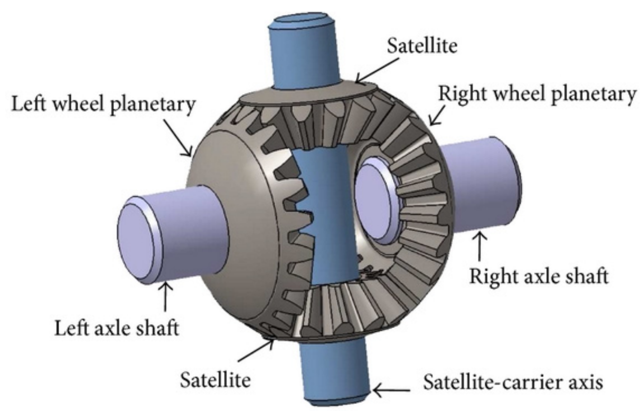


Figure from G. Antoni, *Sci. World J.*, **2014**, 523281 (2014), adapted with permission. Both satellite gears may rotate freely about their common carrier axis.

Solutions:

(i) Let us consider one component **F** of the external forces, applied to the body at a point with a radius vector **r**, as measured from the center of mass O. Since the net force equals zero, any reference frame centered at point O (but not rotating with the body) is inertial, so in that frame, we may use Eq. (1.21) of the lecture notes for the elementary work of the force on the body: $d\mathscr{W} = \mathbf{F}\times d\mathbf{r}$; hence, its instantaneous power is

$$\mathscr{P} \equiv \frac{d\mathscr{W}}{dt} = \frac{\mathbf{F}\cdot d\mathbf{r}}{dt} \equiv \mathbf{F}\cdot\mathbf{v}.$$

Since, according to Eq. (4.9), in this reference frame, $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$, we get[77]

$$\mathscr{P} = \mathbf{F}\cdot(\boldsymbol{\omega}\times\mathbf{r}) \equiv \boldsymbol{\omega}\cdot(\mathbf{r}\times\mathbf{F}).$$

[77] Here the second step uses the operand rotation rule MA Eq. (7.6).

But according to Eq. (1.34), the vector product in the last parentheses is just the torque $\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F}$ of the force $\mathbf{F}$, we get a very simple final result:

$$\mathscr{P} \equiv \boldsymbol{\omega}\cdot\boldsymbol{\tau}. \qquad (*)$$

Since the angular velocity $\boldsymbol{\omega}$ of a rigid body is the same for all its points, the summation of this result over all force components shows that the net power of the forces and their net torque are also related by Eq. (*).

(ii) For a rigid body rotating about a fixed (say, z-) axis, we may get the same result even simpler in another way. If the body is not under the effect of any other forces (besides those contributing to the net torque $\boldsymbol{\tau}$), the power $\mathscr{P}$ we need may be calculated from the work-energy principle (1.21), as dT_{rot}/dt, where T_{rot} is the rotational energy (4.36):

$$\mathscr{P} = \frac{dT_{\text{rot}}}{dt} = \frac{d}{dt}\frac{I_{zz}\omega_z^2}{2} = I_{zz}\omega_z\dot{\omega}_z.$$

But according to Eq. (4.38), in this case, the product $I_{zz}\dot{\omega}_z$ is equal to the applied torque's component τ_z. So we arrive at a result similar to Eq. (*), but this time, for just the Cartesian components of these vectors:[78]

$$\mathscr{P} = \omega_z\tau_z. \qquad (**)$$

(iii) With this general result on hand, the analysis of the differential gear assembly[79] is simpler than one might expect from the first glance at its picture. Indeed, let us fix the satellite-carrier axis' direction first. Then, regardless of the relation of the diameters of the planetary and satellite gears (but only if they are the same for each pair), the device obviously guarantees that the axels may only rotate with the same angular speed, but in opposite directions:

$$\omega_{\text{L}} = -\omega_{\text{R}} \equiv -\omega,$$

where all angular velocities are along the common symmetry axis of both axle shafts. Now if, in addition to the mutual rotation of the axels, the satellite-carrier axis is rotated about the same axis, rotating the whole device with it, it just adds this angular velocity (say, Ω) to those discussed above, so

$$\omega_{\text{L}} = -\omega + \Omega, \qquad \omega_{\text{R}} = \omega + \Omega.$$

(In most practical applications, $|\,\omega\,| << |\,\Omega\,|$, so $\omega_{\text{L}}, \omega_{\text{R}} \approx \Omega$.) Eliminating ω from this simple system of two equations, we get the basic kinematic relation describing the differential assembly:

$$\omega_{\text{L}} + \omega_{\text{R}} = 2\Omega. \qquad (***)$$

Now let Ω be due to the torque τ of external forces – in typical applications, provided by the engine of the machine – say, of a car whose driving wheels are attached to the left and right axels. Then,

[78] Note also that in this case, the requirement for the net exerted force to vanish may be dropped because this force, as well as two other Cartesian components of its torque, are canceled by the reaction of the support system keeping the body's rotation axis fixed.

[79] This keystone device (also called either *differential gearbox*, or *automotive differential gear*, or just *differential*), without that not only cars/trucks but also many other mechanical systems would be impracticable, was apparently known already in Ancient Greece, then re-invented in China in the 3rd century AD, and then again in Europe – by J. Williamson in 1720 and by O. Pecqueur in 1827.

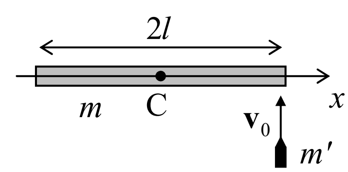
according to Eq. (**), the power input from the engine is $\mathscr{P} = \Omega\tau$, while the power outputs from the gearbox are, respectively, $\mathscr{P}_L = \omega_L\tau_L$ and $\mathscr{P}_R = \omega_R\tau_R$. But if the power losses (i.e. the energy dissipation) inside the box are negligible, the energy conservation requires $\mathscr{P}_L + \mathscr{P}_R = \mathscr{P}$, i.e.

$$\omega_L\tau_L + \omega_R\tau_R = \Omega\tau .$$

But for arbitrary ω_L and ω_R, this relation is compatible with Eq. (***) only if $\tau_L = \tau_R = \tau/2$.

The last equality ensures the usual function of the differential gearbox: to distribute the engine-produced torque equally between two axles even if they rotate at different speeds – say, when a car makes a turn, or if one of its wheels is stuck in mud.

Problem 4.7. The end of a uniform thin rod of length $2l$ and mass m, initially at rest, is hit by a bullet of mass m', flying with a velocity $\mathbf{v}_0$ perpendicular to the rod (see the figure on the right), which gets stuck in it. Use two different approaches to calculate the velocity of the opposite end of the rod right after the collision.



Solution:

Approach 1. In the standard approach to collision problems, we make all calculations from the point of view of the center of mass of the whole system (rod plus bullet). Immediately before and immediately after the collision, the center is located on the axis of the rod, closer to its bullet-hit end, and its distance x_{com} from the rod's center C may be found from Eq. (3.32) of the lecture notes, in this case taking the following form:

$$x_{com} = \frac{m'l}{M},$$

where $M \equiv m + m'$ is the total mass of the system. One more length we will need is the distance of the bullet from the center of mass:

$$l - x_{com} = l\left(1 - \frac{m'}{M}\right) = l\frac{m}{M}.$$

Now we may write the laws of conservation of the total linear and angular momenta of the system at the collision[80] as

$$m'v_0 = Mv_{com}, \qquad m'v_0(l - x_{com}) = I_{com}\omega,$$

where I_{com} is the system's moment of inertia for rotation about the axis passing through the center of mass of the system, and ω is the angular velocity of the rotation resulting from the hit. To calculate I_{com}, we have to account for the rotational inertia of both the bullet and the rod. The latter may be calculated either by direct integration or using the axis shift theorem (4.29), giving

$$I_{com} = m'(l - x_{com})^2 + I_C + mx_{com}^2, \qquad (*)$$

where I_C is the moment of inertia of the rod alone about its geometric center C. For a uniform rod, an elementary integration yields

[80] The mechanical energy is not conserved at this "inelastic" collision.

$$I_C = 2\int_0^l \frac{dm}{dl} x^2 dx = 2\int_0^l \frac{m}{2l} x^2 dx = \frac{ml^2}{3}.$$

Plugging this formula and the above expressions for x_{com} and $(l - x_{com})$ into Eq. (*), we get

$$I_{com} = m'\left(\frac{ml}{M}\right)^2 + \frac{ml^2}{3} + m\left(\frac{m'l}{M}\right)^2 \equiv ml^2\left(\frac{1}{3} + \frac{m'}{M}\right),$$

so the angular momentum conservation law yields

$$\omega = \frac{m'v_0(l - x_{com})}{I_{com}} = \frac{v_0}{l}\frac{1}{1 + M/3m'} \equiv \frac{v_0}{l}\frac{1}{1 + (1 + m/m')/3}. \qquad (**)$$

Now it is straightforward to calculate the velocity of the opposite end of the rod as

$$v = v_{com} - \omega(l + x_{com}).$$

Plugging in the values of x_{com} and ω found above, we get

$$v = -\frac{2}{3}v_0 \frac{1}{1 + (1 + m/m')/3}.$$

This result shows that the opposite end of the rod starts moving, after the hit, in the direction opposite to that of the bullet, at any mass ratio m/m', with the magnitude raging from $v_0/2$ if $m << m'$ (a relatively heavy bullet) to $(2m'/m)v_0 \to 0$ in the opposite limit.

This is all very straightforward, but the reader should agree that the calculation has been unpleasantly bulky for such a simple problem.

Approach 2. Surprisingly enough, the calculation becomes much easier if we continue to consider the bullet and rod as two different bodies even after the collision, related only by the following kinematic condition for the bullet's velocity after the collision: $v' = v_C + l\omega$, where v_C is the velocity of the rod's (not of the full system's!) center C. In this approach, the laws of momenta conservation at collision take the form

$$m'v_0 = mv_C + m'v', \qquad m'v_0 l = I_C\omega + m'v'l.$$

Solving this system of two equations, and using the already calculated value of I_C, we get the same Eq. (**) for ω, plus the following result for v_C:

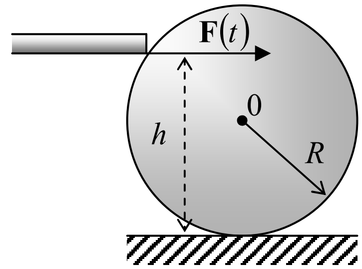
$$v_C = \frac{v_0}{3}\frac{1}{1 + (1 + m/m')/3}. \qquad (***)$$

(This velocity ranges from $v_0/4$ for a relatively heavy bullet, $m << m'$, to $(m'/m)v_0 \to 0$ in the opposite limit.) Now the velocity of the opposite end of the stick may be calculated as

$$v = v_C - \omega l,$$

immediately giving, with Eqs. (**) and (***), the same result as in Approach 1. This is a good illustration of how useful it may be to be flexible at solving a physical (or any other :-) problem, though if there is any doubt in the validity of a special approach, it is prudent to follow the well-beaten path.

Problem 4.8. A ball of radius R, initially at rest on a horizontal surface, is hit with a billiard cue in the horizontal direction, at height h above the surface – see the figure on the right. Using the Coulomb approximation for the kinetic friction force between the ball and the surface ($|F_f| = \mu N$), calculate the final linear velocity of the rolling ball as a function of h. Would it matter if the hit point is shifted horizontally (normally to the plane of the drawing)?



Hint: As in most solid body collision problems, during the short time of the cue hit, all other forces exerted on the ball may be considered negligibly small.

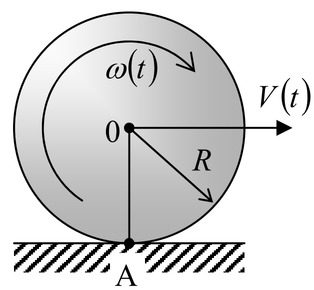
Solution: Per the *Hint*, integration of the 2nd Newton's law for the ball's center of mass (see Eq. (4.44) of the lecture notes) over the time interval Δt of the hit shows that it acquires the following horizontal linear momentum:

$$P = \int_{\Delta t} F(t)dt,$$

where $F(t)$ is the force exerted on the ball during the hit. According to Eq. (4.33), the resulting angular momentum of rotation about the center of mass 0 is directed normally to the plane of the drawing and its magnitude is

$$L = \int_{\Delta t} \tau(t)dt = \int_{\Delta t} F(t)(h-R)dt = (h-R)P.$$

As a result, immediately after the hit, the ball has the following c.o.m.'s and angular velocities (the directions accepted positive are indicated by arrows in the figure on the right):



$$V(0) = \frac{P}{M}, \qquad \omega(0) = \frac{L}{I} = (h-R)\frac{P}{I},$$

where M is the ball's mass, and I is its moment of inertia. (Note that the rotation's actual direction depends on whether the hit point is above or below its center of mass.) Hence the initial velocity of the ball's point A contacting the horizontal surface is

$$v_A(0) = V(0) - \omega(0)R = \frac{P}{M}\left[1-(h-R)\frac{MR}{I}\right].$$

If $v_A(0) > 0$, which happens if[81]

$$\left[1-(h-R)\frac{MR}{I}\right] > 0, \qquad \text{i.e. if } \; h < h_c \equiv R\left(1+\frac{I}{MR^2}\right),$$

then the lower side of the ball slips along the c.o.m. velocity, and the kinetic friction force $F_f = \mu Mg$ exerted on the contact point by the surface is directed against it. This force decelerates the ball's linear motion, while its torque $\tau_f = F_f R$ *accelerates* the ball's rotation:

$$V(t) = V(0) - \mu g t = \frac{P}{M} - \mu g t, \qquad \omega(t) = \omega(0) + \frac{\mu M g R}{I}t = (h-R)\frac{P}{I} + \frac{\mu M g R}{I}t.$$

This process continues until the velocity $v_A(t) = V(t) - \omega(t)R$ vanishes, i.e. until the moment t_0 when

[81] For a uniform ball, $I = (2/5)MR^2$ (see, e.g., the solution of Problem 4.1), so the critical height $h_c = (7/5)R$.

$$\left(\frac{P}{M} - \mu M g t_0\right) - \left[(h-R)\frac{P}{I} + \frac{\mu M g}{I} t_0\right] = 0 .$$

This equation readily yields

$$t_0 = \frac{P}{\mu M g}\left(1 - \frac{h}{h_c}\right), \qquad \text{for } h < h_c .$$

At $t = t_0$, the ball's slippage stops, and it continues to roll with the c.o.m.'s velocity

$$V_0 = V(t_0) = \frac{P}{M}\frac{h}{h_c}, \qquad (*)$$

which is lower than $V(0)$. In the game of billiards, such a cue hit (with $h < h_c$) is called the *low shot*.

In the opposite case of a *high shot* ($h > h_c$), the lower side of the ball slips to the side opposite to the c.o.m.'s velocity, and the kinetic friction force is exerted in the direction of $V(t)$. Hence the force *accelerates* the linear motion of the ball while decelerating its rotation. Since in the Coulomb approximation, the magnitude μMg of this force does not depend on the slippage velocity, we may use the above formulas for $V(t)$, $\omega(t)$, and $v_A(t)$ with the opposite sign before the coefficient μ. For the rotation stoppage time t_0, this sign reversal yields

$$t_0 = \frac{P}{\mu M g}\left(\frac{h}{h_c} - 1\right), \qquad \text{for } h > h_c ,$$

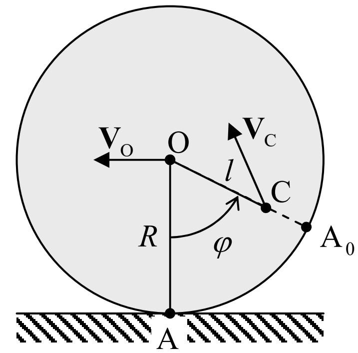
and for the eventual pure roll velocity, the same Eq. (*), now giving $V_0 > V(0)$. In the borderline case $h = h_c$, we get $t_0 = 0$, i.e. the ball starts the pure roll immediately after the hit.

In the game of billiards, very low cue hits (with $h < R$) are also called *draw shots*, because they may be used for the following trick. If a ball hits another similar ball, initially at rest, centrally and elastically, it transfers to it (completely) its linear momentum, but does not affect the angular one. As our analysis has shown, just after a draw shot, the initial ("cue") ball rotates in the direction opposite to its eventual no-slippage roll – in the figures above, counterclockwise. Hence if it hits another ball before this initial rotation has changed its direction, the cue ball starts rolling backward after it.

Finally, a horizontal shift of the hit point does not change the initial value of the linear momentum **P** but gives the initial vector **L** a vertical component, i.e. causes an additional rotation of the ball about its vertical axis. However, this rotation does not affect the velocity of the contact point A, and hence any of the above results.

Problem 4.9. A round cylinder of radius R and mass M may roll, without slippage, over a horizontal surface. The mass density distribution inside the cylinder is not uniform, so its center of mass is at some distance $l \neq 0$ from its geometrical axis, and the moment of inertia I (for rotation about the axis parallel to the symmetry axis but passing through the center of mass) is different from $MR^2/2$, where M is the cylinder's mass. Derive the equation of motion of the cylinder under the effect of the uniform vertical gravity field, and use it to calculate the frequency of small oscillations of the cylinder near its stable equilibrium position.

Solution: The equilibrium position of the cylinder obviously corresponds to its center of mass C right above the point A of its contact with the surface, i.e. right below the geometric center O. If the cylinder has already roll-shifted by angle φ from that position without slippage, the horizontal distance between the current location of the contact point A and its equilibrium position equals the length of the arc AA_0, i.e. $R\varphi$ – see the figure on the right. Hence the only (horizontal) component of the velocity $\mathbf{V}_O$ of the geometric center O of the cylinder's cross-section, as observed from the lab reference frame, is $(-R\dot{\varphi})$. As the same figure shows, the horizontal distance of the center of mass C from the vertical line OA is $l\sin\varphi$, so the horizontal component of its velocity $\mathbf{V}_C$ is



$$V_h = V_O + \frac{d}{dt}(l\sin\varphi) = -R\dot{\varphi} + l\cos\varphi\,\dot{\varphi} \equiv -(R - l\cos\varphi)\dot{\varphi},$$

while the vertical component of the velocity is[82]

$$V_v = \dot{h}_C = \frac{d}{dt}(-l\cos\varphi + \text{const}) \equiv l\sin\varphi\,\dot{\varphi}.$$

Hence, according to Eqs. (4.14) and (4.36) of the lecture notes, the kinetic energy of the cylinder is

$$T = \frac{M}{2}V_C^2 + \frac{I}{2}\omega^2 = \frac{M}{2}\left[(R - l\cos\varphi)^2 + (l\sin\varphi)^2\right]\dot{\varphi}^2 + \frac{I}{2}\dot{\varphi}^2 \equiv \frac{M}{2}\left(a^2 + R^2 + l^2 - 2Rl\cos\varphi\right)\dot{\varphi}^2,$$

where the constant a is defined by the equality $I \equiv Ma^2$. (For a uniform cylinder, we would have $a = R/\sqrt{2}$.) With the evident expression for the potential energy of the cylinder in the uniform gravity field,

$$U = Mgh_C = Mg(-l\cos\varphi + \text{const}),$$

we get the following Lagrangian function:

$$L \equiv T - U = \frac{M}{2}\left(a^2 + R^2 + l^2 - 2Rl\cos\varphi\right)\dot{\varphi}^2 + Mgl\cos\varphi + \text{const}.$$

Using this function to spell out Eq. (2.19) for the only generalized coordinate of this system, the angle φ, we obtain the following equation of motion:

$$\frac{d}{dt}\left[\left(a^2 + R^2 + l^2 - 2Rl\cos\varphi\right)\dot{\varphi}\right] + Rl\sin\varphi\,\dot{\varphi}^2 + gl\sin\varphi = 0.$$

For small oscillations near the equilibrium position $\varphi = 0$, in the linear approximation in $\varphi \to 0$, we may take $\cos\varphi \approx 1$ and ignore the term proportional to $\sin\varphi\dot{\varphi}^2$, so the equation is reduced to

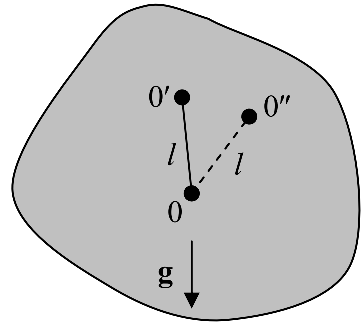
$$\left(a^2 + R^2 + l^2 - 2Rl\right)\ddot{\varphi} + gl\sin\varphi = 0,$$

i.e. to the standard equation of motion of a linear ("harmonic") oscillator with the frequency

$$\omega = \left(\frac{gl}{a^2 + R^2 + l^2 - 2Rl}\right)^{1/2} \equiv \left[\frac{Mgl}{I + M(R-l)^2}\right]^{1/2} \equiv \left(\frac{Mgl}{I_A}\right)^{1/2}, \quad \text{where } I_A \equiv I + M(R-l)^2.$$

According to the shift theorem (4.29), I_A so defined is just the moment of inertia for the rotation about the axis passing through the contact point A, so our result formally coincides with Eq. (4.41).

[82] The vector $\mathbf{V}_C$ as a whole is tangential to the trajectory of point C – a *curate trochoid*.



Problem 4.10. A body may rotate about a fixed horizontal axis – see Fig. 4.5 of the lecture notes, partly reproduced on the right. Find the frequency of its small oscillations in a uniform gravity field, as a function of distance l of the axis from the body's center of mass 0, and analyze the result.

Solution: As follows from the discussion at the beginning of Sec. 4.3 of the lecture notes and in the solution of Problem 4, all equations of rotation about a fixed but arbitrary axis coincide with those of the free-body rotation about a principal axis. In particular, the axis shift theorem (4.28) may be used in its simplified form (4.29). As a result, for our current problem (with the horizontal rotation axis passing through point 0*'*), we may write

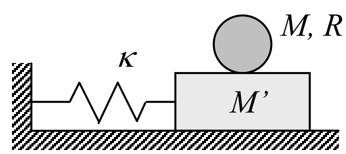
$$I' = I_0 + Ml^2,$$

where I_0 is the moment of inertia about the axis with a parallel (i.e. also horizontal) direction, but passing through the center of mass 0. Plugging this expression into the first form of Eq. (4.41), we get

$$\Omega^2 = \frac{Mgl}{I'} = \frac{Mgl}{I_0 + Ml^2} \equiv \frac{Mg}{I_0}\frac{l}{l_m^2 + l^2}, \qquad \text{with } l_m \equiv \left(\frac{I_0}{M}\right)^{1/2}.$$

The second fraction in the last formula for Ω^2, as a function of the distance l between the rotation axis and the center of mass (at a fixed l_m, i.e., a fixed I_0), has its maximum at $l = l_m$, so the oscillation frequency is the largest at this distance.

Note that, a bit counter-intuitively, Ω does not depend on the axis *location* at fixed l (say, would be the same for small oscillations about the parallel axes 0*'* and 0*"* shown in the figure above), though it may depend on the axis *direction* (via the magnitude of I_0).



Problem 4.11. Calculate the frequency, and sketch the mode of oscillations[83] of a round uniform cylinder of radius R and the mass M, that may roll, without slippage, on a horizontal surface of a block of mass M'. The block, in turn, may move in the same direction, without friction, on an immobile horizontal surface, being connected to it with an elastic spring – see the figure on the right.

Solution: A convenient choice of generalized coordinates of this system are the horizontal positions of the centers of mass of the bodies (say, X for the cylinder and X' for the block), as measured in the lab reference frame. In these coordinates,

$$L \equiv T - U = \frac{M}{2}\dot{X}^2 + \frac{I}{2}\omega^2 + \frac{M'}{2}\dot{X}'^2 - \frac{\kappa}{2}X'^2,$$

where I is the cylinder's moment of inertia, and ω is its angular velocity. From the kinematic no-slippage condition, we have

[83] The term *mode* usually refers to the spatial pattern of oscillations; it will be repeatedly discussed in later chapters.

$$\dot{X}' = \dot{X} + \omega R, \qquad \text{i.e. } \omega = \frac{\dot{X}' - \dot{X}}{R}. \tag{*}$$

Using this relation, and the well-known result $I = MR^2/2$, to eliminate ω, we get

$$L = \frac{3M}{4}\dot{X}^2 + \left(\frac{M'}{2} + \frac{M}{4}\right)\dot{X}'^2 - \frac{M}{2}\dot{X}\dot{X}' - \frac{\kappa}{2}X'^2.$$

The differentiations prescribed by Eq. (2.19) of the lecture notes give the following Lagrange equations of motion:

$$\frac{3}{2}M\ddot{X} - \frac{M}{2}\ddot{X}' = 0, \qquad \text{i.e. } \ddot{X} = \frac{1}{3}\ddot{X}', \tag{**}$$

$$\left(M' + \frac{M}{2}\right)\ddot{X}' - \frac{M}{2}\ddot{X} + \kappa X' = 0. \tag{***}$$

Plugging the last form of Eq. (**) into Eq. (***), we get

$$\left(M' + \frac{M}{3}\right)\ddot{X}' + \kappa X' = 0.$$

This is the usual differential equation of a linear ("harmonic") oscillator, with the frequency

$$\omega = \left(\frac{\kappa}{M' + M/3}\right)^{1/2}.$$

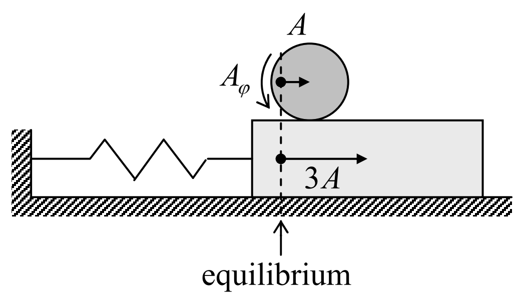
For the oscillation mode, Eq. (**) immediately gives the following first integral of motion,[84]

$$M\dot{X} - \frac{M}{3}\dot{X}' = \text{const}.$$

The constant in this relation has the physical sense of the linear momentum of the cylinder's free roll over the surface of the block. If we are not interested in such a roll, we may set that constant to zero, and immediately get

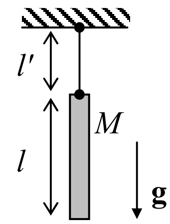
$$3X(t) = X'(t) + \text{const}',$$

where the new constant describes the equilibrium position of the cylinder on the block. This result means that the amplitude A' of the block oscillations (as measured in the lab frame) is three times larger than that (A) of the oscillations of the cylinder's center of mass – see the figure on the right. The amplitude A_φ of the cylinder's rotation angle follows from this result when combined with the no-slipping condition (*):



$$A_\varphi = \frac{A' - A}{R} = \frac{2A}{R}.$$

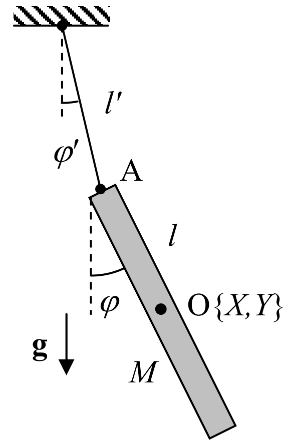
[84] Another first integral of motion for this system is given by the conservation of its full energy $E = T + U$.



Problem 4.12. A thin uniform bar of mass M and length l is hung on a light thread of length l' (like a "chime" bell – see the figure on the right). Derive the equations of the system's motion within a vertical plane passing the suspension point.

Solution: From Eqs. (4.14) and (4.36) of the lecture notes, the Lagrangian function of the system is

$$L = T - U = \frac{M}{2}\left(\dot{X}^2 + \dot{Y}^2\right) + \frac{I}{2}\omega^2 - MgY,$$



where I is the moment of inertia of the bar relative to its center of mass O (whose Cartesian coordinates are X and Y), and ω is the instantaneous angular velocity of the bar's rotation within the plane of the drawing. Just as in a very similar Problem 2.1, the best choice of the generalized coordinates are the angles φ and φ' indicated in the figure on the right, because X, Y, and ω may be simply expressed via them:

$$X = l'\sin\varphi' + \frac{l}{2}\sin\varphi, \qquad Y = -l'\cos\varphi' - \frac{l}{2}\cos\varphi, \qquad \omega = \dot{\varphi}.$$

With these expressions, the Lagrangian function becomes

$$L = \frac{M}{2}\left[l'^2\dot{\varphi}'^2 + \left(\frac{l}{2}\right)^2\dot{\varphi}^2 + ll'\cos(\varphi - \varphi')\dot{\varphi}\dot{\varphi}'\right] + \frac{I}{2}\dot{\varphi}^2 + Mg\left(l'\cos\varphi' + \frac{l}{2}\cos\varphi\right).$$

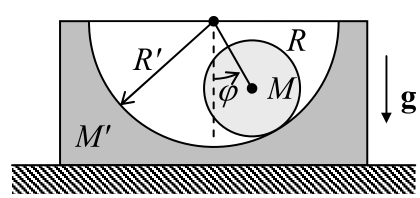
From here, the Lagrange equations of motion are:

$$I_A\ddot{\varphi} + M\frac{l}{2}l'\cos(\varphi - \varphi')\ddot{\varphi}' + M\frac{l}{2}l'\sin(\varphi - \varphi')\dot{\varphi}\dot{\varphi}' + Mg\frac{l}{2}\sin\varphi = 0, \quad \text{with } I_A \equiv I + M\left(\frac{l}{2}\right)^2,$$

$$Ml'^2\ddot{\varphi}' + M\frac{l}{2}l'\cos(\varphi - \varphi')\ddot{\varphi} - M\frac{l}{2}l'\sin(\varphi - \varphi')\dot{\varphi}\dot{\varphi}' + Mgl'\sin\varphi' = 0,$$

where, according to Eq. (4.29), I_A is just the bar's moment of inertia for its rotation about the suspension point A. (An elementary integration yields $I = Ml^2/12$, so $I_A = M\,l^2/12 + M(l/2)^2 \equiv Ml^2/3$.)

An analysis of these equations will be the task of Problem 6.3.



Problem 4.13. A uniform round solid cylinder of mass M can roll, without slippage, over a concave round cylindrical surface of a block of mass M', in a uniform gravity field – see the figure on the right. The block can slide without friction on a horizontal surface. Using the Lagrangian formalism,

(i) find the frequency of small oscillations of the system near the equilibrium, and
(ii) sketch the oscillation mode for the particular case $M' = M$, $R' = 2R$.

Solutions:

(i) Using Eq. (4.36) of the lecture notes, the Lagrangian function of the system may be written as

$$L = T - U = \frac{M'}{2}\dot{X}'^2 + \frac{M}{2}\dot{X}^2 + \frac{M}{2}\dot{Y}^2 + \frac{I}{2}\omega^2 - MgY\,,$$

where *X'* is the (horizontal) coordinate of the block, while *X* and *Y* are the Cartesian coordinates of the cylinder's center of mass (all in the lab reference frame), ω is the cylinder's angular velocity, and *I* is its moment of inertia for rotation about the symmetry axis. With this *X'* and the angle φ defined as shown in the figure above, as generalized coordinates, we may use Eq. (4.48) to write

$$\omega = -\frac{R'-R}{R}\dot{\varphi}, \qquad X = X' + (R'-R)\sin\varphi, \qquad Y = -(R'-R)\cos\varphi\,,$$

and then spell out the Lagrangian function as

$$L = \frac{M'}{2}\dot{X}'^2 + \frac{M}{2}\left[\dot{X}' + (R'-R)\dot{\varphi}\cos\varphi\right]^2 + \frac{M}{2}\left[(R'-R)\dot{\varphi}\sin\varphi\right]^2 + \frac{I}{2}\left(\frac{R'-R}{R}\dot{\varphi}\right)^2 + Mg(R'-R)\cos\varphi\,.$$

The standard procedure of differentiation of *L*, together with the well-known formula $I = MR^2/2$ (which may be derived by an elementary integration), gives us the following Lagrange equations of motion:

$$\frac{d}{dt}\left[(M+M')\dot{X}' + M(R'-R)\dot{\varphi}\cos\varphi\right] = 0,$$

$$\frac{d}{dt}\left[M(R'-R)\dot{X}'\cos\varphi + \frac{3}{2}M(R'-R)^2\dot{\varphi}\right] + M(R'-R)\dot{X}'\dot{\varphi}\sin\varphi + Mg(R'-R)\sin\varphi = 0.$$

Note that the first of these equations immediately yields one first integral of motion,

$$p_{X'} \equiv \frac{\partial L}{\partial \dot{X}'} = (M+M')\dot{X}' + M(R'-R)\dot{\varphi}\cos\varphi = \text{const}\,.$$

(This fact could be noticed directly from the Lagrangian function's form because *L* does not depend explicitly on the coordinate *X'*, so the corresponding generalized momentum $p_{X'}$ is conserved. Physically, this is just the *x*-component of the total momentum of the system, which is conserved because both external forces (gravity and the horizontal surface's reaction) are vertical.[85]) This means that if we are not interested in the common uniform motion of the system, we may assume that we are working in the center-of-mass reference frame, in which $p_{X'} = 0$, so the system has just one "oscillatory" degree of freedom.

Linearizing the equations of motion for small deviations from the equilibrium (with our assumptions, reached at $X' = \varphi = 0$), we get

$$(M+M')\ddot{X}' + M(R'-R)\ddot{\varphi} = 0,$$

$$\ddot{X}' + (3/2)(R'-R)\ddot{\varphi} + g\varphi = 0.$$

Let us express $\ddot{X}'$ from the first equation:

$$\ddot{X}' = -\frac{M}{M+M'}(R'-R)\ddot{\varphi}\,, \qquad (*)$$

and plug it into the second equation. The result may be recast in the standard form

[85] Another first integral of the system's motion is its full mechanical energy $E = T + U$.

$$\ddot{\varphi} + \Omega^2 \varphi = 0, \qquad \text{where } \Omega^2 \equiv \frac{g}{l_{ef}}, \quad l_{ef} \equiv (R' - R)\left(\frac{3}{2} - \frac{M}{M + M'}\right) > 0 .$$

This is of course the equation of a linear ("harmonic") oscillator with frequency Ω – see Eq. (3.12) and its discussion. We see that the small oscillation frequency of the system coincides with that of a point pendulum of the length l_{ef} specified above.

(ii) For sinusoidal oscillations of frequency Ω, the double differentiation over time is equivalent to the multiplication by $(-\Omega^2)$, so Eq. (*) immediately gives us the following relation between the oscillation amplitudes:

$$A_{X'} = -\frac{M}{M + M'}(R' - R)A_\varphi . \qquad (**)$$

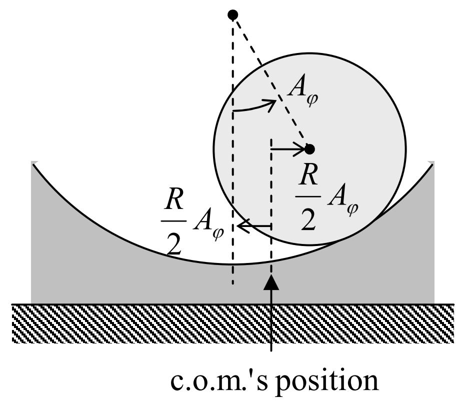
Note also that for small oscillations, the amplitude A_X of the horizontal displacements of the cylinder's center from the center of mass of the system equals

$$A_X = A_{X'} + (R' - R)A_\varphi ,$$

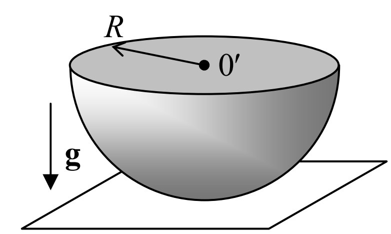
so according to Eq. (**), the $A_X/A_{X'}$ ratio

$$\frac{A_X}{A_{X'}} = -\frac{M'}{M} ,$$

does not depend on the ratio R'/R.



For the particular given case, $M' = M$ and $R' = 2R$, these relations yield $A_{X'} = -A_X = (R/2)A_\varphi$. This oscillation mode is sketched in the figure on the right.

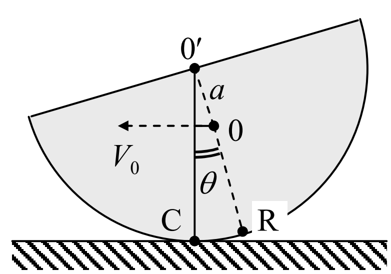


Problem 4.14. A uniform solid hemisphere of radius R and mass M is placed on a horizontal surface – see the figure on the right. Find the frequency of its small oscillations within a vertical plane, for two ultimate cases:

(i) there is *no friction* between the sphere and the horizontal surface;

(ii) the static friction between them is so strong that there is *no slippage*.

Solutions:



(i) In the vertical, equilibrium position, shown in the figure above, the center of mass 0 of the hemisphere is located on its vertical axis, at a certain distance a below the geometric center 0' of the initial sphere. Let us consider a small deviation θ from the equilibrium position – see the figure on the right. The deviation leads to a rise of the center of mass by

$$h = a(1 - \cos\theta) \approx a\frac{\theta^2}{2}, \qquad (*)$$

and hence to increase of its potential energy by

$$U = Mgh \approx Mga\frac{\theta^2}{2}.$$

To calculate the kinetic energy we may notice that due to the absence of friction, no horizontal force is exerted on the hemisphere, and hence its center of mass cannot oscillate horizontally. As a result, it is sufficient to take into account only the kinetic energy of the body's rotation about the center of mass, with the instantaneous angular velocity $\omega = d\theta/dt$:[86]

$$T_1 = \frac{I_0}{2}\omega^2 = \frac{I_0}{2}\dot{\theta}^2,$$

where I_0 is the moment of inertia of the hemisphere for its rotation about the horizontal axis passing through its center of mass (point 0). Now composing the Lagrangian function for small oscillations,

$$L_1 = T_1 - U = \frac{I_0}{2}\dot{\theta}^2 - Mga\frac{\theta^2}{2},$$

we see that its functional form (and hence the Lagrange equation of motion) coincides with that of the usual linear (or "harmonic") oscillator with the frequency

$$\Omega_1 = \left(\frac{Mga}{I_0}\right)^{1/2}.$$

Note that this formula is similar to the solution of Problem 9. It also has the same structure as Eq. (4.41) of the lecture notes for a physical pendulum but with a different meaning of the two participating constants (I_0 and a).

(ii) If there is no slippage, the center of mass of the hemisphere moves horizontally as well: $X_0 = X_{0'} + a\sin\theta = -R\theta - a\sin\theta$. In the linear approximation in small θ, $X_0 = -(R - a)\theta$, so the c.o.m.'s horizontal velocity is $V_0 = -(R - a)\dot{\theta}$, giving a substantial contribution to the total kinetic energy:

$$T_2 = \frac{I_0}{2}\dot{\theta}^2 + \frac{M}{2}V_0^2 = \frac{I_0}{2}\dot{\theta}^2 + \frac{M}{2}(R-a)^2\dot{\theta}^2 \equiv \frac{1}{2}I_R\dot{\theta}^2,$$

where

$$I_R \equiv I_0 + M(R-a)^2. \qquad (**)$$

This expression is similar to that for T_1 (with the replacement $I_0 \to I_R$), while the potential energy is exactly the same as in Task (i). Hence the small oscillation frequency in case (ii) is

$$\Omega_2 = \left(\frac{Mga}{I_R}\right)^{1/2},$$

i.e. is always lower than Ω_1.

Note that the last formula may be also obtained just Eq. (4.41) for a physical pendulum, by writing the equation (4.38) for rotation about point C, induced by the torque $\tau = Mga\sin\theta$ of the vertical gravity force Mg applied to the center of mass 0. This should not be surprising, because the rotation axis shift theorem (4.29) tells us that the I_R defined by Eq. (**) is just the moment of inertia for rotation

[86] Per Eq. (*), the center of mass has the vertical velocity $V_v = dh/dt$ as well, but at $\theta \to 0$, this velocity $d(a\theta^2/2)/dt$ is proportional to θ^2, so the kinetic energy $Mv_v^2/2$ of this motion scales as θ^4, and is negligible in comparison with the terms $O(\theta^2)$ contributing to the small-oscillation frequency.

about a horizontal axis passing through point R, in the small oscillation limit equal to that for rotation about the axis passing through the contact point C.

What remains is to calculate the constants a and I_0 (and then I_R). The first calculation may be readily done by integrations over the hemisphere's volume V, using the constancy of its density ρ.

$$M = \int_V \rho d^3 r = \rho \int_0^R \pi\left(R^2 - z^2\right) dz = \rho \frac{2\pi}{3} R^3, \qquad a = \frac{1}{M}\int_V \rho z d^3 r = \frac{\rho}{M}\int_0^R z\pi\left(R^2 - z^2\right) dz = \frac{\pi}{4}\frac{\rho}{M} R^4 = \frac{3}{8} R.$$

(Here z is the distance of an elementary disk of thickness dz, normal to the symmetry axis of the hemisphere, from the geometrical center 0' of the full sphere, so the disk's area is $\pi(R^2 - z^2)$.)

While I_0 and I_R may be calculated by similar integrations, it is easier to find them by applying the shift theorem (4.29) again, now to the axes passing through the center of mass (point 0) and point 0':

$$I_{0'} = I_0 + Ma^2,$$

because $I_{0'}$ is evidently just half of that of the full sphere, and is hence similarly expressed via the hemisphere's mass: $I_{0'} = (2/5)MR^2$. From here,

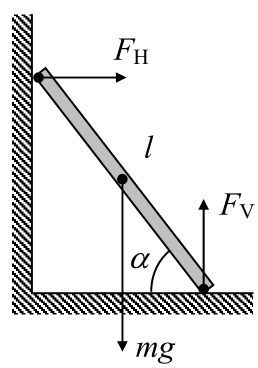
$$I_0 = I_{0'} - Ma^2 = \frac{2}{5} MR^2 - M\left(\frac{3}{8} R\right)^2 = \frac{83}{320} MR^2,$$

$$I_R = I_0 + M\left(R - a\right)^2 = \frac{83}{320} MR^2 + M\left(R - \frac{3}{8} R\right)^2 = \frac{13}{20} MR^2.$$

Using these expressions, we finally get

$$\Omega_1 = \left(\frac{Mga}{I_0}\right)^{1/2} = \left(\frac{120}{83}\frac{g}{R}\right)^{1/2} \approx 1.202\left(\frac{g}{R}\right)^{1/2}, \qquad \Omega_2 = \left(\frac{Mga}{I_R}\right)^{1/2} = \left(\frac{15}{26}\frac{g}{R}\right)^{1/2} \approx 0.760\left(\frac{g}{R}\right)^{1/2},$$

so the oscillation slowdown caused by the static friction is rather substantial.



Problem 4.15. For the "sliding ladder" problem started in Sec. 4.3 of the lecture notes (see Fig. 4.7, reproduced on the right with additions), find the critical value α_c of the angle α at that the ladder loses contact with the vertical wall, assuming that it starts sliding from the vertical position, with a negligible initial velocity.

Solution: From Eqs. (4.51)-(4.52) of the lecture notes, the system's Lagrangian is

$$L \equiv T - U = \frac{1}{6} Ml^2 \dot{\alpha}^2 - \frac{1}{2} Mgl \sin\alpha.$$

It gives the following Lagrange equation of motion:

$$\ddot{\alpha} + \frac{3g}{2l}\cos\alpha = 0. \qquad (*)$$

By using the standard transformation (see, e.g., Sec. 1.2 of the lecture notes),

$$\ddot{\alpha} \equiv \frac{d\dot{\alpha}}{dt} = \frac{d\dot{\alpha}}{d\alpha}\frac{d\alpha}{dt} = \frac{d\dot{\alpha}}{d\alpha}\dot{\alpha} = \frac{1}{2}\frac{d\left(\dot{\alpha}^2\right)}{d\alpha},$$

the equation is brought to the form

$$\frac{d\left(\dot{\alpha}^2\right)}{d\alpha} + \frac{3g}{l}\cos\alpha = 0,$$

and may be readily integrated once over α, giving

$$\dot{\alpha}^2 + \frac{3g}{l}\sin\alpha = C = \text{const}.$$

(Actually, this is just the conservation of the energy given by Eq. (4.53) of the lecture notes.) For our specified initial conditions ($\dot{\alpha} = 0$ at $\alpha = \pi/2$), the constant in the integral is

$$C = \left[\dot{\alpha}^2 + \frac{3g}{l}\sin\alpha\right]_{\alpha = \pi/2,\, \dot{\alpha} = 0} = \frac{3g}{l},$$

so the integral becomes

$$\dot{\alpha}^2 + \frac{3g}{l}\sin\alpha = \frac{3g}{l}. \qquad (**)$$

Now looking at the problem again (see the figure above), the horizontal component of the 2nd Newton law says that the (horizontal) force F_H exerted on the ladder by the wall is

$$F_H = M\ddot{X} = M\frac{d^2}{dt^2}\left(\frac{l}{2}\cos\alpha\right) = \frac{Ml}{2}\left(-\ddot{\alpha}\sin\alpha - \dot{\alpha}^2\cos\alpha\right).$$

The ladder separates from the wall when this force vanishes, i.e. in the moment when

$$\ddot{\alpha}\sin\alpha + \dot{\alpha}^2\cos\alpha = 0.$$

Plugging the $\ddot{\alpha}$ expressed from Eq. (*) and the $\dot{\alpha}^2$ expressed from Eq. (**) into the last formula, we get the following algebraic equation for the threshold angle α_t:

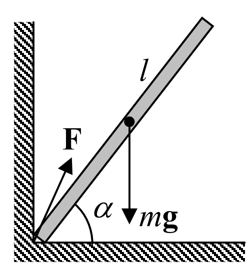
$$\frac{3g}{2l}\sin\alpha_t\cos\alpha_t - \frac{3g}{l}\left(1 - \sin\alpha_t\right)\cos\alpha_t = 0, \qquad \text{i.e.}\left(\frac{3}{2}\sin\alpha_t - 1\right)\cos\alpha_t = 0.$$

As α_t increases from 0, this equation is first satisfied at

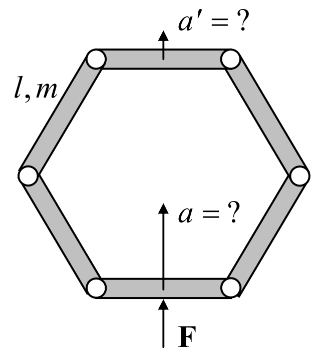
$$\alpha_t = \arcsin\left(2/3\right) \approx 42^\circ.$$

At this angle, the ladder separates from the wall, due to the horizontal motion's inertia accumulated during its initial slide.[87]

Curiously, all this solution is also valid for a different problem: a falling ladder with its lower end in the corner – see the figure on the right. Indeed, due to Eqs. (4.50), which are evidently valid for this new problem as well, the Lagrangian functions and hence the equations of motion of both systems are exactly the same. (In the new problem, F_H and F_V are the Cartesian components of a single force **F**.)

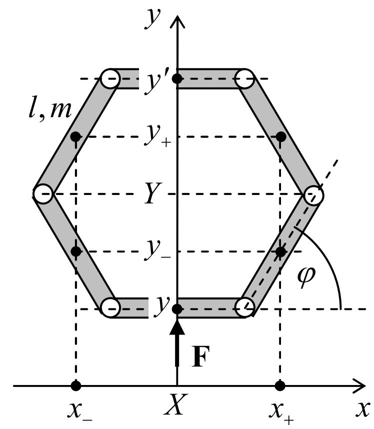


[87] A useful additional exercise: find out whether sometime after this moment, the *lower* end of the ladder would separate from the floor, due to its rotational inertia.



Problem 4.16. Six similar, uniform rods of length l and mass m are connected by light joints so that they may rotate, without friction, versus each other, forming a planar polygon. Initially, the polygon was at rest, and had the correct hexagon shape – see the figure on the right. Suddenly, an external force **F** is applied to the middle of one rod, in the direction of the hexagon's symmetry center. Calculate the accelerations: of the rod to which the force is applied (a) and of the opposite rod (a'), immediately after the application of the force.

Solution: In an inertial reference frame (e.g., in the Cartesian coordinates [x, y] shown in the figure on the right), the system may be described by the Lagrangian function



$$L = T - U, \quad \text{with } U = -Fy,$$

where T is the sum of the kinetic energies of all six rods, and y is the coordinate of the "bottom" rod (to whose middle the force **F** is applied).[88] To calculate the kinetic energies of the rods, we may notice that due to the problem's symmetry, even after the application of the force, the polygon still sustains its mirror symmetry about the lines $x = X$ and $y = Y$, so its geometry is uniquely determined by just two scalar generalized coordinates – for example, the lower rod's coordinate y and the angle φ shown in the figure on the right (where the solid points denote the centers of mass of each rod). In particular, the key coordinates indicated in the figure may be expressed as

$$y_{\pm} = y + (2 \pm 1)\frac{l}{2}\sin\varphi, \qquad y' = y + 4\frac{l}{2}\sin\varphi, \qquad x_{\pm} = X \pm \frac{l}{2}(1 + \cos\varphi).$$

According to Eq. (4.14) of the lecture notes, the kinetic energy of each rod is the sum of the kinetic energy of its center of mass,

$$T_{\text{com}} = \frac{m}{2}\left(\dot{x}_{\text{com}}^2 + \dot{y}_{\text{com}}^2\right),$$

and its rotational energy, in our particular case equal to zero for the two rods parallel to the x-axis, and similar for all four side rods:

$$T_{\text{rot}} = \frac{I}{2}\dot{\varphi}^2, \qquad \text{where } I = \frac{1}{12}ml^2.$$

Summing up all these contributions, we get

$$L = 3m\dot{y}^2 + 6ml\dot{\varphi}\,\dot{y}\cos\varphi + 2ml^2\dot{\varphi}^2\left(\frac{1}{3} + 2\cos^2\varphi\right) + Fy.$$

Now writing the Lagrange equations (2.19) for our two generalized coordinates, y and φ, we get

$$\frac{d}{dt}\left(6m\dot{y} + 6ml\dot{\varphi}\cos\varphi\right) = F,$$
$$\frac{d}{dt}\left[6ml\dot{y}\cos\varphi + 4ml^2\dot{\varphi}\left(\frac{1}{3} + 2\cos^2\varphi\right)\right] = 0. \qquad (*)$$

[88] This U is essentially the Gibbs potential energy of the system – see, e.g., Sec. 1.4 of the lecture notes.

In the first of these equations, we may readily recognize Eq. (4.44) for the motion of the center of mass of the whole system, with the total mass $M = 6m$ and coordinates[89]

$$X = \text{const}, \quad \text{and } Y = y + l\sin\varphi, \qquad \text{so that } \dot{\mathbf{V}} = \mathbf{n}_y \frac{d}{dy}\left(y + l\sin\varphi\right) = \mathbf{n}_y\left(\dot{y} + l\dot{\varphi}\cos\varphi\right).$$

After performing the differentiation, that equation becomes

$$\ddot{y} + l\ddot{\varphi}\cos\varphi - l\dot{\varphi}^2\sin\varphi = \frac{F}{M}. \qquad (**)$$

On the other hand, the zero on the right-hand side of the second Eq. (*) reflects the fact that φ is a cyclic coordinate, because, with our choice of the generalized coordinates, it does not affect the potential energy of the system. So, taking into account that at the initial moment both first derivatives participating in that equation equal zero, we may write the corresponding first integral of motion as

$$2\dot{y}\cos\varphi + \frac{4}{3}l\dot{\varphi}\left(\frac{1}{3} + 2\cos^2\varphi\right) = 0. \qquad (***)$$

The general solution of the system of two nonlinear differential equations (**) and (***) is still not very simple. However, we are only asked about the accelerations *immediately* after the application of the force **F**, i.e. when the rod's coordinates and velocities have had no time to change yet. Hence in the above equations, we may take

$$\varphi = \frac{\pi}{3} \quad \left(\text{i.e. } \cos\varphi = \frac{1}{2}\right), \quad \dot{\varphi} \to 0, \quad \dot{y} \to 0.$$

In this limit, the equations are much simplified:

$$\ddot{y} + \frac{1}{2}l\ddot{\varphi} = \frac{F}{M}, \qquad \dot{y} + \frac{10}{9}l\dot{\varphi} = 0, \quad \text{i.e. } \ddot{y} + \frac{10}{9}l\ddot{\varphi} = 0,$$

and may be readily solved for the second derivatives:

$$\ddot{y} = \frac{20}{11}\frac{F}{M}, \qquad l\ddot{\varphi} = -\frac{18}{11}\frac{F}{M},$$

so the requested accelerations of the rods parallel to the x-axis are

$$a \equiv \ddot{y} = \frac{20}{11}\frac{F}{M} = \frac{10}{33}\frac{F}{m}, \qquad a' \equiv \ddot{y}' = \frac{d^2}{dt^2}\left(y + 2l\sin\varphi\right)\Big|_{\cos\varphi = 1/2,\,\dot{\varphi}=0} = \ddot{y} + l\ddot{\varphi} = \frac{2}{11}\frac{F}{M} = \frac{1}{33}\frac{F}{m}.$$

It is rather impressive how small the latter acceleration is: it just 1/10 of that of the rod to which the force is directly applied. (Indeed, with no torque transferred by the joints, a' is different from zero only due to the inertia of the side rods.)

Problem 4.17. A uniform rectangular cuboid (parallelepiped) with sides a_1, a_2, and a_3, and mass M, is rotated with a constant angular velocity ω about one of its space diagonals – see the figure below. Calculate the torque $\boldsymbol{\tau}$ necessary to sustain this rotation.

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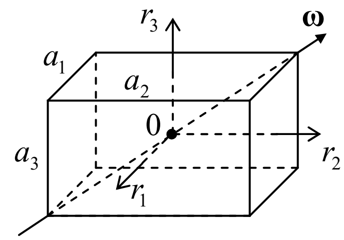
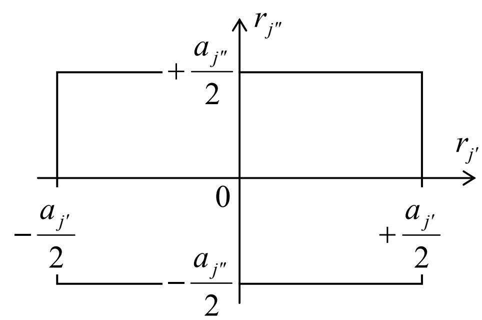
[89] We could of course write this equation directly from the 2nd Newton law, i.e. without using the Lagrange formalism, but this formalism is the easiest way to derive the second of Eqs. (*), so we needed to calculate L anyway.



Solution: Let us introduce the reference frame with the origin in the cuboid's center-of-mass 0, and the axes r_j (j = 1, 2, 3) parallel to the corresponding sides (edges) – see the figure on the right. In this frame (rotating with the body, and hence non-inertial one), Cartesian components of the angular velocity vector $\boldsymbol{\omega}$ are time-independent:

$$\omega_j = \frac{a_j}{d}\omega = \frac{a_j}{\left(a_1^2 + a_2^2 + a_3^2\right)^{1/2}}\omega, \qquad (*)$$

where d is the space diagonal's length. Since, due to the body's symmetry, the axes r_j are the principal ones, Cartesian components τ_j of the necessary torque may be readily found from the Euler equations (4.66) with $\dot{\omega}_j = 0$:



$$\tau_j = \left(I_{j''} - I_{j'}\right)\omega_{j'}\omega_{j''}, \qquad (**)$$

where the indices j, j', and j'' are all different and follow in the "right" order: {1, 2, 3}, etc.

What remains is to calculate the principal moments of inertia I_j. This is simple to do – see Eq. (4.24) of the lecture notes and the figure on the right:

$$I_j = \frac{M}{V}\int_V \left(r_{j'}^2 + r_{j''}^2\right)d^3r = \frac{M}{a_1 a_2 a_3}\int_{-a_j/2}^{+a_j/2} dr_j \int_{-a_{j'}/2}^{+a_{j'}/2} dr_{j'} \int_{-a_{j''}/2}^{+a_{j''}/2} dr_{j''}\left(r_{j'}^2 + r_{j''}^2\right) = M\frac{a_{j'}^2 + a_{j''}^2}{12}.$$

With this result and Eq. (*) on hand, Eq. (**) yields

$$\tau_j = \left(I_{j''} - I_{j'}\right)\omega_{j'}\omega_{j''} = \frac{M\omega^2}{12}\frac{a_{j'}a_{j''}}{a_1^2 + a_2^2 + a_3^2}\left(a_{j'}^2 - a_{j''}^2\right).$$

Note that for a cube ($a_1 = a_2 = a_3$), all components of the necessary torque vanish, i.e. the whole vector $\boldsymbol{\tau}$ equals zero. This is natural, because for any spherical top, including a uniform cube, any axis passing through the center of mass is a principal one, and a rotation about it does not require torque. For *any* other relation of cuboid dimensions, $\boldsymbol{\tau} \neq 0$.

Note also that the calculated vector $\boldsymbol{\tau}$ is constant only in the reference frame rotating with the body, while in the lab frame, it rotates about its rotation axis, with the same angular velocity ω. Superficially, it looks like such a rotating torque is a hard thing to arrange, but actually, any hardware axis fixed in the lab frame does the job automatically! As a sanity check, per Eq. (4.36) of the lecture notes, such a rotation does not change the body's kinetic energy and does not require any external work.

Problem 4.18. A uniform round ball rolls, without slippage, over a "turntable": a horizontal plane rotated about a vertical axis with a time-independent angular velocity Ω. Derive a self-consistent equation of motion of the ball's center, and discuss its solutions.

Solution: The vertical forces of gravity and of the support by the turntable's surface, exerted on the ball, do not create any torque $\boldsymbol{\tau}$ relative to its center. Such torque, however, is created by the horizontal friction force $\mathbf{F}$. This force is applied to the ball's lowest point, which contacts the turntable's surface and is located right below the center, at the distance R from it, where R is the ball's radius. Hence, directing the z-axis vertically up, we may use Eq. (1.34) of the lecture notes to write $\boldsymbol{\tau} = -R\mathbf{n}_z\times\mathbf{F}$. Also, due to the ball's spherical symmetry, we may represent its angular momentum just as $\mathbf{L} = I\boldsymbol{\omega}$, where $\boldsymbol{\omega}$ is its angular velocity and I is the (only) moment of inertia.[90]

As a result, for our ball, Eqs. (4.33) and (4.44) of the lecture notes (valid in any inertial reference frame) have the form

$$I\dot{\boldsymbol{\omega}} = -R\mathbf{n}_z \times \mathbf{F}, \qquad M\dot{\mathbf{V}} = \mathbf{F},$$

where $\mathbf{V}$ is the linear velocity of the ball's c.o.m, and enable a ready elimination of $\mathbf{F}$, giving

$$I\dot{\boldsymbol{\omega}} = -MR\mathbf{n}_z \times \dot{\mathbf{V}}.$$

Integrating both parts of this equation over time, and multiplying them by the ratio R/I, we get

$$R\boldsymbol{\omega} = -\frac{1}{\alpha}\mathbf{n}_z \times \mathbf{V} + \mathbf{C}, \qquad (*)$$

where $\alpha \equiv I/MR^2$ is a numerical factor (for a uniform ball, equal to 2/5), while the integration constant $\mathbf{C}$ is some time-independent vector.

This relation is valid regardless of the ball-to-surface contact type. Now let us combine it with the given no-slippage condition, i.e. the requirement that the net velocity of the ball's lowest point equals to that of the contact point of the turntable. Using, for both these velocities, the main kinematic relation Eq. (4.10), we get

$$\mathbf{V} + \boldsymbol{\omega} \times (-\mathbf{n}_z R) = \boldsymbol{\Omega} \times \boldsymbol{\rho}, \qquad (**)$$

where $\Omega = \Omega\mathbf{n}_z$, while $\boldsymbol{\rho}$ is the horizontal component of the contact point's (and hence of the ball center's) radius vector, with the origin at the turntable's axis.[91] Now plugging the product $R\boldsymbol{\omega}$ given by Eq. (*) into Eq. (**), and taking into account that for any horizontal vector $\mathbf{V}$, the double vector product $(\mathbf{n_z}\times\mathbf{V})\times\mathbf{n}_z$ equals simply $\mathbf{V}$, we obtain the result that may be rewritten as

$$\mathbf{V} = \boldsymbol{\Omega}' \times (\boldsymbol{\rho} - \boldsymbol{\rho}_0), \quad \text{where } \boldsymbol{\Omega}' \equiv \frac{\alpha}{1+\alpha}\boldsymbol{\Omega}, \quad \text{while } \boldsymbol{\rho}_0 \equiv \frac{\alpha}{1+\alpha}\mathbf{C} \times \mathbf{n}_z. \qquad (***)$$

(For a uniform ball, the fraction in these expressions equals 2/7. Also note that since both vectors $\mathbf{V}$ and $\boldsymbol{\rho}$ are horizontal, the vector constant $\boldsymbol{\rho}_0$ may also have only a horizontal component.)

Since $\mathbf{V} \equiv \dot{\boldsymbol{\rho}}$, the first of Eqs. (***) is the self-consistent differential equation of motion we were looking for. It may be easily solved by several elementary methods (say, in Cartesian coordinates), but by comparing it with Eq. (4.10), we may immediately say that it describes a circular motion of the ball's center with the angular velocity Ω'.[92] The amplitude and phase of this rotation, as well as its center's position $\boldsymbol{\rho}_0$, are determined by the initial values of the vectors $\boldsymbol{\rho}$ and $\mathbf{V}$.

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[90] As was calculated in the solution of Problem 1(iv), for a uniform ball $I = (2/5)MR^2$, where M is its mass.

[91] Note that instead of $\boldsymbol{\rho}$, we could formally use the whole radius vector, just as we are still keeping the whole angular velocity vector $\boldsymbol{\omega}$, but vertical components of these vectors are time-independent and have no effect on the dynamics of other degrees of freedom of the system.

[92] Several nice videos of experimental demonstrations of this counter-intuitive effect may be found online.

Problem 4.19. Calculate the free precession frequency of a uniform thin round disk rotating with an angular velocity ω about a direction very close to its symmetry axis, from the point of view of:

(i) an observer rotating with the disk, and
(ii) a lab-based observer.

Solution: As we know from the solution of Problem 1, the principal moments of inertia of such a disk are related as $I_1 = I_2 = I_3/2$, where $\mathbf{n}_3$ is its symmetry axis. (Since the relation is valid for each elementary thin round fragment of the disk, it is valid for a disk with an arbitrary axially symmetric mass distribution.)

(i) For an observer rotating with the disk, we may use Eq. (4.68) of the lecture notes, giving

$$\Omega_{\text{pre}} = \omega_3 \approx \omega ,$$

because according to the assignment, the vectors $\boldsymbol{\omega}_3$ and $\boldsymbol{\omega}$ are almost aligned.

(ii) The above result means that since for the lab-based observer, these two rotations add up, then

$$\omega_{\text{pre}} \approx \Omega_{\text{pre}} + \omega \approx 2\omega .$$

This conclusion may be confirmed by using Eq. (4.59): since in our case, the vector $\mathbf{L}$ is almost exactly aligned with the symmetry axis $\mathbf{n}_3$, we may write $L \approx I_3\omega_3 \approx I_3\omega$, so this formula yields

$$\omega_{\text{pre}} \approx \frac{I_3}{I_1}\omega = 2\omega.$$

By the way, in his memoir, Richard Feynman writes that he was dragged into studying physics by observing a manifestation of exactly this relation in a university cafeteria: a dinner plate tossed up by a student was "wobbling" twice faster than it was spinning.

An additional useful exercise: redraw the diagrams of Fig. 4.8 of the lecture notes for this case.

Problem 4.20. Use the Euler equations to prove the fact mentioned in Sec. 4.4 of the lecture notes: free rotation of an arbitrary body ("asymmetric top") about its principal axes with the smallest and largest moments of inertia is stable, while that about the intermediate-I_j axis is not. Illustrate the same fact using the Poinsot construction.

Solution: Let the vector $\boldsymbol{\omega}$ be very close to the j^{th} principal axis, i.e. $|\omega_{j'}|, |\omega_{j''}| << \omega_j \approx \omega$. Then in the linear approximation in small $\omega_{j'}$ and $\omega_{j''}$, in the Euler equations (4.66) for free rotation ($\tau = 0$), we may ignore the product $\omega_{j'}\omega_{j''}$ in the equation for ω_j, because this product is of the second order in small $\omega_{j'}$ and| $\omega_{j''}$. Hence this equation is reduced to just

$$I_j\dot{\omega}_j = 0, \qquad \text{giving } \omega \equiv \omega_j = \text{const.} \qquad (*)$$

However, the similar products in the equations for $\omega_{j'}$ and $\omega_{j''}$ are linear in these small perturbations, and cannot be ignored, so the only simplification we may make there is to use Eq. (*) for ω_j:

$$I_{j'}\dot{\omega}_{j'} + \left(I_j - I_{j''}\right)\omega\omega_{j''} = 0, \qquad I_{j''}\dot{\omega}_{j''} + \left(I_{j'} - I_j\right)\omega\omega_{j'} = 0 . \qquad (**)$$

At time-independent ω, this is a system of two linear equations, which may be readily solved, for example, by differentiation of one of them over time and then plugging the second one into the result. The resulting equations of motion are similar for both small components of the vector $\boldsymbol{\omega}$; for example:

$$\ddot{\omega}_{j'} + \Omega^2 \omega_{j'} = 0, \qquad \text{with } \Omega^2 \equiv \frac{(I_j - I_{j''})(I_j - I_{j'})}{I_{j'} I_{j''}} \omega^2 . \qquad (***)$$

This equation shows that if both parentheses in the numerator of the last fraction have the same sign (as they do if I_j is either the largest or the smallest one of all principal moments of inertia), i.e. if Ω^2 is positive, then small initial deviations $\omega_{j'}$ and $\omega_{j''}$ just oscillate around zero. (Equations (**) show that the phases of these oscillations are shifted by $\pm\pi/2$, so the vector $\boldsymbol{\omega}$ rotates around the j^{th} principal axis, staying close to it.) However, if I_j is the intermediate one of the three moments of inertia, then Ω^2 is negative, and Eq. (***) describes an exponential growth of some linear combination of $\omega_{j'}$ and $\omega_{j''}$ in time, i.e. the fixed point $\omega_j = \omega$ is unstable.

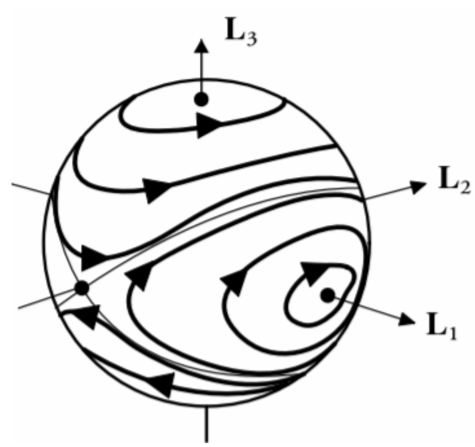
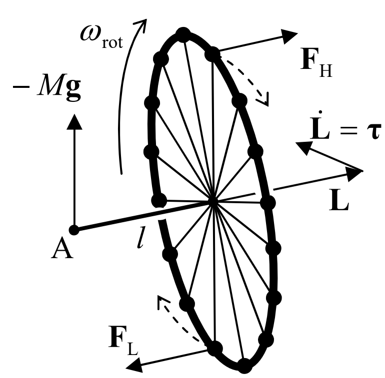


Figure from Prof. Douglas Cline's page at https://phys.libretexts.org; adopted under the CC BY-NC-SA 3.0 license.

This conclusion may be illustrated using the Poinsot construction. Let us represent Eq. (4.60) of the lecture notes for L^2 as a sphere in the Cartesian coordinates aligned with the principal axes $\mathbf{n}_j$ of the body, Eq. (4.61) as an ellipsoid in the same coordinates, and then draw the lines of their intersection for several values of T_{rot}. The figure on the right shows a typical sketch of this construction for a case with $I_1 < I_2 < I_3$. (Its topology does not depend on the exact values of the moments I_j.)

Since at free rotation, both L^2 and T_{rot} are integrals of motion even in this non-inertial reference frame, the vector $\mathbf{L}$ in it may move only along one of these curves. (The directions of such motion, shown by arrows in the figure, follow from the Euler equations, but are unimportant for our discussion.) The figure clearly shows that if the initial direction of the angular momentum $\mathbf{L}$ slightly deviates from the principal axis $\mathbf{n}_1$, this vector continues to move close to this direction, around it. The same is true for the axis $\mathbf{n}_3$. So, the alignments with $\mathbf{L}\|\mathbf{n}_1$ and $\mathbf{L}\|\mathbf{n}_3$ are "orbitally stable" – see Sec. 3.2 of the lecture notes. However, as the same figure shows, the alignment $\mathbf{L}\|\mathbf{n}_2$ is unstable because its minor violation at $t = 0$ leads to very substantial deviations of these vectors at larger times.

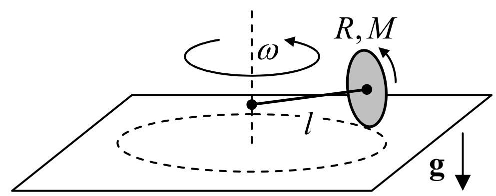
<u>Problem 4.21</u>. Give an interpretation of the torque-induced precession, that would explain its direction, by using a simple system exhibiting this effect, as a model.



Solution: Let us consider the well-known bicycle-wheel demonstration of the precession – see the figure on the right. A wheel, rotating fast about its horizontal axis by inertia, is supported, against the wheel's weight, at a pivot point A offset from the wheel's center by distance l. According to its definition (1.34), the torque $\boldsymbol{\tau}$ of the vertical support force $-M\mathbf{g}$ is perpendicular both to the force itself and to the wheel's axis – and hence to the dominating part of the wheel's orbital momentum **L**. As a result, according to the basic Eq. (1.33), the vector **L** evolves in that direction, i.e. the wheel performs the torque-induced precession in the horizontal plane, with the angular velocity given by Eq. (4.72).

In order to interpret this counter-intuitive effect, let us think of the wheel as a system of many discrete masses – say, those located at the ends of its spokes – see the solid points in the figure above. The torque $\boldsymbol{\tau}$ is transferred to these masses via spokes, which try to rotate the system in the same direction as the support force – see, e.g., the arrows showing the forces $\mathbf{F}_H$ and $\mathbf{F}_L$ exerted on the points that are at the highest and the lowest positions at the considered moment of time. These forces push the masses, respectively, from and toward the pivot point. However, due to the fast rotation of the wheel, these points do not follow the force directions but just slightly deviate from the initial rotation plane – along the trajectories sketched with dashed arrows. As the figure above shows, these deviations result in the wheel's turning horizontally, exactly in the direction predicted by the formal theory of torque-induced precession.

It is easy to verify that this handwaving explanation also gives correct predictions of the major dependencies given by Eq. (4.72). For example, the forces $\mathbf{F}_H$ and $\mathbf{F}_L$ (and hence the precession's velocity) are proportional to the torque Mgl. Also, the deviations of particle trajectories from the initial rotation plane, caused by these forces, during a certain time interval are the smaller the faster the wheel rotates – just as described by the $\omega_{pre} \propto 1/\omega_{rot}$ relation given by that formula.

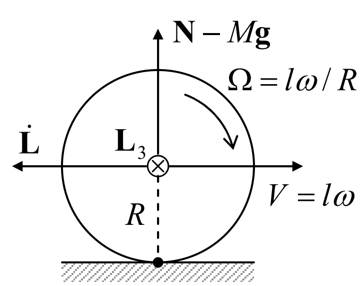


Problem 4.22. One end of a light shaft of length l is firmly attached to the center of a uniform solid disk of radius R and mass M, whose plane is perpendicular to the shaft. Another end of the shaft is attached to a vertical axis (see the figure on the right) so that the shaft may rotate about the axis without friction. The disk rolls, without slippage, over a horizontal surface so that the whole system rotates about the vertical axis with a constant angular velocity ω. Calculate the (vertical) supporting force N exerted on the disk by the surface.

Solution: The instantaneous angular momentum **L** of the disk consists of two mutually perpendicular components: the time-independent vertical component $L_z = I_z\omega$, and the horizontal component of magnitude $L_3 = I_3\Omega$, directed along the instantaneous position $\mathbf{n}_3$ of the shaft. Here Ω is the angular velocity of the disk's rotation around the shaft, and I_3 is its moment of inertia about this principal axis. An easy integration (see the model solution of Task (ii) of Problem 1) yields $I_3 = MR^2/2$, while Ω may be calculated from the no-slippage condition $l\omega - R\Omega = 0$, so $L_3 = MRl\omega/2$.

In the lab reference frame, the axis $\mathbf{n}_3$ rotates about the fixed vertical axis with the angular velocity ω, so the vector $d\mathbf{L}/dt$ is horizontal, perpendicular to $\mathbf{n}_3$, and has the following magnitude:

$$\left|\dot{\mathbf{L}}\right| = L_3\omega = \frac{MRl\omega^2}{2}.$$

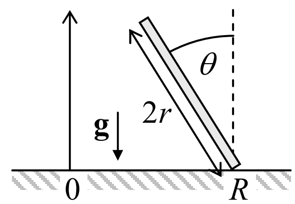


The figure on the right shows the view of the disk and most vectors important for the problem from the instantaneous direction $\mathbf{n}_3$ of its shaft (so the vertical rotation axis is behind the plane of the drawing). The figure shows that due to the system's kinematics, the vector **L** is directed normally to the plane of the drawing, from the viewer, while the vector

$\dot{\mathbf{L}}$ lies within that plane and is directed to the left. Per Eq. (1.34) of the lecture notes, the torque $\boldsymbol{\tau}$ created by the balance ($\mathbf{N} - M\mathbf{g}$) of the vertical forces applied to the disk equals $\mathbf{l}\times(\mathbf{N} - M\mathbf{g})$, where the vector $\mathbf{l}$ is directed from the rotation axis toward the point of the force application, i.e. normally to the plane of the drawing, toward the viewer. As a result, if the difference $N - Mg$ is positive, the vector product $\mathbf{l}\times(\mathbf{N} - M\mathbf{g})$ is directed to the left, i.e. has the same direction as $\dot{\mathbf{L}}$. Hence, in our case, we may write the key equation $\dot{\mathbf{L}} = \boldsymbol{\tau}$ of rotational dynamics in the following scalar form:

$$\frac{MRl\omega^2}{2} = (N - Mg)l, \qquad \text{giving } N = M\left(g + \frac{R\omega^2}{2}\right).$$

The fact the N does not depend on the sign of ω implies that it is unaffected by the direction of the system's rotation – the fact that is easy to verify directly.[93] Hence the rotation of the system, regardless of its direction, always increases N, i.e. presses the disk more to the supporting surface.

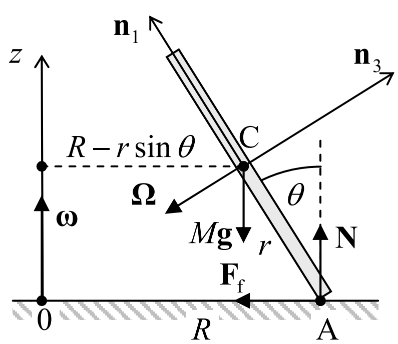


Problem 4.23. A coin of radius r is rolled over a horizontal surface, without slippage. Due to its tilt θ, it rolls around a circle of radius R – see the figure on the right. Modeling the coin as a very thin round disk, calculate the time period of its motion around the circle.

Solution: Let us solve this problem in a reference frame moving around the circle together with the coin, but not rotating with it about its symmetry axis, so the frame rotates only around the center of the circle with the angular velocity $\omega = 2\pi/\mathcal{T}$, where $\mathcal{T}$ is the period we are looking for. In this non-inertial reference frame, the angular momentum $\mathbf{L}$ of the coin does not change in time, so Eq. (4.65) of the lecture notes takes the simple form

$$\boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\tau}, \qquad (*)$$

where $\boldsymbol{\tau}$ is the net torque of the forces exerted on it, about its center of mass C.



In order to spell out this equation, it is convenient to select the orientation of the axes as shown in the figure on the right, where the $\mathbf{n}_2$-axis is directed out of the plane of the drawing, toward the viewer. With this choice, all these axes are principal, ensuring a simple relation between the corresponding Cartesian components of the angular velocity and the angular momentum – see Eq. (4.38).

In our current case, the angular velocity of the coin is the sum of the vertical vector $\boldsymbol{\omega}$ and the vector $\boldsymbol{\Omega}$ of its rotation about the symmetry axis:

$$\boldsymbol{\omega} = \omega\mathbf{n}_z = \omega(\cos\theta\,\mathbf{n}_1 + \sin\theta\,\mathbf{n}_3), \qquad \boldsymbol{\Omega} = -\Omega\mathbf{n}_z,$$

so the coin's angular momentum is

$$\mathbf{L} = I_1\omega\cos\theta\,\mathbf{n}_1 + I_3(\omega\sin\theta - \Omega)\mathbf{n}_3,$$

[93] In the figure above, the change of the rotation's direction alters the direction of vector $\mathbf{L}$ but not of $\dot{\mathbf{L}} = \boldsymbol{\tau}$.

where I_1 and I_3 are the corresponding principal moments of inertia. Since these two angular velocities are related by the no-slipping condition $R\omega = r\Omega$, the expression for **L** may be rewritten as

$$\mathbf{L} = I_1\omega\cos\theta\,\mathbf{n}_1 + I_3\omega\left(\sin\theta - \frac{R}{r}\right)\mathbf{n}_3,$$

so the vector product participating in Eq. (*) has only one (horizontal) component:

$$\boldsymbol{\omega}\times\mathbf{L} = \begin{vmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ \omega\cos\theta & 0 & \omega\sin\theta \\ I_1\omega\cos\theta & 0 & I_3\omega(\sin\theta - R/r) \end{vmatrix} = \omega^2\cos\theta\left[I_1\sin\theta - I_3\left(\sin\theta - \frac{R}{r}\right)\right]\mathbf{n}_2.$$

What remains is the spell out the right-hand side of Eq. (*), i.e. the net torque exerted on the coin about its center of mass C. There is only one force giving a nonvanishing contribution to such torque, the surface's reaction applied to the contact point A, at distance r from point C. Because of the absence of vertical acceleration in our situation, the vertical component **N** of the force has to be equal to $-M\mathbf{g}$, while its horizontal component, the friction force $\mathbf{F}_\text{f}$, has to provide the proper centripetal acceleration of the coin's c.o.m. toward the circle's symmetry axis z:

$$F_\text{f} = Ma = M\omega^2(R - r\sin\theta).$$

As a result, with our choice of coordinate axes, the torque also has only one Cartesian component:

$$\boldsymbol{\tau} = [Nr\sin\theta - F_\text{f}r\cos\theta]\mathbf{n}_2 \equiv [Mgr\sin\theta - \omega^2(R - r\sin)r\cos\theta]\mathbf{n}_2,$$

so the vector equation (*) has only one nontrivial scalar component:

$$\omega^2\cos\theta[I_1\sin\theta - I_3(\sin\theta - R/r)] = Mgr\sin\theta - \omega^2(R - r\sin)r\cos\theta,$$

immediately giving us the final answer:

$$\mathcal{T} = \frac{2\pi}{\omega} = 2\pi\left\{\frac{\cos\theta}{g\sin\theta}\left[R\left(1 + \frac{I_3}{Mr^2}\right) - r\left(1 + \frac{I_3}{Mr^2} - \frac{I_1}{Mr^2}\right)\sin\theta\right]\right\}^{1/2}. \qquad (**)$$

As we know from the solution of Problem 1(ii), if the disk is uniform (which is a very reasonable model for a usual coin), then $I_1/Mr^2 = ¼$ and $I_3/Mr^2 = ½$, so the expression inside the square brackets is

$$R\left(1 + \frac{1}{2}\right) - r\left(1 + \frac{1}{2} - \frac{1}{4}\right)\sin\theta \equiv \frac{3}{2}R - \frac{5}{4}r\sin\theta \equiv \frac{1}{4}(6R - 5r\sin\theta),$$

but Eq. (**) is valid for a thin disk with any radial (i.e. axially symmetric) mass distribution. For example, for a very thin hoop (with $I_1/Mr^2 = ½$, $I_3/Mr^2 = 1$), the expression inside the same square brackets becomes

$$R(1+1) - r\left(1 + 1 - \frac{1}{2}\right)\sin\theta \equiv 2R - \frac{3}{2}r\sin\theta \equiv \frac{1}{2}(4R - 3r\sin\theta),$$

while for a very light wheel with a heavy hub (with I_1/Mr^2, $I_3/Mr^2 \ll 1$) it is simply

$$R - r\sin\theta.$$

In the limit of a very small tilt, $\theta \to 0$, Eq. (**) reduces to a simple formula,

$$\mathcal{T} \approx 2\pi\left[\frac{R\left(1+I_3/Mr^2\right)}{g\theta}\right]^{1/2} \to \infty,$$

which may be readily derived using other methods, for example, the notion of the Coriolis "force" (see, e.g., Sec. 4.6 of the lecture notes) – the additional task highly recommended to the reader.

In the opposite limit, $\theta \to \pi/2$, the predictions given by Eq. (**) should be treated with caution because to be valid, they may require an unrealistically high static friction coefficient μ, to avoid disk slipping. Note also that if the expression in the square brackets of that formula becomes negative, it gives a negative result for ω^2, indicating that the disk's roll motion becomes unstable.

Problem 4.24. Solve the previous problem in the limit when the coin tilt angle θ and the ratio r/R are small, by simpler means, using

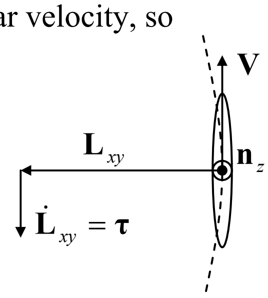
(i) an inertial ("lab") reference frame, and

(ii) the non-inertial reference frame moving with the coin's center but not rotating with it.

Solutions:

(i) Rolling along a circle of radius R with velocity V gives the coin's center of mass the angular velocity $\boldsymbol{\omega}$, directed vertically, with magnitude $\omega = V/R$. As a result, the horizontal component $\mathbf{L}_{xy}$ of the coin's angular momentum has to rotate, in the horizontal plane, with the same angular velocity, so

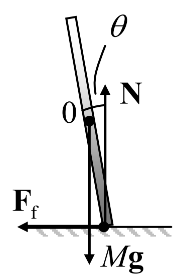
$$\dot{\mathbf{L}}_{xy} = \omega \mathbf{n}_z \times \mathbf{L}_{xy},$$



where $\mathbf{n}_z$ is the unit vertical vector – see, e.g., the figure on the right showing the view of the coin from the top. At $r << R$ and $\theta << 1$, $|L_{xy}|$ may be approximated as $I\Omega$, where $\Omega = V/r$ is the angular velocity of the coin's rotation about his symmetry axis, and I is the corresponding moment of inertia. Per the solution of Problem 1 (ii), for a uniform disk, $I = Mr^2/2$, where M is the coin's mass. With these expressions, the above kinematic relation becomes

$$\left|\dot{\mathbf{L}}_{xy}\right| = \omega I\Omega = \frac{V}{R}\frac{Mr^2}{2}\frac{V}{r} \equiv \frac{MV^2 r}{2R}, \qquad (*)$$

with the direction of that vector shown in the figure above (for our direction of the coin's tilt).



The torque $\boldsymbol{\tau}$ necessary for this time evolution of the vector $\mathbf{L}_{xy}$ is provided by the external forces acting on the coin (see the figure on the right): the vertical gravity force $M\mathbf{g}$, the equal and opposite reaction $\mathbf{N} = -M\mathbf{g}$ of the surface, and the horizontal static friction force $\mathbf{F}_f$ directed to the circle's center and providing the centripetal acceleration $a_{cp} = V^2/R$ of the coin, so $F_f = Ma_{cp} = MV^2/R$. Relatively to the center of mass of the coin (point 0), the torque of the first force vanishes, while the torques created by the two remaining forces evidently have opposite directions, so the net torque $\boldsymbol{\tau}$ has the following magnitude:

$$\tau = Nr\sin\theta - F_f r\cos\theta \approx Mgr\theta - \frac{MV^2 r}{R}. \qquad (**)$$

Now requiring, in accordance with the basic Eq. (1.38) of the lecture notes, the expressions (*) and (**) to be equal, we get[94]

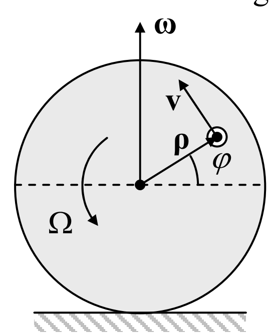
$$\theta = \frac{3}{2}\frac{V^2}{Rg}. \qquad (***)$$

Note that θ does not depend on the coin's mass M, and (more counter-intuitively) on its radius r.[95]

(ii) In the reference frame moving with the coin's center-of-mass but not rotating about its axis, the coin rotates with an angular velocity $\mathbf{\Omega}$ of magnitude $\Omega = V/r$, but since the magnitude and the direction of this velocity are constant in this frame, the net torque of all forces has to equal zero. However, since this reference frame is not inertial, this net torque $\boldsymbol{\tau}_\Sigma = 0$ has to include contributions of the inertial "forces" given by Eq. (4.92) of the lecture notes, exerted on each elementary mass dm of the coin.

The first of them is $-dm\,\mathbf{a}_{0|\text{in lab}}$; for our reference frame, $\mathbf{a}_{0|\text{in lab}}$ is just the centripetal acceleration of the coin's center of mass, directed toward the center of the circle the coin rolls over, of magnitude V^2/R. Since at $r << R$, such forces have virtually the same direction for each elementary mass dm of the coin, they do not produce any net momentum about its center of mass. The second component, which describes the centrifugal force, is $-dm\,\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$, where $\boldsymbol{\omega}$ is the reference frame's rotation vector (directed vertically), of magnitude $\omega = V/R$, and $\mathbf{r}$ is the elementary mass' radius vector – as observed from the moving frame. For the thin-disk model of the coin, the vector $\mathbf{r}$ virtually coincides with the 2D radius vector $\boldsymbol{\rho}$ – see the figure on the right, whose plane coincides with the plane of the coin. The vectors of these elementary "forces" also lie in the plane of the coin (and are directed from the axis $\boldsymbol{\omega}$), and due to the coin's symmetry also do not produce any net torque.

However, this is not true for the third term[96] of Eq. (4.92), $d\mathbf{F}_C = -2dm\,\boldsymbol{\omega}\times\mathbf{v}$, describing the Coriolis forces. Here $\mathbf{v}$ is the velocity of the elementary mass dm, as observed from the moving reference frame, so $v = \Omega\rho = V\rho/R$. As the figure on the right shows, all these elementary forces are directed normally to the coin's plane but their magnitudes depend on the positions $\boldsymbol{\rho}$ of the elementary masses:



$$dF_C = -2dm\,\omega v \sin\varphi = -2dm\frac{V}{R}\frac{V\rho}{r}\sin\varphi,$$

so the elementary torques about the horizontal axis passing through the center of mass (shown with the dashed line in the figure just on the right),

$$d\tau_C = dF_C\,\rho\sin\varphi = -2\frac{V^2}{Rr}\rho^2\sin^2\varphi\, dm,$$

do not cancel at the summation over the coil area $A = \pi r^2$, giving the following net Coriolis torque:

$$\tau_C = \int_A d\tau_C = -2\frac{V^2}{Rr}\int_A \rho^2\sin^2\varphi\, dm = -2\frac{V^2}{Rr}\int_A \rho^2\sin^2\varphi\frac{dm}{dA}d^2r = -2\frac{V^2}{Rr}\frac{M}{A}\int_A \rho^2\sin^2\varphi\, d^2r$$

[94] As a reminder, this expression is valid only if its right-hand side is much smaller than 1.

[95] This is true only if $r\sin\theta << R$ – see the solution of Problem 23.

[96] The fourth term of Eq. (4.92) equals zero in this problem because $\boldsymbol{\omega}$ does not change in time. For an example of when this term is essential, see Problem 35.

$$= -2\frac{V^2}{Rr}\frac{M}{\pi r^2}\int_0^{2\pi}\sin^2\varphi\, d\varphi\int_0^r \rho^3 d\rho = -2\frac{V^2}{Rr}\frac{M}{\pi r^2}\pi\frac{r^4}{4} \equiv -\frac{MV^2 r}{2R}.$$

Adding this inertial-force torque to that of real physical forces, given by Eq. (**), and requiring the sum to equal zero, we get the following equation:

$$\tau_\Sigma = \tau + \tau_C = \left(Mgr\theta - \frac{MV^2 r}{R}\right) - \frac{MV^2 r}{2R} = 0,$$

which immediately yields the same result (***) for the coin's tilt.

Let me hope that the reader enjoys, as much as I do, this amazing feat: in the reference frame where the angular frequency of the coin does not change in time, and hence its moment of inertia I should apparently be irrelevant, the Coriolis force finds its way to smuggle the factor $I = Mr^2/2$ into the result!

Problem 4.25. A symmetric top on a point support (as shown see, e.g., Fig. 4.9 of the lecture notes), rotating around its symmetry axis with a high angular velocity ω_{rot}, is subjected to not only its weight $M\mathbf{g}$ but also an additional force also applied to the top's center of mass, with its vector rotating in the horizontal plane with a constant angular velocity $\omega << \omega_{\text{rot}}$. Derive the system of equations describing the top's motion. Analyze their solution for the simplest case when ω is exactly equal to the frequency (4.72) of the torque-induced precession in the gravity field alone.

Solution: According to the assignment, directing the z-axis up (again as in Fig. 4.9), we may represent the external force applied to the top as

$$\mathbf{F}(t) = \text{Re}\left[A(\mathbf{n}_x + i\mathbf{n}_y)e^{-i\omega t}\right] - Mg\mathbf{n}_z.$$

Generally, the magnitude A of the horizontal force may be complex, its argument representing the force's rotation phase. However, we may always select the time origin to make that phase equal to zero, so we may take

$$\mathbf{F}(t) = A\cos\omega t\,\mathbf{n}_x + A\sin\omega t\,\mathbf{n}_y - Mg\mathbf{n}_z.$$

(If $\omega > 0$, the force's vector rotates in the x-y plane counterclockwise, but this formula and all latter calculations are valid for the opposite case as well.)

Assuming the strong inequality $\omega_{\text{pre}}, \omega << \omega_{\text{rot}}$, we may (as was done at the beginning of Sec. 4.5 lecture notes) attribute the angular momentum $\mathbf{L}$ of the top to ω_{rot} alone, ignoring the contribution due to the precession: $\mathbf{L} = I_3\omega_{\text{rot}}\mathbf{n}_3$. As a result, the torque of the force $\mathbf{F}(t)$ (relative to the top's support point) may be expressed via $\mathbf{L}$:

$$\boldsymbol{\tau}(t) = l\mathbf{n}_3 \times \mathbf{F}(t) = l\frac{\mathbf{L}}{I_3\omega_{\text{rot}}} \times \left(A\cos\omega t\,\mathbf{n}_x + A\sin\omega t\,\mathbf{n}_y - Mg\mathbf{n}_z\right).$$

As a result, the basic equation $\dot{\mathbf{L}} = \boldsymbol{\tau}$ of the angular dynamics (in the inertial reference frame) gives the following self-consistent differential equation for $\mathbf{L}$:

$$\dot{\mathbf{L}} = l\frac{\mathbf{L}}{I_3\omega_{\text{rot}}}\times\left(A\cos\omega t\,\mathbf{n}_x + A\sin\omega t\,\mathbf{n}_y - Mg\mathbf{n}_z\right) = \omega_{\text{pre}}\begin{vmatrix}\mathbf{n}_x & \mathbf{n}_y & \mathbf{n}_z\\ L_x & L_y & L_z\\ a\cos\omega t & a\sin\omega t & -1\end{vmatrix}$$
$$= \omega_{\text{pre}}\left[\mathbf{n}_x\left(-L_z a\sin\omega t - L_y\right) + \mathbf{n}_y\left(L_z a\cos\omega t + L_x\right) + \mathbf{n}_z\left(L_x a\sin\omega t - L_y a\cos\omega t\right)\right]$$

where $\omega_{\text{pre}} \equiv Mgl/I_3\omega_{\text{rot}}$ and $a \equiv A/Mg$. This is the system of three scalar equations:

$$\dot{L}_x = \omega_{\text{pre}}\left(-L_z a\sin\omega t - L_y\right), \quad \dot{L}_y = \omega_{\text{pre}}\left(L_z a\cos\omega t + L_x\right), \quad \dot{L}_z = \omega_{\text{pre}}\left(L_x a\sin\omega t - L_y a\cos\omega t\right). \qquad (*)$$

As a sanity check, in the absence of the rotating force ($a = 0$), these equations are reduced to

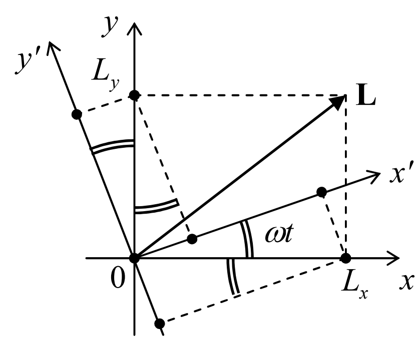
$$\dot{L}_x = -\omega_{\text{pre}}L_y, \quad \dot{L}_y = \omega_{\text{pre}}L_x, \quad \dot{L}_z = 0, \qquad (**)$$

and are easy to solve:

$$L_x = L\sin\theta\cos(\omega_{\text{pre}}t + \varphi), \quad L_y = L\sin\theta\sin(\omega_{\text{pre}}t + \varphi), \quad L_z = L\cos\theta = \text{const}, \qquad (***)$$

where the real constants θ and φ are determined by initial conditions: L, θ, and φ are just the spherical coordinates of the vector $\mathbf{L}$ at $t = 0$. This solution describes the torque-induced precession of the vector $\mathbf{L}$ (and hence of the top's symmetry axis $\mathbf{n}_3$) around the z-axis with the constant frequency ω_{pre} – see Eq. (4.72) of the lecture notes. (If ω_{rot} and hence ω_{pre} are positive, the horizontal component $\mathbf{L}_{x,y}$ of the vector $\mathbf{L}$ rotates counterclockwise, as shown in Fig. 4.9 of the lecture notes.)

However, at $a \neq 0$, Eqs. (*) are inconvenient for analysis because of the oscillating terms on their right-hand sides,. In this situation, the solution (***) gives us a hint of how the equations may be simplified: let us introduce the coordinate system $\{x', y', z'\}$ rotating about the common axis $z' = z$ together with the applied force – see the figure on the right. As the figure shows, in this rotating frame, the vector $\mathbf{L}$ has the following components:



$$L'_x = L_x\cos\omega t + L_y\sin\omega t, \quad L'_y = -L_x\sin\omega t + L_y\cos\omega t, \quad L'_z = L_z.$$

Differentiating these relations over t, then plugging Eqs. (*) into the right-hand-sides of the results, and finally the same relations again, we get

$$\dot{L}'_x = \dot{L}_x\cos\omega t + \dot{L}_y\sin\omega t + \omega\left(-L_x\sin\omega t + L_y\cos\omega t\right)$$
$$= \omega_{\text{pre}}\left(-L_z a\sin\omega t - L_y\right)\cos\omega t + \omega_{\text{pre}}\left(L_z a\cos\omega t + L_x\right)\sin\omega t + \omega L'_y \equiv \left(\omega - \omega_{\text{pre}}\right)L'_y,$$
$$\dot{L}'_y = -\dot{L}_x\sin\omega t + \dot{L}_y\cos\omega t - \omega\left(L_x\cos\omega t + L_y\sin\omega t\right)$$
$$= -\omega_{\text{pre}}\left(-L_z a\sin\omega t - L_y\right)\sin\omega t + \omega_{\text{pre}}\left(L_z a\cos\omega t + L_x\right)\cos\omega t - \omega L'_x \equiv -\left(\omega - \omega_{\text{pre}}\right)L'_x + \omega_{\text{pre}}aL'_z,$$
$$\dot{L}'_z = \dot{L}_z = \omega_{\text{pre}}\left(L_x a\sin\omega t - L_y a\cos\omega t\right) \equiv -\omega_{\text{pre}}aL'_y.$$

The right-hand sides of these equations do not depend on time explicitly, facilitating their solutions – especially in the resonance case when the external force's frequency ω is very close to ω_{pre}. As the equations show, in this case, the rotating field with even a small amplitude,

$$a \sim \left| \frac{\omega - \omega_{\text{pre}}}{\omega_{\text{pre}}} \right| << 1,$$

may significantly change the top's dynamics. In the simplest case of an exact resonance, $\omega_{\text{pre}} = \omega$, these equations become very simple,

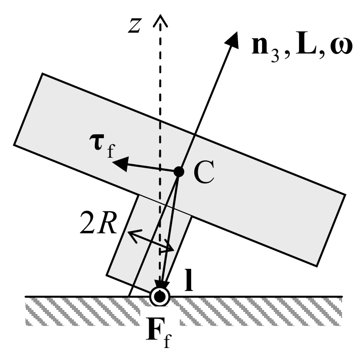
$$\dot{L}'_x = 0, \qquad \dot{L}'_y = \omega a L'_z, \qquad L'_z = -\omega a L'_y, \qquad (****)$$

and completely similar to Eqs. (**) of the "usual" torque-induced precession – besides the axes replacement (now the precession is around the x-axis), and a different frequency: $\omega_{\text{pre}} \rightarrow a\omega = a\omega_{\text{pre}} << \omega_{\text{rot}}$. Hence, for the observer rotating with the field, the angular momentum vector and hence the top's symmetry axis perform a slow rotation in the [y', z'] plane. Naturally, for the observer in the inertial ("lab") frame, this rotation takes place within the plane rotating around the z-axis with frequency ω_{pre} – which also has to be much lower than ω_{rot} for this theory to be quantitatively valid.

This effect may be used for angular momentum manipulation. For example, as Eqs. (****) show, if at $t = 0$, the vector **L** was directed vertically, along the z-axis, a weak ac force's pulse of duration $\mathcal{T} = \pi/2\omega a$ (the so-called $\pi/2$-*pulse*) turns the vector slowly to align with the rotating y'-axis, thus maximizing the amplitude of its much faster precession in the gravity field. Moreover, increasing the pulse length twice, to become the π-*pulse* with $\mathcal{T} = \pi/\omega a$, we may (the top's support arrangement permitting) orient it vertically again, but in the opposite direction.

In mechanical systems, the rotating force assumed in the assignment may be implemented, for example, by rotating the symmetric top's support point with angular velocity ω in the horizontal plane. However, the main motivation for this problem is that according to quantum mechanics (see, e.g., QM Sec. 5.1), the dynamics of the average magnetization **M** of a system of many similar non-interacting spins in an external magnetic field is quantitatively similar to the analyzed dynamics of **L** of a fast-rotating classical top.[97] In particular, the high sensitivity of the nuclear spin magnetization to a small external ac magnetic field of frequency $\omega \approx \omega_{\text{pre}}$, and hence the possibility of its manipulation, is called *magnetic resonance*. This effect is broadly used in chemistry, biology, and medicine – in *magnetic resonance imaging* (MRI) systems. In this case, the rotating magnetic field may be readily induced by two phase-shifted ac currents passed through magnetic coils with their symmetry axes perpendicular to each other and to the direction of the basic, high dc magnetic field. (Technically, the same coils are often used for the magnetization precession signal's pick-up.)

<u>Problem 4.26.</u> Analyze the effect of small friction on a fast rotation of a symmetric top around its axis, using a simple model in that the lower end of the body is a right cylinder of radius R.



Solution: As was discussed in Sec. 4.5 of the lecture notes, the term "fast rotation" means that both vectors **ω** and **L** are virtually aligned with the top's symmetry axis $\mathbf{n}_3$, so we may take $\mathbf{L} = I_3\boldsymbol{\omega}$. Generally, this axis is tilted relative to the vertical z-axis – see the figure on the right. Then the vertical support force $\mathbf{N} = Mg\mathbf{n}_z$ that counter-balances the top's weight $M\mathbf{g} = -Mg\mathbf{n}_z$, applied to its center of mass C, results in the torque $\boldsymbol{\tau}_g = \mathbf{l}\times\mathbf{N}$

[97] See, e.g., QM Sec. 5.1 and Problem 7.13.

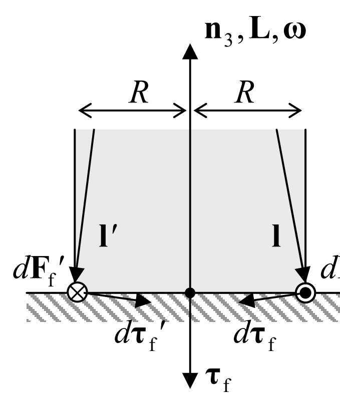
directed normally to the vector **N** and also to the vector **l** connecting point C with the top-to-support contact point. (In the figure on the right, the vector $\boldsymbol{\tau}_g$ is normal to the plane of the drawing, and directed toward the viewer.) This vector is, at each moment, perpendicular to the angular momentum's vector **L**. Hence, according to the basic equation of rotational dynamics, $\dot{\mathbf{L}} = \boldsymbol{\tau}$, it cannot change the **L**'s magnitude but causes its rotation about the vertical axis z, i.e. the torque-induced precession that was discussed in Sec. 4.5 of the lecture notes. Note that in our current case, due to a nonvanishing radius R of the body's lower end, the frequency of this precession is somewhat different from the one given by Eq. (4.72), making it, in particular, dependent on the tilt angle. (If R is small, this difference is small as well.)

The same nonzero radius $R \neq 0$ causes the lowest point of the top to keep slipping on the surface with the linear velocity $v = \omega R$, resulting in a friction force $\mathbf{F}_\text{f}$ directed oppositely to the velocity vector **v**. In the case shown in the figure above, this force is normal to the plane of drawing and directed toward the viewer. This force causes an additional torque $\boldsymbol{\tau}_\text{f} = \mathbf{l}\times\mathbf{F}_\text{f}$ residing in the common plane of the vectors **L** and $\mathbf{n}_z$, and directed from the former to the latter. If the force is small, it does not affect the top's precession directly but causes a decrease of its amplitude with time. In the simplest Coulomb model of the frictional force, at $v \neq 0$, its magnitude $F_\text{f} = \mu N = \mu Mg$ does not depend on v and thence on ω and L, so the magnitude of τ_f is also independent of L. Hence, at small tilt angles, the magnitude of the **L**'s horizontal component, $\mathbf{L}_{xy}$ (see Fig. 49b of the lecture notes) decreases at a nearly constant rate:

$$\frac{d}{dt}L_{xy} \approx -\tau_\text{f} \approx -\mu Mgl\,,$$

so the top straightens up to the vertical position of its symmetry axis[98] during a finite time interval

$$\Delta t \sim \frac{L}{\tau_\text{f}} \sim \frac{I_3\omega}{\mu Mgl}\,.$$



(Getting an exact formula for Δt is not difficult if the position of the top's center of mass C is given.) Comparing this expression with Eq. (4.72), we see that $\omega_\text{pre}\Delta t \sim 1/\mu$, i.e. the visible torque-induced precession may last for many periods only if the friction coefficient is low: $\mu \ll 1$.

As soon the top has reached its vertical position, other points of the body's bottom come into contact with the surface, and the friction forces exerted on its parts become axially symmetric – see the figure on the right, which shows just the lower part of the top, and only two elementary friction forces and their torques. Due to the nonvanishing deviation of the vectors **l** from the vertical direction, each of such torques $d\boldsymbol{\tau}_\text{f} = \mathbf{l}\times d\mathbf{F}_\text{f}$ now has a downward vertical component, so the net torque vector $\boldsymbol{\tau}_\text{f}$ is directed down, i.e. opposite to **L**. As a result, friction leads to a gradual decrease of L and hence of the rotation frequency ω. When ω drops to the threshold value (4.85), the top loses its rotation-enabled stability, so if R is very small, it may fall on a side because of some unavoidable minor asymmetry.

As was mentioned at the end of Sec. 4.5 of the lecture notes, this time hierarchy of the friction effects on a top (its straightening up followed by a rotation slowdown and then by the fall) is common

[98] Note that this straightening up increases the top's potential energy in the gravity field, at the cost of some reduction of the (much higher) kinetic energy of its rotation.

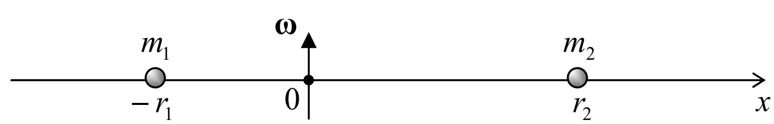
for any symmetric top regardless of the exact shape of its lower end, provided that this tip is small enough.

Problem 4.27. An air-filled balloon is placed inside a water-filled container, which moves by inertia in free space, at negligible gravity. Suddenly, force **F** is applied to the container, pointing in a certain direction. What direction does the balloon move relative to the container?

Solution: The force **F** causes the container's acceleration **a** in the direction of the force. In order to use the 2nd Newton law in the non-inertial reference frame bound to the container, we should add the inertial "force" $-m_j\mathbf{a}$ to all container's components including the water and the air balloon, to the real physical forces acting between components of the system. (The situation is completely equivalent to placing the system into a uniform gravity field $\mathbf{g} = -\mathbf{a}$.) In particular, the inertial forces acting on the air balloon are very small because of low air density, while those applied to the surrounding water create a substantial pressure gradient in the direction opposite to **a** (and hence opposite to **F**). In turn, this pressure gradient would result in the oppositely directed buoyant force. As a result, the balloon starts moving (relatively to the container) in the direction of the force **F** – until it reaches one of its walls. This means that relative to an inertial system, the balloon accelerates even somewhat faster than the container as a whole.

Problem 4.28. Two planets are in a circular orbit around their common center of mass. Calculate the effective potential energy of a much lighter body (say, a spacecraft) rotating with the same angular velocity, on the line connecting the planets. Sketch the radial dependence of U_{ef} and find out the number of so-called *Lagrange points* in which the potential energy has local maxima. Calculate their position explicitly in the limit when one of the planets is much more massive than the other one.

Solution: Let us consider the small body's motion in the non-inertial reference frame rotating with the planets, with the origin in their center of mass – see the figure below. (Note that with this choice, the body's Cartesian coordinate x on the line connecting the planets may be either positive or negative.)

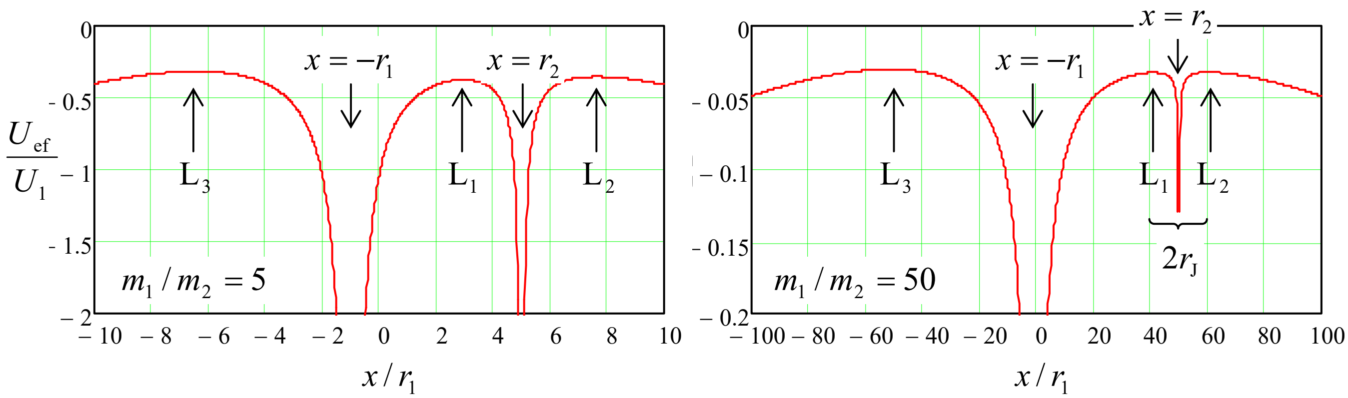


Since in this problem, the angular velocity $\boldsymbol{\omega}$ is fixed, the appropriate effective potential energy is given by Eq. (4.96b) of the lecture notes, with the genuine potential energy being the sum of the gravitation potentials of two planets. Since in our case, the vectors $\boldsymbol{\omega}$ and $\mathbf{r} = \mathbf{n}_x x$ are mutually perpendicular, we get

$$U_{ef} = U - \frac{m}{2}\omega^2 x^2 = -\frac{Gm_1 m}{|x+r_1|} - \frac{Gm_2 m}{|x-r_2|} - \frac{m}{2}\omega^2 x^2 .$$

The figure below shows the function $U_{ef}(x)$, in the units of $U_1 \equiv Gmm_1/r_1$, plotted for two values of the m_2/m_1 ratio, taking into account that its three parameters r_1, r_2, and ω are not independent but are bound by two relations, for example, by the 2nd Newton law spelled out for the circular motion of each planet:[99]

$$m_1 r_1 \omega^2 = G\frac{m_1 m_2}{(r_1 + r_2)^2}, \qquad m_2 r_2 \omega^2 = G\frac{m_1 m_2}{(r_1 + r_2)^2}.$$



As the plots show, the function $U_{ef}(x)$ has maxima in three Lagrange points[100] corresponding to the radial equilibrium of the small mass. If the planet masses $m_{1,2}$ are comparable, the points' distances from the origin are of the order of the inter-planet distance $(r_1 + r_2)$. However, if the masses are very different (as they are, for example, for the Sun-Earth and Earth-Moon systems), one pair of Lagrange points, traditionally called L_1 and L_2, are close to the lighter planet and are on virtually the same distance r_J (called the *Jacobi radius*) from it. In such a limit, say if $m_1 >> m_2$, the radius may be calculated from the above three relations by a straightforward differentiation of the function $U_{ef}(x) \equiv U_{ef}(r_2 \pm r_J)$ over r_J, neglecting r_1 in comparison with the much larger distance $r_2 \approx (m_1/m_2)r_1$:

$$\frac{1}{Gm}\frac{dU_{ef}(r_2 \pm r_J)}{dr_J} = \frac{d}{dr_J}\left[-\frac{m_1}{r_1 + r_2 \pm r_J} - \frac{m_2}{r_J} - \frac{\omega^2}{2G}(r_2 \pm r_J)^2\right] \approx \frac{d}{dr_J}\left[-\frac{m_1}{r_2 \pm r_J} - \frac{m_2}{r_J} - \frac{m_1}{2r_2^3}(r_2 \pm r_J)^2\right]$$

$$= \pm\frac{m_1}{(r_2 \pm r_J)^2} + \frac{m_2}{r_J^2} \mp \frac{m_1}{r_2^3}(r_2 \pm r_J) \approx \pm\frac{m_1}{r_2^2}\left(1 \mp 2\frac{r_J}{r_2}\right) + \frac{m_2}{r_J^2} \mp \frac{m_1}{r_2^2}\left(1 \pm \frac{r_J}{r_2}\right) \equiv -3\frac{m_1}{r_2^3}r_J + \frac{m_2}{r_J^2},$$

and then requiring this derivative to equal zero. The result is

$$r_J \approx \left(\frac{m_2}{3m_1}\right)^{1/3} r_2 = \left(\frac{m_1^2}{3m_2^2}\right)^{1/3} r_1, \qquad \text{for } m_2 << m_1,$$

so $r_1 << r_J << r_2$ – see, e.g., the right panel in the figure above.

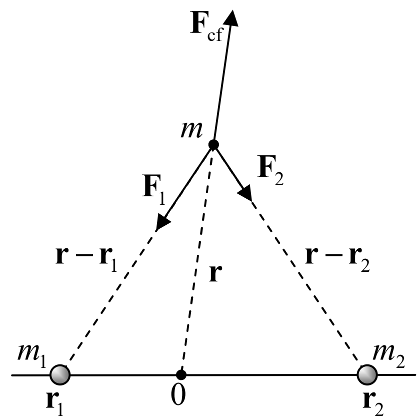
The Lagrange points are frequently used in space research practice, in particular because placing a spacecraft near one of these points enables fuel savings while making multiple observations from virtually the same position. The first example of a space mission using such a point (the L_1 of the Sun-

[99] Multiplying and dividing these two relations, we may represent these two formulas in the equivalent form of two other relations: first, the condition of the center-of-mass being in the origin: $m_1 r_1 = m_2 r_2$, and second, $\omega^2 = G(m_1 + m_2)/(r_1 + r_2)^3$ – essentially the 3rd Kepler law (3.64).

[100] Actually, these three points had been first calculated by L. Euler, a few years before J.-L. Lagrange carried out a general analysis of the problem, finding one more pair of such equilibrium points (L_4 and L_5), also in the plane of planets' motion, but in contrast with L_1-L_3, off the line connecting them – see the next problem.

Earth system) was the International Sun-Earth Explorer 3 (ISEE-3) launched in 1978. Starting from 2010, the point L_2 of the Sun-Earth system, about 1.5×10^6 km from the Earth, has become even more popular, because it is located in our planet's shade. Its recent but already famous tenant is the James Webb Space Telescope launched in 2021. Another example is the L_2 point of the Earth-Moon system, some 65,000 km behind the Moon, used since 2018 by the Chinese satellite Queqiao to relay communications between the Chang-4 robot on the Moon's back side and its Earth-based controllers.

Problem 4.29. Besides the three Lagrange points L_1, L_2, and L_3 discussed in the previous problem, which are located on the line connecting two planets on circular orbits about their mutual center of mass, there are two off-line points L_4 and L_5 – both within the plane of the planets' rotation. Calculate their positions.



Solution: The figure on the right shows the forces and radius vectors of interest on the planets' rotation plane. (The radius vectors are referred to the center-of-mass 0 of the system.) Here $m \ll m_1, m_2$ is a very small mass of a "probe" body placed into one of the Lagrange points, and $\mathbf{r}$ is its radius vector. Using this notation, we may write the following condition of equilibrium of the probe body as the equality to zero of the net force exerted on it, in the non-inertial reference frame in that the planets are at rest:

$$Gmm_1 \frac{\mathbf{r}_1 - \mathbf{r}}{\left|\mathbf{r}_1 - \mathbf{r}\right|^3} + Gmm_2 \frac{\mathbf{r}_2 - \mathbf{r}}{\left|\mathbf{r}_2 - \mathbf{r}\right|^3} + m\omega^2 \mathbf{r} = 0\,. \qquad (*)$$

Here the first two terms represent the real gravity forces $\mathbf{F}_1$ and $\mathbf{F}_2$, while the last term describes the centrifugal "force" $\mathbf{F}_{cf}$ (4.93) directed from the rotation axis (normal to the plane of our drawing) and depending on the angular frequency ω of the planet rotation, that was calculated in the previous problem; in our current (vector) notation:

$$\omega^2 = G\frac{m_1 + m_2}{\left|\mathbf{r}_1 - \mathbf{r}_2\right|^3}\,. \qquad (**)$$

Plugging Eq. (**) into Eq. (*), carrying over all terms with $\mathbf{r}$ in their numerators to its right-hand side, and then canceling the products Gm in the resulting equation, we get

$$m_1 \frac{\mathbf{r}_1}{\left|\mathbf{r}_1 - \mathbf{r}\right|^3} + m_2 \frac{\mathbf{r}_2}{\left|\mathbf{r}_2 - \mathbf{r}\right|^3} = \mathbf{r}\left[\frac{m_1}{\left|\mathbf{r}_1 - \mathbf{r}\right|^3} + \frac{m_2}{\left|\mathbf{r}_2 - \mathbf{r}\right|^3} - \frac{m_1 + m_2}{\left|\mathbf{r}_1 - \mathbf{r}_2\right|^3}\right].$$

Here comes a simple but very powerful argument. Both vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, and hence their linear combination on the left-hand side of this equation, are directed along the line connecting the planets (see the figure above), while the vector in its right-hand part is directed along $\mathbf{r}$. Hence there are only two ways to satisfy this equation for a vector $\mathbf{r} \neq 0$: either to have it directed along the same line (this opportunity is used by the Lagrange points L_1, L_2, and L_3 calculated in the previous problem) or to have both sides of the equation equal to zero. So, for the off-line Lagrange points L_4 and L_5, we need to have

$$m_1 \frac{\mathbf{r}_1}{\left|\mathbf{r}_1 - \mathbf{r}\right|^3} + m_2 \frac{\mathbf{r}_2}{\left|\mathbf{r}_2 - \mathbf{r}\right|^3} = 0, \qquad \frac{m_1}{\left|\mathbf{r}_1 - \mathbf{r}\right|^3} + \frac{m_2}{\left|\mathbf{r}_2 - \mathbf{r}\right|^3} - \frac{m_1 + m_2}{\left|\mathbf{r}_1 - \mathbf{r}_2\right|^3} = 0. \qquad (***)$$

The first of these equations is compatible with the c.o.m.'s definition $m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0$ only if the denominators in its terms are equal, i.e. if

$$\left|\mathbf{r}_1 - \mathbf{r}\right| = \left|\mathbf{r}_2 - \mathbf{r}\right|,$$

i.e. if the Lagrange point is at the same distance from each planet – see the figure above again. On our plane, there are evidently two such points, called L_4 and L_5 – one on each side of the line connecting the planets. The distance of these points from the line may be calculated by plugging the last result into the second of Eqs. (***). Denoting these equal distances as R, we get a very simple equation

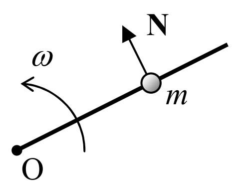
$$\frac{m_1}{R^3} + \frac{m_2}{R^3} - \frac{m_1 + m_2}{\left|\mathbf{r}_1 - \mathbf{r}_2\right|^3} = 0, \qquad \text{i.e.} \quad \frac{m_1 + m_2}{R^3} = \frac{m_1 + m_2}{\left|\mathbf{r}_1 - \mathbf{r}_2\right|^3},$$

with the evident solution

$$R = \left|\mathbf{r}_1 - \mathbf{r}_2\right|.$$

So, rather surprisingly, the result is extremely simple: each of the off-line Lagrange points L_4 and L_5 forms, together with the two planets (more exactly, with their individual centers-of-mass), an equilateral triangle, with all inner angles equal to $\pi/3 = 60°$.

Note that the above result is exactly valid for an arbitrary m_1/m_2 ratio; if the ratio is either very small or very large (as typical for star-planet systems), it may be recast into an even simpler form: both Lagrange points of a planet are virtually on the same orbit around the star as the planet itself, one (traditionally called L_4, or *Greek camp*) moving at the angle $\pi/3$ before it, and its counterpart (L_5, or *Trojan camp*) trailing the planet at a similar angle.[101]



Problem 4.30. The following simple problem may give additional clarity to the physics of the Coriolis "force". A bead of mass m may slide, without friction, along a straight rod that is rotated within a horizontal plane with a constant angular velocity ω – see the figure on the right. Calculate the bead's acceleration and the force **N** exerted on it by the rod, in:

(i) an inertial ("lab") reference frame, and

(ii) the non-inertial reference frame rotating with the rod (but not moving with the bead),

and compare the results.

Solutions:

[101] These names stem from the fact that if the planet mass ratio is larger than ~24.96, the motion of a small body on a so-called *halo orbit* around the points L_4 and L_5 is stable. As a result, small celestial objects, such as dust particles and asteroids, gradually accumulate in the vicinities of these two points, forming two opposite "camps", reminiscent of Greeks and Trojans in Ancient Greek mythology. The camps of Jupiter (the largest planet of our solar system) are especially populous, with more than a million objects larger than one kilometer in size in each of them, while only two Earth's "trojans" of such size have been found so far.

(i) Let us use Cartesian coordinates $\{x, y\}$ with the origin at the rotation center O, so with the proper choice of the time origin, the bead's position at moment t is

$$x(t) = r(t)\cos\omega t, \qquad y(t) = r(t)\sin\omega t,$$

where r is its distance from point O. Differentiating these relations over time twice, we get

$$\ddot{x} = \ddot{r}\cos\omega t - 2\dot{r}\omega\sin\omega t - r\omega^2\cos\omega t, \qquad \ddot{y} = \ddot{r}\sin\omega t + 2\dot{r}\omega\cos\omega t - r\omega^2\sin\omega t. \qquad (*)$$

In the absence of friction, the force **N** exerted on the bead by the rod has to be normal to it, so its Cartesian components may be expressed as

$$N_x = -N\sin\omega t, \qquad N_y = N\cos\omega t.$$

Hence the Cartesian components of the 2nd Newton's law are [102]

$$m\left(\ddot{r}\cos\omega t - 2\dot{r}\omega\sin\omega t - r\omega^2\cos\omega t\right) = -N\sin\omega t,$$
$$m\left(\ddot{r}\sin\omega t + 2\dot{r}\omega\cos\omega t - r\omega^2\sin\omega t\right) = N\cos\omega t.$$

Rewriting these equations as

$$m\left(\ddot{r} - r\omega^2\right)\cos\omega t = -\left(N - 2m\dot{r}\omega\right)\sin\omega t,$$
$$m\left(\ddot{r} - r\omega^2\right)\sin\omega t = \left(N - 2m\dot{r}\omega\right)\cos\omega t,$$

we see that they are compatible only if both terms in the parentheses equal zero. Hence we get the requested results:

$$\ddot{r} = r\omega^2, \qquad N = 2m\dot{r}\omega. \qquad (**)$$

(If needed, the first of these equations may be readily integrated, showing that both r and $\dot{r}$, and hence the force N as well, may be represented as sums of two terms proportional to $\exp\{\pm\omega t\}$, with pre-exponential coefficients determined by initial conditions.)

(ii) Directing one axis (say, X) of the rotating reference frame along the rod, we immediately get $Y(t) = 0$, so we need to write the equation of motion only for the coordinate X – which, at the appropriate choice of its origin, coincides with the distance r. However, according to Eq. (4.92) of the lecture notes, writing the 2nd Newton's law in this non-inertial frame, we have to add, to the real physical force **N** exerted on the bead, two fictitious "inertial forces": the centrifugal force (4.93) and the Coriolis force (4.94):

$$m\mathbf{a} = \mathbf{N} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - 2m\boldsymbol{\omega}\times\mathbf{v}. \qquad (***)$$

Taking into account that at our choice of coordinates,

$$\boldsymbol{\omega} = \mathbf{n}_Z\omega, \qquad \mathbf{r} = \mathbf{n}_X r, \qquad \mathbf{v} = \mathbf{n}_X\dot{r}, \qquad \mathbf{a} = \mathbf{n}_X\ddot{r}, \qquad \mathbf{N} = N\mathbf{n}_Y,$$

and performing the vector multiplications on the right-hand side of Eq. (***), we get

$$m\ddot{r}\mathbf{n}_X = N\mathbf{n}_Y + m\omega^2 r\mathbf{n}_X - 2m\omega\dot{r}\mathbf{n}_Y.$$

[102] The equation of motion for the only degree of freedom of the bead, its distance r from the rotation center, may be obtained even simpler by using the Lagrangian approach discussed in Chapter 2 of the lecture notes, but by construction, it excludes the motion-constraining force **N**, while our assignment explicitly requests it.

Now the requirement that each Cartesian component of this equation is satisfied separately, immediately yields the same results (**) as the first approach.

Hence, the Coriolis "force" is just a physical representation of the mixed radial-angular term in the purely geometric expressions (*) for the Cartesian components of a point's acceleration in a rotating system. Admittedly, Eqs. (*) are just a particular case of the general Eq. (4.90) of the lecture notes, but their derivation is elementary, (hopefully :-) making them more transparent.

Problem 4.31. Analyze the dynamics of the famous *Foucault pendulum* used for spectacular demonstrations of the Earth's rotation. In particular, calculate the angular velocity of the rotation of its oscillation plane relative to the Earth's surface, at a location with a polar angle ("colatitude") Θ. Assume that the pendulum oscillation amplitude is small enough to neglect nonlinear effects, and that its oscillation period is much shorter than 24 hours.

Solution: The Foucault pendulum may be described as the usual point pendulum, with the oscillation decay time[103] long enough to notice the rotation of its oscillation plane due to the Earth's rotation. In the absence of this rotation, it is just the spherical pendulum, which was the subject of Problem 3.9. Since now we are interested only in its small oscillations, with the angular deviations from the vertical position satisfying the condition $|\theta| << 1$, we may approximate its potential energy in the Earth's gravity field as

$$U = -mgl\cos\theta \approx -mgl\left(1-\frac{\theta^2}{2}\right) \approx \frac{mg}{2l}\left(x^2+y^2\right) + \text{const},$$

where $x = l\sin\theta\cos\varphi \approx l\theta\cos\varphi$ and $y = l\sin\theta\sin\varphi \approx l\theta\sin\varphi$ are the 2D Cartesian coordinates of the pendulum in the locally-horizontal plane. In these coordinates, the kinetic energy of the pendulum is also very simple,[104]

$$T = \frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right),$$

so it makes sense to use them for our analysis.

According to the above expressions for T and U, the Lagrange equations of motion of these two degrees of freedom are linear and independent of each other:

$$m\ddot{x} = -\frac{mg}{l}x, \quad m\ddot{y} = -\frac{mg}{l}y,$$

each describing sinusoidal oscillations with the same frequency $\Omega_0 = (g/l)^{1/2}$, and may be merged into one equation for the 2D radius vector $\boldsymbol{\rho} = \{x, y\}$:

$$m\ddot{\boldsymbol{\rho}} = -m\Omega_0^2\boldsymbol{\rho}. \qquad (*)$$

The resulting pendulum's trajectory on the x-y plane is generally an ellipse, but at special initial conditions (if the pendulum is launched, from some point $\boldsymbol{\rho}_0$, with an initial velocity aligned with this radius vector), this ellipse degenerates into a straight line, which retains its spatial orientation.

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103 For a discussion of such a decay, caused by energy dissipation ("damping"), see Sec. 5.1 of the lecture notes.

104 In the limit $|\theta| << 1$, the contribution to T from the velocity's z-component, $\dot{z} = -l\sin\theta\,\dot{\theta} \approx -l\theta\,\dot{\theta}$, is of the order of θ^4, i.e. is negligibly small.

To consider the effect of the Earth's rotation, we need to account for the fact that the Earth-bound reference frame is non-inertial. In order for the 2nd Newton's law to be valid in it, we need to add, to the actual (physical) forces described by the right-hand side of Eq. (*), two fictitious (inertial) "forces": the centrifugal force (4.93) and the Coriolis force (4.94), in our current case with the vector $\boldsymbol{\omega} = \boldsymbol{\omega}_E$ directed along the Earth's polar axis, with the magnitude[105] $\omega_E \approx 2\pi/24\times60\times60 \approx 0.727\times10^{-6}\ \text{s}^{-1}$. In typical implementations of the Foucault pendulum, its swing times are of the order of a few seconds, i.e. $\Omega_0 \sim 1\ \text{s}^{-1}$. This means that $\omega_E << \Omega_0$, so the centrifugal force, proportional to $m\omega_E^2 A$ (where A is the amplitude of oscillations) is much smaller than the Coriolis force, proportional to $m\omega_E v \sim m\omega_E\Omega_0 A$, and may be ignored.[106]

Moreover, for our problem, Eq. (4.94) for the Coriolis force also may be simplified. Indeed, let us recast it as

$$\mathbf{F}_C = -2m\boldsymbol{\omega}_E \times \mathbf{v} = -2m\left(\boldsymbol{\omega}_\rho + \omega_z\mathbf{n}_z\right)\times\left(\dot{\boldsymbol{\rho}} + \dot{z}\mathbf{n}_z\right) \equiv -2m\left(\boldsymbol{\omega}_\rho \times \dot{z}\mathbf{n}_z + \boldsymbol{\omega}_\rho \times \dot{\boldsymbol{\rho}} + \omega_z\mathbf{n}_z \times \dot{\boldsymbol{\rho}}\right).$$

Here the vector $\boldsymbol{\omega}_\rho$, with the meridional direction within the locally horizontal $[x, y]$ plane, has the magnitude $\omega_\rho = \omega_E\sin\Theta$, where Θ is the polar angle ("colatitude") of the pendulum's location on the Earth's surface, while its z-axis counterpart is

$$\omega_z = \omega_E \cos\Theta . \qquad (**)$$

In the final expression for $\mathbf{F}_C$, the vector product involving $\dot{z}$ is negligible because of the smallness of this velocity component at small oscillations (see a footnote above), while the product $\boldsymbol{\omega}_\rho \times \dot{\boldsymbol{\rho}}$ is a vector directed along the z-axis, and the force's component it describes causes only a minor change of the pendulum's support force. As a result, we may keep only the last product, changing Eq. (*) into

$$m\ddot{\boldsymbol{\rho}} = -m\Omega_0^2\boldsymbol{\rho} - 2m\omega_z\left(\mathbf{n}_z \times \dot{\boldsymbol{\rho}}\right). \qquad (***)$$

Using the fact that

$$\mathbf{n}_z \times \dot{\boldsymbol{\rho}} \equiv \mathbf{n}_z \times \left(\mathbf{n}_x\dot{x} + \mathbf{n}_y\dot{y}\right) \equiv \mathbf{n}_y\dot{x} - \mathbf{n}_x\dot{y} ,$$

we may return to the Cartesian coordinates $\{x, y\}$:

$$\ddot{x} = -\Omega_0^2 x + 2\omega_z\dot{y}, \qquad \ddot{y} = -\Omega_0^2 y - 2\omega_z\dot{x} . \qquad (****)$$

Since these differential equations are linear, we may look for their general solution as a linear superposition of particular solutions $a_x e^{-i\Omega t}$, $y = a_y e^{-i\Omega t}$ – see, for example, Eq. (3.13) with $\lambda \equiv -i\Omega$. The resulting homogeneous algebraic equations,

$$-\Omega^2 a_x = -\Omega_0^2 a_x - 2i\omega_z\Omega a_y, \qquad -\Omega^2 a_y = -\Omega_0^2 a_y + 2i\omega_z\Omega a_x ,$$

are compatible only if the determinant of their system equals zero:

$$\begin{vmatrix} \Omega_0^2 - \Omega^2 & 2i\omega_z\Omega \\ -2i\omega_z\Omega & \Omega_0^2 - \Omega^2 \end{vmatrix} = 0, \qquad \text{i.e. if } \left(\Omega_0^2 - \Omega^2\right)^2 - \left(2\omega_z\Omega\right)^2 = 0 .$$

[105] The first sign ≈ reflects the fact that the time of the Earth's rotation in the (nearly) inertial reference frames bound to the Sun and the stars, is slightly (by ~0.3%) shorter than 24 hours.

[106] Actually, its account would result not in any qualitatively new effects, but only in a very small renormalization of the oscillation frequency.

This is an easily solvable quadratic equation for Ω^2; in the limit $\omega_z << \Omega_0$ (again, our Eq. (***) is valid only in this limit), it gives a very simple result:

$$\Omega = \pm\Omega_{\pm}, \qquad \text{where } \Omega_{\pm} \equiv \Omega_0 \pm \omega_z,$$

so the general solution of Eqs. (****) is

$$x(t) = \sum_{\pm} a_x^{\pm} \exp\{-i\Omega_{\pm} t\}, \qquad y(t) = \sum_{\pm} a_y^{\pm} \exp\{-i\Omega_{\pm} t\},$$

where the four coefficients $a_x^{\pm}$ and $a_y^{\pm}$ are determined by the initial position and velocity of the pendulum. The reader should not worry about the apparently complex nature of these formulas for the real functions $x(t)$ and $y(t)$; for any real initial conditions, they always give real results for any t. For example, for simple initial conditions

$$x(0) = A, \quad \dot{x}(0) = 0, \quad y(0) = 0, \quad \dot{y}(0) = 0,$$

these formulas yield

$$x = A\cos\Omega_0 t \cos\omega_z t, \qquad y = -A\cos\Omega_0 t \sin\omega_z t.$$

Since $\omega_z << \Omega_0$, this means that the oscillation trajectory on the $[x, y]$ plane is virtually linear, but this line (i.e. the locally-vertical plane of the pendulum's oscillations) slowly turns with the angular velocity equal to $-\omega_z = -\omega_E \cos\Theta$.[107] For example, on the North Pole ($\Theta = 0$), where $\omega_z = \omega_E$, the oscillation plane turns exactly with the angular speed of the Earth's rotation, in the opposite direction. This fact looks very natural from an inertial reference frame that does not rotate with the planet: for the observer in this frame, the pendulum's oscillation plane retains its initial orientation – the result very natural given the fictitious nature of the Coriolis "force". However, for any Θ but 0 and π, Eq. (**) is somewhat counter-intuitive: it shows (and experiment confirms) that in 24 hours, the pendulum's oscillation plane does not return to its initial orientation, rotating by less than 2π. In particular, a pendulum on the Earth's equator ($\Theta = \pi/2$) oscillates without any rotation of its plane – as observed from the Earth-bound reference frame.

Problem 4.32. A small body is dropped down to the surface of Earth from a height $h << R_E$, without initial velocity. Calculate the magnitude and direction of its deviation from the vertical line, due to the Earth's rotation. Estimate the effect's magnitude for a body dropped from the top floor of the Empire State Building.

Solution: The Earth's rotational effects are relatively weak, $R_E\omega_E^2 << g$, so we can analyze them as small perturbations of a purely vertical fall, with the velocity changing as $v_z(t) = -gt$, and the drop time $\Delta t = (2h/g)^{1/2}$. With the v_z component dominating the body's velocity $\mathbf{v}$, the Coriolis force $\mathbf{F}_C = -2m(\boldsymbol{\omega}_E\times\mathbf{v})$ is directed horizontally (in the Northern hemisphere, eastward), so the deviation x in that direction obeys the following equation:

$$m\ddot{x} = -2m\omega_E v_z(t)\sin\theta = 2m\omega_E gt\sin\theta,$$

where θ is the polar angle (colatitude) of the experiment's location. Integrating the equation over time twice, with zero initial conditions for both x and v_x, we get

[107] This is exactly the effect first demonstrated by Léon Foucault in 1851.

$$v_x(t) = \omega_E g t^2 \sin\theta, \qquad x(t) = \omega_E g \frac{t^3}{3} \sin\theta,$$

so the final eastward deviation is

$$d_E \equiv x(\Delta t) = \frac{1}{3}\omega_E g \left(\frac{2h}{g}\right)^{3/2} \sin\theta .$$

The centrifugal force $\mathbf{F}_c = -m\boldsymbol{\omega}_E\times(\boldsymbol{\omega}_E\times\mathbf{r})$, is perpendicular to the polar axis, i.e. has both the horizontal and the vertical components, but the latter gives just a small correction to the flight time. The horizontal (meridional) component $(\mathbf{F}_c)_y = -F_c\cos\theta = -(m\omega_E^2 R_E \sin\theta)\cos\theta$, is virtually constant in time, so the corresponding equation of motion in the meridional direction,

$$m\ddot{y} = -m\omega_E^2 R_E \sin\theta\cos\theta = \text{const},$$

has the well-known solution

$$y(t) = -\omega_E^2 R_E \sin\theta\cos\theta \frac{t^2}{2},$$

so the final southward deviation is

$$d_S \equiv -y(\Delta t) = \omega_E^2 R_E \sin\theta\cos\theta \frac{h}{g}.$$

For our parameters, $\omega_E \approx 0.727\times10^{-4}$ s^{-1}, $R_E \approx 6.38\times10^6$ m, $g \approx 9.81$ m/s^2, $\theta \approx 49.3°$, and $h = 373$ m (the top floor of the Empire State Building), the southward deviation is very substantial, $d_S \approx 63$ cm. However, it has to be measured from the direction toward the center of the sphere that the Earth *would be* without its rotation, rather than from the static position of a vertically hanging pendulum – which is the standard measure of the local vertical direction. The latter direction aligns itself with the "net force" $\mathbf{F} = m\mathbf{g} - m\boldsymbol{\omega}_E\times(\boldsymbol{\omega}_E\times\mathbf{r})$, so from it, the falling body does not deviate to either North or South. This is why the velocity-dependent eastward deviation d_E, due to the Coriolis force, is much easier to measure, though it is smaller: for our parameters, $d_E \approx 12$ cm.

Another interesting aspect of the solution is that the Coriolis-force-induced deviation d_E is directed *eastward*, while for a horizontal flight from the North Pole, such deviation is directed *westward* – see, e.g., Sec. 4.6 of the lecture notes and in particular Fig. 4.15. The reader is challenged to interpret this difference from the point of view of an inertial-frame observer.

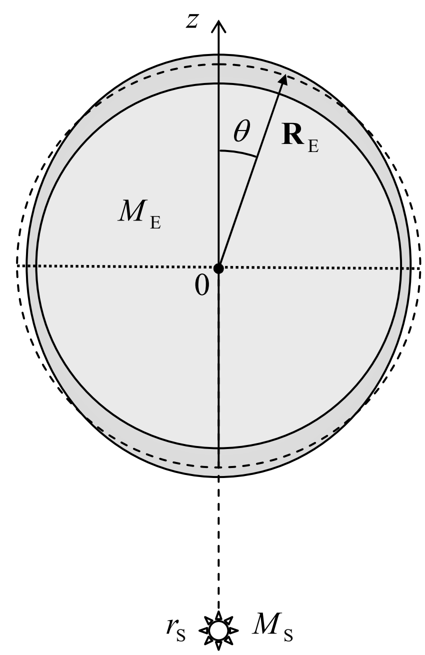
Problem 4.33. Calculate the height of solar tides on a large ocean, using the following simplifying assumptions: the tide period (½ of the Earth's day) is much longer than the period of all ocean waves, the Earth (of mass M_E) is a sphere of radius R_E, and its distance r_S from the Sun (of mass M_S) is constant and much larger than R_E.

Solution: Due to the first of the listed assumptions, we may consider the tides as a stationary configuration in an Earth-based reference frame always oriented to the Sun – see the figure below (in which the tides are dramatically exaggerated for clarity).[108] In such an equilibrium, the surface of the water (or any other liquid) aligns with the equipotential surface – the surface of equal values of the

[108] Naturally, due to the Earth's rotation about its axis, for an Earth-surface-bound observer, this pattern yields the familiar time sequence of two low and two high tides every 24 hours.

effective potential energy of a unit probe mass.[109] To find this surface in our case, in the non-inertial Earth-bound reference frame we need to take into account three contributions to the effective potential: of the real gravity fields (1.15) of the Earth and the Sun, and the effective potential,

$$U_{\rm cf} = -\frac{1}{2}(\boldsymbol{\omega}\times\mathbf{r})^2, \qquad (*)$$



of the centrifugal "inertial force" – see Eq. (4.96b) with $m = 1$, where in our current case, $\boldsymbol{\omega}$ is the angular frequency of the rotation of our reference frame (and hence of the Earth) around the Sun.

Since, as we know from experimental observations (and will be confirmed by as our solution), the tide amplitude h_0 is much smaller than R_E, the Earth gravity field may be taken uniform:

$$U_E \approx gh \equiv G\frac{M_E}{R_E^2}h,$$

where h is the point's height over some sphere of a radius close to R_E.

Next, since $r_S >> R_E$, the Sun's gravity potential

$$U_S(\mathbf{r}) = -G\frac{M_S}{|\mathbf{r}+\mathbf{n}_z r_S|} \approx -G\frac{M_S}{r_S+z},$$

where $z << r_S$ is the distance of the observation point from the plane normal to the direction toward the Sun and passing through Earth's center (shown with a dotted line in the figure on the right), may be also taken in an approximate form:

$$U_S(z) \approx U_S\big|_{z=0} + \frac{\partial U_S}{\partial z}\Big|_{z=0} z + \frac{1}{2}\frac{\partial^2 U_S}{\partial z^2}\Big|_{z=0} z^2 = -G\frac{M_S}{r_S} + G\frac{M_S}{r_S^2}z - G\frac{M_S}{r_S^3}z^2. \qquad (**)$$

The reason why we need to keep all three leading terms of this Taylor series is that the term linear in z in the expression for the centrifugal potential (*),

$$U_{\rm cf} \approx -\frac{1}{2}\omega^2(r_S+z)^2 \equiv -\frac{1}{2}\omega^2 r_S^2 - \omega^2 r_S z - \frac{1}{2}\omega^2 z^2,$$

is exactly equal and opposite to that in Eq. (**), because ω satisfies the equality

$$\omega^2 r_S = G\frac{M_S}{r_S^2}, \qquad (***)$$

which expresses the 2nd Newton law for the Earth's motion on the orbit of radius r_S around the Sun. (It also follows from Eq. (3.52) of the lecture notes.)

As a result, the net potential energy of a unit mass is

[109] Indeed, if a liquid's surface did *not* coincide with an equipotential surface, the potential energy's gradient ∇U (which equals $-\mathbf{F}$) would have a tangential component, and the liquid would follow it – see also Chapter 8.

$$U(h,z) = U_E + U_S + U_{cf} = G\frac{M_E}{R_E^2}h - G\frac{M_S}{r_S^3}z^2 - \frac{1}{2}\omega^2 z^2 + \text{const}.$$

Now using Eq. (***) to eliminate ω, and representing z near the Earth's surface as $R_E\cos\theta$ (where θ is the polar angle referred to the direction to/from the Sun – see the figure above), we may rewrite this expression as

$$U(h,z) = G\frac{M_E}{R_E^2}h - \frac{3}{2}G\frac{M_S}{r_S^3}z^2 + \text{const} \equiv G\frac{M_E}{R_E^2}h - \frac{3}{2}G\frac{M_S}{r_S^3}R_E^2\cos^2\theta + \text{const},$$

so the condition $U = \text{const}$ yields the following result for the water surface height:

$$h = \text{const} + h_0\cos^2\theta, \qquad \text{with } h_0 = \frac{3}{2}\frac{M_S}{M_E}\frac{R_E^4}{r_S^3}.$$

This expression (which is, interestingly, independent of G) shows that in an open ocean, the high tides would correspond to the directions to and from the Sun (i.e. take place at the local noon and midnight) – just as sketched in the figure above. With the numerical values of the astronomical constants ($M_S \approx 1.99\times10^{30}$ kg, $r_S \approx 1.496\times10^{11}$ m, $M_E \approx 0.592\times10^{25}$ kg, $R_E \approx 0.637\times10^{7}$ m), the above expression yields $h_0 \approx$ 0.246 m, thus confirming our assumptions $h_0 \ll R_E$, r_S, and hence justifying the performed truncation of the Taylor series.

The real picture of ocean tides is more complicated, first of all, because of the Moon, which induces its own, similar tides of an approximately twice larger height. These tides have approximately the same period but a slowly drifting phase difference between them and the solar tides, leading to the total tide height's modulation with an approximately two-week period.

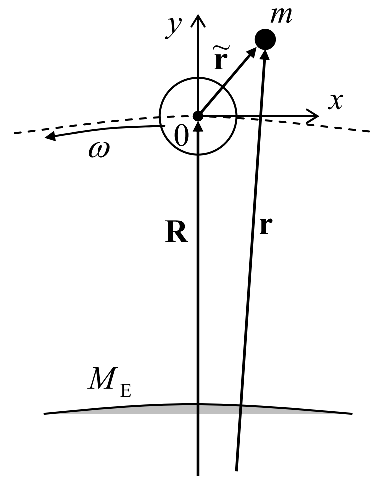
In addition, the first of the simplifying assumptions made in the problem's assignment is not quite realistic. As will be discussed in Sec. 8.4 of the lecture notes, the velocity v of the longest waves on the ocean is approximately $(gd)^{1/2}$, where d is the ocean's depth, so for the average depth $d \sim 5$ km of the Earth's oceans, v is about 200 m/s. While rather impressive on the human scale, this velocity still requires time $t = \pi R_E/v \approx 10^5$s for the wave to travel around half the globe, which is comparable with the half-a-day (43,200-second) tide period. For a uniform ocean, of a constant depth, this delay would only cause a constant phase shift of the tides in comparison with the above result; however, complex depth patterns cause the tide amplitude and phase distributions to be rather complex as well – especially near the shores, where certain bay shorelines lead to a dramatic increase of the tide height – up to 8 meters! (A quantitative analysis of this effect will be the subject of Problem 8.13.)

Problem 4.34. A satellite is on a circular orbit of radius R, around the Earth. Neglecting the gravity field of the satellite,

(i) write the equations of motion of a small body as observed from the satellite and simplify them for the case when the motion is limited to the satellite's close vicinity;

(ii) use these equations to prove that a body may be placed on an elliptical trajectory around the satellite's center of mass, within its plane of rotation around the Earth. Calculate the ellipse's orientation and eccentricity.

Solutions:



(i) Let **r** be the radius vector of the small body under analysis, as measured from the Earth's center, and **R** be that of the satellite's center of mass (point 0 in the figure on the right). Then the radius vector of the body in the reference frame bound to the satellite is

$$\tilde{\mathbf{r}} = \mathbf{r} - \mathbf{R}.$$

Since this reference frame is non-inertial, the body's motion in it should be described by the modified 2nd Newton law (4.92), which in our case takes the form

$$m\ddot{\tilde{\mathbf{r}}} = \mathbf{F} - G\frac{mM_E(\mathbf{R}+\tilde{\mathbf{r}})}{|\mathbf{R}+\tilde{\mathbf{r}}|^3} - m\mathbf{a}_0 - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\tilde{\mathbf{r}}) - 2m\boldsymbol{\omega}\times\dot{\tilde{\mathbf{r}}}, \qquad (*)$$

Here m is the small body's mass, **F** is the sum of real (physical) forces applied to the body (besides the Earth's gravity force, which is spelled out separately), and the vector **ω** is directed perpendicular to the satellite's rotation plane; its magnitude may be readily found from the 2nd Newton law (written in an inertial reference frame) applied to the satellite:

$$G\frac{M_E}{R^2} = a_0 = \omega^2 R, \qquad \text{giving } \omega^2 = \frac{GM_E}{R^3}. \qquad (**)$$

(Since **ω** is constant, in our case, the last term of Eq. (4.92) vanishes.)

Because of the second (gravity) term on the right-hand side of Eq. (*), this equation is nonlinear in $\tilde{\mathbf{r}}$ (even if the additional force **F** is a function of time alone), and hence is hard to solve analytically. However, if the body is sufficiently close to the satellite,

$$\tilde{r} << R,$$

the equation may be simplified by expanding the gravity term into the Taylor series in the Cartesian components of the small ratio $\tilde{\mathbf{r}}/R$ and then dropping all terms higher than the linear ones. Selecting the coordinate axes as shown in the figure above (so, in particular, $\mathbf{a}_0 = -a_0\mathbf{n}_y$, $\mathbf{R} = R\mathbf{n}_y$, and $\boldsymbol{\omega} = \omega\mathbf{n}_z$), we may write

$$\frac{\mathbf{R}+\tilde{\mathbf{r}}}{|\mathbf{R}+\tilde{\mathbf{r}}|^3} \equiv \frac{x\mathbf{n}_x + (R+y)\mathbf{n}_y + z\mathbf{n}_z}{\left[x^2 + (R+y)^2 + z^2\right]^{3/2}}$$

$$\equiv \frac{1}{R^2}\frac{\mathbf{n}_y}{\left[(x/R)^2 + (1+y/R)^2 + (z/R)^2\right]^{3/2}} + \frac{1}{R^3}\frac{x\mathbf{n}_x + y\mathbf{n}_y + z\mathbf{n}_z}{\left[(x/R)^2 + (1+y/R)^2 + (z/R)^2\right]^{3/2}}$$

$$\approx \frac{\mathbf{n}_y}{R^2(1+3y/R)} + \frac{1}{R^3}(x\mathbf{n}_x + y\mathbf{n}_y + z\mathbf{n}_z) \approx \frac{\mathbf{n}_y}{R^2}\left(1 - 3\frac{y}{R}\right) + \frac{1}{R^3}(x\mathbf{n}_x + y\mathbf{n}_y + z\mathbf{n}_z)$$

$$\equiv \frac{\mathbf{n}_y}{R^2} + \frac{1}{R^3}(x\mathbf{n}_x - 2y\mathbf{n}_y + z\mathbf{n}_z).$$

In the same breath, let us also spell out the vector products in the last two terms of Eq. (*):

$$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\tilde{\mathbf{r}}) = \omega^2\mathbf{n}_z\times[\mathbf{n}_z\times(x\mathbf{n}_x + y\mathbf{n}_y + z\mathbf{n}_z)] = \omega^2\mathbf{n}_z\times(x\mathbf{n}_y - y\mathbf{n}_x) = -\omega^2(x\mathbf{n}_x + y\mathbf{n}_y),$$

$$2\boldsymbol{\omega}\times\dot{\tilde{\mathbf{r}}} = 2\omega\mathbf{n}_z\times(\dot{x}\mathbf{n}_x + \dot{y}\mathbf{n}_y + \dot{z}\mathbf{n}_z) = 2\omega(\dot{x}\mathbf{n}_y - \dot{y}\mathbf{n}_x).$$

Plugging these expressions into Eq. (*), and using Eq. (**) to replace the product GM_E with ω^2R^3, we may write separate equations for all Cartesian coordinates:

$$\begin{aligned} m\ddot{x} &= F_x + 2m\omega\dot{y}, \\ m\ddot{y} &= F_y + m(3\omega^2 y - 2\omega\dot{x}), \\ m\ddot{z} &= F_z - m\omega^2 z. \end{aligned} \qquad (***)$$

(ii) In the absence of any other forces (**F** = 0) we may cancel m, getting, for the motion within the $[x, y]$ plane, the following system of two linear, homogeneous differential equations:

$$\ddot{x} = 2\omega\dot{y}, \qquad \ddot{y} = 3\omega^2 y - 2\omega\dot{x}. \qquad (****)$$

It is natural to look for their solution in the sinusoidal form (for more about that, see Chapter 5 below),

$$x = \text{Re}\left[Ae^{-i\Omega t}\right], \quad y = \text{Re}\left[\alpha A e^{-i\Omega t}\right],$$

where α is some (so far, unknown) complex number whose argument is the phase shift between the oscillations of the two coordinates. Plugging this solution into Eqs. (****), we get the following system of equations for α and Ω:

$$-\Omega^2 = -2i\omega\Omega\alpha, \qquad -\Omega^2\alpha = 3\omega^2\alpha + 2i\omega\Omega,$$

with the solution

$$\Omega = \omega, \qquad \alpha = -\frac{i}{2} \equiv \frac{1}{2}e^{-i\pi/2}.$$

Thus, if we select the time origin so that A is real, the body's coordinates evolve as

$$x = A\cos\omega t, \quad y = \frac{A}{2}\cos\left(\omega t + \frac{\pi}{2}\right) \equiv \frac{A}{2}\sin\omega t.$$

These formulas describe the small body's periodic motion around the satellite, with the frequency exactly equal to the angular velocity ω of the satellite's rotation, along an elliptic trajectory with the major semi-axis $a = A$ oriented in the "horizontal" direction (x) and a twice smaller minor semi-axis $b = A/2$ in the "vertical" direction (y). Using Eqs. (3.63) of the lecture notes, we may calculate the eccentricity parameter of such trajectory: $e = (1 - b^2/a^2)^{1/2} = \sqrt{3}/2 \approx 0.866$.

Note that the size of this orbit is determined not by the equations of motion, but only by initial conditions. Note also that according to the last of Eqs. (***), the motion of the free body along the z-axis is also sinusoidal with the same frequency ω,[110] but with independent amplitude and phase.

These results may be also obtained using the inertial reference frame bound to Earth's center. In this case, the substantially elliptic orbital motion discussed above is described as the difference between

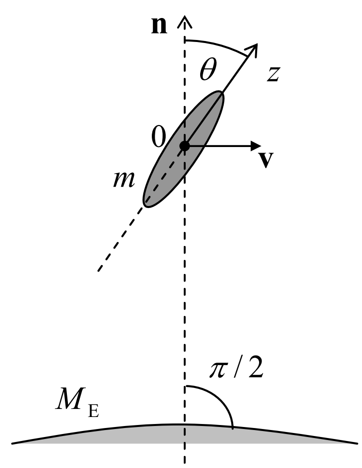
[110] This means, in particular, that even at arbitrary initial conditions, the body will return exactly to the same position (relative to the satellite) after each revolution around the Earth.

a *slightly* elliptical (with $e << 1$) orbit of the body in question around the Earth center and the circular orbit of the satellite, with the same main axis $a = R$ – and hence, according to the 3rd Kepler law (3.64b), the same rotation period. Deriving them from this standpoint is a good exercise, highly recommended to the reader.

Problem 4.35. A non-spherical shape of an artificial satellite may ensure its stable angular orientation relative to the Earth's surface, advantageous for many practical goals. By modeling a satellite as a strongly elongated, axially-symmetric body moving around the Earth on a circular orbit of radius R, find its stable orientation.

Solution: The gravitational potential energy of a small slice dz of the satellite's length (see the figure below, with mass $dm = \mu(z)dz$, may be represented as

$$dU = -GM_\text{E}\frac{1}{r}dm = -GM_\text{E}\frac{1}{\left(R^2 + 2Rz\cos\theta + z^2\right)^{1/2}}\mu(z)dz,$$



where R is the distance of its center-of-mass from the Earth's center, while θ is the angle between the current position of its symmetry axis z (with the origin at the satellite's center of mass) and the "vertical" direction **n** from the Earth's center – see the figure on the right. In the non-inertial reference frame moving together with the satellite's center-of-mass 0, to this energy we need to add the centrifugal potential energy – see Eq. (4.96b) of the lecture notes, so that the total effective potential energy is

$$dU_\text{ef} = -GM_\text{E}\frac{1}{\left(R^2 + 2Rz\cos\theta + z^2\right)^{1/2}}\mu(z)dz - \frac{\omega^2}{2}(R + z\cos\theta)^2\mu(z)dz.$$

If the satellite's size is much less than R, we may expand the fraction on the right-hand side of this expression into the Taylor series in small z/R, and keep only three leading terms of this expansion:[111]

$$dU_\text{ef} \approx -GM_\text{E}\left(\frac{1}{R} - \frac{\cos\theta}{R^2}z + \frac{3\cos^2\theta - 1}{2R^3}z^2\right)\mu(z)dz - \frac{\omega^2}{2}\left(R^2 + 2Rz\cos\theta + z^2\cos^2\theta\right)\mu(z)dz$$

$$= \frac{GM_\text{E}}{R^3}\frac{1 - 2\cos^2\theta}{2}z^2\mu(z)dz + \text{const},$$

where the last step used the 2nd Newton law for the satellite's circular motion

$$ma \equiv m\omega^2 R = G\frac{M_\text{E}m}{R^2}, \qquad \text{giving } \omega^2 = \frac{GM_\text{E}}{R^3},$$

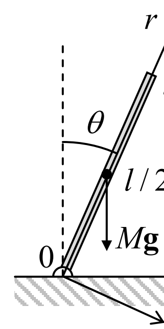
and "const" means the sum of θ-independent terms. (Note that the terms linear in z have canceled; this is actually a good sanity check because their sum gives the effective radial force $-\partial U_\text{ef}/\partial z$, which has to be zero in the reference frame moving with the satellite.) Now integrating this expression along the entire satellite's length, we may calculate its full effective potential energy:

[111] Since R has to be larger than $R_\text{E} \sim 6{,}000$ km, even for the largest artificial satellites (such as the International Space Station), with $a \sim 70$ m, the ratio a/R is below 10^{-5}, so the relative error of this approximation is as small as $(a/R)^3 \sim 10^{-15}$.

$$U_{\text{ef}} = \frac{GM_{\text{E}}}{R^3}\frac{1-2\cos^2\theta}{2}\int \mu(z) z^2 dz + \text{const}.$$

The integral in this expression is just the principal moment of inertia of the satellite's rotation about any axis perpendicular to its symmetry axis z. Since it is always positive, the calculated effective energy has minima at the angles $\theta = 0$ and $\theta = \pi$, so these two positions of the satellite, with the "vertical" orientation of its symmetry axis, are orbitally stable fixed points.[112] (The "horizontal" positions, with $\theta = \pm\pi/2$, corresponding to the maxima of the function $U_{\text{ef}}(\theta)$, are stationary but unstable.)

Problem 4.36. A rigid, straight, uniform rod of length l, with the lower end on a pivot, falls in a uniform gravity field – see the figure on the right. Neglecting friction, calculate the distribution of the bending torque τ along its length, and analyze the result.



Hint: As will be discussed in detail in Sec. 7.5 of the lecture notes, the bending torque's gradient along the rod's length is equal to the rod-normal ("shear") component of the total force between two parts of the rod, mentally separated by its cross-section.

Solution: This is a nice example of the (relatively rare) problems that may be solved by invoking the inertial "force" described by the last term on the right-hand side of Eq. (4.92) of the lecture notes:

$$m\mathbf{a} = \mathbf{F} - m\mathbf{a}_0\big|_{\text{in lab}} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - 2m\boldsymbol{\omega}\times\mathbf{v} - m\dot{\boldsymbol{\omega}}\times\mathbf{r}. \qquad (*)$$

Indeed, let us use a non-inertial reference frame with its origin staying on the pivot but rotating together with the rod – see the figure above. In this case, the vector $\mathbf{r}$ of any small fragment of the rod does not change in time, so $\mathbf{v} \equiv \dot{\mathbf{r}} = 0$, and $\mathbf{a} \equiv \ddot{\mathbf{r}} = 0$. At the same time, the vector $\boldsymbol{\omega}$ is that of the falling rod, so it does change in time. In the figure above, this vector is normal to the plane of the drawing; taking the direction into the plane of the drawing (from the viewer) for the positive one, we have $\omega = \dot{\theta}$.

Now let us spell out the shear (rod-normal) component of Eq. (*) for a small fragment of the rod, of length dr and mass $dm = \mu dr$, where $\mu \equiv dm/dr = M/l = \text{const}$ is the linear density of the rod's mass. The first term on the right-hand side is provided by the gravity force, whose rod-normal component is $(dm)g\sin\theta$, and also by the differential $dF = (dF/dr)dr$ of the shear component F of the force exerted by the top part of the rod on its lower part (mentally separated by the considered cross-section). The second term in Eq. (*) vanishes because the origin of our non-inertial frame has no linear acceleration. The third term (the centrifugal force) has only a longitudinal component we are not currently interested in. The next term, representing the Coriolis force, vanishes because $\mathbf{v} = 0$. However, the last term, $-m\dot{\boldsymbol{\omega}}\times\mathbf{r}$, is a vector normal to the rod (within the plane of its fall), with the magnitude $-(dm)\ddot{\theta}r$, so Eq. (*) yields[113]

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[112] Such *passive* orientation of satellites is frequently called their *gravity-gradient stabilization*; it was used successfully, for the first time, in 1967. Note, however, that the stabilizing torque $\tau = |\partial U_{\text{ef}}/\partial\theta|$ is very small, and for low-orbit satellites may be comparable with that from the atmospheric drag. This is why the desired orientation of most satellites is *actively* maintained, using either small thrusters or internal "momentum wheels".

[113] Alternatively, this equation may be derived in the inertial, "lab" reference frame, as the 2nd Newton law for the product of the elementary mass dm by its linear acceleration $r\ddot{\theta}$, equal to the sum of only actual (physical) force components $(dm)g\sin\theta$ and dF.

$$0 = (dm)g\sin\theta + \frac{dF}{dr}dr - (dm)\ddot{\theta}r,$$

giving the following differential equation for the function $F(r)$:

$$\frac{dF}{dr} = \frac{dm}{dr}\left(-g\sin\theta + \ddot{\theta}r\right) \equiv \mu\left(-g\sin\theta + \ddot{\theta}r\right). \qquad (**)$$

Equation (**) may be readily integrated with the evident boundary condition $F(l) = 0$, giving

$$F(r) = -\int_r^l \frac{dF(r')}{dr'}dr' = \mu\int_r^l \left(g\sin\theta - \ddot{\theta}r'\right)dr' = \mu\left[g\sin\theta(l-r) - \ddot{\theta}\frac{l^2-r^2}{2}\right],$$

and allowing us to calculate the bending torque $\tau(r)$. Indeed, by using the equality given in the *Hint*,

$$\frac{d\tau}{dr} = F(r),$$

with the obvious boundary condition $\tau(l) = 0$ (no torque at the top end of the rod), we get

$$\begin{aligned}\tau(r) &= -\int_r^l F(r')dr' = \mu\int_r^l\left[-g\sin\theta(l-r') + \ddot{\theta}\frac{l^2-r'^2}{2}\right]dr' \\ &= \mu\left[g\sin\theta\left(lr - \frac{r^2}{2} - \frac{l^2}{2}\right) - \ddot{\theta}\left(\frac{l^2r}{2} - \frac{r^3}{6} - \frac{l^3}{3}\right)\right].\end{aligned} \qquad (***)$$

Requiring the torque to vanish at $r \to 0$ as well (because the pivot does not resist rotation), we get[114]

$$\ddot{\theta} = \frac{3}{2}\frac{g}{l}\sin\theta. \qquad (****)$$

Of course, Eq. (****) could be derived much faster by using Eq. (4.38) for the rod's rotation around the pivot, with $I_{zz} = Ml^2/3$ and $\tau_z = (Mgl/2)\sin\theta$ – see the figure above. The advantage of our current approach is that after plugging Eq. (****) into Eq. (***), we immediately get an explicit result for the bending torque's distribution along the rod length:

$$\tau(r) = \mu g\sin\theta\frac{r(l-r)^2}{4l} \leq 0.$$

The negative sign of τ corresponds to the rod being bent backward, with its top end "trying" to lag behind the faster-falling "belly".[115] (This is natural because the uniform gravity force density $\mu g\sin\theta$ alone is insufficient to give the rod's top its larger linear acceleration $r\ddot{\theta}$.) A straightforward differentiation shows that $d\tau/dr = 0$, i.e. the magnitude of τ reaches its maximum, at $r = l/3$. In

[114] Note that this particular result is also valid for the "falling ladder" problem analyzed in Sec. 4.3 of the lecture notes and continued in Problem 15. Note also the following paradoxical fact: the rod's end acceleration, $l\ddot{\theta} = (3/2)g\sin\theta$, becomes larger than g at the tilt angles θ larger than ~42°. This fact is used in popular lecture demonstrations in which a small body (say, a coin), initially placed on the top of a pivoted rod, eventually lags behind it, if the rod is left to fall.

[115] Examining these photos and videos, take into account that in contrast with the simple model considered above, a typical brick chimney's cross-section is somewhat larger at its lower end.

accordance with this result, falling brick chimneys typically break apart at approximately one-third of their height – see the numerous spectacular photos and videos available online.[116]

Problem 4.37. Let **r** be the radius vector of a particle, as measured in a possibly non-inertial but certainly non-rotating reference frame. Taking its Cartesian components for the generalized coordinates, calculate the corresponding generalized momentum $\mathcal{p}$ of the particle and its Hamiltonian function H. Compare $\mathcal{p}$ with $m\mathbf{v}$, and H with the particle's energy E. Derive the Lagrangian equation of motion in this approach, and compare it with Eq. (4.92) of the lecture notes.

Solution: If the moving frame does not rotate, i.e. its motion is translational, we may use Eq. (4.95) with $\boldsymbol{\omega} = 0$ to rewrite the Lagrangian function of the particle as

$$L \equiv T - U = \frac{m}{2}(\mathbf{v}_0 + \mathbf{v})^2 - U \equiv \frac{m}{2}v_0^2 + m\mathbf{v}_0 \cdot \mathbf{v} + \frac{m}{2}v^2 - U, \qquad (*)$$

(where $\mathbf{v}_0$ is the velocity of the moving frame, as measured from the lab frame), so the generalized momentum corresponding to **r** is:[117]

$$\mathcal{p} \equiv \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + m\mathbf{v}_0 \equiv m\mathbf{v}\big|_{\text{in lab}}.$$

Note that just as in the case of rotation discussed at the end of Sec. 4.6 of the lecture notes, this "canonical" momentum $\mathcal{p}$ is different from the "kinetic" momentum $\mathbf{p} \equiv m\mathbf{v}$, and the last expression for $\mathcal{p}$ is similar to that for the case of rotating reference frame – see the discussion of Eq. (4.98) in the lecture notes.

Now calculating the corresponding Hamiltonian function (2.32),

$$H \equiv \mathcal{p} \cdot \mathbf{v} - L = (m\mathbf{v} + m\mathbf{v}_0) \cdot \mathbf{v} - \left(\frac{m}{2}v_0^2 + m\mathbf{v}_0 \cdot \mathbf{v} + \frac{m}{2}v^2 - U\right) = \frac{m}{2}v^2 + U - \frac{m}{2}v_0^2,$$

we see that it differs from the particle's lab-frame energy $E = T + U$:

$$E - H = \left(\frac{m}{2}v_0^2 + m\mathbf{v}_0 \cdot \mathbf{v} + \frac{m}{2}v^2 + U\right) - \left(\frac{m}{2}v^2 + U - \frac{m}{2}v_0^2\right) = \mathbf{v}_0 \cdot m(\mathbf{v} + \mathbf{v}_0) \equiv \mathbf{v}_0 \cdot \mathbf{p}\big|_{\text{in lab}}.$$

Note a full analogy of this relation with Eq. (4.102) of the lecture notes (valid for a pure rotation of the reference frame). So the difference between E and H exists not only at rotation.

The Lagrange equation of motion (or rather the vector aggregate of the three equations of motion of the three Cartesian components of **r**) in this approach is

$$\frac{d}{dt}(m\mathbf{v} + m\mathbf{v}_0) - (-\nabla U) = 0,$$

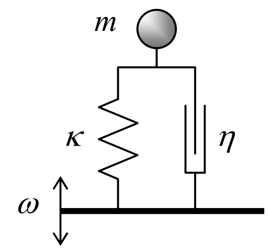
and may be represented either as $d\mathcal{p}/dt = -\nabla U = \mathbf{F}$, or in the form

[116] Brick structures resist the stretching stress caused by bending (see Sec. 7.5 of the lecture notes) much worse than the compression stress, so they cannot sustain even relatively weak bending.
[117] I am using the same shorthand as in Eq. (4.98) of the lecture notes. Note also that in this approach, $\mathbf{v}_0$ is considered a fixed function of time, independent of **r** and **v**.

$$m\mathbf{a} = \mathbf{F} - m\dot{\mathbf{v}}_0 .$$

This is just Eq. (4.92) for our particular case $\boldsymbol{\omega} = 0$. So the equations of motion of a particle are independent of our choice of its generalized coordinates – as they should be.

Chapter 5. Oscillations



Problem 5.1. A body of mass m is connected to its support not only with an elastic spring but also a *damper* (say, an air brake) that provides a drag force obeying Eq. (5.5) of the lecture notes – see the figure on the right.

(i) How to select the constants κ and η to minimize the body's vibrations caused by vertical oscillations of its support with frequency ω?

(ii)* What if the oscillations are random?

Solutions:

(i) Let us denote the body's deviation from its equilibrium height (as measured in an inertial reference frame) as $z(t)$. Then the 2nd Newton's law gives the following equation of the body's motion:

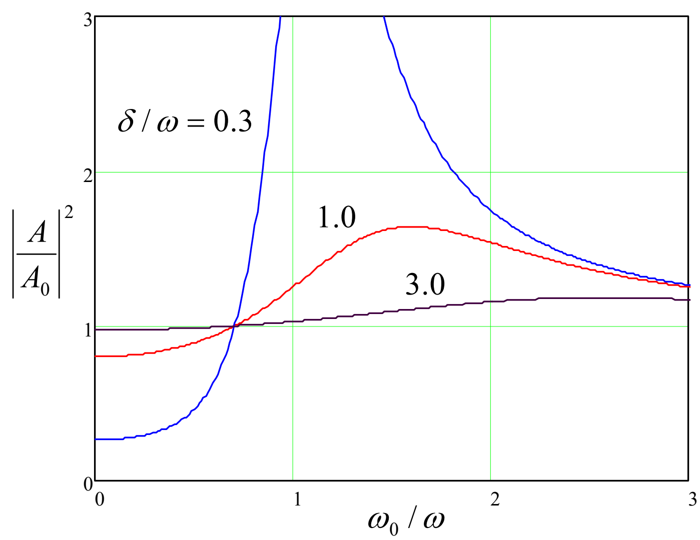
$$m\ddot{z} + \eta\left[\dot{z} - \dot{z}_0(t)\right] + \kappa\left[z - z_0(t)\right] = 0,$$

where the function $z_0(t)$ describes the support's oscillations. Rewriting this equation in the form

$$\ddot{z} + 2\delta\dot{z} + \omega_0^2 z = 2\delta\,\dot{z}_0(t) + \omega_0^2 z_0(t), \qquad \text{where } \omega_0^2 \equiv \frac{\kappa}{m},\ \delta \equiv \frac{\eta}{2m}, \tag{*}$$

we see that it is similar to Eq. (5.13), although the net external force described by the right-hand side depends on the parameters κ and η, and hence cannot be considered fixed at the system's optimization. Taking the support oscillations in the form $z_0(t) = A_0\cos\omega t$, and looking for the solution of Eq. (*) in the usual form $z = \text{Re}(Ae^{-i\omega t})$, which is adequate for the description of forced oscillations, we readily get the following response function:

$$r(\omega) \equiv \left|\frac{A}{A_0}\right|^2 = \frac{\omega_0^4 + 4\delta^2\omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4\delta^2\omega^2}. \tag{**}$$

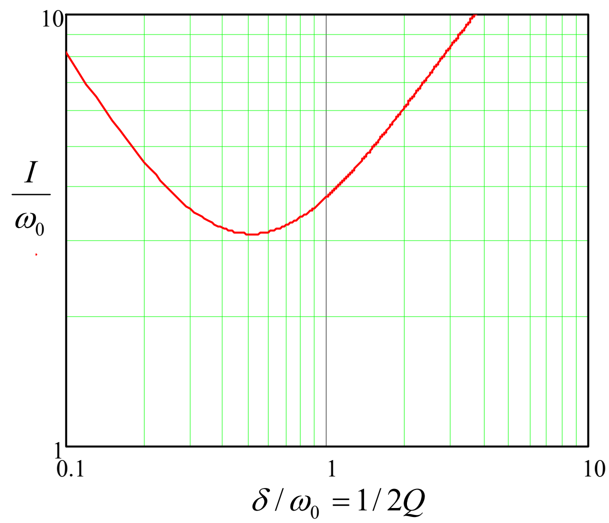


The figure on the right shows this result as a function of ω_0 at a fixed oscillation frequency ω. It shows that in this case, the body's vibrations may be reduced by reducing the spring constant κ so that the own oscillation frequency $\omega_0 \equiv (\kappa/m)^{1/2}$ becomes well below the threshold value $\omega_t \equiv \omega/\sqrt{2}$ (curiously, at that threshold, the result is independent of δ), and reducing the damping factor δ as much as possible.

(ii) In most practical systems of this kind (say, car suspension systems) the oscillation frequency ω may vary and is not known in advance – for example, it may depend on the velocity of the car's motion over rough pavement and the statistics of this roughness. The simplest but very reasonable model of such a random function $z_0(t)$ is the so-called *white noise* – the superposition of fluctuations with all frequencies, with their spectral density $S_0(\omega)$ being constant.[118] The basic mathematical statistics of random functions (see, for example, SM Sec. 5.4) shows that the most appropriate measure of the intensity of the resulting fluctuations $z(t)$, their variance $\langle z^2\rangle$, may be calculated as

$$\langle z^2\rangle = 2\int_0^\infty S_z(\omega)d\omega = 2\int_0^\infty S_0(\omega)r(\omega)d\omega \equiv 2S_0\omega_0 I, \qquad \text{where } I \equiv \int_0^\infty r(\omega)d\omega.$$

The figure on the right shows the dimensionless ratio I/ω_0, calculated for the response $r(\omega)$ given by Eq. (**), as a function of the normalized damping factor δ, i.e. of the reciprocal Q-factor (5.11), on the appropriate log-log scale. One can see that its minimum, and hence the minimum of intensity $\langle z^2\rangle$ of the body's vibrations, at a given ω_0, is achieved at $\delta/\omega_0 \approx 0.5$, i.e. at $Q \equiv \omega_0/2\delta \approx 1$. (According to Eq. (5.8), this damping is still somewhat lower than needed for the oscillation-relaxation crossover $\delta = \omega_0$.)



Finally, note that this problem is important not only for the car industry but also for the proper design of all physics experiments that require high isolation from seismic and/or acoustic vibrations. (In some of these cases, the function $S_0(\omega)$ may be very different from a constant.) For example, a significant initial underestimate of seismic vibrations has delayed the now-famous LIGO experiment from reaching the planned sensitivity, and hence the first observation of gravitational waves, for quite a few years. Similarly, only insufficient vibration isolation has presented the NIST team led by Russell Young, the inventor of scanning-probe microscopy, from achieving atomic resolution in the early 1970s – and hence from winning the Nobel Prize that eventually went to G. Binning and H. Rohrer for similar but more carefully designed experiments carried out a decade later. In many cases, sufficient isolation requires multistage damping systems much more complex than the simplest example discussed in this problem – see, e.g., the book recommended in Sec. 5.1 of the lecture notes.

Problem 5.2. For a system with the response function given by Eq. (5.17) of the lecture notes:

(i) prove Eq. (5.26), and

(ii) use an approach different from the one used in Sec. 5.1, to derive Eq. (5.34).

Hint: You may like to use the *Cauchy integral theorem* and the *Cauchy integral formula* for analytic functions of a complex variable.[119]

Solution: First, we need to prove that the following integral,

[118] For the definition of the spectral density see, e.g., SM Eq. (5.58).

[119] See, e.g., MA Eqs. (15.1)-(15.2).

$$I \equiv \int_{-\infty}^{+\infty} \chi(\omega) e^{-i\omega\tau} d\omega = \int_{-\infty}^{+\infty} \frac{1}{\left(\omega_0^2 - \omega^2\right) - 2i\omega\delta} e^{-i\omega\tau} d\omega \ ,$$

vanishes for any $\tau < 0$. It is straightforward to verify that the denominator of the fraction under this integral may be represented as a product $-(\omega - \omega_+)(\omega - \omega_-)$, where $\omega_\pm = i\lambda_\pm$, and $\lambda_\pm$ are the characteristic equation roots given by Eq. (5.8) of the lecture notes, so

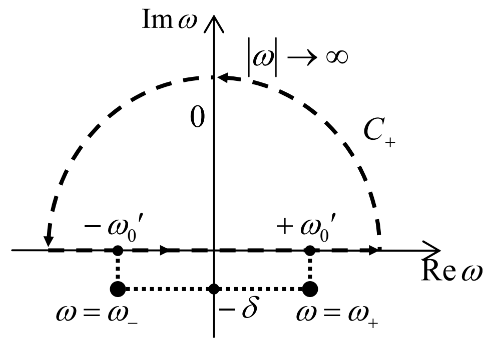
$$\omega_\pm = \pm\omega_0' - i\delta, \quad \text{with } \omega_0' \equiv \left(\omega_0^2 - \delta^2\right)^{1/2}.$$

Now let us represent the fraction as a sum of two simple $1/x$-type singularities (called *single poles*):

$$\frac{1}{(\omega_0^2 - \omega^2) - 2i\omega\delta} = -\frac{1}{(\omega - \omega_+)(\omega - \omega_-)} \equiv \frac{1}{2\omega_0'(\omega - \omega_-)} - \frac{1}{2\omega_0'(\omega - \omega_+)}.$$

As a result, our initial integral may be divided into the sum of two simpler integrals:

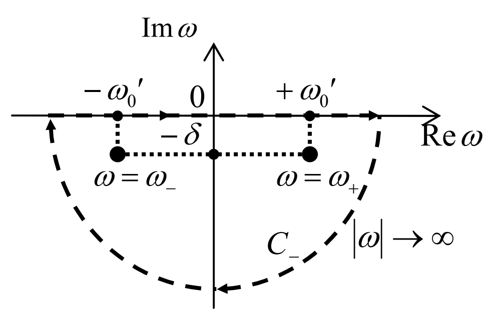
$$I = \frac{1}{2\omega_0'}\left(\int_{-\infty}^{+\infty} e^{-i\omega\tau} \frac{d\omega}{\omega - \omega_-} - \int_{-\infty}^{+\infty} e^{-i\omega\tau} \frac{d\omega}{\omega - \omega_+} \right),$$



so the expression under each of them, considered a function of the complex argument **ω** (so that $\omega = \text{Re}\,\boldsymbol{\omega}$), has a single pole in the lower half-plane – see the figure on the right. At $\tau < 0$, the exponent under the integrals,

$$\exp\{-i\omega\tau\} = \exp\{-i(\text{Re}\,\omega + i\,\text{Im}\,\omega)\tau\} \equiv \exp\{-i\tau\,\text{Re}\,\omega\}\exp\{\tau\,\text{Im}\,\omega\},$$

tends to zero at $\text{Im}\omega \to +\infty$. Hence, our integrals (which have to be taken along the real axis of the complex plane **ω**) would not change if we add to each of them a similar integral over a very large semi-circle in the upper half-plane, i.e. replace each of them with an integral over a closed contour C_+ shown with the dashed line in the figure above. But on the area inside the contour, the functions under the integrals do not have any poles (are analytic), so according to the Cauchy integral theorem, each of the integrals equals zero, q.e.d.[120]



Now a similar operation, but with the contour C_- closed in the *lower* half-plane (see the figure on the right) may be readily used to calculate Green's function $G(\tau)$ at $\tau > 0$, when the exponent $\exp\{(\text{Im}\omega)\tau\}$ quenches the contribution from the *lower* semi-circle. Applying the Cauchy integral formula to each component integral, we get

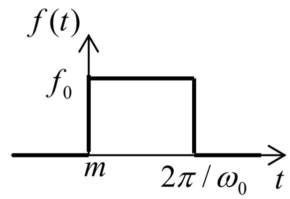
$$G(\tau) \equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty} \chi(\omega) e^{-i\omega\tau} d\omega = \frac{1}{4\pi\omega_0'}\left(\int_{-\infty}^{+\infty} \frac{e^{-i\omega\tau} d\omega}{\omega - \omega_-} - \int_{-\infty}^{+\infty} \frac{e^{-i\omega\tau} d\omega}{\omega - \omega_+} \right)$$

[120] This is the acronym for *quod erat demonstrandum* (Lat.), meaning "which had to be proved". This expression is frequently met in mathematic literature (with its clinical sequences of postulates, lemmas, and theorems :-) but not so much in physics, where QED usually stands for "quantum electrodynamics" – see, e.g., QM Secs. 9.1-9.4.

$$= \frac{1}{4\pi\omega_0'}\left(-\oint_{C_-} \frac{e^{-i\omega\tau} d\omega}{\omega - \omega_-} + \oint_{C_-} \frac{e^{-i\omega\tau} d\omega}{\omega - \omega_+} \right) = \frac{1}{4\pi\omega_0'} 2\pi i \left(-e^{-i\omega_-\tau} + e^{-i\omega_+\tau} \right)$$

$$= \frac{i}{2\omega_0'}\left(-e^{i\omega_0'\tau} e^{-\delta\tau} + e^{-i\omega_0'\tau} e^{-\delta\tau} \right) \equiv \frac{\sin \omega_0'\tau}{\omega_0'} e^{-\delta\tau}, \qquad \text{for } \tau > 0.$$

This is exactly Eq. (5.34) of the lecture notes.



Problem 5.3. A square-wave pulse of force (see the figure on the right) is exerted on a damping-free linear oscillator of frequency ω_0, initially at rest. Calculate the law of motion $q(t)$, sketch it, and interpret the result.

Solution: Using the Green's function approach, we may express the solution as

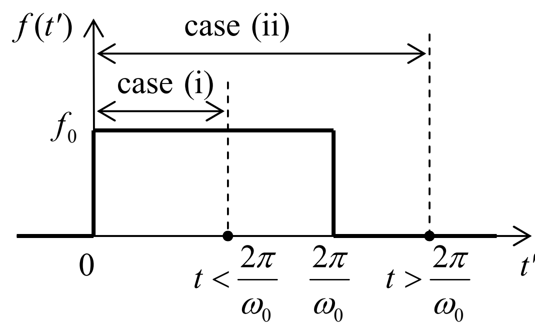
$$q(t) = \int_{-\infty}^{t} f(t')G(t-t')dt' = \int_{0}^{\infty} f(t-\tau)G(\tau)d\tau, \quad \text{where} \quad \tau \equiv t - t'. \qquad (*)$$

The Green's function $G(\tau)$ of such an oscillator has been calculated in Sec. 5.1 of the lecture notes – see Eq. (5.34) – and again in the previous problem. In the limit of negligible damping, it is reduced to

$$G(\tau) = \frac{\sin \omega_0\tau}{\omega_0}.$$

Due to the piecewise-constant character of the function $f(t)$ in our current problem, the non-zero part of the integral (*) has different limits (and hence gives different final results) for two cases:

(i) $0 < t < 2\pi/\omega_0$, and
(ii) $2\pi/\omega_0 < t$.



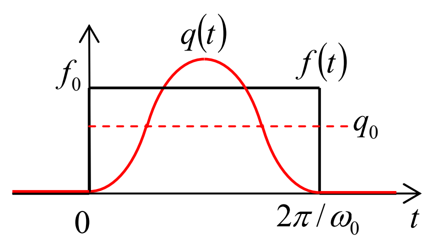
– see the figure on the right. As a result, in case (i),

$$q(t) = f_0 \int_0^t G(t-t')dt' = f_0 \int_0^t G(\tau)d\tau$$

$$= q_0(1 - \cos \omega_0 t), \quad \text{where} \quad q_0 \equiv \frac{f_0}{\omega_0^2} = \frac{F_0}{\kappa},$$

while in case (ii)

$$q(t) = f_0 \int_0^{2\pi/\omega_0} G(t-t')dt' = f_0 \int_{t-2\pi/\omega_0}^{t} G(\tau)d\tau = 0.$$



The final result is sketched with a red line in the figure on the right. Its physics is simple: the front step of the force applied at $t = 0$ shifts the equilibrium position of the oscillator from 0 to $q_0 \equiv F_0/\kappa$. Hence at $t = +0$ the oscillator finds itself shifted from this new equilibrium point, and starts sinusoidal oscillations around the new equilibrium position, with amplitude $A = q_0$. The equal and opposite back step of force, arriving at time $t = 2\pi/\omega_0$, quenches these oscillations completely. (If either the time interval between the two steps was not exactly a multiple of the oscillation period, or the force was not strictly constant between the steps, this quenching would not be complete.)

Problem 5.4. A linear oscillator with frequency ω_0 and damping δ was at rest at $t \leq 0$. At $t = 0$, an external force $F(t) = F_0\cos\omega t$ starts to be exerted on it.

(i) Derive the general expression for the time evolution of the oscillator's displacement, and interpret the result.

(ii) Spell out the result for the exact resonance ($\omega = \omega_0$) in an oscillator with low damping ($\delta << \omega_0$) and explore the limit $\delta \to 0$.

Solutions:

(i) We need to solve Eq. (5.13b) of the lecture notes, with the following right-hand side:

$$f(t) = \begin{cases} 0, & \text{for } t < 0, \\ f_0 \cos\omega t, & \text{for } t > 0, \end{cases}$$

where $f_0 \equiv F_0/m$. Due to the zero initial conditions at $t = 0$, we may calculate $q(t)$ using Eq. (5.27) with the Green's function (5.34):

$$q(t) = \int_0^\infty f(t-\tau)\,G(\tau)d\tau = f_0\int_0^t \cos\omega(t-\tau)\frac{1}{\omega_0'}e^{-\delta\tau}\sin\omega_0'\tau\,d\tau, \qquad \text{with } \omega_0' \equiv \left(\omega_0^2 - \delta^2\right)^{1/2}.$$

Using the well-known relations between trigonometric functions,[121] we may readily transform the above expression to a sum of four integrals of the type (5.36), which may be worked out similarly – see Eq. (5.37):

$$q(t) = \frac{f_0}{2\omega_0'}\times\left\{\begin{aligned}&\cos\omega t\left[\int_0^t \sin(\omega_0'+\omega)\tau\, e^{-\delta\tau}d\tau + \int_0^t \sin(\omega_0'-\omega)\tau\, e^{-\delta\tau}d\tau\right]\\ &+\sin\omega t\left[-\int_0^t \cos(\omega_0'+\omega)\tau\, e^{-\delta\tau}d\tau + \int_0^t \cos(\omega_0'-\omega)\tau\, e^{-\delta\tau}d\tau\right]\end{aligned}\right\}$$

$$= \frac{f_0}{2\omega_0'}\left[\cos\omega t\left(\operatorname{Im}I_+ + \operatorname{Im}I_-\right) + \sin\omega t\left(-\operatorname{Re}I_+ + \operatorname{Re}I_-\right)\right] \equiv \frac{f_0}{2\omega_0'}\operatorname{Im}\left(I_+e^{-i\omega t} + I_-e^{i\omega t}\right),$$

where

$$I_\pm \equiv \int_0^t e^{(i\omega_0' \pm i\omega - \delta)\tau}d\tau = \frac{e^{(i\omega_0' \pm i\omega - \delta)t} - 1}{i(\omega_0' \pm \omega) - \delta}. \qquad (*)$$

These formulas show that $q(t)$ may be represented as a sum of two different processes: free oscillations with the damping-renormalized own frequency ω_0' of the oscillator, decaying in time as $e^{-\delta t}$, which arise because of the sudden turn-on of the force, and forced oscillations with the frequency ω of the external force, with constant amplitude and phase. (Simple algebra shows that their amplitude is given exactly by Eq. (5.18) of the lecture notes.) Such representation is especially convenient if both the

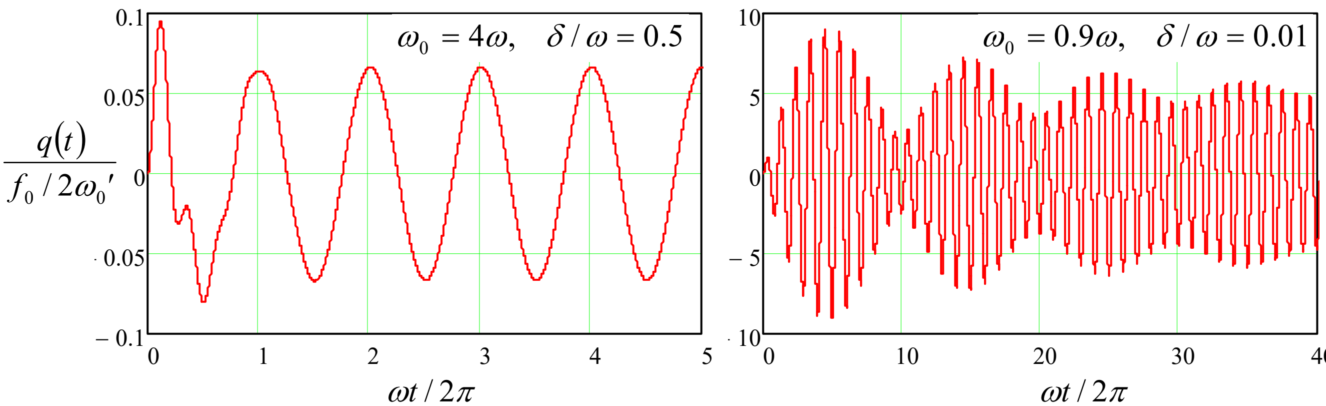
[121] See, e.g., MA Eq. (3.1)-(3.3).

damping δ and the difference between frequencies ω and ω_0 are relatively high – see, e.g., the example on the left panel of the figure below.

On the other hand, if the damping is low and $\omega \approx \omega_0$, the integral I_- evolves in time much slower than the "carrier" waveform of frequency ω, and $|I_+| << |I_-|$, so the same result for $q(t)$ may be more adequately interpreted as the oscillations of the external frequency ω, with their amplitude and phase modulated with the "beat" frequency

$$\omega_b \equiv |\omega - \omega_0|,$$

with the modulation depth slowly decaying with time ($\propto e^{-\delta t}$) – for example, see the right panel in the figure.



(ii) This duality of the result's representation is especially evident in the simple case when $\omega = \omega_0$ and $\delta << \omega_0$, so there is virtually no difference between ω_0' and ω_0. In this case, the above result is reduced to

$$q(t) = \frac{f_0}{2\omega_0\delta}\sin\omega_0 t - \frac{f_0}{2\omega_0\delta}e^{-\delta t}\sin\omega t \equiv \frac{f_0}{2\omega_0\delta}\sin\omega t\left(1 - e^{-\delta t}\right). \qquad (**)$$

Again, the first form of this expression may be interpreted as a sum of stationary forced oscillations (with the due phase shift $\Delta\varphi = \pi/2$ relative to the force[122]) and of decaying free oscillations. On the other hand, the (mathematically identical) second form, as well the plot of the whole process (see the figure below), allow the result to be also interpreted as a single-frequency process with a time-dependent amplitude,

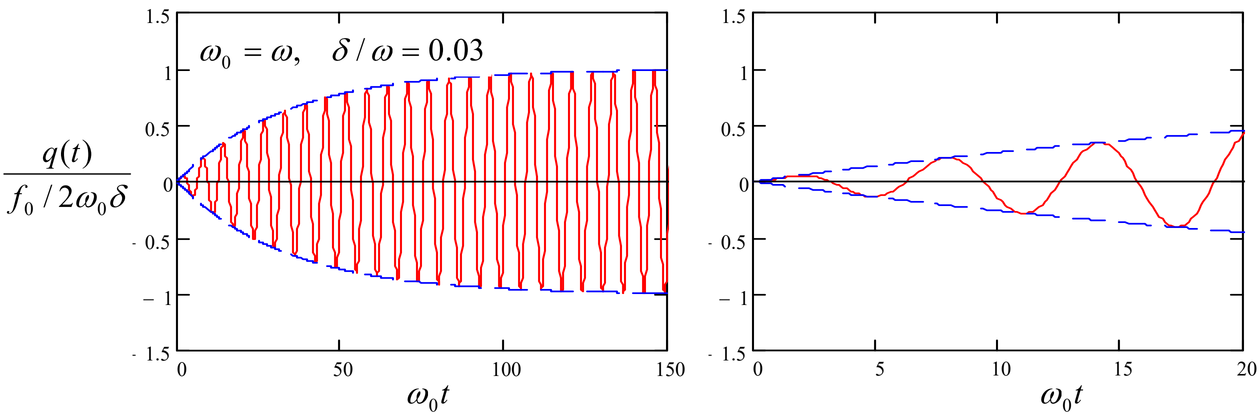
$$A(t) = \frac{f_0}{2\omega_0\delta}\left(1 - e^{-\delta t}\right),$$

which starts from zero at $t = 0$, and gradually approaches the stationary value $f_0/2\omega_0\delta$.[123] (See the "envelope" $\pm A(t)$ plotted with the blue dashed lines in the figure below.)

Note that according to Eq. (**), the beginning of the transient process (at times $t << 1/\delta$) is independent of δ, and describes oscillations with a linearly growing amplitude:

[122] See, e.g., the right panel in Fig. 5.1 of the lecture notes.
[123] This value evidently complies, for this particular case, with Eq. (5.18) of the lecture notes.

$$q(t) \approx \frac{f_0}{2\omega_0} t \sin \omega t, \quad \text{at } \delta t << 1. \qquad (**)$$



At $\delta \to 0$, both the duration of this initial stage and the finite amplitude of oscillations tend to infinity, making Eq. (**) the full solution of the problem (for zero initial conditions). This result enables the following interpretation of the stationary amplitude's divergence at $\omega = \omega_0$: after the resonant external force has been turned on, the oscillation amplitude A grows linearly in time, without saturation, with the force supplying more and more energy to the oscillator.

Problem 5.5. A pulse of external force $F(t)$, with a finite duration $\mathcal{T}$, is exerted on a linear oscillator with negligible damping, initially at rest in its equilibrium position. Use two different approaches to calculate the resulting change of the oscillator's energy.

Solution: The oscillator's energy is given by Eq. (5.1) of the lecture notes:

$$E = T(\dot{q}) + U(q) = \frac{m}{2}\dot{q}^2 + \frac{\kappa}{2}q^2 \equiv \frac{m}{2}\left(\dot{q}^2 + \omega_0^2 q^2\right).$$

In order to calculate the final value of the energy at $t = \mathcal{T}$, we may use the expression for $q(t)$ following from Eqs. (5.27) and (5.34), with $\delta = 0$ (in particular, giving $\omega_0' = \omega_0$):

$$q(t) = \frac{1}{m\omega_0}\int_0^t F(t')\sin \omega_0 (t - t')\, dt'$$

(where $t = 0$ corresponds to the beginning of the pulse), and the resulting expression for the velocity:

$$\dot{q}(t) \equiv \frac{dq(t)}{dt} = \frac{1}{m}\int_0^t F(t')\cos \omega_0 (t - t')\, dt', \qquad (*)$$

getting

$$E(\mathcal{T}) = \frac{1}{2m}\left\{\left[\int_0^{\mathcal{T}} F(t)\cos \omega_0 (\mathcal{T} - t)\, dt\right]^2 + \left[\int_0^{\mathcal{T}} F(t)\sin \omega_0 (\mathcal{T} - t)\, dt\right]^2\right\}, \qquad (**)$$

where the integration variable t' is replaced with t for brevity.

Alternatively, the energy may be calculated as the work $\mathscr{W}$ done by the force $F(t)$ on the system, by integrating its instantaneous power $\mathscr{P}(t)$ over time:

$$E(\mathcal{T}) = \int_0^{\mathcal{T}} \mathscr{P}(t)dt = \int_0^{\mathcal{T}} F(t)\dot{q}(t)dt \,.$$

Plugging in Eq. (*), we get

$$E(\mathcal{T}) = \frac{1}{m}\int_0^{\mathcal{T}} F(t)dt \int_0^{t} F(t')\cos\omega_0(t-t')dt' \,. \qquad (***)$$

It may look like these two results are different. However, Eq. (**) may be readily reduced to Eq. (***) by representing the integrals squared as double integrals, merging them, and then using the well-known trigonometric formula:[124]

$$E(\mathcal{T}) = \frac{1}{2m}\left[\int_0^{\mathcal{T}} F(t)\cos\omega_0(\mathcal{T}-t)dt \int_0^{\mathcal{T}} F(t')\cos\omega_0(\mathcal{T}-t')dt' + \int_0^{\mathcal{T}} F(t)\sin\omega_0(\mathcal{T}-t)dt \int_0^{\mathcal{T}} F(t')\sin\omega_0(\mathcal{T}-t')dt'\right]$$

$$= \frac{1}{2m}\int_0^{\mathcal{T}} F(t)dt \int_0^{\mathcal{T}} F(t')dt' \left[\cos\omega_0(\mathcal{T}-t)\cos\omega_0(\mathcal{T}-t') + \sin\omega_0(\mathcal{T}-t)\sin\omega_0(\mathcal{T}-t')\right]$$

$$= \frac{1}{2m}\int_0^{\mathcal{T}} F(t)dt \int_0^{\mathcal{T}} F(t')dt' \cos\omega_0(t-t'),$$

bringing us to Eq. (***) again.

<u>Problem 5.6</u>. A bead may slide, without friction, in a vertical plane along a parabolic curve $y = \alpha x^2/2$, with $\alpha > 0$, in a uniform gravity field $\mathbf{g} = -g\mathbf{n}_y$. Calculate the change of frequency of its small free oscillations as a function of their amplitude A, in the first nonvanishing approximation in $A \to 0$, by using two different approaches.

Solution: The kinetic energy of the bead is

$$T = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) = \frac{m}{2}\left[\dot{x}^2 + \left(\frac{dy}{dx}\frac{dx}{dt}\right)^2\right] \equiv \frac{m}{2}\left[1 + \left(\frac{dy}{dx}\right)^2\right]\dot{x}^2 = \frac{m}{2}\left(1+\alpha^2 x^2\right)\dot{x}^2,$$

while its potential energy is $U = mgy = mg\alpha x^2/2$, so the Lagrangian function is

$$L \equiv T - U = \frac{m}{2}\left(1+\alpha^2 x^2\right)\dot{x}^2 - \frac{mg\alpha}{2}x^2 \,.$$

<u>Approach 1</u>. Per Eq. (2.19a) of the lecture notes, this Lagrangian function yields the following Lagrange equation of motion for the (only) generalized coordinate of this system, x:

$$\frac{d}{dt}\left[m\left(1+\alpha^2 x^2\right)\dot{x}\right] - \left(m\alpha^2\dot{x}^2 x - mg\alpha x\right) = 0 \,.$$

This equation may be rewritten in the canonical form (5.38):

[124] See, e.g., MA Eq. (3.1a) with the bottom sign.

$$\ddot{x}+\omega^2 x = f \equiv 2\xi\omega x - \alpha^2\left(\dot{x}^2 x + \ddot{x}x^2\right),$$

where $\xi \equiv \omega - \omega_0$, and $\omega_0 \equiv (g\alpha)^{1/2}$. In the limit $A \to 0$, we may solve this equation by the harmonic balance method (see Sec. 5.2 of the lecture notes), by plugging into its right-hand side the 0^{th}-order approximation $q^{(0)}(t) = A\cos\Psi$, where $\Psi \equiv \omega t - \varphi$:

$$f \approx f^{(0)} = 2\xi\omega A\cos\Psi - \alpha^2 A^3\omega^2\left(\sin^2\Psi\cos\Psi - \cos^3\Psi\right).$$

In the last (nonlinear) term of this expression, let us separate the terms with frequency ω from those of the higher (in this particular case, 3^{rd}) harmonic:

$$\begin{aligned}\sin^2\Psi\cos\Psi - \cos^3\Psi &= (1-\cos^2\Psi)\cos\Psi - \cos^3\Psi \equiv \cos\Psi - 2\cos^3\Psi \\ &= \cos\Psi - 2\left(\frac{3}{4}\cos\Psi + \frac{1}{4}\cos 3\Psi\right) \equiv -\frac{1}{2}\cos\Psi + \text{a } 3^{\text{rd}} \text{ harmonicterm.}\end{aligned}$$

Now, requiring the net amplitude of the component of frequency ω of the function $f^{(0)}$ to vanish, we get the following harmonic-balance equation:

$$2\xi\omega A + \frac{\alpha^2\omega^2}{2}A^3 = 0.$$

Since this analysis is valid only asymptotically at $\alpha^2 A^2 \to 0$ and $|\xi| << \omega$, the difference between the frequencies ω and ω_0 is negligible in all terms except for the detuning $\xi \equiv \omega - \omega_0$. As a result, we may write

$$\omega - \omega_0 \equiv \xi \approx -\frac{\alpha^2 A^2}{4}\omega_0\,.$$

Approach 2. Since for this system, $\partial L/\partial t = 0$, its Hamiltonian function

$$H \equiv \frac{\partial L}{\partial \dot{x}}\dot{x} - L = m\left(\dot{x} + \alpha^2 x^2\dot{x}\right)\dot{x} - \left[\frac{m}{2}\left(1+\alpha^2 x^2\right)\dot{x}^2 - \frac{mg\alpha}{2}x^2\right] \equiv \frac{m}{2}\left(1+\alpha^2 x^2\right)\dot{x}^2 + \frac{mg\alpha}{2}x^2 = T + U \equiv E$$

is an integral of motion. Using this integral to find the generalized velocity,

$$\dot{x} = \pm\left(\frac{2E/m - g\alpha x^2}{1+\alpha^2 x^2}\right)$$

and taking into account that it is an even function of x, we may find the oscillation period $\mathcal{T}$ as

$$\mathcal{T} = \oint_{\substack{\text{one}\\ \text{period}}} dt = \oint_{\substack{\text{one}\\ \text{period}}} \frac{dx}{\dot{x}} = 4\int_0^A \left(\frac{1+\alpha^2 x^2}{2E/m - g\alpha x^2}\right)^{1/2} dx\,,$$

where A is the oscillation amplitude determined as the value of x at $\dot{x} = 0$, i.e. $2E/m - g\alpha A^2 = 0$, so

$$\mathcal{T} = 4\int_0^A \left(\frac{1+\alpha^2 x^2}{g\alpha A^2 - g\alpha x^2}\right)^{1/2} dx \equiv \frac{4}{A\omega_0}\int_0^A \left(\frac{1+\alpha^2 x^2}{1 - x^2/A^2}\right)^{1/2} dx, \qquad \text{with } \omega_0 \equiv (g\alpha)^{1/2}.$$

For $\alpha = 0$ but $\omega_0 \neq 0$, this integral may be readily calculated (see, e.g., Eq. (3.28) of the lecture notes), giving an amplitude-independent period

$$\mathcal{T} = \frac{2\pi}{\omega_0},$$

i.e. the amplitude-independent frequency $\omega = \omega_0$. To capture the frequency's dependence on the frequency at small A, we may expand the numerator of the function under the integral into the Taylor series in small $(\alpha x)^2 \sim (\alpha A)^2$, and keep only two leading terms:

$$\mathcal{T} = \frac{4}{A\omega_0}\int_0^A \frac{\left(1+\alpha^2 x^2\right)^{1/2} dx}{\left(1 - x^2/A^2\right)^{1/2}} \approx \frac{4}{A\omega_0}\int_0^A \frac{1+\alpha^2 x^2/2}{\left(1-x^2/A^2\right)^{1/2}} dx \equiv \frac{4}{\omega_0}\left[\int_0^1 \frac{d(x/A)}{\left(1-x^2/A^2\right)^{1/2}} + \frac{\alpha^2 A^2}{2}\int_0^1 \frac{(x/A)^2 d(x/A)}{\left(1-x^2/A^2\right)^{1/2}}\right].$$

Both integrals may be readily worked out using the same substitution $x/A \equiv \sin\xi$, so $(1 - x^2/A^2)^{1/2} = \cos\xi$, and

$$\int_0^1 \frac{d(x/A)}{\left(1-x^2/A^2\right)^{1/2}} = \int_0^1 \frac{d(\sin\xi)}{\cos\xi} = \int_0^{\pi/2} d\xi = \frac{\pi}{2},$$

$$\int_0^1 \frac{(x/A)^2 d(x/A)}{\left(1-x^2/A^2\right)^{1/2}} = \int_0^1 \frac{\sin^2\xi \, d(\sin\xi)}{\cos\xi} = \int_0^{\pi/2} \sin^2\xi \, d\xi = \frac{1}{2}\int_0^{\pi/2}\left(1-\cos 2\xi\right)d\xi = \frac{\pi}{4},$$

finally giving

$$\mathcal{T} \approx \frac{4}{\omega_0}\left(\frac{\pi}{2} + \frac{\alpha^2 A^2}{2}\frac{\pi}{4}\right) \equiv \frac{2\pi}{\omega_0}\left(1 + \frac{\alpha^2 A^2}{4}\right),$$

so the oscillation frequency

$$\omega \equiv \frac{2\pi}{\mathcal{T}} \approx \frac{\omega_0}{1+\alpha^2 A^2/4} \approx \omega_0\left(1 - \frac{\alpha^2 A^2}{4}\right).$$

This is the same result as was obtained using Approach 1. Note, however, that Approach 2 was based on the conservation of energy, while Approach 1, based on the harmonic balance method, is applicable even in the absence of such first integral of motion, though being fundamentally conditioned by the small amplitude limit.

Problem 5.7. For a system with the Lagrangian function

$$L = \frac{m}{2}\dot{q}^2 - \frac{\kappa}{2}q^2 + \varepsilon\dot{q}^4,$$

with small parameter ε, use the harmonic balance method to find the frequency of free oscillations as a function of their amplitude.

Solution: According to Eq. (2.19a) of the lecture notes, the given Lagrangian function yields the following equation of motion,

$$\frac{d}{dt}\left(m\dot{q} + 4\varepsilon\dot{q}^3\right) + \kappa q = 0,$$

which may be rewritten in the standard form (5.38):

$$\ddot{q} + \omega^2 q = f \equiv 2\xi\omega q - \frac{12\varepsilon}{m}\dot{q}^2\ddot{q},$$

where $\xi \equiv \omega - \omega_0$, and $\omega_0 \equiv (\kappa/m)^{1/2}$. Following the general recipe of the harmonic balance method, we plug into the right-hand part of this equation the 0^{th}-order solution $q^{(0)}(t) = A\cos\Psi$, with $\Psi \equiv \omega t - \varphi$:

$$f \to f^{(0)} = 2\xi\omega A\cos\Psi + \frac{12\varepsilon\omega^4}{m}A^3\sin^2\Psi\cos\Psi.$$

Then, in the nonlinear term of this expression, we have to separate the components of the basic frequency ω from those of the higher (in this particular case, 3^{rd}) harmonic:

$$\sin^2\Psi\cos\Psi = (1-\cos^2\Psi)\cos\Psi \equiv \cos\Psi - \cos^3\Psi = \frac{1}{4}\cos\Psi + \text{the } 3^{\text{rd}} \text{ harmonic}.$$

Now, requiring the net amplitude of the component of frequency ω of the function $f^{(0)}$ to vanish, we get the harmonic-balance equation

$$2\xi\omega A + \frac{3\varepsilon\omega^4}{m}A^3 = 0.$$

Since all our analysis is valid only asymptotically at $\varepsilon \to 0$, and at $\varepsilon = 0$ we have $\omega = \omega_0$, the difference between these two frequencies is negligible in all terms except for the detuning $\xi = \omega - \omega_0$. As a result, we may write

$$\omega = \omega_0 + \xi \approx \omega_0\left(1 - \frac{3\varepsilon\omega_0^2}{2m}A^2\right). \qquad (*)$$

This asymptotic expression is quantitatively only valid when the magnitude of the second term in the parentheses is much less than 1.

Note that since for this system, $\partial L/\partial t = 0$, its Hamiltonian function,

$$H \equiv \frac{\partial L}{\partial \dot{q}}\dot{q} - L = \frac{m}{2}\dot{q}^2 + \frac{\kappa}{2}q^2 + 3\varepsilon\dot{q}^4, \qquad (**)$$

is an integral of motion, which may be used, just as it was done in the previous problem's solution, for an alternative way to derive Eq. (*).[125]

In this context, let me make one more important remark. Nothing prevents us from rewriting our Lagrangian function as the usual difference

$$L = T - U, \qquad \text{with } T = \frac{m}{2}\dot{q}^2 + \varepsilon\dot{q}^4, \quad U = \frac{\kappa}{2}q^2,$$

and interpreting T as the kinetic energy of the system, and U its potential energy. However, this does not mean that their sum,

$$T + U = \frac{m}{2}\dot{q}^2 + \frac{\kappa}{2}q^2 + \varepsilon\dot{q}^4, \qquad (***)$$

(i.e. the apparent total mechanical energy of the system) is conserved. Indeed, the conservation of this sum would contradict the already proved conservation of the H that is given by Eq. (**) and differs from Eq. (***) by a coefficient in the term proportional to ε. Formally, this fact is not surprising, because this T is not a quadratic-homogeneous function of the generalized velocity, and hence H may be different from $T + U$. Physically, such a situation is met, for example, in special relativity, where, in the limit of

[125] See, e.g., EM Problem 9.9.

small velocities ($v^2 \equiv \dot{q}^2 << c^2$), Eq. (**) rather than Eq. (***) gives the genuine energy of a 1D particle in a quadratic potential well – or rather its difference from the rest energy mc^2.[126]

Problem 5.8. Use a different approach to derive Eq. (5.49) of the lecture notes for the frequency of free oscillations of the system described by the Duffing equation (5.43) with $\delta = 0$, in the first nonvanishing approximation in the small parameter $\alpha A^2/\omega_0^2 << 1$.

Solutions: In the absence of damping and an external force, Eq. (5.43) is reduced to

$$\ddot{q} = -\omega_0^2 q + \alpha q^3,$$

and is identical to the equation (3.2) of motion of a 1D particle of mass m in the following time-independent potential well:

$$U(q) = \frac{m\omega_0^2}{2} q^2 - \frac{m\alpha}{4} q^4.$$

Since the Lagrangian function of this system,

$$L \equiv T - U = \frac{m\dot{q}^2}{2} - \frac{m\omega_0^2}{2} q^2 + \frac{m\alpha}{4} q^4,$$

does not explicitly depend on time, its Hamiltonian function H

$$H \equiv \frac{\partial L}{\partial \dot{q}} \dot{q} - L = \frac{m}{2}\dot{q}^2 + \frac{m\omega_0^2}{2} q^2 - \frac{m\alpha}{4} q^4,$$

(which, in this case, is also the system's energy $E \equiv T + U$), is the first integral of motion: $dH/dt = 0$. Using this integral to calculate the generalized velocity as a function of the coordinate,

$$\dot{q} = \pm\left(\frac{2H}{m} - \omega_0^2 q^2 + \frac{\alpha q^4}{2}\right)^{1/2},$$

and taking into account that it is an even function of q, we may find the oscillation period $\mathcal{T}$ as

$$\mathcal{T} = \oint_{\substack{\text{one}\\\text{period}}} dt = \oint_{\substack{\text{one}\\\text{period}}} \frac{dq}{\dot{q}} = 4\int_0^A \frac{dq}{\left(2H/m - \omega_0^2 q^2 + \alpha q^4/2\right)^{1/2}}.$$

Here A is the oscillation amplitude, which is determined by the condition $\dot{q} = 0$ at $q = A$, giving

$$\frac{2H}{m} = \omega_0^2 A^2 - \frac{\alpha A^4}{2},$$

so

$$\mathcal{T} = 4\int_0^A \frac{dq}{\left[\omega_0^2\left(A^2 - q^2\right) - \alpha\left(A^4 - q^4\right)/2\right]^{1/2}} \equiv \frac{4}{\omega_0}\int_0^1 \frac{d\xi}{\left[\left(1-\xi^2\right) - \alpha A^2\left(1-\xi^4\right)/2\omega_0^2\right]^{1/2}}, \qquad (*)$$

where $\xi \equiv q/A$.

[126] In that particular case, $\varepsilon = m/8c^2$, where c is the speed of light – see, e.g., EM Sec. 9.3.

If $\alpha = 0$, this integral may be readily calculated – for example as in Eq. (3.28) of the lecture notes, e.g., by using the variable substitution $\xi \equiv \sin\alpha$:

$$\int_0^1 \frac{d\xi}{\left(1-\xi^2\right)^{1/2}} = \int_0^1 \frac{d(\sin\alpha)}{\cos\alpha} = \int_0^{\pi/2} d\alpha = \frac{\pi}{2}.$$

This gives us the amplitude-independent $\mathcal{T} = 2\pi/\omega_0$, i.e. the amplitude-independent oscillation frequency $\omega \equiv 2\pi/\mathcal{T} = \omega_0$. In order to capture the frequency's dependence on the amplitude at small but non-zero α, we may expand the denominator of the function under the integral in Eq. (*) into the Taylor series in small $\alpha A^2/\omega_0^2 << 1$, and keep only two leading terms:

$$\mathcal{T} = \frac{4}{\omega_0}\int_0^1 \frac{d\xi}{\left(1-\xi^2\right)^{1/2}\left[1-\left(\alpha A^2/2\omega_0^2\right)\left(1+\xi^2\right)\right]^{1/2}} \approx \frac{4}{\omega_0}\int_0^1 \frac{d\xi}{\left(1-\xi^2\right)^{1/2}}\left[1+\frac{\alpha A^2}{4\omega_0^2}\left(1+\xi^2\right)\right] =$$
$$\equiv \frac{4}{\omega_0}\int_0^1 \frac{d\xi}{\left(1-\xi^2\right)^{1/2}} + \frac{\alpha A^2}{\omega_0^3}\int_0^1 \frac{\left(1+\xi^2\right)}{\left(1-\xi^2\right)^{1/2}} d\xi.$$

The first of these integrals has already been calculated and the remaining one may be worked out using the same substitution, $\xi \equiv \sin\alpha$:

$$\int_0^1 \frac{\left(1+\xi^2\right)d\xi}{\left(1-\xi^2\right)^{1/2}} = \int_0^{\pi/2}\left(1+\sin^2\alpha\right)d\alpha = \int_0^{\pi/2}\left(\frac{3}{2}-\frac{1}{2}\cos 2\alpha\right)d\alpha = \frac{3\pi}{4},$$

finally giving

$$\mathcal{T} \approx \frac{2\pi}{\omega_0} + \frac{3\pi\alpha A^2}{4\omega_0^3} \equiv \frac{2\pi}{\omega_0}\left(1+\frac{3\alpha A^2}{8\omega_0^2}\right),$$

so the oscillation frequency

$$\omega \equiv \frac{2\pi}{\mathcal{T}} \approx \frac{\omega_0}{1+3\alpha A^2/8\omega_0^2} \approx \omega_0 - \frac{3\alpha A^2}{8\omega_0}.$$

This is the same result as given by Eq. (5.49). (The difference between ω and ω_0 in the second, already small, term is beyond the accuracy of our first approximation.) Note, however, that the harmonic-balance approach to this problem, used in Sec. 5.2 of the lecture notes, is more general, because it may be used even in the absence of a first integral of motion – say, for dissipative and/or externally-driven systems. (On the other hand, Eq. (*) is valid for an arbitrary oscillation amplitude – or, more exactly, for any $\alpha A^2/\omega_0 < 1$ – the range where the Duffing equation has periodic solutions.)

Problem 5.9. On the plane $[a_1, a_2]$ of two real parameters a_1 and a_2, find the regions in which the fixed point of the following system of equations,

$$\dot{q}_1 = a_1(q_2 - q_1),$$
$$\dot{q}_2 = a_2 q_1 - q_2,$$

is unstable, and sketch the regions of each fixed point type – stable and unstable nodes, focuses, etc.

Solution: The only fixed point of this system of equations is the trivial one, $q_1 = q_2 = 0$, and since both equations are already linear, they do not need an additional linearization in the vicinity of this point. Looking for the solution of the system in the usual form $q_{1,2} \propto \exp\{\lambda t\}$, we get the characteristic equation

$$\begin{vmatrix} -a_1 - \lambda & a_1 \\ a_2 & -1-\lambda \end{vmatrix} \equiv \lambda^2 + \lambda(a_1 + 1) + a_1(1 - a_2) = 0 .$$

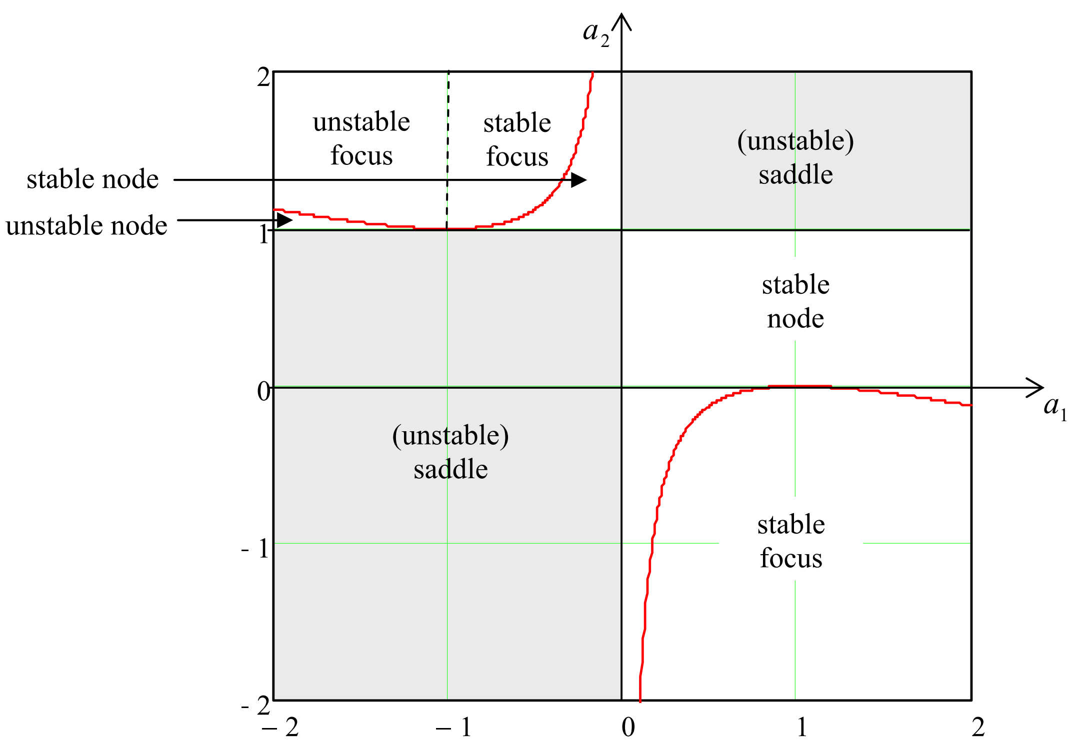
Solving this quadratic equation, we get the following two roots:

$$\lambda_\pm = -\frac{a_1+1}{2} \pm \left[\left(\frac{a_1+1}{2}\right)^2 + a_1(a_2 - 1)\right]^{1/2} \equiv -\frac{a_1+1}{2} \pm \frac{1}{2}\left[(a_1-1)^2 + 4a_1a_2\right]^{1/2}. \qquad (*)$$

The fixed point's type depends, first of all, on whether the expression under the square root is positive or negative. As the last form of Eq. (*) shows, it is negative (and hence both roots $\lambda_\pm$ are complex, i.e. the fixed point is a focus) if

$$4a_1a_2 < -(a_1 - 1)^2 .$$

This relation is satisfied if either $a_1 > 0$ and $a_2 < f(a_1) \equiv -(a_1 - 1)^2/4a_1$, or $a_1 < 0$ and $a_2 > f(a_1)$. This means that in the figure below, the point $[a_1, a_2]$ is somewhere outside the region limited by the two red curves, which show the function $f(a_1)$.



Whether the focus is stable or unstable is determined by the real part of the roots, i.e. by the expression $-(a_1 + 1)/2$ before the square root in Eq. (*). In particular, the focus is unstable (Re $\lambda_\pm > 0$), if $-(a_1 + 1)/2 > 0$, meaning that $a_1 < -1$, i.e. on the left of the vertical dashed line shown in the figure above.

If the point $[a_1, a_2]$ is located between the two red curves, both roots $\lambda_\pm$ are real, and the stability depends on the sign of the largest root, λ_+. It is positive (i.e. the fixed point is unstable) if the square root in Eq. (*) is larger than $|-(a_1 + 1)/2|$; according to the first form of Eq. (*), this is true if

$$a_1(a_2 - 1) > 0,$$

i.e. if a_1 and $(a_2 - 1)$ have the same sign. In the figure above, these regions are shaded; in them, the fixed point is a saddle – which is always unstable. In the region $a_1 < -1$, i.e. to the left of the dashed vertical line in the figure below (but still between the red lines), the smallest root, λ_-, is also positive, so the fixed point is an unstable node.

The figure legend summarizes the resulting picture of the fixed point types. Finally, let me note that this problem is not occasional, but is a particular case (with $q_3(t) \equiv 0$) of the famous system (9.1) of three nonlinear equations, whose study by E. Lorenz in the early 1960s has triggered a wave of studies of *deterministic chaos* effects, to be discussed in Chapter 9. The instability of the trivial point within certain regions of the parameter space, which was discussed above, implies (though does not prove) nontrivial dynamic properties of the full system.

Problem 5.10. Solve Problem 4(ii) by using the reduced equations (5.57), and compare the result with the exact solution.

Solutions: For a linear oscillator, we may reuse, for example, the reduced equations Eqs. (5.60) for the nonlinear oscillator (5.43) by taking $\alpha = 0$, so, per Eq. (5.48), in our current case when $\omega = \omega_0$, the function $\xi(A)$ equals zero for any A and the equations are reduced to

$$\dot{A} = -\delta A + \frac{f_0}{2\omega}\sin\varphi, \qquad \dot{\varphi} = \frac{f_0}{2\omega A}\cos\varphi. \qquad (*)$$

In the very beginning of the transient, when the amplitude A of the oscillations is very small, the right-hand side of the second of these equations is very large, so it describes a very fast relaxation of the phase φ to the stable fixed point $\varphi_+ = \pi/2$. (See, e.g., Fig. 5.5 of the lecture notes and its discussion.)[127] With this value, the first of Eqs. (*) takes a simple linear form,

$$\dot{A} = -\delta A + \frac{f_0}{2\omega},$$

and may be readily integrated – for example by taking $A(t) \equiv A_0 + A_1(t)$, where $A_0 \equiv f_0/2\omega\delta = \text{const}$ so that $-\delta A_0 + f_0/2\omega = 0$. With this variable replacement, the differential equation becomes homogeneous,

$$\dot{A}_1 = -\delta A_1, \qquad \text{with } A_1(0) = -A_0 \equiv -\frac{f_0}{2\omega\delta},$$

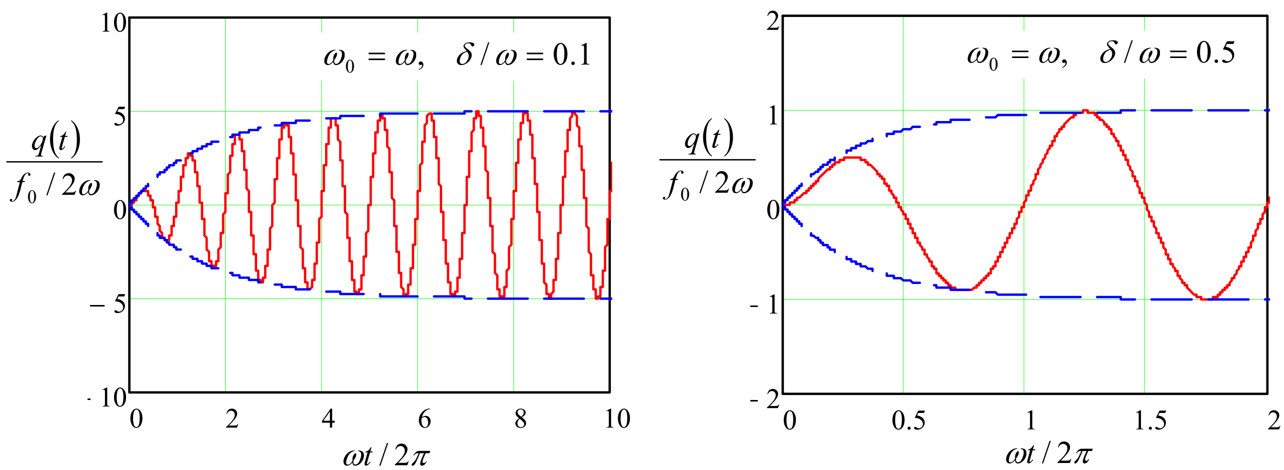
and has the well-known exponential solution $A_1(t) = A_1(0)e^{-\delta t}$, finally giving

[127] Actually, if the initial value of the amplitude is exactly zero, there is no transient at all: the oscillations just start to grow with the "correct" phase φ_+. The best way to prove that is to rewrite the reduced equations (*) in the Cartesian form (5.57c), just as this was done in Sec. 5.5 of the lecture notes for the parametric oscillation analysis – see Eq. (5.79). Let me leave this simple calculation for the reader's additional exercise.

$$A(t) = \frac{f_0}{2\omega\delta}\left(1 - e^{-\delta t}\right). \tag{**}$$

But taking into account that in our case $\omega = \omega_0$, this is exactly the result that was obtained in the solution of Problem 4(ii), in the similar limit $\delta << \omega$.

It is also instructive to use this simple problem to see how critical is the smallness of a typical "small parameter" (in this case, δ) for the reduced equations' accuracy. The red lines in the figure below show the plots given by the exact formula for $q(t)$, derived in the solution of Problem 4(i), for the exact resonance ($\omega = \omega_0$), while the dashed blue lines show the functions $\pm A(t)$ following from Eq. (**) (which are supposed to give the oscillations' envelope), for two values of the δ/ω_0 ratio.



The plots show that the reduced equations work quite reasonably even for δ/ω_0 as large as 0.5, i.e. just twice smaller than the critical value at which the renormalized frequency ω_0' vanishes, so the oscillator turns into a relaxator. Thus the van der Pol approximation shares the nice common feature of all good asymptotic methods: usually, they give practically acceptable accuracy far beyond their supposed validity range.

Problem 5.11. Use the reduced equations to analyze forced oscillations in an oscillator with weak nonlinear damping, described by the following equation:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^{\ 2} q + \beta\dot{q}^3 = f_0 \cos\omega t,$$

with $\omega \approx \omega_0$; $\beta, \delta > 0$; and $\beta\omega A^2 << 1$. In particular, find the stationary amplitude of the forced oscillations and analyze their stability. Discuss the effect(s) of the nonlinear term on the resonance.

Solution: Calculating the right-hand sides of the reduced equations (5.57a) for this case, we get

$$\dot{A} = -\delta(A)A + \frac{f_0}{2\omega}\sin\varphi,$$
$$A\dot{\varphi} = \xi A + \frac{f_0}{2\omega}\cos\varphi, \quad \text{where} \ \ \delta(A) \equiv \delta + \frac{3}{8}\beta\omega^2 A^2. \tag{*}$$

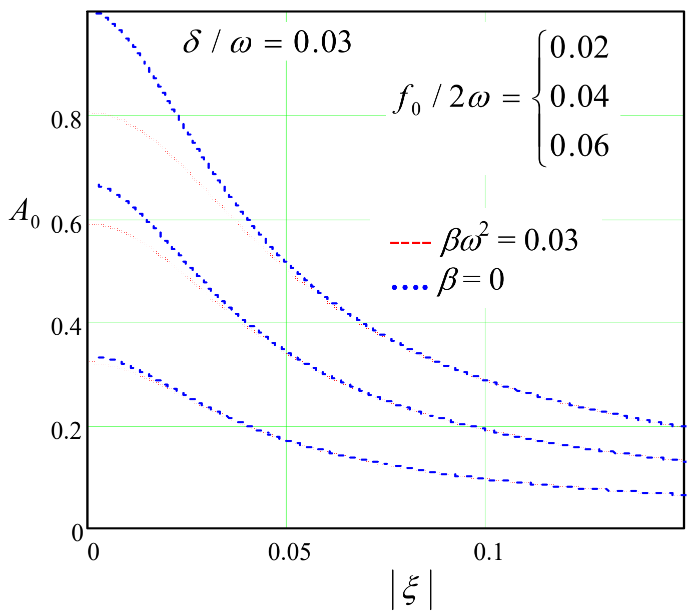
Here the function $\delta(A)$, which was already used in Eq. (5.63a) in the lecture notes, has the meaning of the effective damping coefficient, which grows with the oscillation amplitude. With this notation, the only fixed point $\{A_0, \varphi_0\}$ of the reduced equations (*) satisfies the system of algebraic equations formally similar to those describing the linear oscillator:

$$\delta(A_0)A_0 = \frac{f_0}{2\omega}\sin\varphi_0, \qquad \xi A_0 = -\frac{f_0}{2\omega}\cos\varphi_0, \qquad (**)$$

from which the oscillation phase may be readily eliminated:

$$\left(\frac{f_0}{2\omega}\right)^2 = A_0^2\left[\delta^2(A_0) + \xi^2\right].$$

This result, which may be readily used to write an explicit expression for resonant curves in the reciprocal form $\xi(A_0)$, shows that at $\xi \approx 0$ (i.e., at $\omega \approx \omega_0$), the system exhibits the usual symmetric resonance but for larger values of f_0, the curves $A_0(\xi)$ are somewhat flattened on the top – see the figure on the right for an example.



In order to analyze the stability of this fixed point, we can, as usual, plug into Eqs. (*) the following variable replacement:

$$A = A_0 + \tilde{A}(t), \quad \varphi = \varphi_0 + \tilde{\varphi}(t),$$

and then linearize the resulting differential equations with respect to small variations. The resulting linear equations,

$$\dot{\tilde{A}} = -\delta'(A_0)A_0\tilde{A} - \delta(A_0)\tilde{A} + \frac{f_0}{2\omega}\cos\varphi_0\,\tilde{\varphi},$$
$$A_0\dot{\tilde{\varphi}} = -\frac{f_0}{2\omega}\sin\varphi_0\,\tilde{\varphi} - \frac{f_0}{2\omega A_0}\cos\varphi_0\tilde{A},$$

where $\delta'(A) \equiv d\delta(A)/dA$, may be substantially simplified using Eqs. (**):

$$\dot{\tilde{A}} = -\left[\delta'(A_0)A_0 + \delta(A_0)\right]\tilde{A} - \xi A_0\,\tilde{\varphi},$$
$$A_0\dot{\tilde{\varphi}} = \xi\,\tilde{A} - \delta(A_0)A_0\,\tilde{\varphi},$$

As a result, the characteristic equation has a simple form:

$$\begin{vmatrix} -\left[\delta(A_0) + \delta'(A_0)A_0\right] - \lambda & -\xi \\ \xi & -\delta(A_0) - \lambda \end{vmatrix} = 0,$$

and its roots may be expressed as

$$\lambda_\pm = -\left[\delta(A_0) + \frac{1}{2}\delta'(A_0)A_0\right] \pm \left[\left(\frac{1}{2}\delta'(A_0)A_0\right)^2 - \xi^2\right]^{1/2}.$$

There can be two cases here. If the detuning is small, $|\xi| < \delta'(A_0)A_0/2$, i.e. if we are close to the resonance, then the expression under the square root is positive, and both roots $\lambda_\pm$ are real. Note, however, that because of the term $\xi^2 > 0$, the square root's modulus is always less than $\delta'(A_0)A_0/2$. Since the latter term is positive (indeed, by differentiating $\delta(A)$ over A, we get $\delta'(A_0)A_0 = (3/4)\beta\omega^2 A_0^2 > 0$), both roots $\lambda_\pm$ are negative, so the fixed point is a stable node.

For larger values of detuning, $|\xi| > \delta'(A_0)A_0/2$, the expression under the square root is negative, and both roots $\lambda_\pm$ are complex, with equal and negative real parts, so the fixed point is a stable focus. (This case is close to the forced oscillations in a *linear* oscillator, with $\beta = 0$ and hence $\delta(A) = \delta$ and $\delta'(A) = 0$, when $\lambda_\pm = -\delta \pm i\xi$, so the fixed point is a stable focus for *any* ξ.)

Thus for an oscillator with weak nonlinearity of the drag, the fixed point corresponding to the forced oscillations is stable for any values of ξ and f_0, i.e. for any frequency and amplitude of the driving force. However, if the (potential) returning force is nonlinear, such as in the case described by the Duffing equation (5.43), the situation is quite different – see Problem 13 below.

Problem 5.12. Within the approach discussed in Sec. 5.4 of the lecture notes, calculate the average frequency of a self-oscillator outside of the range of its phase locking by a weak sinusoidal force.

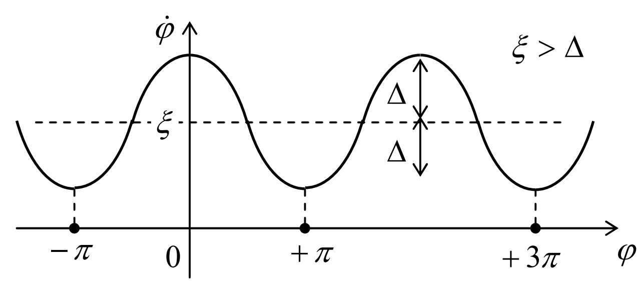
Solution: According to Eq. (5.41) of the lecture notes, which was used for the analysis in Sec. 5.4, the average frequency of the quasi-sinusoidal oscillations $q(t)$ may be calculated as

$$\overline{\omega}_q \equiv \omega - \overline{\dot\varphi} \equiv \omega_0 + \xi - \overline{\dot\varphi}, \qquad (*)$$

where the averaging is carried out over a sufficiently long time interval. Within the van der Pol approximation used in Sec. 5.4, the time evolution of the phase shift φ is given by Eq. (5.68):

$$\frac{d\varphi}{dt} = \xi + \Delta\cos\varphi, \qquad \text{i.e. } dt = \frac{d\varphi}{\xi + \Delta\cos\varphi}. \qquad (**)$$

For notation simplicity, let us assume that $\Delta > 0$ (the final result does not depend on the sign of this parameter), and start with the case $\xi \geq \Delta$. (As a reminder, at $|\xi| \leq \Delta$, Eq. (**) yields φ = const, so the phase locking is complete: $\omega_q = \omega$.) The figure on the right shows (schematically) the phase plane of Eq. (**) for this case. From this diagram, it is clear that phase φ increases monotonically, but with a periodically increasing and decreasing rate $d\varphi/dt$.[128] Taking into account that, according to Eq. (**), $d\varphi/dt$



[128] An additional exercise for the reader: explore this motion in more detail, by using the fact that Eq. (**) may be analytically integrated even in indefinite limits – see, e.g., MA Eq. (6.3c).

is an even and 2π-periodic function of φ, the time $\mathcal{T}$ during which the phase φ increases by 2π may be calculated, from that equation, as

$$\mathcal{T} = \int_{-\pi}^{+\pi} \frac{d\varphi}{\xi + \Delta\cos\varphi} = 2\int_{0}^{\pi} \frac{d\varphi}{\xi + \Delta\cos\varphi}.$$

Integrals of this type may be worked out by the standard substitution $s \equiv \tan(\varphi/2)$, giving $\cos\varphi = (1 - s^2)/(1 + s^2)$, and $d\varphi = 2ds/(1 + s^2)$. As a result, we get

$$\mathcal{T} = 2\int_0^\infty \frac{2ds/\left(1+s^2\right)}{\xi + \Delta\left(1-s^2\right)/\left(1+s^2\right)} \equiv 4\int_0^\infty \frac{ds}{(\xi+\Delta)+s^2(\xi-\Delta)} \equiv \frac{4}{\left(\xi^2-\Delta^2\right)^{1/2}}\int_0^\infty \frac{d\zeta}{1+\zeta^2},$$

where $\zeta \equiv [(\xi - \Delta)/(\xi + \Delta)]^{1/2}s$. The last integral is well-known,[129] and equal to $\pi/2$, so we finally have

$$\mathcal{T} = \frac{2\pi}{\left(\xi^2-\Delta^2\right)^{1/2}}.$$

From here, the average speed of the phase increase is

$$\overline{\dot\varphi} = \frac{2\pi}{\mathcal{T}} = \left(\xi^2-\Delta^2\right)^{1/2},$$

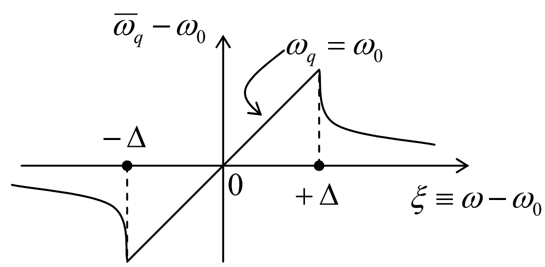
so Eq. (*) yields

$$\overline{\omega}_q = \omega_0 + \xi - \overline{\dot\varphi} = \omega_0 + \xi - \left(\xi^2-\Delta^2\right)^{1/2}, \qquad \text{for } \xi \geq +\Delta.$$

An absolutely similar calculation for negative values of the detuning yields

$$\overline{\omega}_q = \omega_0 + \xi - \overline{\dot\varphi} = \omega_0 + \xi + \left(\xi^2-\Delta^2\right)^{1/2}, \qquad \text{for } \xi \leq -\Delta.$$

Together with the result obtained in Sec. 5.4 of the lecture notes ($\omega_q = \omega$ at $|\xi| \leq \Delta$), we get the general picture shown (schematically) in the figure on the right. It shows that the average frequency of the self-oscillations is a continuous function of the detuning $\xi \equiv \omega - \omega_0$, and as the magnitude of ξ is increased well beyond the phase-locking range Δ, the frequency gradually returns to its unperturbed value ω_0.



Problem 5.13.* Use the reduced equations to analyze the stability of the forced nonlinear oscillations described by the Duffing equation (5.43). Relate the result to the slope of the resonance curves (Fig. 5.4).

Solution: Acting just in Problem 11, we look for the solution of the corresponding reduced equations (5.60) in the form

$$A = A_n + \tilde{A}(t), \quad \varphi = \varphi_n + \tilde{\varphi}(t),$$

where each fixed point $\{A_n, \varphi_n\}$ satisfies the harmonic balance equations (5.47):

[129] See, e.g., MA Eq. (6.5a).

$$2\xi(A_n)\omega A_n + f_0\cos\varphi_n = 0, \qquad 2\delta\omega A_n - f_0\sin\varphi_n = 0, \qquad \text{where } \xi(A_n) \equiv \xi + \frac{3}{8}\frac{\alpha A_n^2}{\omega}. \qquad (*)$$

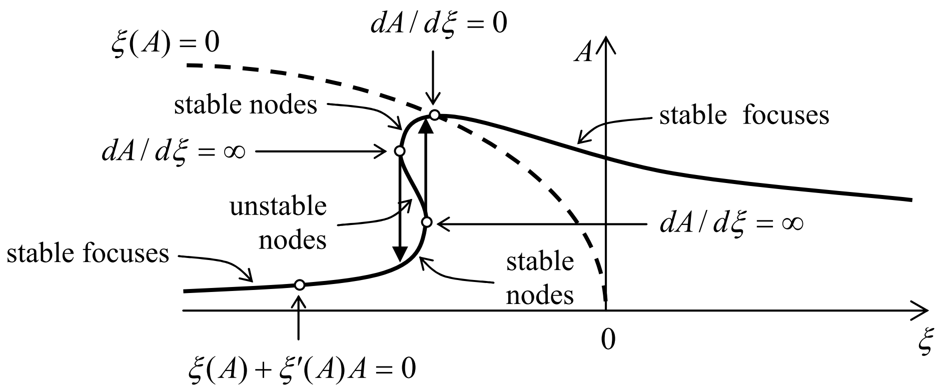
The linearized system yields the following characteristic equation:

$$\begin{vmatrix} -\delta-\lambda & -\xi(A_n) \\ \xi(A_n)+\xi'(A_n)A_n & -\delta-\lambda \end{vmatrix} = 0, \qquad \text{where } \xi'(A) \equiv \frac{d\xi(A)}{dA},$$

with two roots,

$$\lambda_\pm = -\delta \pm i\{\xi(A_n)[\xi(A_n)+\xi'(A_n)A_n]\}^{1/2}, \qquad (**)$$

for each fixed point. (As a reminder: depending on the parameters, the system may have either one or three of them – see Fig. 5.4 in the lecture notes and the crude sketch in the figure below.)



To analyze Eq. (**), let us assume that $\alpha > 0$. (In the opposite case, the situation is mirror-symmetric with respect to the point $\xi = 0$.) In this case, $\xi'(A) \equiv (3\alpha/4\omega)A$ is positive for all fixed points. As a result, for all positive $\xi(A)$, i.e. for all fixed points on the right side of the "skeleton curve" $\xi(A) = 0$ (shown with the dashed line in the figure above), the expression under the square root in Eq. (**) is positive, so the fixed points are stable focuses. Moving to the left of the skeleton curve, we are changing the sign of $\xi(A)$, and hence that of the expression under the square root, thus making both roots $\lambda_\pm$ real, and turning the fixed point into a node. For small negative $\xi(A)$, these nodes are stable. However, one of the roots λ becomes positive (and hence the node unstable) as soon as the modulus of the square root becomes equal to δ, i.e. at the point where

$$\xi(A_n)[\xi(A_n)+\xi'(A_n)A_n] + \delta^2 = 0.$$

To interpret this condition, let us consider the differential of the first of Eqs. (5.48),

$$A_n^2 = \frac{f_0^2}{4\omega^2}\frac{1}{\xi^2(A_n)+\delta^2}, \qquad (***)$$

which follows from Eqs. (*) at fixed f_0 and other parameters of the system besides the detuning ξ and the stationary value A_n of amplitude A:

$$2A_n dA_n = -\frac{f_0^2}{4\omega^2}\frac{d[\xi^2(A_n)]}{[\xi^2(A_n)+\delta^2]^2} = -\frac{f_0^2}{4\omega^2}\frac{2\xi(A_n)[d\xi+\xi'(A_n)dA_n]}{[\xi^2(A_n)+\delta^2]^2} = -A_n^2\frac{2\xi(A_n)[d\xi+\xi'(A_n)dA_n]}{\xi^2(A_n)+\delta^2},$$

where for the last step, Eq. (***) was used again. Solving the resulting equation for dA_n, we get

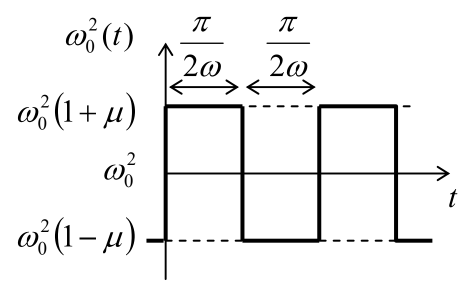
$$dA_n = -\frac{\xi(A_n)A_n}{\xi(A_n)\left[\xi(A_n) + \xi'(A_n)A_n\right] + \delta^2} d\xi .$$

This expression shows that derivative $dA_n/d\xi$ diverges, i.e. the resonance curve has the vertical slope (see the figure above) exactly as the fixed point becomes unstable. Hence, as could be expected, in the region of frequencies where the system has three fixed points, only two of them are stable, and are separated by an unstable point at the negative-slope branch of the resonance curve $A(\xi)$. (This fixed point topology is very typical for bi- and multi-stable dynamic systems.) As a result, if the resonance curve is passed by slowly changing the detuning at fixed f_0, the system experiences the amplitude jumps shown by the bold vertical arrows in the figure above, i.e. is its resonance curve is *hysteretic*.

Just to complete our analysis, in the region below the lowest instability point, we see a new change from stable nodes to stable focuses, now due to the change of sign of the second factor under the square root in Eq. (**), i.e. at the point where

$$\xi(A_n) + \xi'(A_n)A_n = 0 .$$

Thus for large negative values of the detuning, the system again behaves qualitatively as a linear oscillator, with the (only) fixed point being a stable focus.



Problem 5.14. Use the van der Pol method to find the condition of parametric excitation of an oscillator described by the following equation:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2(t)q = 0,$$

where $\omega_0^2(t)$ is a periodic square-wave function shown in the figure on the right, with $\omega \approx \omega_0$.

Solution: To solve the problem, we need to calculate the right-hand sides of the reduced equations (5.57) for the following function:

$$f^{(0)} = 2\delta\omega A\sin(\omega t - \varphi) + A\cos(\omega t - \varphi)\left[2\xi\omega - \mu\omega_0{}^2 S(t)\right],$$

where

$$S(t) \equiv \begin{cases} +1, & \text{for } \pi n < \omega t < \pi(n+1/2), \\ -1, & \text{for } \pi(n+1/2) < \omega t < \pi(n+1), \end{cases}$$

for any integer n. For the $\{u, v\}$ form, which is most suitable for the parametric excitation analysis, straightforward integration of the right-hand sides of Eqs. (5.57c) over the time period $2\pi/\omega$ (with breaking it into four $\pi/2\omega$-long segments, each with a constant value of the function $S(t) = \pm 1$) yields

$$\dot{u} = -\delta u - \xi v + \frac{\mu\omega}{\pi}u,$$
$$\dot{v} = -\delta v + \xi u - \frac{\mu\omega}{\pi}v.$$

These linear equations have a structure reminding Eqs. (5.79) that describe the sinusoidal parameter modulation case, but still differ from them, so it is prudent to redo the stability analysis. The characteristic equation now takes the form

$$\begin{vmatrix} -\delta + \dfrac{\mu\omega}{\pi} - \lambda & -\xi \\ \xi & -\delta - \dfrac{\mu\omega}{\pi} - \lambda \end{vmatrix} \equiv (\lambda + \delta)^2 - \left(\frac{\mu\omega}{\pi}\right)^2 + \xi^2 = 0,$$

and the expression for its roots,

$$\lambda_\pm = -\delta \pm \left[\left(\frac{\mu\omega}{\pi}\right)^2 - \xi^2\right]^{1/2},$$

is similar to Eq. (5.83) for the sinusoidal modulation, but with a larger coefficient before μ, so at the exact tuning ($\xi = 0$) the excitation condition (5.86) is replaced with

$$\mu > \frac{\pi\delta}{\omega} = \frac{\pi}{2Q} \approx \frac{1.57}{Q}. \qquad (*)$$

This increase of modulation efficiency (by the factor of $4/\pi \approx 1.27$) is due to square-wave modulation law – as could be expected from the discussion of the pendulum's length modulation at the beginning of Sec. 5.5. Note, however, that at a fair comparison ($\Delta l/l \leftrightarrow 2\mu$), this modulation is still somewhat less effective than the similar change of the pendulum's length at optimum time moments, illustrated in Fig. 5.6. The reasons for this difference have been discussed in Sec. 5.5 of the lecture notes, right after Eq. (5.86).

Problem 5.15. Use the van der Pol method to analyze the parametric excitation of an oscillator with weak nonlinear damping, described by the following equation:

$$\ddot{q} + 2\delta\dot{q} + \beta\dot{q}^3 + \omega_0^{\ 2}(1 + \mu\cos 2\omega t)q = 0,$$

with $\omega \approx \omega_0$; β, $\delta > 0$; and μ, $\beta\omega A^2 << 1$. In particular, find the amplitude of stationary oscillations and analyze their stability.

Solution: At the specified conditions μ, $\beta\omega A^2 << 1$, and also $|\xi| << \omega$, the given equation may be rewritten in the canonical form (5.38), with the following right-hand side:

$$f = -2\delta\dot{q} + \beta\dot{q}^3 + 2\xi\omega q - \mu\omega_0^2 \cos 2\omega t\ q, \qquad \text{where } \xi \equiv \omega - \omega_0.$$

Spelling out Eqs. (5.57a) of the lecture notes for the particular function f, we get the following system of the reduced equations[130]

$$\begin{aligned} \dot{A} &= -\delta(A)A - \frac{\mu\omega}{4}A\sin 2\varphi, \\ A\dot{\varphi} &= \xi A - \frac{\mu\omega}{4}A\cos 2\varphi, \qquad \text{with } \delta(A) \equiv \delta + \frac{3}{8}\beta\omega^2 A^2. \end{aligned} \qquad (*)$$

[130] Alternatively, these equations may be obtained directly by the natural synthesis of Eqs. (5.63) and (5.77), taking into account the similarity of the initial equations – respectively, Eqs. (5.62) and (5.75).

Since at $A \to 0$, $\delta(A) \to \delta$, the damping's nonlinearity does not affect the condition (5.84) of the parametric excitation, derived in Sec. 5.5 of the lecture notes:

$$\left(\frac{\mu\omega}{4}\right)^2 > \delta^2 + \xi^2 . \qquad (**)$$

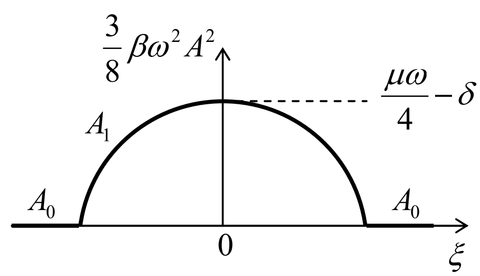
Within this region, the trivial stationary solution of Eqs. (*), with $A_0 = 0$, is unstable, so we should focus on its nontrivial fixed point,[131] with amplitude $A_1 \neq 0$, which may be found by canceling A in both parts of the stationary versions of Eqs. (*):

$$-\delta(A_1) - \frac{\mu\omega}{4}\sin 2\varphi_{1,2} = 0, \qquad \xi - \frac{\mu\omega}{4}\cos 2\varphi_{1,2} = 0. \qquad (***)$$

If we are not interested in the oscillations' phase, we may just require the sum of squares of the expressions for $\sin 2\varphi_{1,2}$ and $\cos 2\varphi_{1,2}$, following from these relations, to equal 1. This immediately gives us a simple equation for A_1:

$$\left(\frac{\mu\omega}{4}\right)^2 = [\delta(A_1)]^2 + \xi^2, \qquad \text{i.e.} \quad \left(\frac{\mu\omega}{4}\right)^2 = \left(\delta + \frac{3}{8}\beta\omega^2 A_1^2\right)^2 + \xi^2 . \qquad (****)$$

First of all, note that the first form of this equation coincides with the boundary of the parametric excitation region (**), but with the replacement $\delta \to \delta(A_1)$. This fact has a simple physical meaning: if the excitation condition (*) is satisfied, the oscillations grow until the growth makes the effective (nonlinear) damping $\delta(A)$ so large that the system is brought to the excitation region's border. The second form of Eq. (****) shows that on the [A^2, ξ] plane, the nontrivial stationary value of amplitude corresponds to a part of an ellipse (or, after a proper normalization, just to a circular arc – see the figure on the right), with the arc's ends (where $A_1 = 0$) corresponding exactly to the boundary of the excitation region (**).[132]



Thus, the system has just one stationary amplitude (either $A_0 = 0$ or $A_1 \neq 0$) for any set of its parameters. It is almost evident that each of the corresponding fixed points is stable as soon as it exists. For the trivial fixed point (in which $\delta(A) = \delta$), the proof of this statement was a result of the analysis carried out in Sec. 5.6 of the lecture notes – see in particular Fig. 5.10, which summarizes its results. In order to confirm this conclusion for the nontrivial fixed point, we need to plug the amplitude and phase, in the form

$$A = A_1 + \tilde{A}, \quad \varphi = \varphi_{1,2} + \tilde{\varphi} ,$$

[131] Strictly speaking, there are two nontrivial points with the same amplitude A_1, but with opposite phases: φ_1 and $\varphi_2 = \varphi_1 + \pi$. Indeed, as it follows from the invariance of Eqs. (*) to the phase shift $\varphi \to \varphi + \pi$, the dynamics of oscillations excited with any of two distinguishable phases, mutually shifted by π, is absolutely similar, and hence they have exactly the same stationary amplitude. This fact, common for any degenerate parametric excitation, was discussed at the very end of Sec. 5.6 of the lecture notes.

[132] Sometimes such dependence of A on ξ, i.e. on the excitation frequency $\omega \equiv \omega_0 + \xi$, is called the *parametric resonance*. In my humble view, this term is very unfortunate, because it ignores the principal differences between the parametric excitation and the forced oscillations.

into the reduced equations (*) and linearize them with respect to small variations $\tilde{A}$ and $\tilde{\varphi}$, as was discussed in Sec. 5.6 of the lecture notes. With the account of Eqs. (***), the resulting linear equations have a simple form:

$$\dot{\tilde{A}} = -\delta'(A_1)A_1\tilde{A} - 2\xi A_1\tilde{\varphi},$$
$$\dot{\tilde{\varphi}} = -2\delta'(A_1)\tilde{\varphi},$$

where $\delta'(A) \equiv d\delta(A)/dA$ is the same function as was used in the model solution of Problem 11. Solving the characteristic equation of this linear system,

$$\begin{vmatrix} -\delta'(A_1)A_1 - \lambda & -2\xi A_1 \\ 0 & -\delta'(A_1)A_1 - \lambda \end{vmatrix} = 0,$$

we get two equal roots, $\lambda_\pm = \lambda \equiv -\delta'(A_1)A_1$. Since for our system, $\delta'(A_1)A_1 \equiv [d\delta(A_1)/dA_1]A_1 = (3/4)\beta\omega^2 A_1^2 > 0$, this λ is always negative, so the nontrivial stationary solution is always stable – within the parameter interval where it exists.

<u>Problem 5.16</u>. Upon adding the nonlinear term αq^3 to the left-hand side of Eq. (5.75) of the lecture notes,

(i) find the corresponding addition to the reduced equations,
(ii) calculate the stationary amplitude A of the parametric oscillations,
(iii) find the type and stability of each fixed point of the reduced equations,
(iv) sketch the Poincaré phase plane of the system in major parameter regions.

Solutions:

(i) Looking for the oscillations in the standard form (5.41), $q = A\cos\Psi$ with $\Psi \equiv \omega t - \varphi$, we get the following additional contributions to the right-hand sides of the reduced equations (5.57a), respectively, for $\dot{A}$ and $\dot{\varphi}$:

$$-\frac{1}{\omega}\overline{[\alpha(A\cos\Psi)^3]\sin\Psi} = 0, \qquad \frac{1}{\omega}\overline{[\alpha(A\cos\Psi)^3]\cos\Psi} = -\frac{3}{8}\frac{\alpha A^3}{\omega},$$

so Eqs. (5.77) are changed to

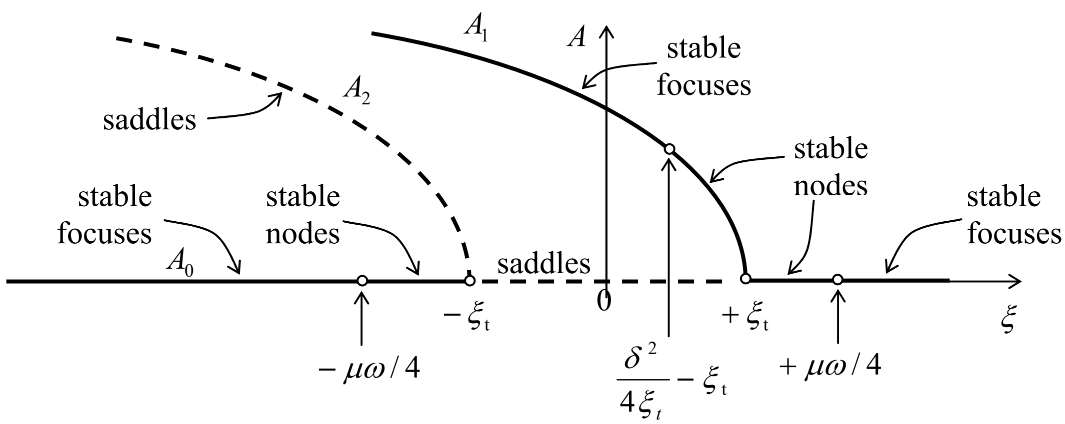
$$\dot{A} = -\delta A - \frac{\mu\omega}{4}A\sin 2\varphi, \qquad A\dot{\varphi} = \xi(A)A - \frac{\mu\omega}{4}A\cos 2\varphi, \qquad \text{with} \quad \xi(A) \equiv \xi + \frac{3}{8}\frac{\alpha A^2}{\omega}, \qquad (*)$$

– cf. the second of Eqs. (5.48) for the forced oscillations in a system with this nonlinearity.

(ii) At every fixed point, the time derivatives of A and φ have to vanish. Solving Eqs. (*) for this case, we get three possible stationary amplitudes:

$$A_0 = 0, \qquad A_{1,2}^2 = \frac{8\omega}{3\alpha}(\xi \pm \xi_t), \qquad \text{where} \quad \xi_t \equiv \left[\left(\frac{\mu\omega}{4}\right)^2 - \delta^2\right]^{1/2}. \qquad (**)$$

The first of them, A_0, describes the system at rest (no parametric excitation), while the functions $A_{1,2}(\xi)$ describing the excitation are sketched in the figure below for the case when $\alpha > 0$ and $\mu\omega/4 > \delta$. (If $\alpha < 0$, the situation is similar, besides that the curves $A_{1,2}(\xi)$ are bent toward the positive rather than negative direction of the ξ-axis. If $(\mu\omega/4)^2 < \delta^2$, then $A_0 = 0$ is the only solution, i.e. there is no parametric excitation at any ξ.)



The physics of these curves is similar to that of the resonance curve bending at forced oscillations, discussed in Sec. 5.2 of the lecture notes (see, in particular, Fig. 5.4): the growth of the excited parametric oscillations changes the effective own frequency of the oscillator in accordance with Eq. (5.49):

$$\omega_0(A) = \omega_0 - \frac{3}{8}\frac{\alpha A^2}{\omega}.$$

As a result, the effective detuning $\xi(A) \equiv \omega - \omega_0(A)$ changes, and eventually carries the system to the excitation threshold, where $|\,\xi(A)\,| = \xi_t$.

An interesting feature of Eqs. (**) is that the reduced equations cannot give the absolute maximum of the parametric excitation amplitude: $A_{1,2} \to \infty$ at $\xi \to -\infty$. (Actually, the maximum does exist, but only at $|\,\xi\,| \sim \omega$, i. e. beyond the range of validity of the van der Pol approximation we are using.) Also, note that per Eqs. (*), the following relations are valid for the nontrivial fixed points $A_{1,2}$:

$$\xi(A_{1,2}) = \mp\xi_t, \qquad \sin 2\varphi_{1,2} = \delta/(\mu\omega/4), \qquad \cos 2\varphi_{1,2} = \mp\xi_t/(\mu\omega/4).$$

These expressions will simplify the forthcoming analysis of the fixed point stability.

(iii) Since at $A \to 0$, the nonlinear effects are negligible, for the stability of type of the *trivial* fixed point $A_0 = 0$, we can immediately use the results that were obtained in Sec. 5.6 of the lecture notes – by using the Cartesian slow variables $u \equiv A\cos\varphi$ and $v \equiv A\sin\varphi$. As a reminder (see Fig. 5.10), if the modulation depth μ is sufficiently large, $\mu\omega/4 > \delta$, there is a finite range of frequencies, $-\xi_t < \xi < +\xi_t$, where the largest characteristic root λ_+ is positive, i.e. the trivial point is unstable and parametric oscillations get self-excited. In this case, the fixed point is a saddle.

On the contrary, at

$$\xi_t < |\,\xi\,| < \frac{\mu\omega}{4},$$

(i.e. outside of, but adjacent to the self-excitation range), both roots are real and negative, so we are dealing with a stable node. Finally, at even larger detuning values, $|\xi| > \xi_t$, both roots λ have a negative real part ($\text{Re}\lambda_\pm = -\delta$) and finite imaginary parts, so the trivial fixed point is a stable focus.

Near the *non-trivial* fixed points ($A_{1,2} \neq 0$), the reduced equations are regular in both forms, $\{A, \varphi\}$ and $\{u, v\}$, and we can use either of them for the stability analysis, but calculations are a bit simpler in the $\{A, \varphi\}$ form. Linearizing the equations (*) of motion near $A_{1,2}$ as usual, we get

$$\dot{\tilde{A}} = -2\xi(A_{1,2})A_{1,2}\tilde{\varphi},$$
$$A_{1,2}\dot{\tilde{\varphi}} = \xi'(A_{1,2})A_{1,2}\tilde{A} - 2\delta A_{1,2}\tilde{\varphi},$$

where $\xi'(A) \equiv d\xi(A)/dA = (3/4)\alpha A/\omega$ is the same function as has appeared in the model solution of Problem 11. Writing and solving the characteristic equation for this linear system, we get

$$\lambda_\pm = -\delta \pm \left[\delta^2 - 2\xi(A_{1,2})\xi'(A_{1,2})A_{1,2}\right]^{1/2}. \qquad (***)$$

Again, let us assume that $\alpha > 0$. (For the opposite case, the fixed point topology is similar.) In this case $\xi'(A_{1,2})A_{1,2} = (3/4)\alpha A_{2,3}^2/\omega > 0$. According to Eq. (**), for the lower branch of fixed points ($A = A_2$, see the dashed curve in the figure above), $\xi(A_2) = -\xi_t < 0$, so the addition to δ^2 under the square root in Eq. (***) is positive, i.e. the square root is larger than δ. Hence both roots $\lambda_\pm$ are real, and one of them is positive while another one is negative. Thus these fixed points are (unstable) saddles.

For the upper branch (shown with the solid curve in the figure above), $\xi(A_1) = +\xi_t > 0$. Because of this, the addition to δ^2 under the square root of Eq. (***) is always negative, so $\text{Re}\lambda_\pm = -\delta < 0$, and these fixed points are always stable. The fixed point type here depends on the detuning: if $2\xi(A_1)\xi'(A_1)A^2 > \delta^2$, the expression under the square root is negative and the points are stable focuses. Since $\xi'(A)A = 2[\xi(A) - \xi]$, and $\xi(A_2) = +\xi_t$, the condition for that may be rewritten as

$$\xi < \frac{\delta^2}{4\xi_t} - \xi_t .$$

In the opposite case, the fixed points are stable nodes.

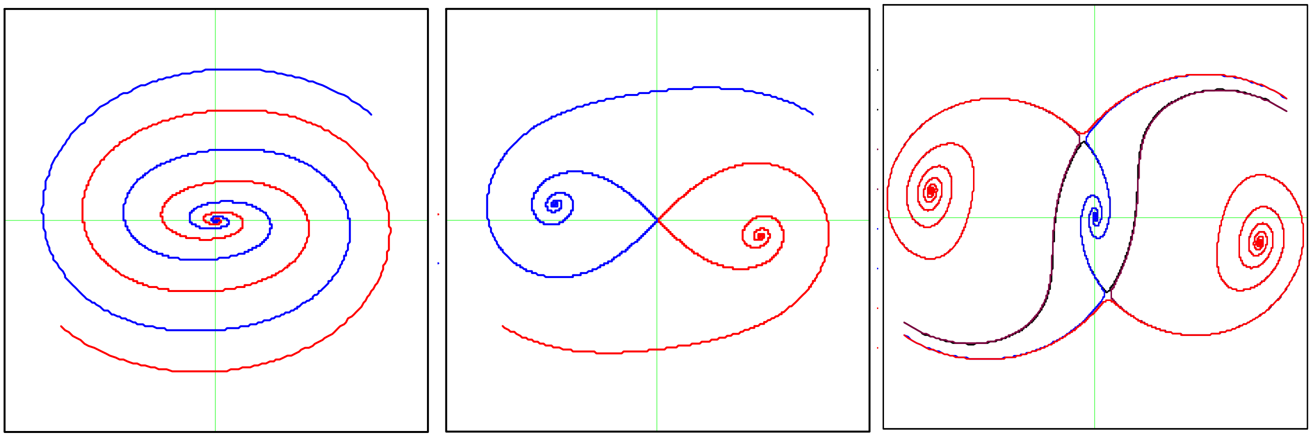
The summary classification of all fixed points is shown in the figure above.

(iv) The Poincare plane $[u, v]$ looks differently in three ranges of detuning.

– The case $\xi > +\xi_t$ is simple: no parametric excitation, just a stable focus or node in the origin – see the left panel in the figure below (obtained by numerical integration of the reduced equations (*), for $\xi = +1.5\xi_t$).

– Within the range $-\xi_t < \xi < +\xi_t$, we have to combine a saddle structure at $A_0 = 0$ (it was discussed in Sec. 5.5; see, in particular, Fig. 5.8b) with two stable focuses (or stable nodes) near two non-trivial fixed points $\{A_1, \varphi_1\}$ and $\{A_1, \varphi_1 + \pi\}$. As a result, we get a pattern like the one shown on the middle panel of the figure below (computed for $\xi = 0$).

– Finally, if $\xi < -\xi_t$, we should accommodate five fixed points: a stable focus (or node) at the origin, two (unstable) saddles corresponding to the same $A_2 \neq 0$ but two π-shifted values of the oscillation phase, and two stable focuses corresponding to $A_1 > A_2$ (also for two π-shifted values of phase) – see the right panel of the figure below (computed for $\xi = -1.5\xi_t$).



Note how involved (and beautiful) these patterns are – for such a simple system!

Problem 5.17. Use the van der Pol method to find the condition of parametric excitation of a linear oscillator with simultaneous weak modulation of the effective mass: $m(t) = m_0(1 + \mu_m \cos 2\omega t)$ and the effective spring constant: $\kappa(t) = \kappa_0[1 + \mu_\kappa \cos(2\omega t - \psi)]$, with the same frequency $2\omega \approx 2\omega_0$, for arbitrary modulation depths ratio μ_m/μ_κ and phase shift ψ. Interpret the result in terms of modulation of the oscillator's instantaneous frequency $\omega(t) \equiv [\kappa(t)/m(t)]^{1/2}$ and impedance $Z(t) \equiv [\kappa(t)m(t)]^{1/2}$.

Solution: Generally, the equations of motion of an oscillator depend on the exact way of the parameter variation in time. Let us explore the most natural model described (in the absence of damping) by the following Lagrangian function:

$$L = T - U = \frac{m(t)}{2}\dot{q}^2 - \frac{\kappa(t)}{2}q^2,$$

and hence by the following Lagrange equation of motion:

$$\frac{d}{dt}\left[m(t)\dot{q}\right] + \kappa(t)q = 0 .$$

For the parameter modulation laws specified in the assignment, this equation becomes

$$\frac{d}{dt}\left(1 + \mu_m \cos 2\omega t\right)\dot{q} + \omega_0^2\left[1 + \mu_\kappa \cos(2\omega t - \psi)\right]q = 0, \quad \text{with} \quad \omega_0^2 \equiv \frac{\kappa_0}{m_0}.$$

This equation may be rewritten in the canonical form (5.38), with the right-hand side

$$f = 2\xi\omega q - \mu_m\left(\ddot{q}\cos 2\omega t - 2\omega\dot{q}\sin 2\omega t\right) - \mu_\kappa\omega_0^2 q\cos(2\omega t - \psi).$$

Within the 0th-order approximation $q(t) \approx q_0(t) \equiv A\cos\Psi \equiv A\cos(\omega t - \varphi)$, this function reduces to

$$f \to f^{(0)} = A\cos\Psi\left[2\xi\omega + \mu_m\omega^2\cos 2\omega t - \mu_\kappa\omega_0^2\cos(2\omega t - \psi)\right] - A\sin\Psi\left[2\mu_m\omega^2\sin 2\omega t\right].$$

Carrying out the multiplication of the trigonometric functions, and replacing (as usual in the van der Pol approximation) ω_0 with ω everywhere but in the detuning $\xi \equiv \omega - \omega_0$, we get

$$\begin{aligned} f^{(0)} &= 2\xi\omega A\cos(\omega t - \varphi) - \frac{A}{2}\omega^2\left[\mu_m \cos(\omega t + \varphi) + \mu_\kappa \cos(\omega t + \varphi - \psi)\right] + 3^{\text{rd}} \text{ harmonic} \\ &= 2\xi\omega A\cos(\omega t - \varphi) - \frac{A}{2}\omega^2 \mu_{\text{ef}} \cos(\omega t + \varphi + \text{const}) + 3^{\text{rd}} \text{ harmonic}, \end{aligned} \qquad (*)$$

where μ_{ef} is the effective parameter modulation amplitude:

$$\mu_{\text{ef}}^2 \equiv \mu_m^2 + \mu_\kappa^2 + 2\mu_m \mu_\kappa \cos\psi\,. \qquad (**)$$

Since the phase constant in the last parentheses of the final form of Eq. (*) is not significant for the parametric excitation (it may be always brought to zero by the appropriate choice of the time origin), a comparison of Eq. (*) with Eq. (5.76) of the lecture notes shows that our current problem is reduced to the one solved in Sec. 5.5 the lecture notes, with the replacement $\mu \to \mu_{\text{ef}}$. In particular, if the two parameters of the oscillator are modulated in phase ($\psi = 0$), the effects of their modulation add up: $\mu_{\text{ef}} = \mu_m + \mu_\kappa$, as could be expected. However, if the two modulations are in anti-phase ($\psi = \pi$), then their effects, a bit counter-intuitively, subtract: $\mu_{\text{ef}} = |\mu_m - \mu_\kappa|$, so at the equal modulation depths, μ_{ef} vanishes, and the parametric excitation is impossible at all.

This result may be recast in a more transparent form by transferring from the parameters $m(t)$ and $\kappa(t)$ to the more physical notions of the instantaneous frequency $\omega_0(t)$ of the oscillator and its impedance $Z(t)$:[133]

$$\begin{aligned} \omega_0^2(t) &\equiv \kappa(t)/m(t) = \omega_0^2\left[1 + \mu_\kappa \cos(2\omega t - \psi)\right]/(1 + \mu_m \cos 2\omega t), \\ Z^2(t) &\equiv \kappa(t) \times m(t) = Z_0^2\left[1 + \mu_\kappa \cos(2\omega t - \psi)\right] \times (1 + \mu_m \cos 2\omega t), \end{aligned}$$

where $Z_0 \equiv (\kappa_0 m_0)^{1/2}$ is the impedance in the absence of modulation. Since our result (**) is only valid in the linear approximation in small μ_κ and μ_m, the above formulas may be rewritten, with the same accuracy, as

$$\begin{aligned} \omega_0^2(t) &\approx \omega_0^2\left[1 + \mu_\kappa \cos(2\omega t - \psi) - \mu_m \cos 2\omega t\right], \\ Z^2(t) &\approx Z_0^2\left[1 + \mu_\kappa \cos(2\omega t - \psi) + \mu_m \cos 2\omega t\right]. \end{aligned}$$

But the amplitude of the sum of the two time-dependent terms in the last formula is exactly the μ_{ef} defined above. Hence (at least within our model of parameter modulation[134]) the parametric excitation may be interpreted as a result of the modulation of the oscillator's *impedance* alone. This conclusion is important not as much for practical parametric systems[135] as for the general understanding of the key role of the oscillator's impedance for its dynamics.

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[133] The notion of impedance, briefly mentioned in Sec. 5.5 of the lecture notes, becomes especially crucial for analysis of wave propagation – see, e.g., Chapter 6.

[134] It is curious that in the (physically, less plausible) model described by an alternative equation, $m(t)\ddot{q} + \kappa(t)q = 0$, the frequency and impedance modulation roles are swapped. (Proving this is a straightforward but useful exercise, highly recommended to the reader.)

[135] In parametric systems of any physical nature, typically, only one of the parameters (equivalent to either m or κ) is modulated. Due to that fact, to the best of my knowledge, the important role of the oscillator's impedance modulation had evaded the attention of virtually all textbook and review authors until very recently – see B. Zeldovich, *Physics-Uspekhi* **51**, 465 (2008).

Problem 5.18.* Find the condition of parametric excitation of a nonlinear oscillator described by the following equation:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q + \gamma q^2 = f_0 \cos 2\omega t,$$

with sufficiently small δ, γ, f_0, and ξ, where $\xi \equiv \omega - \omega_0$.

Solution: Following the discussion in Sec. 5.8 of the lecture notes, let us look for a solution to this differential equation in the form

$$q(t) = A_2 \cos\Psi_2 + q_1 + q_n, \quad \Psi_2 = 2\omega t - \varphi_2,$$

with constant A_2 and φ_2. Here the first term describes the forced oscillations at frequency 2ω (far from the resonance), q_1 reflects the parametric excitation at the near-resonance frequency $\omega \approx \omega_0$, and q_n is reserved for the description of other possible effects (e.g., the second harmonic generation) arising from the system's quadratic nonlinearity, so q_n does not have an ω-component. Since the problem's assignment specifies that the nonlinearity is weak, and we are interested only in the condition of *excitation* of the parametric oscillations, we may assume that q_1, $q_n \to 0$, and hence neglect the quadratic term $\gamma(q_1 + q_n)^2$. As a result, plugging our solution into the equation of motion, we get

$$\left[\left(-4\omega^2 + \omega_0^{\ 2}\right)A_2 \cos\Psi_2 + \left(-4\delta\omega\right)A_2 \sin\Psi_2 - f_0 \cos 2\omega t\right] + \left[\ddot{q}_1 + 2\delta\dot{q}_1 + \omega_0^{\ 2} q_1 + 2\gamma q_1 A_2 \cos\Psi_2\right]$$
$$+ \left[\ddot{q}_n + 2\delta\dot{q}_n + \omega_0^{\ 2} q_n + 2\gamma q_n A_2 \cos\Psi + \gamma A_2^2 \left(1 + \cos 2\Psi_2\right)/2\right] = 0.$$

Here the terms collected in different square brackets differ by their frequency: 2ω in the first bracket, ω and 3ω in the second bracket, and all other harmonics $n\omega$ (including $n = 0$) in the third bracket.[136] For the equation to be satisfied at all times, each of these brackets has to equal zero separately. The first of the three resulting equations gives the usual formula for the complex amplitude $a_2 \equiv A_2\exp\{i\varphi_2\}$ of the forced oscillations at frequency 2ω – see Eqs. (5.16)-(5.17) of the lecture notes:

$$a_2 = \frac{f_0}{(-4\omega^2 + \omega_0^{\ 2}) - i4\omega\delta} \approx -\frac{f_0}{3\omega^2}.$$

The third of the resulting equations allows the small $q_n(t)$ to be expressed via that amplitude (this is not our major concern right now), while the second one may be rewritten in the form

$$\ddot{q}_1 + 2\delta\dot{q}_1 + \omega_0^2\left[1 + \mu\cos\left(2\omega t - \Psi_2\right)\right]q_1 = 0,$$

with the effective parameter modulation amplitude

$$\mu = \frac{2\gamma A_2}{\omega_0^{\ 2}} \approx -\frac{2\gamma f_0}{3\omega_0^{\ 4}}.$$

Apart from an inconsequential phase shift, this equation for q_1 coincides with the canonical form (5.75) of the equation describing the parametric excitation. Hence we may plug the above expression for μ into Eq. (5.84) to get the final condition of the parametric excitation in our case:

$$\frac{\gamma f_0}{6\omega_0^{\ 3}} > \left(\delta^2 + \xi^2\right)^{1/2}.$$

[136] Strictly speaking, the term $2\gamma q_n A_2 \cos\Psi$ also may have a 2ω-component but at small γ and q_n, it is negligible.

Problem 5.19. Find the condition of stability of the equilibrium point $q = 0$ of a parametric oscillator described by Eq. (5.75) of the lecture notes, in the limit when $\delta << |\omega_0| << \omega$ and $\mu << 1$. Use the result to analyze the stability of the Kapitza pendulum mentioned in Sec. 5.5.

Solution: Similarly (but not identically) to the solution of the previous problem, let us look for the solution of our differential equation,

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 \left(1 + \mu \cos 2\omega t\right) q = 0, \qquad (*)$$

in the following frequency-separated form:

$$q(t) = q_0 + A_2 \cos \Psi_2 + q_n, \quad \Psi_2 = 2\omega t - \varphi_2,$$

where q_0 describes the component changing slowly (on the time scale of $1/|\omega_0| >> 1/\omega$), the second term represents oscillations of frequency 2ω, and q_n denotes the sum of terms with all other frequencies. Plugging this form into Eq. (*), opening the parentheses, and transforming the product $\cos 2\omega t \cos\Psi_2$ into a sum of two cosine functions,[137] we get

$$\left(\ddot{q}_0 - 4A_2\omega^2 \cos \Psi_2 + \ddot{q}_n\right) + 2\delta\left(\dot{q}_0 - 2A_2\omega \sin \Psi_2 + \dot{q}_n\right) + \omega_0^2\left(q_0 + A_2 \cos\Psi_2 + q_n\right)$$

$$+ \mu\omega_0^2 \left\{ \cos 2\omega t\, q_0 + \frac{A_2}{2}\left[\cos\left(4\omega t - \varphi\right) + \cos\varphi\right] + \cos 2\omega t\, q_n \right\} = 0.$$

Let us discuss the magnitude of each term in the limit given in the assignment, in order to keep only the terms of the largest order in the listed small parameters. First of all, the terms proportional to q_n may arise due only to the term proportional to $\mu\cos(4\omega t - \varphi)$ and hence may have only this frequency. The only way for these terms to interact with the terms of our other frequencies (~0 and 2ω) is via the term listed last, but it has an additional small factor $\mu << 1$. Hence, for the analysis of slow dynamics of the system, all terms of frequency 4ω may be ignored. Next, because of the smallness of $|\omega_0|$ in comparison with ω, the second term in the third parentheses is negligible in comparison with the second term in the first parentheses. Finally, due to the condition $\delta << |\omega_0| << \omega$ and the fact that this system is not resonant, all terms proportional to δ are also negligible. As a result, our equation is reduced to

$$\ddot{q}_0 - 4A_2\omega^2 \cos\Psi_2 + \omega_0^2 q_0 + \mu\omega_0^2 \cos 2\omega t\, q_0 + \frac{\mu\omega_0^2}{2} A_2 \cos\varphi = 0. \qquad (**)$$

Only two terms of this equation (the second and the fourth ones) have frequency 2ω, and for the equation to be satisfied at any t, their sum has to vanish. This condition immediately yields

$$A_2 = \frac{\mu\omega_0^2}{4\omega^2} q_0, \quad \text{and} \quad \Psi_2 = 2\omega t, \quad \text{i.e. } \varphi = 0\,.$$

Plugging this result back into the last term of Eq. (**), the remaining part of this equation (or, if you like, into the whole equation after its averaging over fast oscillations of frequency 2ω), becomes

$$\ddot{q}_0 + \omega_0^2 q_0 + \frac{\mu^2\omega_0^4}{8\omega^2} q_0 = 0.$$

[137] See, e.g., MA Eq. (3.3a).

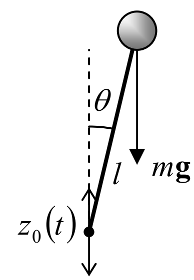
This is the usual equation of a damping-free linear oscillator with the frequency modified by the parameter modulation:

$$\omega_{\text{ef}}^2 = \omega_0^2 + \frac{\mu^2 \omega_0^4}{8\omega^2}.$$

Since the last term is positive for any sign of ω_0^2 and μ, it may make ω_{ef}^2 positive (and hence the equilibrium position $q_0 = 0$ stable) even if $\omega_0^2 < 0$, provided that the modulation coefficient is sufficiently large:

$$\mu^2 > \frac{8\omega^2}{|\omega_0^2|}. \qquad (***)$$

One example of such an oscillator is the usual point-mass pendulum near its top position – see the figure on the right. The modulation of its parameter may be implemented not only in the way shown in Fig. 5.6 of the lecture notes (for a stiff supporting rod that may be technically tricky), but also simpler: by moving its support point z_0 up and down – say, sinusoidally, with $z_0(t) = Z_0 \cos 2\omega t$. Indeed, this system is very similar to that considered in Problem 2.3, and its equation of motion may be obtained similarly, by using the Lagrangian formalism – the exercise highly recommended to the reader. However, after our discussion in Sec. 4.6, we may obtain it even simpler by using the non-inertial reference frame moving together with the support point, without rotation. As a result, of all the inertial "forces" contributing to the right-hand side of Eq. (4.92), only the first one does not vanish:



$$\mathbf{F}_{\text{in}} = -m\mathbf{a}_0\big|_{\text{in lab}} = -m\ddot{z}_0(t)\mathbf{n}_z = 4m\omega^2 Z_0 \cos 2\omega t\, \mathbf{n}_z.$$

Hence, for the simplest case of a 1D motion of the pendulum (in a vertical plane containing the support point), the equation of motion of its angular deviation from the vertical position is

$$ml\ddot{\theta} = (m\mathbf{g} + \mathbf{F}_{\text{in}})_\theta = (mg + 4m\omega^2 Z_0 \cos 2\omega t)\sin\theta \approx (mg + 4m\omega^2 Z_0 \cos 2\omega t)\theta,$$

where the last approximation is valid only for small deviations from the vertical position: $\theta \to 0$. Comparing this equation with Eq. (*) with $\delta \to 0$ and $q \equiv \theta$, we see that they coincide provided that

$$\omega_0^2 = -\frac{g}{l} \quad \text{and} \quad \mu\frac{g}{l} = \frac{4\omega^2 Z_0}{l}, \text{ i.e. } \mu^2 = 16\frac{\omega^4 Z_0^2}{g^2}.$$

As a result, for the Kapitza pendulum, the stability condition (***) becomes[138]

$$2\omega^2 Z_0^2 > gl.$$

Finally, with the standard Mathieu equation's notation discussed in Sec. 4.5 of the lecture notes, $(\omega_0/\omega)^2 \equiv a$, $\mu/2 \equiv -b/a$, Eq. (***) reads $a > -b^2/2$. As the lowest curve in Fig. 5.7 of the lecture notes shows, this result (strictly valid only asymptotically at $b \to 0$) works very reasonably at least up to $b \sim 1$.

[138] This result was first obtained in 1951 by P. Kapitza on the basis of an essentially similar calculation, though in terms of the effective potential energy rather than the effective force – see, e.g., Sec. 30 in *Mechanics* by Landau and Lifshitz.

Problem 5.20.* Use numerical simulation to explore phase-plane trajectories $[q, \dot{q}]$ of an autonomous pendulum described by Eq. (5.42) of the lecture notes with $f_0 = 0$, for both low and high damping, and discuss their most significant features.

Solution: For the purposes of numerical simulations and interpretation of their results, the second-order differential equation describing the pendulum's dynamics,

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 \sin q = 0,$$

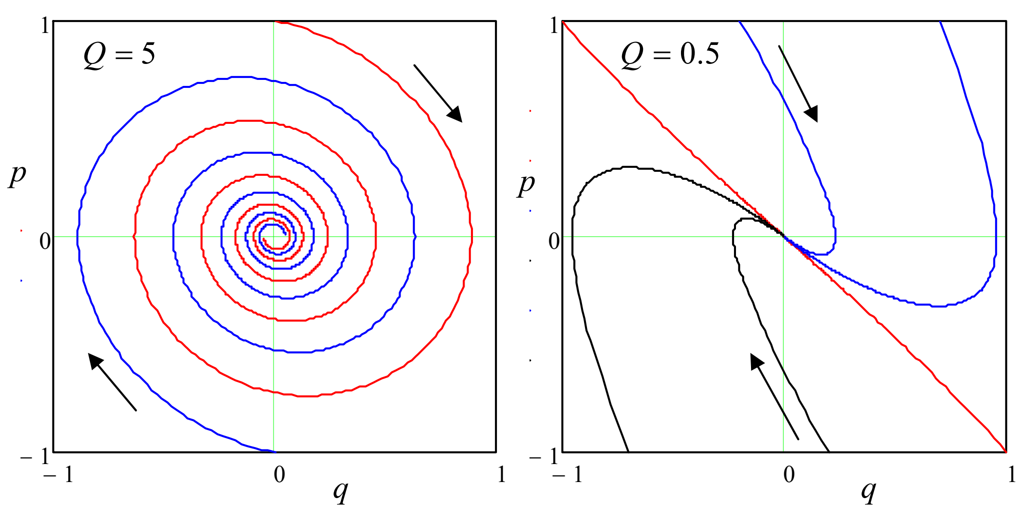
may be rewritten as a system of two first-order equations for its generalized coordinate q and normalized momentum $p \equiv \dot{q}/\omega_0$:

$$\frac{dq}{d\ell} = p, \qquad \frac{dp}{d\ell} = -\frac{p}{Q} - \sin q, \qquad (*)$$

where $Q \equiv \omega_0/2\delta$ is the pendulum's Q-factor for its small oscillations near equilibrium – see Eq. (5.11) of the lecture notes, while $\ell \equiv \omega_0 t$ is the naturally normalized time. (The convenience of such normalization is that p, q, and ℓ are all dimensionless, and the only explicit parameter of Eqs. (*) is Q, also a dimension-free constant.) The figures below show several typical examples of the phase plane trajectories (each defined by a specific set of initial conditions), calculated from Eqs. (*) using the standard 4th-order Runge-Kutta routine (5.99).

Since the autonomous pendulum with nonvanishing damping loses its initial energy with time, it eventually has to approach one of is (physically, similar[139]) equilibrium points $\{p = 0, q = 2\pi n\}$, with integer n. Hence it makes sense to analyze the phase-plane trajectories on two scales.

1. A close vicinity of a stable fixed point corresponding to the equilibrium, for example $\{p = 0, q = 0\}$. The two figures below show that here the phase plane patterns are similar to those of the linear systems discussed in Sec. 5.6 of the lecture notes – see in particular Fig. 5.8 and its discussion.



Indeed, in the limit $q \to 0$, we may approximate $\sin q$ with q, so Eqs. (*) are reduced to

139 Note that even though these points are *similar*, they are not *indistinguishable* – the fact of significant importance for quantum mechanics of such periodic systems – see, e.g., QM Secs. 2.7-2.8.

$$\frac{dq}{d\ell} = p, \qquad \frac{dp}{d\ell} = -\frac{p}{Q} - q.$$

Looking for the solution of this system of linear homogeneous differential equations in the standard form (5.88): p, $q \propto \exp\{\lambda\ell\}$, we get the following characteristic equation for λ:

$$\lambda^2 + \frac{\lambda}{Q} + 1 = 0,$$

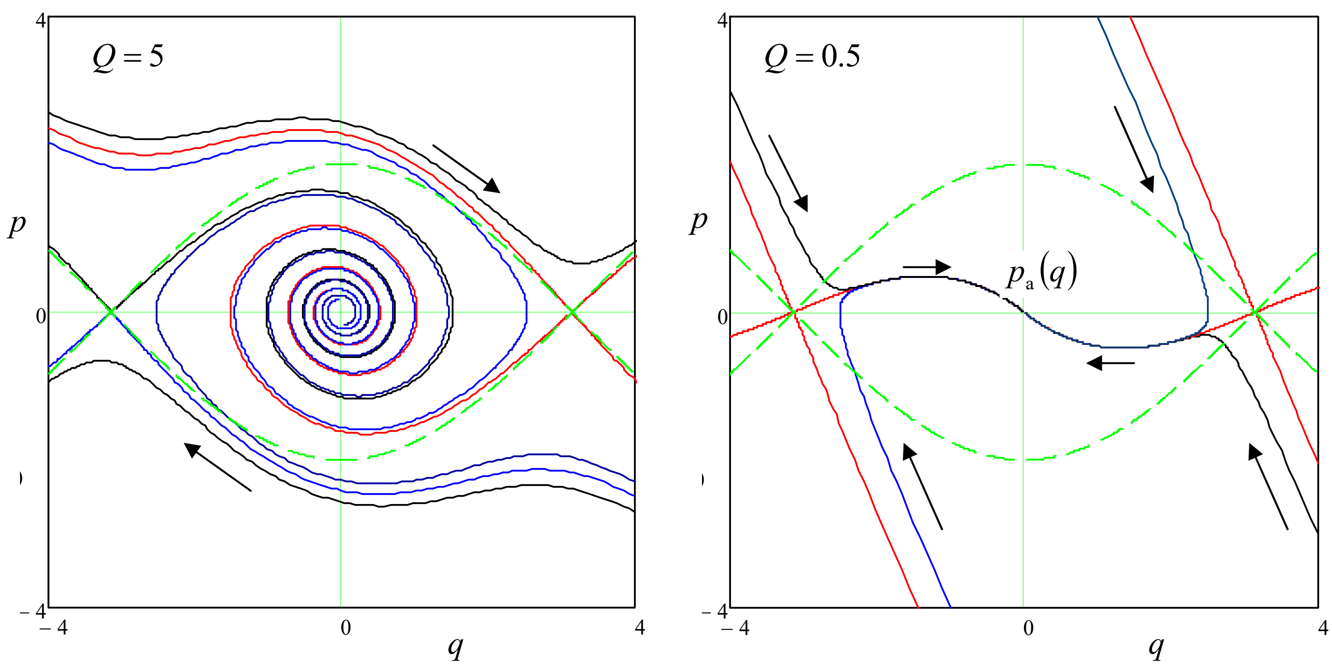
with two roots[140]

$$\lambda_\pm = \frac{1}{2Q}\left[-1 \pm \left(1 - 4Q^2\right)^{1/2}\right].$$

This formula shows that at low damping ($Q > ½$), the fixed point at the origin is a stable focus – see the left figure above, to be compared with Fig. 5.8c of the lecture notes. Physically, this means that the system loses its energy (which, in this approximation, is proportional to the sum $p^2 + q^2$, i.e. to the squared distance from the origin) relatively slowly, in the course of many oscillations. On the other hand, at $Q < ½$ (the right figure above corresponds to this critical value) the fixed point is a stable node. In this case, all trajectories tend to a straight-line separatrix (red line) and continue along it to the origin. This means that the system just relaxes to the equilibrium position without oscillations.

2. The "global" topology of the phase plane is more complicated – and more interesting – see the figure below. Its left panel shows that at low damping when the pendulum loses its initial energy only slowly, its phase-plane trajectories again lean to the constant-energy lines, but at $q \sim 1$ these lines differ from the circles. Indeed, at $Q \to \infty$, the system (*) may be readily integrated once, giving

$$\frac{p^2}{2} + (1 - \cos q) = \text{const} \propto E. \qquad (**)$$



[140] Actually, this result was (in a different notation) obtained in Sec. 5.1 of the lecture notes – see Eqs. (5.7)-(5.8).

As an example, the dashed lines in the figures above show the constant-energy lines (**) for the critical case when the pendulum's energy E is barely sufficient for swinging past its top positions $\pi + 2\pi n$. Naturally, these lines serve as separatrices between the trajectories corresponding to the pendulum's oscillations about a certain equilibrium point and those corresponding to its continuous rotation.

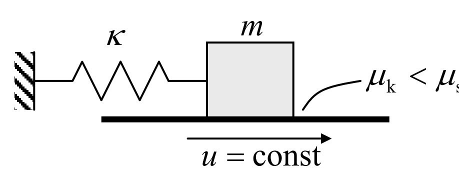
If the energy is dissipated ($Q < \infty$), it gradually decreases and a rotating pendulum eventually gets trapped in the "attraction domain" of one of the stable fixed points. In this case, the separatrices (such as the ones shown with the red lines in the figures above) separate the trajectories trapped in adjacent attraction domains. At $Q >> 1$, the separatrices lean to the lines (**), but at higher damping ($Q \leq ½$), they go directly to the unstable (saddle) fixed points $\{p = 0, q = \pi + 2\pi n\}$ – see the right figure.

In the latter case, the phase-plane trajectories have one more interesting feature: they rapidly converge to an asymptote that, for this nonlinear system, is a curve rather than a straight line as in the linear approximation – see, e.g., Fig. 5.8a,b in the lecture notes. Indeed, as the second of Eqs. (*) shows, at $Q << 1$, at the first stage of the transient (with the time scale $\sim Q/\omega_0$) the momentum p relaxes very rapidly to a q-dependent value p_a making the right-hand side of that equation very small:

$$p \to p_a(q) \equiv -Q \sin q \, .$$

After that, the system proceeds much more slowly (on the time scale $\sim 1/Q\omega_0$) along this asymptote toward the final equilibrium point $\{p = 0, q = 2\pi n\}$. This transient process hierarchy[141] is important for physical statistics[142] and also for practical applications including (but not limited to) vibration-isolating systems for sensitive physical measurements – see also Problem 1 above.

Problem 5.21. Analyze relaxation oscillations of the system shown in the figure on the right. Here an elastic spring prevents a block of mass m from being carried away by a horizontal conveyor belt moving with a constant velocity u. Assume that the coefficient μ_k of the kinematic friction between the block and the belt is lower than the static friction coefficient μ_s.



Solution: The assignment implies the usual Coulomb approximation, $|F| = \mu N$, for the horizontal friction force exerted on the block by the moving belt, where N is its normal reaction, in our current problem equal simply to mg.[143] However, here it is important that the meaning of this formula is different for the static and kinetic cases. In the former case, when the block moves with the belt without slippage, $\mu_s N$ is just the *largest magnitude* of the friction force F_f, whose actual value and direction should be found from the 2nd Newton's law. In our current case of the belt's uniform motion, i.e. of zero acceleration, it is

$$F_f - \kappa q = 0, \qquad \text{for } \dot{q} = u \text{ and } |F_f| \leq \mu_s m, \quad \text{i.e. for } |q| \leq q_{max} \equiv \frac{\mu_s mg}{\kappa}, \qquad (*)$$

where κ is the spring constant, and q is the block's displacement from its equilibrium position.

[141] It was briefly mentioned in Sec. 5.1 of the lecture notes.
[142] See, e.g., SM Secs. 5.6 and 5.7.
[143] See, also Problems 1.4 and 4.5.

However, in the opposite, kinetic case, when the block is slipping on the belt, i.e. its velocity (as measured in the lab frame) is different from u, the Coulomb approximation gives the *actual magnitude* of the friction force:

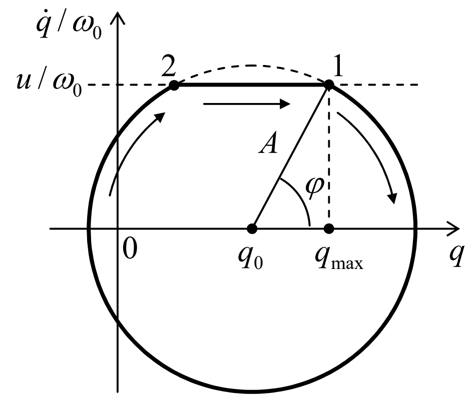
$$F_f = -\mu_k mg\,\mathrm{sgn}(\dot{q}-u), \qquad \text{for } \dot{q} \neq u\,. \qquad (**)$$

On the time intervals when the relative velocity $(\dot{q}-u)$ is negative, Eq. (**) yields the following equation of the block's motion:

$$m\ddot{q} = \mu_k mg - \kappa q, \qquad \text{i.e. } m\ddot{\tilde{q}} + \kappa\tilde{q} = 0, \quad \text{where } \tilde{q} \equiv q - q_0, \quad \text{with } q_0 \equiv \frac{\mu_k mg}{\kappa}.$$

Besides the time-independent displacement q_0, this is just the standard equation of motion of a 1D harmonic oscillator without damping,[144] with the well-known solution (5.3),

$$\tilde{q} = A\cos\Psi, \quad \text{i.e. } q = q_0 + A\cos\Psi, \qquad \dot{q} \equiv \dot{\tilde{q}} = -\omega_0 A\sin\Psi, \quad \text{with } \dot{\Psi} = \omega_0 = (\kappa/m)^{1/2}. \qquad (***)$$



The amplitude A of these oscillations has to be found from their matching with the periods of the non-slip motion described by Eq. (*). Perhaps the simplest way to do this is to plot the result (***) on the phase plane of the system, with $\dot{q}$ normalized to ω_0 – cf. Fig. 5.9 of the lecture notes. On this plane (see the figure on the right), Eq. (***) describes a clockwise rotation of the representing point on a circle of radius A, displaced from the origin by q_0. However, before drawing the circle, let us mark the vertical line corresponding to the q_{max} given by Eq. (*) and the horizontal line corresponding to the belt's motion. If the block moves with the belt, then the representing point moves to the right along that horizontal line until its deviation q from equilibrium reaches the value q_{max} – see point 1 in the figure above. (Note that by the problem's assignment, $q_{max} > q_0$.) At this point, the block starts to slip, and hence to obey the law (***). Hence, the representing point starts moving around the circle of the following radius:

$$A = \left(q_{max}^2 + \frac{u^2}{\omega_0^2}\right)^{1/2}.$$

This motion (physically, a backswing of the block) continues until point 2 where its velocity matches that of the belt again so that the block gets captured by static friction and starts moving with the belt. This process repeats again and again, providing a classical example of relaxation oscillations.[145]

The full period $\mathcal{T}$ of these oscillations may be calculated by adding the time of the block's motion with the belt,

$$\mathcal{T}_{2\to1} = \frac{2(q_{max}-q_0)}{u} \equiv 2(\mu_s - \mu_k)\frac{mg}{\kappa u} \equiv 2(\mu_s - \mu_k)\frac{g}{u\omega_0^2},$$

---

[144] This lack of damping, at substantial kinematic friction, is an artifact of the simple Coulomb approximation, in which the force (**) is velocity-independent. In reality, kinematic friction grows with velocity.

[145] As the figure shows, at $\mu_s < \mu_k$, i.e. at $q_{max} < q_0$, such oscillations are impossible, and an initial transient process leads to the block's oscillations about its equilibrium position $q = q_0$.

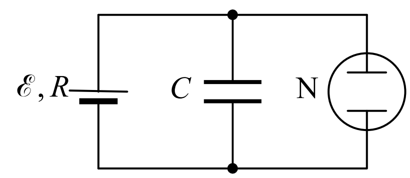
to that of the backswing (see the figure again):

$$\mathcal{T}_{1\to2} = \frac{\pi + 2\varphi}{\omega_0} = \frac{1}{\omega_0}\left[\pi + 2\tan^{-1}\frac{u}{\omega_0(q_{\max} - q_0)}\right], \qquad \text{so } \frac{\pi}{\omega_0} \le \mathcal{T}_{1\to2} \le \frac{2\pi}{\omega_0}.$$

As these formulas show, the relation between these two time intervals, as well as the oscillation waveform, depends on the dimensionless ratio

$$r \equiv \frac{u}{\omega_0(q_{\max} - q_0)} \equiv \frac{u\omega_0}{(\mu_s - \mu_k)g}.$$

If this ratio is much smaller than 1, then $\mathcal{T} \approx \mathcal{T}_{2\to1}$, i.e. the oscillation period is dominated by the relatively long interval of uniform motion of the block with the belt, followed by its fast swing back by $\Delta q \approx -2A \approx -2(q_{\max} - q_0)$. In the opposite limit of a relatively fast belt's motion ($r >> 1$), $\mathcal{T} \approx \mathcal{T}_{1\to2}$ and the oscillation waveform is nearly sinusoidal, with an amplitude close to u/ω_0, and each period interrupted by just a short time interval of the block's being captured by the belt.

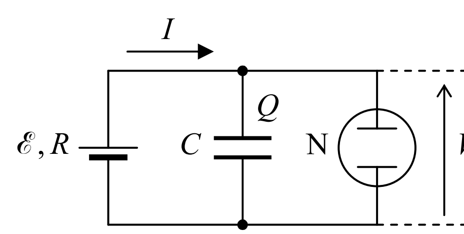


Problem 5.22. The figure on the right shows the circuit of the simplest electronic relaxation oscillator. N is a bistable circuit element that switches very rapidly from its very-high-resistance state to a very-low-resistance state as the voltage across it is increased beyond some value V_t, and switches back as the voltage is decreased below another value $V_t' < V_t$.[146] Calculate the waveform and the time period of voltage oscillations in the circuit.

Hint: The solution of this problem requires a very basic understanding of electric circuits, including such notions as the e.m.f. $\mathcal{E}$ and the internal resistance R of a dc current source – e.g., of an electric battery.

Solution: First, let the bistable device N be in its "off" (high-resistance) state. Then the electric current flowing through it is negligible, so all the dc current I from its source flows into the capacitor C, changing its electric charge Q – see the figure below. Combining the basic relations describing the circuit elements,

$$V = \mathcal{E} - IR, \quad Q = CV, \quad \text{and} \quad \frac{dQ}{dt} = I,$$



we get a simple linear differential equation of the first order:

$$C\frac{dV}{dt} = \frac{\mathcal{E} - V}{R}, \quad \text{i.e.} \quad \frac{d\tilde{V}}{dt} = -\frac{\tilde{V}}{\tau},$$

where

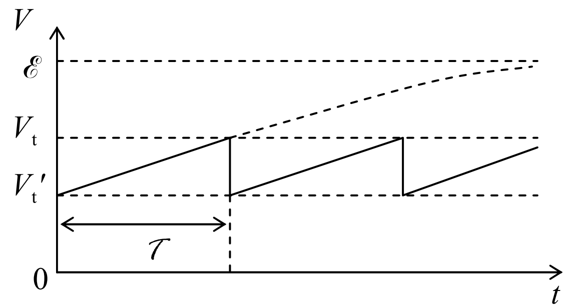
$$\tilde{V} \equiv V - \mathcal{E} \text{ and } \tau \equiv RC,$$

This equation may be readily solved, giving

[146] This is a good model for many two-terminal gas-discharge devices (such as *glow lamps*), whose effective resistance may drop by up to 5 orders of magnitude when the discharge has been ignited by voltage $V > V_t$. In the usual neon glow lamps, the discharge stops at a voltage V_t' that is about 30% lower than V_t.

$$\tilde{V}(t) = \tilde{V}(0)\exp\left\{-\frac{t}{\tau}\right\}, \quad \text{i.e. } V(t) = \mathcal{E} + \left[V(0) - \mathcal{E}\right]\exp\left\{-\frac{t}{\tau}\right\}. \qquad (*)$$

This solution describes an exponential approach, with the time constant $\tau = RC$, of the voltage V to the battery's e.m.f. $\mathcal{E}$. If $\mathcal{E} < V_t$, this is the end of the story, but if the opposite relation is true, then at the moment when V reaches this threshold value, the bistable device "turns on", i.e. switches into its low-resistance state. As soon as this has happened, the capacitor discharges through it very fast, until the voltage V drops to the turn-off value V_t'. At this point, the device switches back into its high-resistance state, and the whole process repeats periodically – see the figure on the right. Note that if the switching threshold difference $V_t - V_t'$ is much smaller than the difference $\mathcal{E} - V_t'$, the voltage rise is virtually linear in time, i.e. the oscillations have a triangular "sawtooth" waveform.

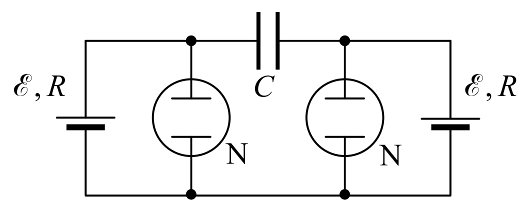


In order to calculate the oscillation period $\mathcal{T}$, we may take the moment of the voltage drop from V_t to V_t' for $t = 0$, i.e. use Eq. (*) with $V(0) = V_t'$ and $V(\mathcal{T}) = V_t$, so it yields

$$V_t = \mathcal{E} + \left(V_t' - \mathcal{E}\right)\exp\left\{-\frac{\mathcal{T}}{\tau}\right\}, \quad \text{i.e. } \mathcal{T} = \tau \ln\frac{\mathcal{E} - V_t'}{\mathcal{E} - V_t}.$$

Note that if $\mathcal{E}$ is relatively close to V_t, this period is rather sensitive to the e.m.f.'s value – the fact used to design simple *voltage-controlled oscillators*.

As an additional exercise, the reader is challenged to analyze similar oscillations in the slightly more complex circuit shown in the figure on the right. Such circuits with two similar neon glow lamps were used, in particular, in blinking-eye figurines of owls and witches, popular some time ago.[147]

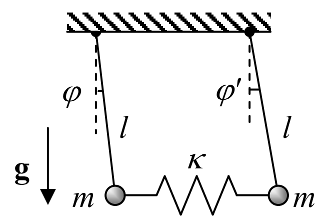


[147] It is interesting that a natural extension of this circuit to $N > 2$ similar lamps, each biased by its own current source, leads to chaotic dynamics (see Chapter 9 below): at a modest bias $\mathcal{E} > V_t$, only one lamp can glow at a time, and the prediction of which one would turn on next is impossible.

Chapter 6. From Oscillations to Waves

For each of the systems specified in Problems 1-6:

(i) introduce convenient generalized coordinates q_j of the system,
(ii) calculate the frequencies of its small harmonic oscillations near the equilibrium,
(iii) calculate the corresponding distribution coefficients, and
(iv) sketch the oscillation modes.



Problem 6.1. Two elastically coupled pendula confined to a vertical plane that contains both suspension points, with the parameters shown in the figure on the right (see also Problems 1.8 and 2.9).

Solution: The derivation of the equations of motion of this system was the subject of Problems 1.8 and 2.9. By introducing the partial frequency $\Omega \equiv (g/l)^{1/2}$ of oscillations of each pendulum in the absence of the coupling spring and the frequency $\omega_0 \equiv (\kappa/m)^{1/2}$ of oscillations of a single mass m under the effect of the spring alone, these equations may be recast into a simpler form:

$$\ddot{\varphi} + \left(\omega_0^2 + \Omega^2\right)\varphi - \omega_0^2\varphi' = 0,$$
$$-\omega_0^2\varphi + \ddot{\varphi}' + \left(\omega_0^2 + \Omega^2\right)\varphi' = 0.$$

Now looking for a partial solution of this system of two homogeneous linear differential equations in the usual form given by Eq. (6.6) of the lecture notes, with $\omega \equiv i\lambda$,

$$\varphi = ae^{-i\omega t}, \quad \varphi' = a'e^{-i\omega t},$$

we get a system of two linear algebraic equations for the complex amplitudes a and a':

$$\begin{aligned}\left(-\omega^2 + \omega_0^2 + \Omega^2\right)a - \omega_0^2 a' = 0,\\ -\omega_0^2 a + \left(-\omega^2 + \omega_0^2 + \Omega^2\right)a' = 0.\end{aligned} \qquad (*)$$

For these equations to be consistent, the determinant of the system's matrix must equal zero:

$$\begin{vmatrix} -\omega^2 + \omega_0^2 + \Omega^2 & -\omega_0^2 \\ -\omega_0^2 & -\omega^2 + \omega_0^2 + \Omega^2 \end{vmatrix} = 0.$$

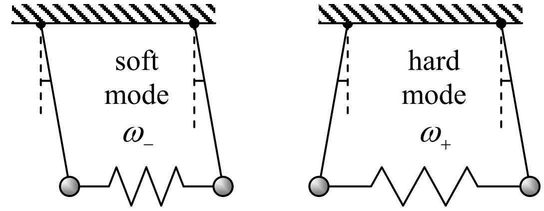
Solving the resulting quadratic characteristic equation, we get two roots for ω^2:

$$\omega_-^2 = \Omega^2 \equiv \frac{g}{l}, \qquad \omega_+^2 = \Omega^2 + 2\omega_0^2 \equiv \frac{g}{l} + \frac{2\kappa}{m} \geq \omega_-^2.$$

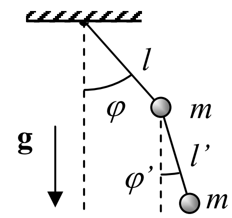
Plugging the frequencies, one by one, back into any of Eqs. (*), we get the corresponding distribution coefficients:

$$\left(\frac{a'}{a}\right)_- = +1, \qquad \left(\frac{a'}{a}\right)_+ = -1.$$

These modes are sketched in the figure on the right. In the "soft" mode with frequency ω_-, the oscillations are in phase: $\varphi(t) \equiv \varphi'(t)$, and the spring is not engaged at all. On the other hand, in the second, "hard" mode with frequency ω_+, the oscillations are in anti-phase, $\varphi(t) \equiv -\varphi'(t)$, so the spring is being most fully stretched/compressed, resulting in a higher oscillation frequency.



Problem 6.2. The double pendulum confined to the vertical plane containing the support point (which was the subject of Problem 2.1), with $m' = m$ and $l = l'$ – see the figure on the right.



Solution: In order to retain some means for sanity checking (see below), it is always prudent to carry out the calculations for arbitrary parameters of the system – in our current case, for arbitrary masses and lengths of the pendula – up to the point where this starts to lead to overly bulk expressions. With the angles φ and φ' shown in the figure on the right used as the generalized coordinates, the equations of motion are as follows (see the model solution of Problem 2.1):

$$\frac{d}{dt}\left[(m+m')l^2\dot{\varphi} + m'll'\dot{\varphi}'\cos(\varphi-\varphi')\right] - \left[-(m+m')gl\sin\varphi - m'll'\dot{\varphi}\dot{\varphi}'\sin(\varphi-\varphi')\right] = 0,$$
$$\frac{d}{dt}\left[m'l'^2\dot{\varphi}' + m'll'\dot{\varphi}\cos(\varphi-\varphi')\right] - \left[-m'gl'\sin\varphi' + m'll'\dot{\varphi}\dot{\varphi}'\sin(\varphi-\varphi')\right] = 0.$$

Linearizing these equations with respect to small φ and φ', we get a set of two linear equations

$$(m+m')l^2\left(\ddot{\varphi} + \Omega^2\varphi\right) + m'll'\ddot{\varphi}' = 0,$$
$$m'll'\ddot{\varphi} + m'l'^2\left(\ddot{\varphi}' + \Omega'^2\varphi'\right) = 0,$$

where $\Omega \equiv (g/l)^{1/2}$ and $\Omega' \equiv (g/l')^{1/2}$. Looking for the solution of this system in the usual form

$$\varphi = ae^{-i\omega t}, \quad \varphi' = a'e^{-i\omega t},$$

we get the following set of two linear algebraic equations for the complex amplitudes a and a':

$$(m+m')l^2(-\omega^2 + \Omega^2)a - m'll'\omega^2 a' = 0,$$
$$-m'll'\omega^2 a + m'l'^2\left(-\omega^2 + \Omega'^2\right)a' = 0.$$

The usual condition of self-consistency of this system gives the following characteristic equation:

$$\left(-\omega^2 + \Omega^2\right)\left(-\omega^2 + \Omega'^2\right) - \frac{m'}{m+m'}\omega^4 = 0. \qquad (*)$$

Since the solution of this quadratic equation for ω^2 leads to a somewhat bulky formula, this is a good point to take a pause and check whether our results make sense for simple particular cases. If $m'/m \to 0$, while $l \sim l'$, then the roots of Eq. (*) are obvious: $\omega_\pm = \{\Omega, \Omega'\}$, i.e. the pendula are independent. This is natural, because the heavy upper pendulum, due to its large mass, does not "feel" the lower light pendulum. At the same time, slow oscillations of the upper pendulum do not affect the fast oscillations of its lower counterpart.

On the other hand, if $m'/m \to \infty$, the result is quite different. For the lowest-frequency ("soft") oscillation mode, we may replace the factor $m'/(m + m')$ in Eq. (*) with 1, and readily get $\omega_-^2 = g/(l + l')$. This is also very natural because, in this limit, the main mode of oscillations is the aligned motion of both pendula, with the total length equal to $(l + l')$. (A good additional exercise: find and interpret the hard mode of oscillations in this limit.)

Now when we are sure that our general results are sensible, let us move to our particular case ($m' = m$, and $l' = l$, i.e. $\Omega' = \Omega$). In this case, Eq. (*) reduces to

$$\left(-\omega^2 + \Omega^2\right)^2 - \frac{1}{2}\omega^4 = 0,$$

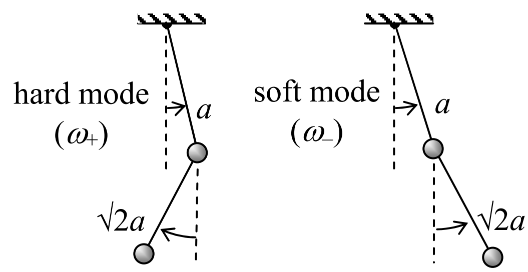
and may be readily solved to give the following values of the "hard" (sign +) and "soft" (sign –) frequencies of the system:

$$\omega_\pm^2 = \Omega^2\left(2 \pm \sqrt{2}\right), \qquad \text{i.e. } \frac{\omega_+}{\omega_-} = \sqrt{2} + 1 \approx 2.41.$$

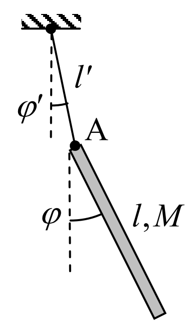
Plugging these values of ω^2, one by one, back into the system of linear equations for a and a', for the distribution coefficients we get:

$$\frac{a'_\pm}{a_\pm} = \mp\sqrt{2} \approx \mp 1.41.$$

so the oscillation modes look as sketched in the figure on the right.



Problem 6.3 The chime bell considered in Problem 4.12 (see the figure on the right), for the particular case $l = l'$.



Solution: The general equations of motion of the bell were derived in the model solution of Problem 4.12:

$$I_A\ddot{\varphi} + M\frac{l}{2}l'\cos(\varphi - \varphi')\ddot{\varphi}' + M\frac{l}{2}l'\sin(\varphi - \varphi')\dot{\varphi}\dot{\varphi}' + Mg\frac{l}{2}\sin\varphi = 0,$$

$$Ml'^2\ddot{\varphi}' + M\frac{l}{2}l'\cos(\varphi - \varphi')\ddot{\varphi} - M\frac{l}{2}l'\sin(\varphi - \varphi')\dot{\varphi}\dot{\varphi}' + Mgl'\sin\varphi' = 0,$$

where φ and φ' are the angles shown in the figure on the right, while $I_A \equiv Ml^2/3$. These equations are significantly simplified by their linearization near the fixed point $\varphi = \varphi' = 0$, which describes the equilibrium position:

$$I_A\ddot{\varphi} + M\frac{l}{2}l'\ddot{\varphi}' + Mg\frac{l}{2}\varphi = 0,$$

$$Ml'^2\ddot{\varphi}' + M\frac{l}{2}l'\ddot{\varphi} + Mgl'\varphi' = 0,$$

Looking for the solution of the equations of motion in the usual form

$$\varphi = ae^{-i\omega t}, \quad \varphi' = a'e^{-i\omega t},$$

we get a system of linear equations for oscillation amplitudes a and a':

$$\left(Mg\frac{l}{2}-\omega^2 I_{\rm A}\right)a+\left(-\omega^2 M\frac{l}{2}l'\right)a'=0,$$
$$\left(-\omega^2 M\frac{l}{2}l'\right)a+\left(Mgl'-\omega^2 Ml'^2\right)a'=0, \qquad (*)$$

This is a good moment to introduce a more transparent notation, for example,

$$\Omega^2 \equiv \frac{Mg(l/2)}{I_{\rm A}}=\frac{3}{2}\frac{g}{l} \quad \text{and} \quad \Omega'^2 \equiv \frac{g}{l'},$$

where, according to Eq. (4.41) of the lecture notes, Ω has the physical sense of the oscillation frequency of the bar if it is suspended by its endpoint A (i.e. if φ' = const), while Ω' is the oscillation frequency the system would have if all the bar's mass was concentrated at point A. In this notation, Eqs. (*) take the form

$$\left(\Omega^2-\omega^2\right)a+\left(-\omega^2\frac{\Omega^2}{\Omega'^2}\right)a'=0,$$
$$\left(-\omega^2\frac{3}{4}\frac{\Omega'^2}{\Omega^2}\right)a+\left(\Omega'^2-\omega^2\right)a'=0. \qquad (**)$$

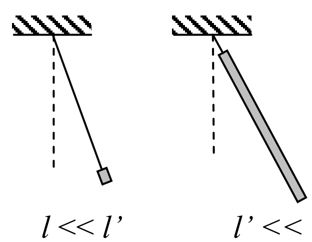
The condition of their compatibility is

$$\begin{vmatrix} \Omega^2-\omega^2 & -\omega^2\dfrac{\Omega^2}{\Omega'^2} \\ -\omega^2\dfrac{3}{4}\dfrac{\Omega'^2}{\Omega^2} & \Omega'^2-\omega^2 \end{vmatrix} \equiv \frac{1}{4}\left[\left(\omega^2\right)^2-4\omega^2\left(\Omega^2+\Omega'^2\right)+4\Omega^2\Omega'^2\right]=0.$$

This quadratic equation for ω^2 has two roots:

$$\omega_\pm^2=2\left\{\left(\Omega^2+\Omega'^2\right)\pm\left[\left(\Omega^2+\Omega'^2\right)^2-\Omega^2\Omega'^2\right]^{1/2}\right\}. \qquad (***)$$

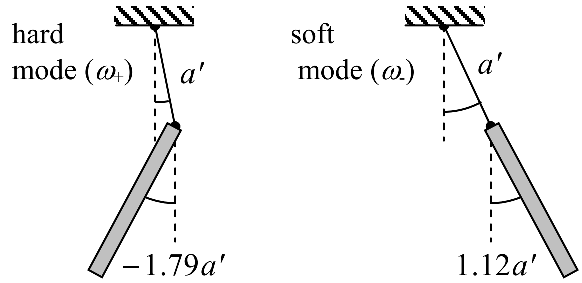
As a sanity check, if $l \ll l'$ (the bar is shrunk to a point), then $\Omega \gg \Omega'$, and Eq. (***) yields $\omega_- \approx \Omega'$, while in the opposite limit, $l' \ll l$, $\omega_- \approx \Omega$, correctly reflecting the nature of the lowest oscillation mode in these two limits – see the figure on the right.



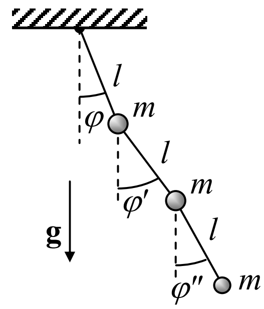
In our particular case, $l = l'$, so $\Omega'^2 = (2/3)\Omega^2$ and $\omega_\pm^2 = (2/3)(5 \pm \sqrt{19})\Omega^2$, i.e. $\omega_+/\omega_- \approx 3.821$. Plugging these values, one by one, back into any equation of the system (**), we get

$$\left(\frac{a}{a'}\right)_\pm=\frac{5\pm\sqrt{19}}{1-2\left(5\pm\sqrt{19}\right)/3}\approx\begin{cases}-1.786,\\+1.120.\end{cases}$$

The most important feature of this result is the opposite signs of these ratios, so the oscillation modes look (approximately) as shown in the figure on the right.



It is very informative to compare this result with the solution of the previous problem (the double pendulum with point masses), for the same case $l = l'$. The modes are qualitatively similar, but in the double pendulum, the magnitude of both distribution coefficients is the same ($\sqrt{2} \approx 1.41$), and the own frequency ratio is lower (~2.41 instead of ~3.82), i.e. the contrast between the hard and soft modes is somewhat less pronounced.



Problem 6.4. The triple pendulum shown in the figure on the right, with the motion confined to a vertical plane containing the support point.

Hint: In this problem, you may use any (e.g., numerical) method to calculate the characteristic equation's roots.

Solution: Using the pendula displacement angles φ, φ', and φ'' (see the figure on the right) as the generalized coordinates, and acting exactly as at the model solution of Problem 2.1, we get similar kinematic relations:

$$\begin{aligned}
x &= l\sin\varphi, & y &= -l\cos\varphi,\\
x' &= l(\sin\varphi + \sin\varphi'), & y' &= -l(\cos\varphi + \cos\varphi'),\\
x'' &= l(\sin\varphi + \sin\varphi' + \sin\varphi''), & y'' &= -l(\cos\varphi + \cos\varphi' + \cos\varphi'');
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= l\dot{\varphi}\cos\varphi, & \dot{y} &= l\dot{\varphi}\sin\varphi,\\
\dot{x}' &= l(\dot{\varphi}\cos\varphi + \dot{\varphi}'\cos\varphi'), & \dot{y}' &= l(\dot{\varphi}\sin\varphi + \dot{\varphi}'\sin\varphi'),\\
\dot{x}'' &= l(\dot{\varphi}\cos\varphi + \dot{\varphi}'\cos\varphi' + \dot{\varphi}''\cos\varphi''), & \dot{y}'' &= l(\dot{\varphi}\sin\varphi + \dot{\varphi}'\sin\varphi' + \dot{\varphi}''\sin\varphi''),
\end{aligned}$$

so the Lagrangian function of the system is

$$\begin{aligned}
L &= \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{x}'^2 + \dot{y}'^2 + \dot{x}''^2 + \dot{y}''^2\right) - mg\,(y + y' + y'')\\
&= \frac{ml^2}{2}\Big[(\dot{\varphi}\cos\varphi)^2 + (\dot{\varphi}\sin\varphi)^2 + (\dot{\varphi}\cos\varphi + \dot{\varphi}'\cos\varphi')^2 + (\dot{\varphi}\sin\varphi + \dot{\varphi}'\sin\varphi')^2\\
&\quad + (\dot{\varphi}\cos\varphi + \dot{\varphi}'\cos\varphi' + \dot{\varphi}''\cos\varphi'')^2 + (\dot{\varphi}\sin\varphi + \dot{\varphi}'\sin\varphi' + \dot{\varphi}''\sin\varphi'')^2\Big]\\
&\quad + mgl\big[(\cos\varphi) + (\cos\varphi + \cos\varphi') + (\cos\varphi + \cos\varphi' + \cos\varphi'')\big].
\end{aligned}$$

Since this expression is already bulky, and our task is limited to small oscillations, it makes sense to simplify L already at this stage, keeping only its terms of the second order in the small displacement angles and their time derivatives – which are responsible for the linear terms in the Lagrange equations of motion:

$$L \approx \frac{ml^2}{2}\left[(\dot\varphi)^2 + (\dot\varphi + \dot\varphi')^2 + (\dot\varphi + \dot\varphi' + \dot\varphi'')^2\right] - \frac{mgl}{2}\left[(\varphi^2) + (\varphi^2 + \varphi'^2) + (\varphi^2 + \varphi'^2 + \varphi''^2)\right]$$
$$\equiv \frac{ml^2}{2}\left(3\dot\varphi^2 + 2\dot\varphi'^2 + \dot\varphi''^2 + 4\dot\varphi\dot\varphi' + 2\dot\varphi\dot\varphi'' + 2\dot\varphi'\dot\varphi''\right) - \frac{mgl}{2}\left(3\varphi^2 + 2\varphi'^2 + \varphi''^2\right).$$

The corresponding equations of motion are:

$$\text{for } \varphi: \quad 3\ddot\varphi + 2\ddot\varphi' + \ddot\varphi'' + 3\Omega^2\varphi = 0,$$
$$\text{for } \varphi': \quad 2\ddot\varphi' + 2\ddot\varphi + \ddot\varphi'' + 2\Omega^2\varphi' = 0,$$
$$\text{for } \varphi'': \quad \ddot\varphi'' + \ddot\varphi + \ddot\varphi' + \Omega^2\varphi'' = 0,$$

where $\Omega \equiv (g/l)^{1/2}$ is a unit of frequency, natural for this problem. Looking for their solution in the usual form,

$$\varphi = ae^{-i\omega t}, \qquad \varphi' = a'e^{-i\omega t}, \qquad \varphi'' = a''e^{-i\omega t},$$

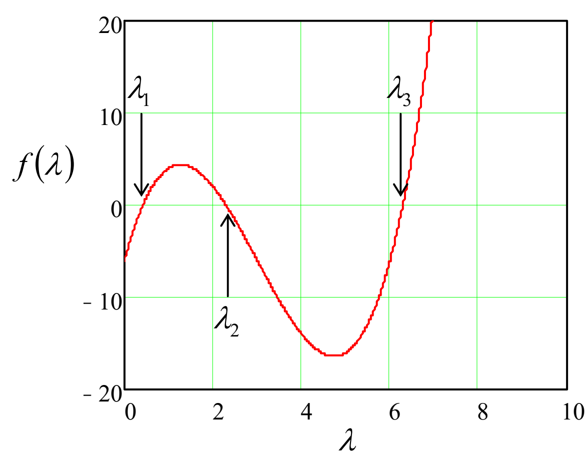
and dividing all terms by Ω^2, we get the following system of three linear algebraic equations:

$$\begin{aligned} &3(\lambda - 1)a + 2\lambda a' + \lambda a'' = 0,\\ &2\lambda a + 2(\lambda - 1)a' + \lambda a'' = 0, \\ &\lambda a + \lambda a' + (\lambda - 1)a'' = 0, \end{aligned} \qquad (*)$$

where $\lambda \equiv \omega^2/\Omega^2$. The condition of their consistency is

$$f(\lambda) \equiv \begin{vmatrix} 3(\lambda-1) & 2\lambda & \lambda \\ 2\lambda & 2(\lambda-1) & \lambda \\ \lambda & \lambda & \lambda - 1 \end{vmatrix} \equiv \lambda^3 - 9\lambda^2 + 18\lambda - 6 = 0 \;.$$

The function $f(\lambda)$ is plotted in the figure on the right. The plot shows that the characteristic equation $f(\lambda) = 0$ has three real roots λ_j, all positive. Their numerical calculation yields:[1]



$$\lambda_1 = 0.4158..., \qquad \lambda_2 = 2.2943..., \qquad \lambda_3 = 6.289...,$$

giving the following three eigenfrequencies $\omega_j \equiv \lambda_j^{1/2}\Omega$:

$$\omega_1 \approx 0.6448\,\Omega, \quad \omega_2 \approx 1.5147\,\Omega, \quad \omega_3 \approx 2.508\,\Omega.$$

Now we should find the corresponding values of the distribution coefficients, for example, a'/a and a''/a. General expressions for these ratios may be calculated from any two equations of the system (*). For example, by solving the system of the first two equations, we readily get

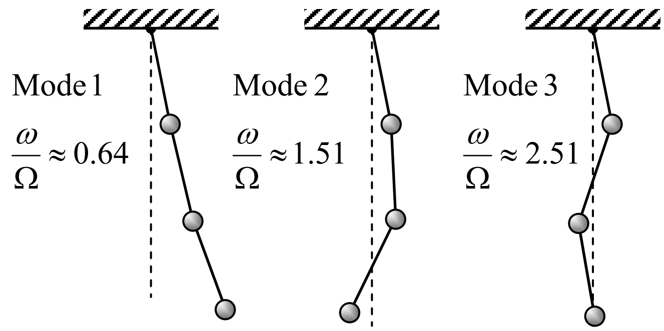
[1] For this problem, with no free parameters, using bulky quasi-analytical Tartaglia-Cardano formulas for the cubic equation roots has no real advantage over its purely numerical solution. (Ready routines for that are available in any of the numerical packages listed in MA Sec. 16iv.)

$$\frac{a'}{a}=\frac{3-\lambda}{2}, \qquad \frac{a''}{a}=\frac{\lambda^2-6\lambda+3}{\lambda}.$$

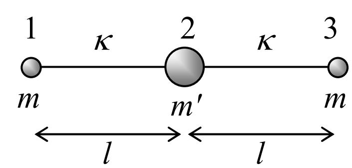
Plugging into these formulas the numerical values of the three eigenvalues of λ_j, we get

$$\left(\frac{a'}{a}\right)_1\approx 1.292,\ \left(\frac{a''}{a}\right)_1\approx 1.631; \qquad \left(\frac{a'}{a}\right)_2\approx 0.353,\ \left(\frac{a''}{a}\right)_2\approx -2.398; \qquad \left(\frac{a'}{a}\right)_3\approx -1.645,\ \left(\frac{a''}{a}\right)_3\approx 0.767.$$

The figure on the right shows sketches of these modes; they remind transverse standing waves in a distributed system, with the "rigid" boundary condition on the top end and the "open" boundary condition on the bottom end – see the discussion in Sec. 6.5 of the lecture notes and, in particular, Fig. 6.10b. (This analogy will be pursued further in Problem 18.)



Problem 6.5. The symmetric three-particle system shown in the figure on the right, where the connections between the particles not only act as usual elastic springs (giving potential energies $U=\kappa(\Delta l)^2/2$), but also resist bending, giving additional potential energy $U'=\kappa'(l\theta)^2/2$, where θ is the (small) bending angle.[2]



Solution: Since small longitudinal displacements $\tilde{x}$ (along the system's axis) do not affect either the kinetic or potential energy of small transverse displacements and vice versa, their oscillations may be considered separately.

The *longitudinal* displacements of the particles (from the equilibrium positions shown in the figure above) are described by the following Lagrangian function:

$$L_1=\frac{m}{2}\dot{\tilde{x}}_1^2+\frac{m'}{2}\dot{\tilde{x}}_2^2+\frac{m}{2}\dot{\tilde{x}}_3^2-\frac{\kappa}{2}\left(\tilde{x}_1-\tilde{x}_2\right)^2-\frac{\kappa}{2}\left(\tilde{x}_2-\tilde{x}_3\right)^2.$$

One combination of these three shifts, namely the center-of-mass displacement

$$\tilde{X}=\frac{m\tilde{x}_1+m'\tilde{x}_2+m\tilde{x}_3}{M}, \quad \text{where } M\equiv 2m+m',$$

describes the possible uniform longitudinal motion of the system as a whole rather than its oscillations. Assuming that it equals 0 (which may be always ensured by the proper choice of an inertial reference frame), we may eliminate one argument, for example,

$$\tilde{x}_2=-\frac{m}{m'}\left(\tilde{x}_1+\tilde{x}_3\right), \qquad (*)$$

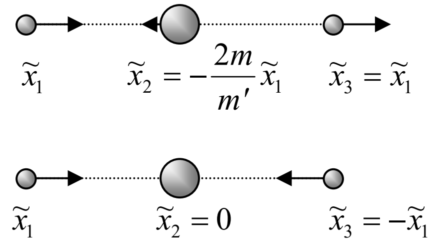
from the Lagrangian function, getting

[2] This is a reasonable model for small oscillations of linear molecules such as the now-infamous CO_2.

$$L_1 = \frac{m}{2}\dot{\tilde{x}}_1^2 + \frac{m^2}{2m'}\left(\dot{\tilde{x}}_1 + \dot{\tilde{x}}_3\right)^2 + \frac{m}{2}\dot{\tilde{x}}_3^2 - \frac{\kappa}{2}\left[\tilde{x}_1 + \frac{m}{m'}\left(\dot{\tilde{x}}_1 + \dot{\tilde{x}}_3\right)\right]^2 - \frac{\kappa}{2}\left[\tilde{x}_3 + \frac{m}{m'}\left(\dot{\tilde{x}}_1 + \dot{\tilde{x}}_3\right)\right]^2. \qquad (**)$$

Now we could, in a regular way, use Eq. (2.19) of the lecture notes to derive the Lagrange equations of motion for the two remaining coordinates – and for an asymmetric system, this would be the best way to proceed. However, due to the evident symmetry of the function L_l (reflecting that of the physical system), it is immediately clear that for the two longitudinal oscillation modes in the system – see the figure below:[3]

$$\tilde{x}_3 = \begin{cases} +\tilde{x}_1, & \text{(symmetric mode)} \\ -\tilde{x}_1. & \text{(antisymmetric mode)} \end{cases}$$



Plugging these relations, one by one, into Eq. (2), we get:

$$\left(L_l\right)_\text{s} = m\left(1 + \frac{2m}{m'}\right)\dot{\tilde{x}}_1^2 - \kappa\left(1 + \frac{2m}{m'}\right)^2 \tilde{x}_1^2, \qquad \left(L_l\right)_\text{a} = m\dot{\tilde{x}}_1^2 - \kappa\tilde{x}_1^2.$$

These are the usual Lagrangian functions of harmonic oscillators (see, e.g., Eq. (5.1) of the lecture notes), with frequencies, respectively,

$$\omega_\text{s} = \left[\frac{\kappa}{m}\left(1 + \frac{2m}{m'}\right)\right]^{1/2} \equiv \left(\frac{\kappa M}{mm'}\right)^{1/2}, \qquad \omega_\text{a} = \left(\frac{\kappa}{m}\right)^{1/2} \leq \omega_\text{sym}.$$

The simplicity of the latter relation is natural, because according to Eq. (*), in the antisymmetric mode, $\tilde{x}_2 = 0$, i.e. the central particle does not move – see the figure above. The symmetric mode's frequency may be also recast in a more physically transparent form:

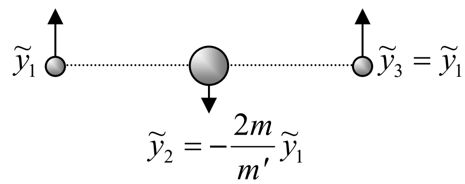
$$\omega_\text{s} = \left(\frac{2\kappa}{m_\text{ef}}\right)^{1/2}, \qquad \text{with } \frac{1}{m_\text{ef}} \equiv \frac{1}{2m} + \frac{1}{m'},$$

showing that these are just the oscillations of the reduced mass m_ef, given by Eq. (3.35) with $m_1 = 2m$ and $m_2 = m'$, under the joint effect of the two springs. Note also that since $M \equiv 2m + m' \geq m'$, the symmetric mode is always "harder" (has a higher frequency) than the antisymmetric one.

Now proceeding to the *transverse* motion of the particles (in the directions perpendicular to the static system's axis x), we may note that it may include not only bending oscillations, but also the rotation of the system about two perpendicular axes y and z, and also to the translational motion of the system as the whole along these directions. In the absence of these non-oscillatory motions, the particles stay within one immobile plane (say, [x, y]), and their total linear momentum component P_y and the angular momentum component L_z equal zero. These equalities impose two conditions upon three transverse displacements of the particles:

[3] Note that due to the central particle's motion, the first of these modes may be called symmetric only in a limited sense of the word.

$$m\dot{\tilde{y}}_1 + m'\dot{\tilde{y}}_2 + m\dot{\tilde{y}}_3 = 0, \qquad ml\dot{\tilde{y}}_1 - ml\dot{\tilde{y}}_3 = 0. \qquad (***)$$



(Note that the second condition immediately yields $\dot{\tilde{y}}_1 = \dot{\tilde{y}}_3$, showing that the bending oscillations are always symmetric – see the figure on the right.) The conditions (***), integrated over time with zero integration constants (which would describe inconsequential linear and angular shifts), enable the elimination of two of three degrees of freedom, for example

$$\tilde{y}_1 = \tilde{y}_3 = -\frac{m'}{2m}\tilde{y}_2,$$

from the general Lagrangian of the transverse motion within the [x, y] plane,

$$L_\text{t} = \frac{m}{2}\dot{\tilde{y}}_1^2 + \frac{m'}{2}\dot{\tilde{y}}_2^2 + \frac{m}{2}\dot{\tilde{y}}_3^2 - \frac{\kappa' l^2}{2}\theta^2, \quad \text{with } \theta = \theta_1 + \theta_2 = \frac{\tilde{y}_2 - \tilde{y}_1}{l} + \frac{\tilde{y}_3 - \tilde{y}_2}{l} \equiv \frac{\tilde{y}_1 + \tilde{y}_3 - 2\tilde{y}_2}{l},$$

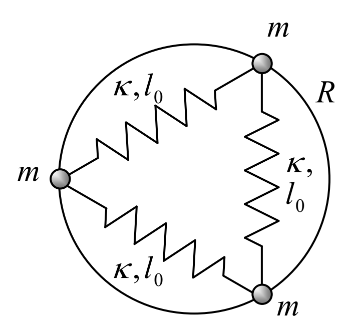
reducing it to

$$L_\text{t} = \frac{m'}{2}\left(1 + \frac{m'}{2m}\right)\dot{\tilde{y}}_2^2 - \frac{\kappa'}{2}\left(2 + \frac{m'}{m}\right)^2 \tilde{y}_2^2.$$

This is, again, the standard Lagrangian function of a harmonic oscillator with the frequency

$$\omega_\text{t} = \left[\kappa'\left(2 + \frac{m'}{m}\right)^2 \Big/ m'\left(1 + \frac{m'}{2m}\right)\right]^{1/2} \equiv \left(\frac{2\kappa' M}{mm'}\right)^{1/2} \equiv \left(\frac{4\kappa'}{m_\text{ef}}\right)^{1/2}.$$

Note, however, that since the orientation of the bending oscillation plane (taken above for [x, y]) is arbitrary, with the only condition that it contains the system's axis x, it is common to represent the oscillation displacement vectors as a sum of their components in two fixed, mutually perpendicular planes, say [x, y] and [x, z], and speak about two independent bending modes that are "degenerate" in the sense that they have the same frequency ω_t. Such representation becomes convenient, in particular, if the bending rigidity coefficient κ' is different in the two planes, "lifting" the mode degeneracy, i.e. creating a small between their frequencies.



Problem 6.6. Three similar beads of mass m, which may slide along a round ring of radius R without friction, connected with similar springs with elastic constants κ and equilibrium lengths l_0 (not necessarily equal to $\sqrt{3}R$) – see the figure on the right.

Solution: The potential energy of the spring connecting the beads number j and number j' is

$$U_{jj'} = \frac{\kappa}{2}\left(l_{jj'} - l_0\right)^2,$$

where $l_{jj'}$ is the spring's length. The length may be readily calculated from the system's geometry:

$$l_{jj'} = 2R\sin\frac{\varphi_{jj'}}{2},$$

where $\varphi_{jj'} \equiv \varphi_{j'} - \varphi_j > 0$ is the angular distance between the adjacent beads. Combining these two relations, we get

$$U_{jj'} = \frac{\kappa}{2}\left(2R\sin\frac{\varphi_{jj'}}{2} - l_0\right)^2.$$

In the symmetric stationary state, all three angles are equal: $\varphi_{jj'} = \varphi_0$. Expanding $U_{jj'}$ in the Taylor series in small deviations $\tilde{\varphi}_{jj'} \equiv \varphi_{jj'} - \varphi_0$ from this equilibrium, and dropping all terms higher than $O(\tilde{\varphi}_{jj'}^2)$, we get

$$U_{jj'} \approx \frac{\kappa}{2}\left[2R\left(\sin\frac{\varphi_0}{2} + \frac{1}{2}\cos\frac{\varphi_0}{2}\,\tilde{\varphi}_{jj'} - \frac{1}{8}\sin\frac{\varphi_0}{2}\,\frac{\tilde{\varphi}_{jj'}^2}{2}\right) - l_0\right]^2 = \frac{\kappa}{2}\left(\Delta l_0 + R\cos\frac{\varphi_0}{2}\,\tilde{\varphi}_{jj'} - \frac{R}{4}\sin\frac{\varphi_0}{2}\,\tilde{\varphi}_{jj'}^2\right)^2$$

$$\approx \frac{\kappa}{2}(\Delta l_0)^2 + \frac{\kappa}{2}2(\Delta l_0)\left(R\cos\frac{\varphi_0}{2}\right)\tilde{\varphi}_{jj'} + \frac{\kappa}{2}\left[\left(R\cos\frac{\varphi_0}{2}\right)^2 - 2(\Delta l_0)\left(\frac{R}{4}\sin\frac{\varphi_0}{2}\right)\right]\tilde{\varphi}_{jj'}^2,$$

where $\Delta l_0 \equiv [2R\sin(\varphi_0/2) - l_0]$ is the spring's extension at equilibrium. The first term in the last expression for $U_{jj'}$ is a constant, and the sum of the second terms over all springs is also motion-independent because $\tilde{\varphi}_{12} + \tilde{\varphi}_{23} + \tilde{\varphi}_{31} \equiv (\tilde{\varphi}_2 - \tilde{\varphi}_1) + (\tilde{\varphi}_3 - \tilde{\varphi}_2) + (\tilde{\varphi}_1 - \tilde{\varphi}_3) \equiv 0$. The last, quadratic term may be rewritten as

$$\tilde{U}_{jj'} = \frac{\kappa'}{2}R^2\tilde{\varphi}_{jj'}^2,$$

where κ' is the effective spring constant:

$$\kappa' \equiv \frac{\kappa}{R^2}\left[\left(R\cos\frac{\varphi_0}{2}\right)^2 - 2(\Delta l_0)\left(\frac{R}{4}\sin\frac{\varphi_0}{2}\right)\right] \equiv \kappa\left(\cos^2\frac{\varphi_0}{2} - \frac{\Delta l_0}{2R}\sin\frac{\varphi_0}{2}\right).$$

The constant is positive, and hence the potential energy of the spring grows with the deviation from equilibrium, if

$$\Delta l_0 < 2R\frac{\cos^2(\varphi_0/2)}{\sin(\varphi_0/2)}, \quad \text{i.e. if } l_0 > 2R\left[\sin(\varphi_0/2) - \frac{\cos^2(\varphi_0/2)}{\sin(\varphi_0/2)}\right].$$

In our particular case of three beads[4]

$$\varphi_0 \equiv \frac{2\pi}{3}, \qquad \text{so that} \quad \sin\frac{\varphi_0}{2} = \frac{\sqrt{3}}{2}, \quad \cos\frac{\varphi_0}{2} = \frac{1}{2},$$

and the above conditions take the form

$$\Delta l_0 < \frac{R}{\sqrt{3}} \approx 0.577\,R, \quad \text{i.e. if } l_0 > \left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)R \approx 1.155\,R.$$

This is the condition of stability of the symmetric stationary state; note that it is always fulfilled if $\Delta l_0 \leq 0$, i.e. if the springs are either pre-compressed or not stretched in the equilibrium state.[5] If this

[4] Note that before this point, the analysis is valid for an arbitrary number ($N > 1$) of beads, with $\varphi_0 = 2\pi/N$.

[5] The only requirement on the pre-compression is to avoid spring "buckling" (which would break our implicit assumption that the springs are always straight).

condition is fulfilled,[6] i.e. if $\kappa' > 0$, we may analyze small oscillations around the equilibrium by writing the system's Lagrangian function

$$L = T - U = (T_1 + T_2 + T_3) - (U_{12} + U_{23} + U_{31}) = \frac{m}{2}(v_1^2 + v_2^2 + v_3^2) - (\tilde{U}_{12} + \tilde{U}_{23} + \tilde{U}_{31})$$
$$= \frac{m}{2}R^2\left(\dot{\tilde{\varphi}}_1^2 + \dot{\tilde{\varphi}}_2^2 + \dot{\tilde{\varphi}}_3^2\right) - \frac{\kappa'}{2}R^2\left(\tilde{\varphi}_{12}^2 + \tilde{\varphi}_{23}^2 + \tilde{\varphi}_{31}^2\right)$$
$$\equiv \frac{m}{2}R^2\left(\dot{\tilde{\varphi}}_1^2 + \dot{\tilde{\varphi}}_2^2 + \dot{\tilde{\varphi}}_3^2\right) - \frac{\kappa'}{2}R^2\left[(\tilde{\varphi}_2 - \tilde{\varphi}_1)^2 + (\tilde{\varphi}_3 - \tilde{\varphi}_2)^2 + (\tilde{\varphi}_3 - \tilde{\varphi}_1)^2\right],$$

with

$$\kappa' = \kappa\left(\cos^2\frac{\varphi_0}{2} - \frac{\Delta l_0}{2R}\sin\frac{\varphi_0}{2}\right)_{\varphi_0 = 2\pi/3} = \frac{\kappa}{4}\left(1 - \sqrt{3}\frac{\Delta l_0}{R}\right).$$

With the angle deviations $\tilde{\varphi}_j$ (with j = 1, 2, 3) taken for the generalized coordinates q_j, the standard differentiation of the function (if necessary, see Eq. (2.19a) of the lecture notes again) yields three similar differential equations of motion:

$$m\ddot{\tilde{\varphi}}_j + \kappa'\left(2\tilde{\varphi}_j - \tilde{\varphi}_{j+1} - \tilde{\varphi}_{j-1}\right) = 0.$$

With its solution taken in the usual form $\tilde{\varphi}_j = c_j e^{-i\omega t}$, this system reduces to that of three linear algebraic equations:

$$-m\omega^2 c_j + \kappa'\left(2c_j - c_{j+1} - c_{j-1}\right) = 0. \qquad (*)$$

Now there are two different ways to proceed to find the eigenvalues ω^2, which eventually yield the same result. The general way is to write the condition of self-consistency of the system (*) by equating its determinant to zero (see Eq. (6.20) of the lecture notes), and solving the resulting characteristic equation to find three roots for ω^2. A smarter way is to use the system's symmetry, which expresses itself in the similarity of Eq. (*) for all three values of the index j, by looking for the solution in the form of a traveling wave: $c_j = ae^{i\alpha j}$ – see Eq. (6.26). This substitution immediately yields the characteristic equation,

$$-m\omega^2 + \kappa'\left(2 - e^{i\alpha} - e^{-i\alpha}\right) = 0,$$

absolutely similar to Eq. (6.27) of the lecture notes, and hence the dispersion relation (6.30),

$$\omega = \pm 2\left(\frac{\kappa'}{m}\right)^{1/2}\sin\frac{\alpha}{2}.$$

This relation was discussed in detail in Secs. 6.3-6.5 of the lecture notes, both for the infinite and finite number N of particles – see, in particular, Figs. 6.5 and 6.11. In our current case, N = 3, and the system is looped into a ring, so the wave assumption $c_j = e^{i\alpha j}$ may be valid only if we impose the

[6] A useful additional exercise: analyze what will happen to the system if the condition is violated. (Hint: spontaneous symmetry breaking.)

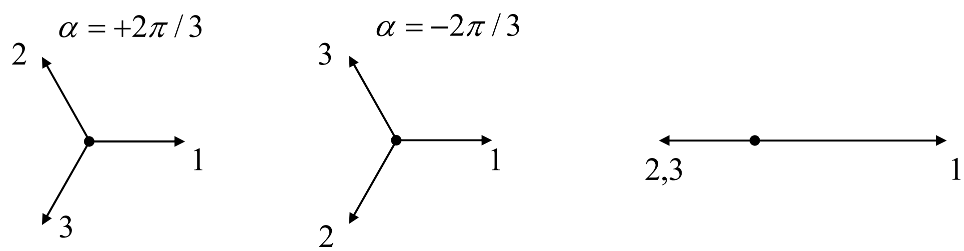
additional periodicity condition $c_{j+3} = c_j$.[7] This condition yields $3\alpha = 2\pi n$, with any integer n, so there are only three physically different (i.e. located on a single 2π-segment, say $-\pi \leq \alpha < \pi$) values of the constant α:[8]

$$\alpha_1 = 0, \qquad \alpha_{2,3} = \pm\frac{2\pi}{3},$$

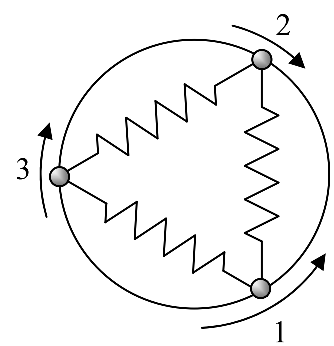
corresponding to just two physically different frequencies:

$$\omega_1 = 0, \qquad \omega_{2,3} = 2\left(\frac{\kappa'}{m}\right)^{1/2} \sin\frac{\pi}{3} \equiv \left(\frac{3\kappa'}{m}\right)^{1/2}.$$

The first solution is the formal description of the evident fact that an arbitrary displacement of the three beads from their equilibrium state by an arbitrary similar shift $\Delta\varphi(t)$ (for example, a uniform rotation of the whole system with any angular frequency) does not disturb the mutual force balance, and hence does not result in oscillations. The second and third solutions describe oscillations that have the same frequency but are physically different by the sign of the phase shift α between the oscillations of any fixed pair of beads. The two left panels in the figure below show their "phasor diagrams", i.e. the sets of the complex amplitudes c_j of the oscillations (represented as 2D vectors) of each bead.



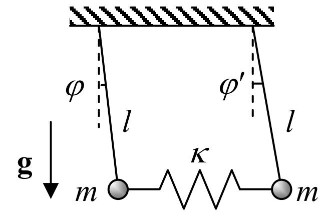
Note that due to the linearity of the equation of motion (for small oscillations only!), any linear superposition of these two basic phasor diagrams, with arbitrary complex amplitudes a, also represents a possible motion mode of the system. In particular, if the initial conditions are such that the amplitudes a of these two modes are equal, we should just sum up the vectors of both basic diagrams, getting the new diagram shown on the right panel of the figure above. (Physically, this summation corresponds to a merger of two traveling waves, with opposite angular directions of motion, into a single standing wave – exactly as was discussed, for a linear 1D system, in Sec. 6.5.) This phasor corresponds to a rather curious mode of oscillations, in which beads 2 and 3 oscillate in phase but in antiphase to bead 1 (with a twice smaller amplitude), so the length of one spring, l_{23}, does not change – see the figure on the right. Evidently, there are three such modes, which differ only by bead numbers.



[7] Note that this system may be considered as the physical implementation of the Born-Karman boundary conditions discussed in the end of Sec. 6.5.

[8] The fact that the wave assumption gives 3 different oscillation modes means that covers all modes which might be calculated using the more general method mentioned above.

Problem 6.7. On the example of the model considered in Problem 1, explore free oscillations in a system of two similar and weakly coupled linear oscillators.



Solution: As was discussed in the model solution of Problem 1, the linearized equations of motion of this system, which are valid at $|\varphi|$, $|\varphi'| << 1$ (see the figure on the right), may be represented as

$$\ddot{\varphi} + \left(\omega_0^2 + \Omega^2\right)\varphi - \omega_0^2\varphi' = 0,$$
$$-\omega_0^2\varphi + \ddot{\varphi}' + \left(\omega_0^2 + \Omega^2\right)\varphi' = 0,$$

where $\Omega \equiv (g/l)^{1/2}$ and $\omega_0 \equiv (\kappa/m)^{1/2}$, and their general solution is given by any sum ("linear superposition") of two sinusoidal oscillations with the following "normal" (or "own", or "eigen-") frequencies

$$\omega_-^2 = \Omega^2 \equiv \frac{g}{l}, \quad \text{and} \quad \omega_+^2 = \Omega^2 + 2\omega_0^2 > \omega_-^2,$$

and some complex amplitudes a_+ and a_-, depending on initial conditions. As a result, we may write

$$\varphi(t) = \text{Re}\left(a_- \exp\{-i\omega_- t\} + a_+ \cos\{-i\omega_+ t\}\right), \qquad \varphi'(t) = \text{Re}\left(a'_- \exp\{-i\omega_- t\} + a'_+ \exp\{-i\omega_+ t\}\right). \quad (*)$$

At two special types of initial conditions, providing either $a_+ = 0$ or $a_- = 0$, this solution describes purely sinusoidal oscillations of both φ and φ'. These are the normal modes of motion; in this symmetric system, the distribution coefficients (also calculated in the solution of Problem 1) and hence the oscillation modes are very simple:

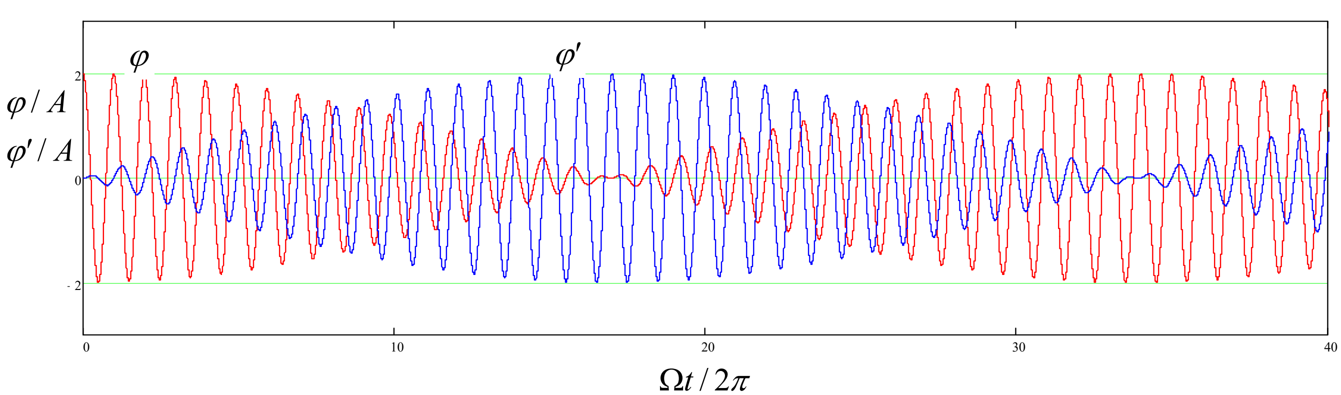
$$\varphi'(t) = \varphi(t) \times \begin{cases} +1, & \text{for the soft mode (frequency } \omega_-), \\ -1, & \text{for the hard mode (frequency } \omega_+ > \omega_-). \end{cases}$$

(See their sketches in the model solution of Problem 1.)

For any other initial conditions, the functions $\varphi(t)$ and $\varphi'(t)$ include sinusoidal components of both normal frequencies. For revealing the physical picture of such linear superpositions at weak coupling ($\omega_0^2 << \Omega^2$), the exact values of a_+ and a_- are not important, so let us consider a particular but representative case when $a_+ = a_- = A$, with Im $A = 0$. (The last condition may be always ensured by a proper choice of the time origin.) Then Eqs. (*) reduce to

$$\varphi(t) = A\left(\cos\omega_- t + \cos\omega_+ t\right) \qquad \varphi'(t) = A\left(\cos\omega_- t - \cos\omega_+ t\right). \qquad (**)$$

The figure below shows a plot of these two functions for a particular coupling value $\omega_0^2/\Omega^2 = 0.03 << 1$.



A bit counter-intuitive, each oscillation waveform looks almost sinusoidal, but with a slowly oscillating amplitude. However, this is natural, because Eqs. (**) may be equivalently represented as[9]

$$\varphi(t) = A(t)\cos\frac{\omega_- + \omega_+}{2}t, \qquad \text{with } A(t) \equiv 2A\cos\frac{\omega_+ - \omega_-}{2}t,$$

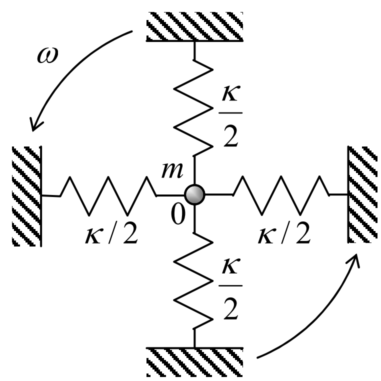
$$\varphi'(t) = A'(t)\sin\frac{\omega_- + \omega_+}{2}t, \qquad \text{with } A'(t) \equiv 2A\sin\frac{\omega_+ - \omega_-}{2}t.$$

At weak coupling, the frequency of the time-dependent amplitudes $A(t)$ and $A'(t)$ is much lower than the carrier wave frequency:

$$\frac{\omega_+ - \omega_-}{2} = \frac{1}{2}\left[\left(\Omega^2 + 2\omega_0^2\right)^{1/2} - \Omega\right] \approx \Omega\frac{\omega_0^2}{2\Omega^2} << \Omega, \qquad \text{while } \frac{\omega_- + \omega_-}{2} \approx \Omega.$$

Such slow periodic "beats", and the resulting "repumping" of the oscillation energy between the components, is a general property of all systems of two weakly coupled oscillators, though, in less symmetric systems than ours, the repumping may be incomplete, with the energy of each oscillator never vanishing completely.

Note that this effect has a close analog in quantum mechanics – the so-called *quantum oscillations* of the probabilities to find a system in any of its two weakly coupled states.[10]



Problem 6.8. A small body is held by four similar elastic springs as shown in the figure on the right. Analyze the effect of rotation of the system as a whole about the axis normal to its plane, on the body's small oscillations within this plane. Assume that the oscillation frequency is much higher than the angular velocity ω of the rotation. Discuss the physical sense of your results and possible ways of using such systems for measurement of the rotation.

Solution: Let us write the equation of motion of the body within the [x, y] plane of the drawing. If its displacements from the equilibrium are small, and the strings are not pre-stretched, the normal component of the elastic force exerted on the body by each spring is proportional to its displacement squared and in the first approximation may be ignored (cf. Eq. (6.24) of the lecture notes), so we may approximate the total force exerted by the springs as

$$\mathbf{F} = -\kappa\mathbf{\rho} \equiv -m\Omega_0^2\mathbf{\rho}, \qquad (*)$$

where $\mathbf{\rho} \equiv \mathbf{n}_x x + \mathbf{n}_y y$ is the 2D radius vector of the body, as measured from its equilibrium position, and $\Omega_0 = (\kappa/m)^{1/2}$. Now, there are two different ways to proceed.

On one hand, we may analyze the motion in the reference frame connected to the rotating system. Since this frame is non-inertial, in order for the 2nd Newton's law to be valid in it, we need to add, to the actual (physical) force (*), two fictitious (inertial) "forces": the centrifugal force (4.93) and the Coriolis force (4.94), in our current geometry, both with $\boldsymbol{\omega} = \mathbf{n}_z\omega$. In our case when $\omega << \Omega_0$, the

[9] If you need to, consult MA Eqs. (3.3a) and (3.3b).
[10] See, e.g., QM Secs. 2.6, 4.6, 5.1, and on.

former force, proportional to $m\omega^2 A$ (where A is the amplitude of oscillations) is much smaller than the latter one, proportional to $m\omega v \sim m\omega\Omega_0 A$, and may be ignored.[11] Hence the body's motion, in the rotating frame, obeys the following equation:

$$m\ddot{\boldsymbol{\rho}} = -m\Omega_0^2\boldsymbol{\rho} - 2m\omega\left(\mathbf{n}_z \times \dot{\boldsymbol{\rho}}\right). \qquad (**)$$

This is the same equation (with the replacement $\omega_E \to \omega$) that was obtained and solved in the model solution of Problem 4.30 (on the Foucault pendulum), so we could just borrow its results. However, in view of the discussion of normal oscillation modes in Secs. 6.1-2 of the lecture notes, it makes sense to rewrite this solution in a different form that sheds new light on the physics of the system.

Rewritten in the Cartesian coordinates $\{x, y\}$, Eq. (**) reads

$$\ddot{x} = -\Omega_0^2 x + 2\omega\dot{y}, \qquad \ddot{y} = -\Omega_0^2 y - 2\omega\dot{x}. \qquad (***)$$

Since these differential equations are linear, we may look for their general solution as a linear superposition of the usual particular solutions (6.6) with $\lambda \equiv -i\Omega$: $x = a_x e^{-i\Omega t}$, $y = a_y e^{-i\Omega t}$. The resulting homogeneous algebraic equations

$$-\Omega^2 a_x = -\Omega_0^2 a_x - 2i\omega\Omega a_y, \qquad -\Omega^2 a_y = -\Omega_0^2 a_y + 2i\omega\Omega a_x. \qquad (****)$$

are compatible only if the determinant of their system equals zero:

$$\begin{vmatrix} \Omega_0^2 - \Omega^2 & 2i\omega\Omega \\ -2i\omega\Omega & \Omega_0^2 - \Omega^2 \end{vmatrix} = 0, \qquad \text{i.e. if } \left(\Omega_0^2 - \Omega^2\right)^2 - \left(2\omega\Omega\right)^2 = 0.$$

This is an easily solvable quadratic equation for Ω^2; in the limit $\omega << \Omega_0$ (and again, our Eq. (*) is valid only in this limit), it gives an extremely simple result for the normal mode frequencies Ω:

$$\Omega = \Omega_\pm \equiv \Omega_0 \pm \omega.$$

In order to reveal the physics of this result, let us see how the normal modes look; for finding them, we need, as usual (see, e.g., Sec. 6.1 of the lecture notes), to plug the frequencies $\Omega_\pm$, one by one, back into any of Eqs. (****) and find the corresponding ratios a_x/a_y, i.e. the oscillation distribution coefficients. The result, in the same linear approximation in the small ratio $\omega/\Omega_0 << 1$, is also extremely simple:

$$\left(\frac{a_x}{a_y}\right)_\pm = \pm i, \quad \text{i.e.} \left(\frac{a_y}{a_x}\right)_\pm = \mp i.$$

Let us have a look at how do these modes look, taking (as we always may) the complex amplitude $(a_x)_\pm$ in the form $A_\pm \exp\{i\varphi_\pm\}$, where $A_\pm$ and $\varphi_\pm$ are real; then $(a_y)_\pm = \mp iA_\pm \exp\{i\varphi_\pm\}$ and

$$\begin{aligned}\boldsymbol{\rho}_\pm &\equiv \text{Re}\left(\mathbf{n}_x x + \mathbf{n}_y y\right)_\pm = \text{Re}\left[\exp\{-i\Omega_\pm t\}\left(\mathbf{n}_x a_x + \mathbf{n}_y a_y\right)_\pm\right] = A_\pm \,\text{Re}\left[\exp\{i(\varphi_\pm - \Omega_\pm t)\}\left(\mathbf{n}_x \pm i\mathbf{n}_y\right)\right] \\ &\equiv A_\pm\left[\mathbf{n}_x \cos(\Omega_\pm t - \varphi_\pm) \mp \mathbf{n}_y \sin(\Omega_\pm t - \varphi_\pm)\right].\end{aligned}$$

[11] Actually, its account would result not in any new effects, but only in a small renormalization of the oscillation frequency: $\Omega_0^2 \to \Omega_0^2 - \omega^2$.

This means that the vector $\boldsymbol{\rho}_+$ rotates clockwise with the angular velocity $\Omega_+ = \Omega + \omega$, while $\boldsymbol{\rho}_-$ rotates counterclockwise with the angular velocity $\Omega_- = \Omega - \omega$, each retaining its initial length. Such 2D oscillations are called *circularly polarized*; their natural extension to waves plays a very important role in electrodynamics[12] – in particular, in optics.

Now let us see what this result means for the oscillations that are initially (say, before the rotation starts) linearly polarized – for example, along the x-axis. We may describe them as a linear superposition of the two circularly polarized modes $\boldsymbol{\rho}_\pm$ with equal amplitudes: $A_+ = A_- \equiv A/2$ and phases: $\varphi_+ = \varphi_- \equiv \varphi$:

$$\boldsymbol{\rho} = \boldsymbol{\rho}_+ + \boldsymbol{\rho}_- \equiv \frac{A}{2}\left\{\mathbf{n}_x\left[\cos(\Omega_+ t - \varphi) + \cos(\Omega_- t - \varphi)\right] + \mathbf{n}_y\left[-\sin(\Omega_+ t - \varphi) + \sin(\Omega_- t - \varphi)\right]\right\}.$$

Indeed, in the absence of rotation (when $\Omega_+ = \Omega_- = \Omega_0$), the resulting radius vector is

$$\boldsymbol{\rho} = \mathbf{n}_x A\cos(\Omega_0 t - \varphi),$$

i.e. the body simply oscillates along the x-axis, with $y(t) = 0$. However, at nonvanishing rotation, simple trigonometry[13] enables us to rewrite our result as

$$\boldsymbol{\rho} = A\left[\mathbf{n}_x \cos\left(\frac{\Omega_+ + \Omega_-}{2}t - \varphi\right)\cos\frac{\Omega_- - \Omega_+}{2}t + \mathbf{n}_y \cos\left(\frac{\Omega_+ + \Omega_-}{2}t - \varphi\right)\sin\frac{\Omega_- - \Omega_+}{2}t\right]$$
$$\equiv A\cos(\Omega_0 t - \varphi)\left(\mathbf{n}_x \cos\omega t - \mathbf{n}_y \sin\omega t\right).$$

This means that the oscillations remain linearly polarized, but their polarization axis rotates in the direction *opposite* to the system's rotation as measured in the lab frame – in the case $\omega > 0$, shown by the arrow in the figure above, clockwise. This result is very natural: it means that in the "lab" (meaning any inertial) reference frame, the oscillation polarization direction is not affected by the rotation at all This fact motivates us to have another look at our problem, from such a reference frame. From this standpoint, the net elastic spring force (*) is the only one acting on the body, and it may be described by the potential energy

$$U = \frac{\kappa}{2}\rho^2 \equiv \frac{\kappa}{2}\left(x^2 + y^2\right).$$

Since this potential is axially symmetric, the particle's motion in it cannot be affected by the system's rotation about its symmetry axis.

This does not mean, however, that our (much longer) analysis from the point of view of the rotating reference frame was in vain: it helps to discuss possible applications of such systems for the detection of rotation and measurement of its parameters. Conceptually, the observation of the rotation of the polarization line of free oscillations is sufficient not only for the measurement of ω but even for direct observation of the total angle of rotation during the given time interval. However, all real oscillators have nonvanishing damping (a finite Q-factor) and need an external drive of frequency $\Omega \approx \Omega_0$ to sustain the oscillations - see, e.g., Sec. 5.1 of the lecture notes. In the simplest case when such a drive force is applied in one (say, x-) direction, the equations describing the body's motion may be readily obtained by the evident generalization of Eqs. (***):

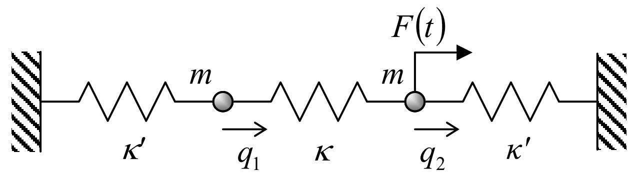
[12] See, e.g., EM Sec. 7.1 and on.
[13] See, e.g., MA Eqs. (3.2a) and (3.2c).

$$\ddot{x} = -2\delta\dot{x} - \Omega_0^2 x + 2\omega\dot{y} + f_0 \cos\Omega t, \qquad \ddot{y} = -2\delta\dot{y} - \Omega_0^2 y - 2\omega\dot{x}.$$

Even without a direct solution and analysis of these equations (which are left for the reader's exercise), it is clear that in the absence of rotation ($\omega = 0$) the second of them describes a gradual full decay of the y-component of oscillations in time. The first of these equations shows that after this transient process, the x-component of the displacement performs the usual forced oscillations of frequency Ω, with the amplitude largest at the resonance $\Omega = \Omega_0$. Now a slow rotation causes the last term of the second equation to oscillate with that frequency, playing the role of a near-resonant driving force for the y-component of the oscillations, with an amplitude proportional to ω and Q. Measurement of the amplitude and/or phase of these oscillations in several devices oriented along three mutually perpendicular axes enables a full recovery of both the speed and the direction of their rotation.

Such devices (usually called *MEMS gyroscopes*) have become very popular after the development, since the late 1980s, of an industrial technology for the fabrication of micro-electro-mechanical systems (MEMS), which may be very compact and inexpensive.[14] As a result, MEMS gyroscopes are currently used even in consumer gadgets such as smartphones. Their accuracy $\delta\omega$, in the best implementations (with the Q-factor of the order of 10^5) approaching 10^{-7} s^{-1}, is 3 to 4 orders of magnitude worse than that of the (much more complex and expensive) gyroscopes based on optical interferometers,[15] but is sufficient for most everyday purposes.

Problem 6.9. An external longitudinal force $F(t)$ is applied to the right particle of the system shown in Fig. 6.1 of the lecture notes, with $\kappa_L = \kappa_R = \kappa'$ and $m_1 = m_2 \equiv m$ (see the figure on the right), and the response $q_1(t)$ of the left particle to this force is being measured.



(i) Calculate the temporal Green's function for this response.

(ii) Use this function to calculate the response to the following force:

$$F(t) = \begin{cases} 0, & \text{for } t < 0, \\ F_0 \sin\omega t, & \text{for } 0 \le t, \end{cases}$$

with constant amplitude F_0 and frequency ω.

Solutions:

(i) Due to the clear physical meaning of Eqs. (6.5) describing the system (as the set of 2nd Newton laws for each particle), we may readily generalize them to the case of the additional force applied to the right particle:

$$m\ddot{q}_1 + m\Omega^2 q_1 = \kappa q_2,$$

$$m\ddot{q}_2 + m\Omega^2 q_2 = \kappa q_1 + F(t).$$

Since for our current symmetric system, Eq. (6.3b) of the lecture notes yields $\kappa_1 = \kappa_2 = \kappa + \kappa'$, and Eq. (6.2) becomes $\Omega^2 = (\kappa + \kappa')/m$, this system of differential equations may be rewritten in a simpler form,

[14] See, e.g., C. Liu, *Foundations of MEMS*, 2nd ed. Pearson, 2016.

[15] For a recent review see, e.g., V. M. N. Passaro *et al*., *Sensors* **17**, 2283 (2017).

$$\ddot{q}_1 + \Omega^2 q_1 - \Omega_0^2 q_2 = 0, \qquad q_2 + \Omega^2 q_2 - \Omega_0^2 q_1 = f(t),$$

where $f \equiv F/m$ and $\Omega_0^{\ 2} \equiv \kappa/m$. As was discussed in Sec. 5.1 of the lecture notes, the temporal Green's function $G(\tau)$ is just the response of this system, with t replaced with τ, to the delta-functional right-hand side, $f(\tau) = \delta(\tau)$, with zero initial conditions: $q_1(\tau = 0) = q_2(\tau = 0) = 0$. Repeating the argumentation which has led us to Eqs. (5.33) of the lecture notes, the problem may be reduced to the solution, for $\tau > 0$, of the corresponding homogeneous system of equations,

$$\frac{d^2 q_1}{d\tau^2} + \Omega^2 q_1 - \Omega_0^2 q_2 = 0, \qquad \frac{d^2 q_2}{d\tau^2} + \Omega^2 q_2 - \Omega_0^2 q_1 = 0, \tag{*}$$

with the initial conditions

$$q_1(0) = 0, \quad \frac{dq_1}{d\tau}(0) = 0, \quad q_2(0) = 0, \quad \frac{dq_2}{d\tau}(0) = 1. \tag{**}$$

As was discussed in Sec. 6.1, the general solution of this system is any linear superposition of harmonic oscillations:

$$G(\tau) \equiv q_1(\tau) = \text{Re}\left[(c_1)_+ \exp\{-i\omega_+ \tau\} + (c_1)_- \exp\{-i\omega_- \tau\}\right],$$
$$q_2(\tau) = \text{Re}\left[(c_2)_+ \exp\{-i\omega_+ \tau\} + (c_2)_- \exp\{-i\omega_- \tau\}\right],$$

with the two "normal" frequencies $\omega_\pm$ given by Eq. (6.10), which in our particular case is reduced to

$$\omega_\pm^2 = \Omega^2 \pm \Omega_0^2 = \frac{\kappa + \kappa' \pm \kappa}{m}, \qquad \text{i.e. to } \omega_+ = \left(\frac{\kappa' + 2\kappa}{m}\right)^{1/2}, \quad \omega_- = \left(\frac{\kappa'}{m}\right)^{1/2} < \omega_+ .$$

Plugging the basic sinusoidal solutions, one by one, into any of Eqs. (*), we get the following result for the (generally, complex) distribution coefficients c:[16]

$$\left(\frac{c_2}{c_1}\right)_\pm = \mp 1,$$

so our solution reduces to

$$G(\tau) \equiv q_1(\tau) = \text{Re}\left[c_+ \exp\{-i\omega_+ \tau\} + c_- \exp\{-i\omega_- \tau\}\right],$$
$$q_2(\tau) = \text{Re}\left[-c_+ \exp\{-i\omega_+ \tau\} + c_- \exp\{-i\omega_- \tau\}\right].$$

Now the coefficients $c_\pm$ may be found by plugging these solutions into the initial conditions (**). The result of the solution of this (easy) system of equations is $c_\pm = \mp i/2\omega_\pm$, so finally we get, in particular:

$$G(\tau) = \text{Re}\left[-\frac{i}{2\omega_+}\exp\{-i\omega_+ \tau\} + \frac{i}{2\omega_+}\exp\{-i\omega_- \tau\}\right] \equiv -\frac{1}{2\omega_+}\sin\omega_+ \tau + \frac{1}{2\omega_-}\sin\omega_- \tau . \tag{***}$$

This expression for the system of two coupled oscillators may be compared with Eq. (5.34) for a single oscillator (with negligible damping, i.e. with $\delta = 0$ and $\omega_0' = \omega_0$):

---

[16] Actually, these relations could be guessed just by looking at the figure above: in this symmetric system, the hard mode with frequency ω_+ corresponds to equal and opposite displacements $q_1(t)$ and $q_2(t)$, while the soft mode with frequency ω_-, to the motion with $q_1(t) = q_2(t)$, without deformation of the middle spring.

$$G_s(\tau)=\frac{1}{\omega_0}\sin\omega_0\tau .$$

A major difference between these functions is that at small τ, $G_s(\tau)$ grows linearly, reflecting the motion of a body that has received a sharp "kick" (an impulse of force) at $\tau = 0$. In contrast, the $G(\tau)$ given by Eq. (***) starts much more gently:

$$G(\tau)\Big|_{\omega_\pm\tau<<1}\approx-\frac{1}{2\omega_+}\left(\omega_+\tau-\frac{1}{3!}\omega_+^3\tau^3\right)+\frac{1}{2\omega_-}\left(\omega_-\tau-\frac{1}{3!}\omega_-^3\tau^3\right)\equiv\frac{1}{12}\left(\omega_+^2-\omega_-^2\right)\tau^3,$$

because it describes the motion of a particle other than that receiving the kick, reflecting the additional inertia of the oscillator coupling. (At larger τ, this Green's function features "beats" between the oscillations with frequencies ω_+ and ω_-, similar to those discussed in the solution of Problem 7.)

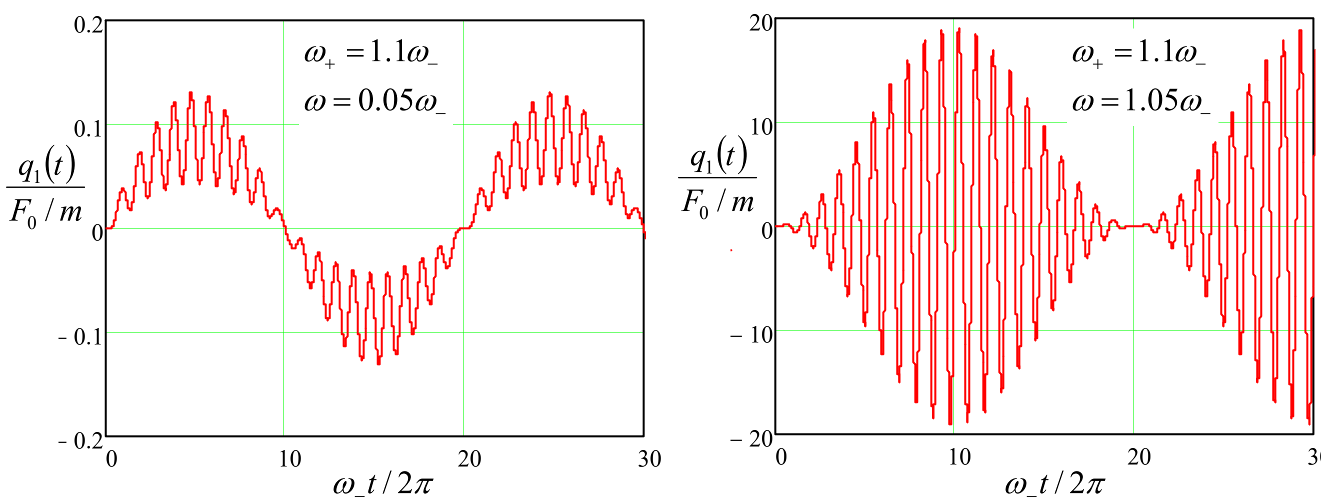
(ii) Using Eq. (5.27) of the lecture notes (essentially, the definition of the temporal Green's function), and Eq. (***), we get

$$q_1(t)=\int_{-\infty}^{t}f(t')\,G(t-t')dt'=\frac{F_0}{m}\int_0^t\sin\omega t'\;G(t-t')dt'$$
$$=-\frac{F_0}{2\omega_+m}\int_0^t\sin\omega t'\sin\omega_+(t-t')dt'\;+\frac{F_0}{2\omega_-m}\int_0^t\sin\omega t'\sin\omega_-(t-t')dt'.$$

Now using MA Eqs. (3.2c) and then carrying out elementary integrations, we finally get

$$q_1(t)=\frac{F_0}{2m}\left[\frac{1}{\omega_+^2-\omega^2}\left(\frac{\omega}{\omega_+}\sin\omega_+t-\sin\omega t\right)-\frac{1}{\omega_-^2-\omega^2}\left(\frac{\omega}{\omega_-}\sin\omega_-t-\sin\omega t\right)\right].$$

The figure below shows the plots of this result for two representative values of the external force's frequency: $\omega << \omega_\pm$ (left panel) and $\omega \approx \omega_\pm$ (right panel).[17]



[17] Due to the initially gradual growth of the external force, this response grows, at small times, even slower than the Green's function: $q_1(t) \propto t^5$.

So, as might be expected, the response $q_1(t)$ is a sum of stationary forced oscillations, with the external force's frequency ω, and free oscillations at the own frequencies $\omega_\pm$ of the system, excited by the sudden turn-on of the external force. At weak coupling, i.e. at $\omega_+ \approx \omega_-$, the latter components form the "beats" pattern with a slowly oscillating envelope. (At any nonvanishing damping, this pattern eventually decays, so the oscillations become purely sinusoidal.)

Problem 6.10. Use the Lagrangian formalism to re-derive Eqs. (6.25) of the lecture notes for both the longitudinal and the transverse oscillations in the system shown in Fig. 6.4a.

Solution: Let the deviation $\mathbf{q}(t)$ of each particle from the equilibrium position have all three Cartesian components, $q_x(t)$, $q_y(t)$, and $q_z(t)$. Then its kinetic energy is clearly separable to three independent quadratic components,

$$T = \frac{m}{2}\left(\dot{q}_x^2 + \dot{q}_y^2 + \dot{q}_z^2\right),$$

so we need to analyze only the potential energy of the system. For each pre-stretched spring connecting two adjacent particles, with deviations $\mathbf{q} = \{q_x, q_y, q_z\}$ and $\mathbf{q}' = \{q_x', q_y', q_z'\}$,

$$U = \frac{\kappa}{2}(D-d)^2 + \mathscr{T}(D-d), \qquad (*)$$

where D is the full distance between the points, i.e. the current spring's length,

$$D = \left[(q_x - q_x')^2 + (q_y - q_y')^2 + (d + q_z - q_z')^2\right]^{1/2},$$

and $\mathscr{T} > 0$ is the spring tension's magnitude. The second term on the right-hand side of Eq. (*) takes into account the tension's work on the system at the spring's extension.[18] Its sign may be verified by realizing that at the fixed pre-stretch of the spring, increasing its length requires the system to perform a positive work against the tension force source, i.e. increases its potential energy.

Now expanding D in the Taylor series with respect to small deviations $\mathbf{q}$ and $\mathbf{q}'$ (with $|\mathbf{q}|, |\mathbf{q}'| << d$), and dropping all terms higher than quadratic ones, we get

$$D - d \approx \frac{(q_x - q_x')^2}{2d} + \frac{(q_y - q_y')^2}{2d} + (q_z - q_z'),$$

so, with the similar accuracy, Eq. (*) yields

$$U \approx \frac{\kappa}{2}(q_z - q_z')^2 + \frac{\mathscr{T}}{d}\left[\frac{(q_x - q_x')^2}{2} + \frac{(q_y - q_y')^2}{2}\right] + \mathscr{T}(q_z - q_z').$$

After the summation over all springs, the last terms in this expression cancel each other, so the Lagrangian function per each unit (a particle plus an adjacent spring) is a sum of three independent quadratic terms, with a similar structure for each Cartesian component:

$$L = \left[\frac{m}{2}\dot{q}_x^2 - \frac{\mathscr{T}}{2d}(q_x - q_x')^2\right] + \left[\frac{m}{2}\dot{q}_y^2 - \frac{\mathscr{T}}{2d}(q_y - q_y')^2\right] + \left[\frac{m}{2}\dot{q}_x^2 - \frac{\kappa}{2}(q_y - q_z')^2\right].$$

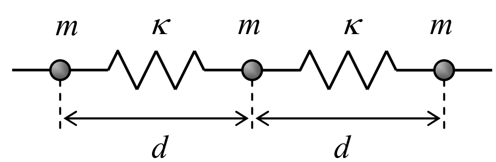
[18] Hence this U is essentially the Gibbs potential energy U_G – see Sec. 1. 4 of the lecture notes.

As a result, we get three similar, independent Lagrange equations for each Cartesian component of the oscillations. To derive them correctly, let us write the full Lagrangian for the whole system of N particles, keeping just one Cartesian component and replacing the coordinate index with the particle's number:

$$L = \sum_{j=1}^{N}\left[\frac{m}{2}\dot{q}_j^2 - \frac{\kappa_{\text{ef}}}{2}\left(q_j - q_{j+1}\right)^2\right] \qquad \text{where } \kappa_{\text{ef}} \equiv \begin{cases} \kappa' \equiv \mathscr{T}/d, & \text{for } q_x \text{ and } q_y, \\ \kappa, & \text{for } q_z. \end{cases} \qquad (**)$$

Note that when performing the Lagrangian differentiation over q_j, we should take into account that this coordinate participates not only in the j^{th} term spelled out in Eq. (**), but also in the similar, adjacent term, proportional to $(q_{j-1} - q_j)^2/2$. As a result, we arrive at Eq. (6.25) of the lecture notes, which was derived in Sec. 6.3 using the Newton law.

So the small oscillations/waves in the system may be indeed decomposed into three independent modes: two transverse (in our notation, q_x and q_y) and one longitudinal (q_z).



Problem 6.11. Calculate the energy (per unit length) of a sinusoidal traveling wave propagating in the 1D system shown in Fig. 6.4a of the lecture notes, reproduced on the right. Use your result to calculate the average power flow created by the wave, and compare it with Eq. (6.49) in the acoustic wave limit.

Solution: Each term in the Lagrangian function, given by Eq. (**) of the model of the previous problem, is the difference between the kinetic energy T_j of the j^{th} particle and the potential energy U_j of the spring on the right of it. Hence the energy of the system may be represented as the sum over j of energies $E_j = T_j + U_j$ of each such unit:

$$E = \sum_j E_j, \quad \text{where } E_j = \frac{m}{2}\dot{q}_j^2 + \frac{\kappa_{\text{ef}}}{2}\left(q_{j+1} - q_j\right)^2.$$

Plugging into this expression the sinusoidal wave solution (6.28), rewritten in the form

$$q_j(t) = \frac{a}{2}e^{i\Psi} + \frac{a^*}{2}e^{-i\Psi}, \quad \text{where } \Psi \equiv kz_j \mp \omega t,$$

so

$$q_{j+1}(t) - q_j(t) = \frac{a}{2}e^{i\Psi}\left(e^{ikd} - 1\right) + \frac{a^*}{2}e^{-i\Psi}\left(e^{-ikd} - 1\right), \qquad \dot{q}_j(t) = \mp i\omega\frac{a}{2}e^{i\Psi} \pm i\omega\frac{a^*}{2}e^{-i\Psi}, \quad (*)$$

we get

$$E_j = \frac{m\omega^2}{2}aa^* + a^2e^{2i\Psi}\left[\frac{m\omega^2}{8} + \frac{\kappa_{\text{ef}}}{8}\left(e^{ikd} - 1\right)^2\right] + \left(a^*\right)^2e^{-2i\Psi}\left[\frac{m\omega^2}{8} + \frac{\kappa_{\text{ef}}}{8}\left(e^{-ikd} - 1\right)^2\right],$$

where I have used the fact that according to the dispersion relation (6.30), $\kappa_{\text{ef}}(e^{ikd} - 1)(e^{-ikd} - 1) \equiv 4\kappa_{\text{ef}}\sin^2(kd/2) = m\omega^2$. The two last terms, proportional to $e^{\pm 2i\Psi} \propto e^{\pm 2i\omega t}$, describe fast (with frequency 2ω) oscillations of the energy between its T_j and U_j components of the same cell and also between adjacent cells, and do not contribute to the time-averaged energy

$$\overline{E_j} = \frac{m\omega^2}{2}aa^*.$$

This expression shows that the average energy is independent of the cell number j, so the wave energy per unit length of the periodic structure is

$$\frac{\overline{E}}{l}=\frac{\overline{E_j}}{d}=\frac{m\omega^2}{2d}aa^*.$$

In the acoustic wave limit $kd \to 0$, the wave propagates with the dispersion-free velocity $\pm v$, where according to Eq. (6.32), $v = (\kappa_{\rm ef}/m)^{1/2}d$, creating the average power flow

$$\overline{\mathscr{P}}=\pm\frac{\overline{E}}{l}v=\pm\frac{m\omega^2}{2d}aa^*\left(\frac{\kappa_{\rm ef}}{m}\right)^{1/2}d=\pm\frac{\omega^2(\kappa_{\rm ef}m)^{1/2}}{2}aa^*=\pm\frac{\omega^2 Z}{2}aa^*,$$

where $Z \equiv (\kappa_{\rm ef}m)^{1/2}$ is the wave impedance of the system – see Eq. (6.47). This expression coincides with the one following from the general Eq. (6.49) of the lecture notes (derived there in a different way), if we take $df_\pm/dt \equiv \partial q_\pm/\partial t$ from Eq. (*), and average the result over time.

<u>Problem 6.12</u>. Calculate spatial distributions of the kinetic and potential energies in a standing sinusoidal 1D acoustic wave and analyze their evolution in time.

Solution: In the acoustic limit, we may use the wave equation (6.40) of the lecture notes,

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial z^2}\right)q(z,t)=0, \qquad (*)$$

which describes, for example, the 1D system shown in Fig. 6.4a, at $\omega << \omega_{\rm max}$. As was discussed in Sec. 6.4 of the lecture notes, a particular solution of this equation is the sinusoidal standing wave (6.66). Counting time, for the notation simplicity, from the beginning of one of the oscillation periods, and dropping the index n, we may write

$$q = A\sin\omega t\sin kz, \qquad \text{so} \quad \frac{\partial q}{\partial t}=A\omega\cos\omega t\sin kz, \quad \frac{\partial q}{\partial z}=Ak\sin\omega t\cos kz.$$

where A is the real amplitude of the particle displacement. In the acoustic wave limit, in which Eq. (*) is valid, the wavelength $\lambda = 2\pi/k$ is much longer than the distance d between the adjacent particles, so we may calculate the kinetic energy dT of a small segment dz, with $d << dz << \lambda$, by neglecting the difference of the displacement velocities on it:

$$dT=\sum_{dj=dz/d}\frac{m}{2}\dot{q}_j^2\approx\frac{m}{2}\left(\frac{\partial q}{\partial t}\right)^2 dj=\frac{m}{2}\left(\frac{\partial q}{\partial t}\right)^2\frac{dj}{dz}dz=\frac{\mu}{2}\left(\frac{\partial q}{\partial t}\right)^2 dz,$$

where $\mu \equiv m(dj/dz) = m/d$ is the mass of the system per unit length. Hence the kinetic energy's density is

$$\frac{dT}{dz}=\frac{\mu}{2}\left(\frac{\partial q}{\partial t}\right)^2=\frac{\mu}{2}\omega^2A^2\cos^2\omega t\sin^2 kz. \qquad (**)$$

Similarly, the expression for the potential energy dU of the system on the small segment dz,

$$dU=\sum_{dj=dz/d}\frac{\kappa_{\rm ef}}{2}(q_{j+1}-q_j)^2,$$

may be simplified, in the acoustic limit, in the same way as was used in the lecture notes for the derivation of Eq. (6.44): $(q_{j+1} - q_j) \approx \partial q/\partial j = \partial q/\partial(z/d) = d\,(\partial q/\partial z)$, so

$$dU \approx \frac{\kappa_{\text{ef}}}{2}\left(d\frac{\partial q}{\partial z}\right)^2 \frac{dj}{dz}dz = \frac{\kappa_{\text{ef}}d}{2}\left(\frac{\partial q}{\partial z}\right)^2 dz = \frac{\mu v^2}{2}\left(\frac{\partial q}{\partial z}\right)^2 dz\ ,$$

where the last step used Eq. (6.31): $v = (\kappa_{\text{ef}}/m)^{1/2}d$. Hence the spatial density of the standing wave's potential energy is

$$\frac{dU}{dz} = \frac{\mu v^2}{2}\left(\frac{\partial q}{\partial z}\right)^2 = \frac{\mu}{2}(vk)^2 A^2 \sin^2 \omega t \cos^2 kz\ . \qquad (***)$$

The results (**) and (***) show that the kinetic and potential energies oscillate both in time and space, and the phases of these oscillations are shifted by $\pi/2$. Moreover, due to the linear dispersion law (6.31) of the acoustic waves, $\omega = \pm\, vk$, the maximum values of the energy densities are equal. These facts may be interpreted by saying that the standing wave's energy is periodically "re-pumped" in time between its kinetic and potential components, simultaneously "wiggling" in space between the displacement's nodes and maxima. Superficially, this picture is similar to the familiar periodic exchange between $T(t)$ and $U(t)$ in a simple ("lumped") harmonic oscillator; however, in that case, the full energy $E = T(t) + U(t)$ remains constant, while in the standing-wave oscillator even the full energy's density oscillates both in time and space:

$$\frac{dE}{dz} = \frac{dT}{dz} + \frac{dU}{dz} = \frac{\mu}{2}\omega^2 A^2\left(\cos^2 \omega t \sin^2 kz + \sin^2 \omega t \cos^2 kz\right) \equiv \frac{\mu}{4}\omega^2 A^2\left(1 - \cos 2\omega t\, \cos 2kz\right),$$

so only its average over either temporal (π/ω) or spatial (π/k) half-periods of the oscillations (and hence over any of their multiples) is constant.

Problem 6.13. The midpoint of a guitar string of length l has been slowly pulled off sideways by a distance $h << l$ from its equilibrium position, and then let go. Neglecting energy dissipation, use two different approaches to calculate the midpoint's displacement as a function of time.

Hint: You may like to use the following series: $\sum_{m=1}^{\infty}\frac{\cos(2m-1)\xi}{(2m-1)^2} = \frac{\pi^2}{8}\left(1 - \frac{\xi}{\pi/2}\right)$, for $0 \le \xi \le \pi$.

Solution:

Approach 1. According to Eq. (6.71) of the lecture notes, any small displacement (in our case, the transverse one) of a string, with rigidly fixed ends at $z = 0$ and $z = l$, may be represented as an expansion over all possible standing waves:

$$q(z,t) = \text{Re}\sum_{n=1}^{\infty} a_n \exp\{-i\omega_n t\}\sin k_n z \equiv \sum_{n=1}^{\infty} u_n \cos\omega_n t \sin k_n z + \sum_{n=1}^{\infty} v_n \sin\omega_n t \sin k_n z \qquad (*)$$

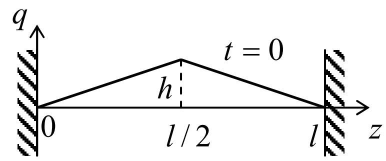
(where $u_n \equiv \text{Re}\, a_n$ and $v_n \equiv \text{Im}\, a_n$ are real constants), with the spectra of their wave numbers and frequencies given by Eqs. (6.64)-(6.65):

$$k_n = \frac{\pi}{l}n, \qquad \omega_n = vk_n = \frac{\pi v}{l}n \equiv \frac{2\pi}{\mathcal{T}}n, \qquad \text{with } n = 1, 2, 3,\ldots,$$

where $\mathcal{T} \equiv 2l/v$ is the period of the fundamental standing wave mode. (Since the frequency spectrum is equidistant, the periods of all other modes, $\mathcal{T}_n \equiv 2\pi/\omega_n$, are integer fractions of $\mathcal{T}$, so this it is also the time period of *any* transverse oscillations of the system.) The displacement velocities may be readily found from Eq. (*) by differentiation:

$$\frac{\partial q}{\partial t}(z,t) = -\sum_{n=1}^{\infty} u_n \omega_n \sin \omega_n t \sin k_n z + \sum_{n=1}^{N} v_n \omega_n \cos \omega_n t \sin k_n z \,. \qquad (**)$$

The (formally, infinite) sets of the coefficients u_n and v_n should be found from the requirement that at $t = 0$, Eqs. (*)-(**) correctly describe the initial distributions of q (see the figure on the right[19]) and its time derivative, over the string's length:



$$q(z,0) \equiv \sum_{n=1}^{\infty} u_n \sin k_n z = h\left(1 - \frac{|z - l/2|}{l/2}\right), \qquad \frac{\partial q}{\partial t}(z,0) \equiv \sum_{n=1}^{N} v_n \omega_n \sin k_n z = 0, \qquad \text{for } 0 \le z \le l\,.$$

The second of these initial conditions is evidently satisfied by the set of all $v_n = 0$, and since the expansion of any function into a series over a full set of orthogonal functions (such as our sin$k_n z$) is unique, this is the only possible set of v_n. The coefficients u_n may be found from the first of the initial conditions by the standard trick: the multiplication of both sides of this equality by sin$k_{n'}t$, and their integration over z from 0 to l. At this interval, such functions are mutually orthogonal:

$$\int_0^l \sin k_{n'} z \sin k_n z \, dz = \delta_{n,n'} \int_0^l \sin^2 k_n z \, dz = \delta_{n,n'} \frac{l}{2},$$

so all terms of the sum in the initial condition, besides the one with $n' = n$, vanish, and (replacing n' with n for the notation simplicity), we get

$$u_n \frac{l}{2} = \int_0^l \sin k_n z \; q(z,0) dz \equiv h \int_0^l \left(1 - \frac{|z - l/2|}{l/2}\right) \sin k_n z \, dz \,.$$

Since the function $q(z, 0)$ is symmetric with respect to the midpoint $z = l/2$ (see the figure above again), this integral equals zero for all even $n = 2m$, while for any odd $n = 2m - 1$ (where $m = 1, 2, \ldots$) the integrals over two halves of the string ([0, l/2] and [l/2, l]) are equal to each other, so we may write

$$u_{2m-1} \frac{l}{2} = 2h \int_0^{l/2} \left(1 - \frac{|z - l/2|}{l/2}\right) \sin k_{2m-1} z \, dz \equiv 2h \int_0^{l/2} \frac{z}{l/2} \sin \frac{\pi(2m-1)z}{l} z \, dz \equiv \frac{4hl}{\pi^2 (2m-1)^2} \int_0^{\pi(2m-1)/2} \xi \sin \xi \, d\xi,$$

where $\xi \equiv \pi(2m - 1)\, z/l$. This dimensionless integral may be readily worked out by parts. In the limits specified above, it is just $(-1)^{m-1}$, and we finally get

$$u_{2m-1} = \frac{8}{\pi^2} \frac{(-1)^{m-1}}{(2m-1)^2} h,$$

so the full solution (*) is reduced to

[19] Note that in all sketches of this solution, string deformations are grossly exaggerated for clarity. As a reminder, Eqs. (*)-(**) for transverse waves are only valid if $|\, q \,| << l$ at all points.

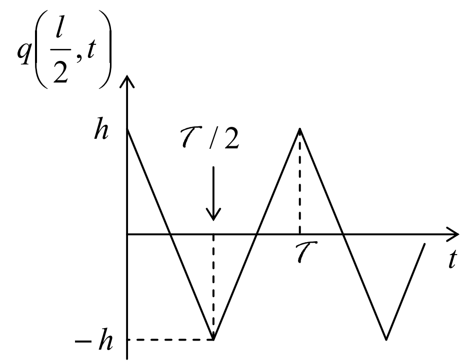
$$q(z,t) = \frac{8h}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m-1)^2}\cos\omega_{2m-1}t\sin k_{2m-1}z\,. \qquad (***)$$

In particular, at the midpoint, where $\sin k_{2m-1}z = \sin(k_{2m-1}l/2) = \sin[\pi(2m-1)/2] = (-1)^{m-1}$, we get

$$q\left(\frac{l}{2},t\right) = \frac{8h}{\pi^2}\sum_{m=1}^{\infty}\frac{\cos\omega_{2m-1}t}{(2m-1)^2} \equiv \frac{8h}{\pi^2}\sum_{m=1}^{\infty}\frac{\cos[(2m-1)2\pi t/\mathcal{T}]}{(2m-1)^2},$$

so using the series supplied in the *Hint*, with $\xi = 2\pi t/\mathcal{T}$, we get a surprisingly simple result:

$$q\left(\frac{l}{2},t\right) = h\left(1-\frac{t}{\mathcal{T}/4}\right), \qquad \text{for } 0 \le t \le \frac{\mathcal{T}}{2}.$$



Now, the fact that the $q(z, t)$ given by Eq. (***) is a symmetric and $\mathcal{T}$-periodic function of time (for any z) enables us to extend this solution to further times, getting the triangular pattern shown in the figure on the right.

Approach 2. Since this result is somewhat counter-intuitive, it is prudent to obtain it in a different way. For this, let us first consider the solution of the wave equation (6.40b),

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right)q(z,t) = 0\,,$$

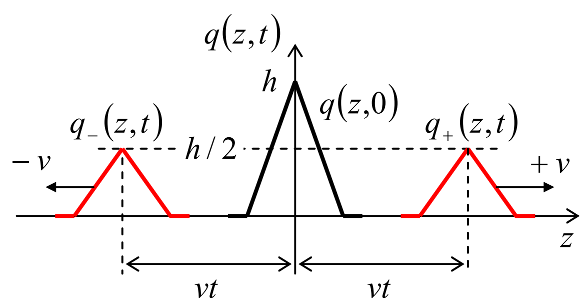
on the infinite axis, $-\infty < z < +\infty$, with the following initial conditions:

$$q(z,0) = f(z), \qquad \frac{\partial q}{\partial t}(z,0) = 0\,, \qquad (****)$$

where the function $f(z)$ may be different from zero at any finite segment. It is straightforward to verify that this problem is satisfied by the following solution:[20]

$$q(z,t) = q_+(z,t) + q_-(z,t), \quad \text{with } q_\pm(z,t) = \frac{1}{2}f(z \mp vt),$$

which is just a linear superposition of the two traveling wave solutions (6.41), with similar waveforms. The figure on the right shows this solution for an initial condition of our type, with a triangular waveform $f(z)$.

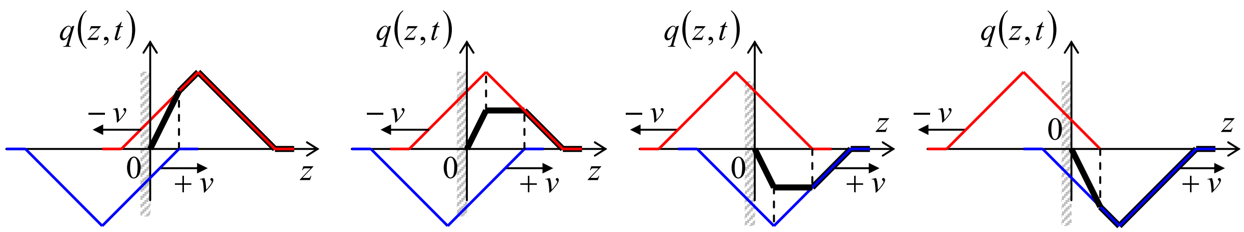


This solution is evidently valid even for a system of finite size until one of the partial "pulses" has reached one of its ends. Let us see what happens when it does, for example when one of these pulses, say the partial wave $q_-(z, t)$, runs into a rigid wall at $z = 0$. It is virtually evident, and may be readily verified by

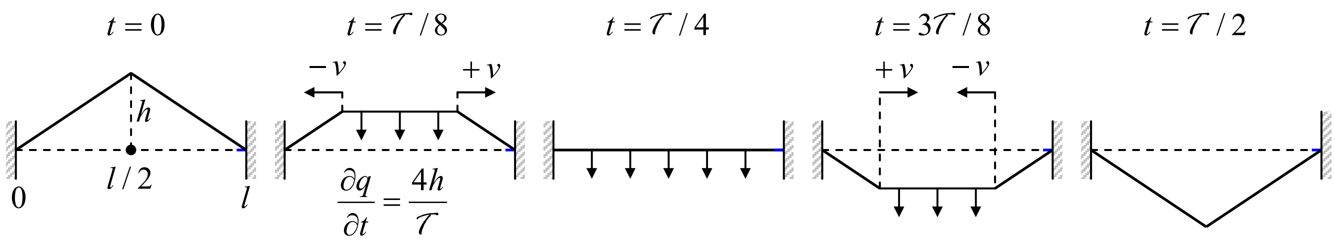
[20] Actually, this is just a particular case of the so-called *d'Alembert solution* of the wave equation, valid when Eq. (****) is generalized to include an arbitrary distribution of the initial velocity, $\partial q/\partial t(z, 0) = g(z)$:

$$q(z,t) = \frac{1}{2}[f(z-vt) + f(z+vt)] + \frac{1}{2v}\int_{z-vt}^{z+vt} g(z')dz'\,.$$

substitution, that the wave equation and the boundary condition $q(0, t) = 0$ are satisfied by the superposition of the incident pulse $q_-(z, t) \equiv f(z + vt)/2$ and an inverted pulse of the similar waveform, moving in the opposite direction – see the figure below, drawn for four sequential moments of time. (The red lines show the incident pulse, the blue lines, the inverted pulse, while the bold black lines represent their linear superposition, i.e. the actual waveform.) The plots show that the net displacement waveform first disappears,[21] just to appear imminently, with the reversed polarity and velocity's direction.



Combining these two pictures for our initial triangular waveform (which occupies the whole string's length, so the outgoing partial waveforms $q_\pm$ immediately start reflecting from the corresponding walls), we get the wave evolution shown in the figure below. (At $t = \mathcal{T}/2$, the string starts a similar motion in the opposite direction, and after $t = \mathcal{T}$, the whole process repeats periodically.)



This picture is evidently compatible with the midpoint pattern for $q(l/2, t)$ that was sketched above. Moreover, it provides an explanation of the piecewise-constant velocity $\partial q/\partial t = \pm 4h/\mathcal{T}$ of the deformation. Indeed, at all straight parts of the string, $\partial^2 q/\partial z^2 = 0$. For them, the wave equation (*) yields $\partial^2 q/\partial t^2 = 0$, which is the case only if $\partial q/\partial t$ is constant in time – equal to zero at the "sloped" side parts of the string and to a non-zero constant at its "horizontal" middle part, which contains the midpoint at all times.

The comparison of the two used approaches[22] may be completed by performing a straightforward Fourier expansion of the function $q(z, t)$ sketched above, which gives Eq. (***) again. (This additional exercise is highly recommended to the reader.)

<u>Problem 6.14</u>. Spell out the spatial-temporal Green's function (6.82) for waves in a 1D uniform system of N points, with rigid boundary conditions (6.62). Explore the acoustic limit of your result.

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[21] At the instance when the displacement q and hence the potential energy of the wave vanish at all z, the pulse still exists, in the form of the string's velocity $\partial q/\partial t$, i.e. of its purely kinetic energy.
[22] The first of them is frequently associated with the name of J. Bernoulli, and the second, with the name of J. d'Alembert – both already mentioned in Sec. 2.1.

Solution: As was discussed in Sec. 6.5 of the lecture notes, for this system

$$Z_n(z_j) = s_n \sin k_n z_j, \qquad \text{with } k_n = \frac{\pi n}{l} \equiv \frac{\pi n}{Nd}, \quad z_j = jd, \quad n = 1,2,...,N.$$

In order to use Eqs. (6.82), the constants s_n should be calculated from the normalization condition

$$\sum_{j=1}^{N} \left[Z_n(z_j)\right]^2 \equiv s_n^2 \sum_{j=1}^{N} \sin^2 \frac{\pi nj}{N} \equiv s_n^2 \sum_{j=1}^{N} \frac{1}{2}\left(1 - \cos\frac{2\pi nj}{N}\right) \equiv s_n^2 \frac{N}{2} = 1,$$

so $s_n^2 = 2/N$. As a result, the second of Eqs. (6.82) reads

$$\mathcal{G}(z_j, z_{j'}, \tau) \equiv \frac{2}{N} \sum_{n=1}^{N} G_n(\tau) \sin\frac{\pi nj}{N} \sin\frac{\pi nj'}{N}, \qquad (*)$$

where the temporal Green's functions $G_n(\tau)$ are given by Eq. (5.34) with the replacement $\omega_0 \to \omega_n$:

$$G_n(\tau) = \frac{\sin \omega_n' \tau}{\omega_n'} \exp\{-\delta\tau\}, \qquad \text{with } \omega_n' \equiv \left(\omega_n^2 - \delta^2\right)^{1/2}, \quad \omega_n = v k_n, \quad v \equiv \left(\frac{\kappa_{\text{ef}}}{m}\right)^{1/2} d.$$

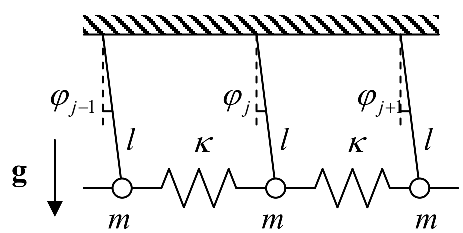
Since the acoustic limit is equivalent to the continuity approximation ($N \to \infty$, $d \to 0$), all sums over j should be replaced with integrals over the system's length $l = Nd$:

$$\frac{1}{N}\sum_{j=1}^{N} \varphi(z_j) \to \frac{1}{N}\int_0^N \varphi(z_j)\, dj \equiv \frac{1}{N}\int_0^N \varphi(z_j)\, d\left(\frac{z_j}{d}\right) \equiv \frac{1}{l}\int_0^l \varphi(z)\, dz,$$

for any function $\varphi(z_j)$. As a result, Eqs. (6.82), with our spatial-temporal Green's function (*), become

$$q(z,t) = \int_0^l dz' \int_0^\infty d\tau\, f(z', t-\tau)\, g(z, z', \tau), \qquad \text{where } g(z,z',\tau) \equiv \frac{2}{l}\sum_{n=1}^{\infty} G_n(\tau) \sin\frac{\pi nz}{l} \sin\frac{\pi nz'}{l},$$

where $f(z, t)$ is the linear density of the external force divided by the linear mass density of the system.



Problem 6.15. Calculate the dispersion law $\omega(k)$ and the highest and lowest frequencies of small longitudinal waves in a long chain of similar, spring-coupled pendula – see the figure on the right.

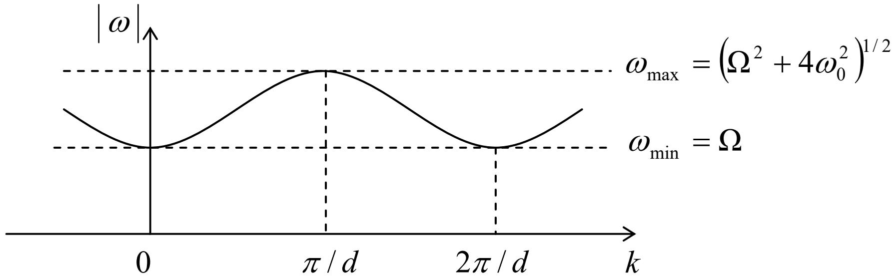
Solution: The linearized equation of motion of the j^{th} pendulum is an evident generalization of the equations derived in the solution of Problem 1.8:

$$\ddot{\varphi}_j = -\Omega^2 \varphi_j + \omega_0^2(\varphi_{j-1} - \varphi_j) + \omega_0^2(\varphi_{j+1} - \varphi_j),$$

where $\Omega \equiv (g/l)^{1/2}$ and $\omega_0 \equiv (\kappa/m)^{1/2}$. Looking for a particular solution of this equation in the standard form of a sinusoidal traveling wave (see Eq. (6.28) of the lecture notes), $\varphi_j = \text{Re}[a \exp\{i(kz_j - \omega t)\}]$, with $z_j = jd$, where d is the structure period, we get the following dispersion relation

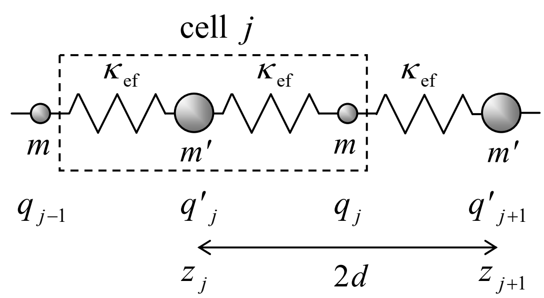
$$\omega = \pm\left(\Omega^2 + 4\omega_0^2 \sin^2 \frac{kd}{2}\right)^{1/2}.$$

The function $\omega(k)$ is sketched in the figure below; it shows that the system may sustain waves of any frequency between $\omega_{\min} = \Omega$ and $\omega_{\max} = (\Omega^2+4\omega_0^2)^{1/2} \geq \Omega$. The former value is achieved at $kd = 2\pi n$ (where n is an integer), i.e. at $\exp\{ikd\} = +1$, when all pendula oscillate in phase, and springs are not strained at all; the latter value is attained at $kd = (2n + 1)\pi$, i.e. at $\exp\{ikd\} = -1$, when the adjacent pendula oscillate in anti-phase so the springs are most involved.



Note the absence of acoustic branches; this is very typical for systems of particles that are elastically bound to their equilibrium positions.

Problem 6.16. Calculate and analyze the dispersion relation $\omega(k)$ for small waves in a long 1D chain of elastically coupled particles with alternating masses – see the figure on the right. In particular, discuss the dispersion relation's period Δk, and its evolution at $m' \to m$.



Solution: Just as in the uniform case (see Sec. 6.3 of the lecture notes), the system may be represented as a set of identical unit cells. However, in contrast with the case $m = m'$, each elementary cell of the current system (see the dashed line in the figure above) consists of two particles rather than one. Using either the appropriate Lagrangian function or the 2nd Newton law directly, we can write the following equations of motion of these two particles:

$$\begin{aligned} m\ddot{q}_j &= \kappa_{\text{ef}}(q'_j - q_j) + \kappa_{\text{ef}}(q'_{j+1} - q_j), \\ m'\ddot{q}'_j &= \kappa_{\text{ef}}(q_{j-1} - q'_j) + \kappa_{\text{ef}}(q_j - q'_j). \end{aligned} \qquad (*)$$

Let us look for the solution in almost the same traveling-wave form as for $m = m'$ (see Eq. (6.28) of the lecture notes), but now allowing for different amplitudes of oscillations of the different particles – just like has been done for two coupled oscillators, see Eq. (6.6):

$$q_j(t) = \text{Re}\left[ae^{i(kz_j - \omega t)}\right], \qquad q'_j(t) = \text{Re}\left[a'e^{i(kz_j - \omega t)}\right],$$

where z_j is the position of the whole unit cell (with two different particles), so $z_{j+1} - z_j = 2d$. The substitution of these relations into Eq. (*) shows that this is indeed a particular solution of the equations of motion, provided that the constants a, a', and kd satisfy the following system of equations:

$$-m\omega^2 a = \kappa_{\text{ef}}(a' - a) + \kappa_{\text{ef}}\left(a' e^{i2kd} - a\right),$$
$$-m'\omega^2 a' = \kappa_{\text{ef}}\left(a e^{-i2kd} - a'\right) + \kappa_{\text{ef}}(a - a'). \qquad (**)$$

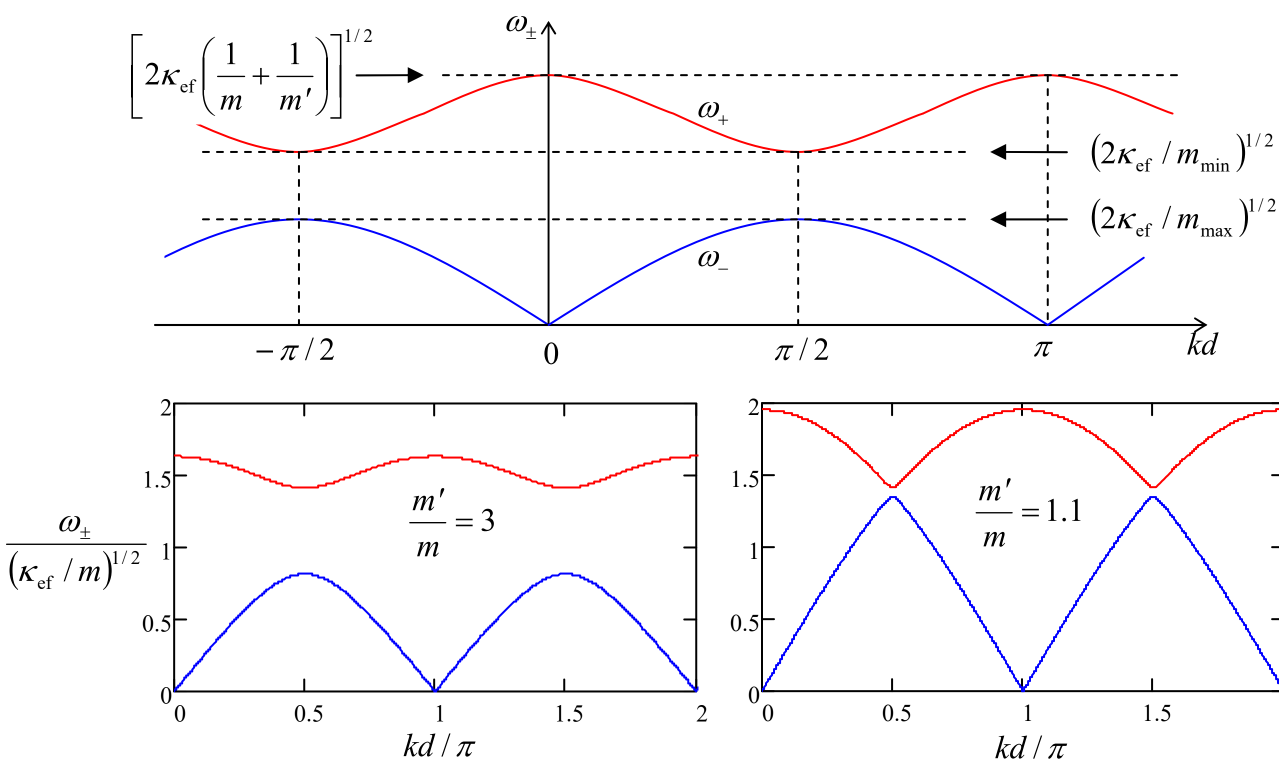
These two homogeneous linear equations are compatible if the determinant of their matrix equals zero:

$$\begin{vmatrix} m\omega^2 - 2\kappa_{\text{ef}} & \kappa_{\text{ef}}\left(1 + e^{i2kd}\right) \\ \kappa_{\text{ef}}\left(e^{-i2kd} + 1\right) & m'\omega^2 - 2\kappa_{\text{ef}} \end{vmatrix} = 0 \cdot$$

This is a quadratic equation for ω^2, whose roots may be represented in either of two equivalent forms – each one more convenient for a certain part of the result's analysis:

$$\omega_\pm^2 = \kappa_{\text{ef}}\left\{\left(\frac{1}{m} + \frac{1}{m'}\right) \pm \left[\left(\frac{1}{m} - \frac{1}{m'}\right)^2 + \frac{4}{mm'}\cos^2 kd\right]^{1/2}\right\} \equiv \kappa_{\text{ef}}\left\{\left(\frac{1}{m} + \frac{1}{m'}\right) \pm \left[\frac{1}{m^2} + \frac{1}{m'^2} + \frac{2}{mm'}\cos 2kd\right]^{1/2}\right\}.$$

This dispersion relation is shown in the figure below – schematically sketched on the top panel, and numerically plotted (for two values of the m'/m ratio) on the bottom two panels.



Note the following remarkable features of the result:

(i) The dispersion law has two branches ("frequency bands"); in quantum mechanics, the top band corresponds to the elementary excitations called *optical phonons*, and the lower one, to *acoustic phonons*. The first term is of historic origin: it stems from the fact that the very high phase and group velocities, $(v_{\text{ph}})_+ = \omega_+/k$ and $(v_{\text{gr}})_+ = d\omega_+/dk$, possible for the optical *phonons* in solids, enable their effective interaction with optical *photons* with velocities close to c that is much higher than the typical phonon velocity scale, $\omega_\pm d$. The lower branch's name is also due to convention only because only its lowest parts correspond to the genuinely acoustic (i.e. dispersion-free) waves, with the velocity

$$v = \left|\frac{d\omega_-}{dk}\right|_{\omega\to 0} = \left(\frac{\kappa_{\text{ef}}}{m_{\text{ave}}}\right)^{1/2} d, \quad \text{where } m_{\text{ave}} \equiv \frac{m+m'}{2}.$$

Just as for the particular case $m = m'$ that was discussed in Sec. 6.3 of the lecture notes, the physics of these results may be revealed by plugging the calculated values of ω^2 back into the algebraic equations (**) for the distribution coefficients a and a'. In particular, let us consider the long-wave limit, $kd \to 0$. In this limit, at the "optical" branch, with

$$\omega_+ \approx \left[2\kappa_{\text{ef}}\left(\frac{1}{m}+\frac{1}{m'}\right)\right]^{1/2},$$

we get $m'a' = -ma$, i.e. the neighboring particles, with different masses, oscillate in anti-phase. (Note that for $m = m'$, we had a similar phenomenon taking place at $kd = \pi(2n+1)$ – see Fig. 6.5 of the lecture notes and its discussion.) It is only natural that this result involves the effective particle mass similar to that participating in the planetary problems – cf. Eq. (3.35) of the lecture notes. On the contrary, at the "acoustic" branch, with

$$\omega_- \approx \pm vk,$$

the particles move in phase: $a' \approx a$, with the spring effects felt only at relatively large distances $\Delta z \sim 1/k >> d$.

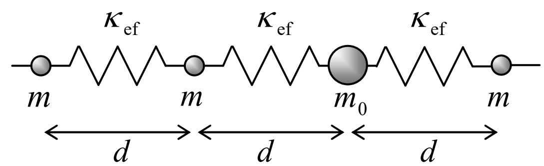
Even more interesting is the situation at the adjacent edges of the "frequency gap",[23] i.e. at $2kd = \pi(2n+1)$, for example at $kd = \pm\pi/2$. Here either a or a' vanishes, i.e. the subsystem of all particles of one mass stays still, while the particles of the other mass oscillate about it, with the nearest particles of this type moving in antiphase – so the total momentum of the system is still conserved. This motion simplicity makes the expressions for $\omega_\pm$ on the gap's edges also simple – see the labels in the figure above.

(ii) The period of both functions $\omega_\pm(k)$ is $\Delta k = \pi/d$. This is very natural because the spatial period Δz of the structure is now $2d$ rather than d, so $\Delta k = 2\pi/\Delta z$, just as in the uniform chain. In the degenerate case $m' \to m$, we have $m_{\text{min}} \to m_{\text{max}}$, the bandgap closes (cf. the right bottom plot in the figure above), and the dispersion relation may be considered a set of two almost independent curves, each describing the dispersion law (6.30), with the double period $\Delta k = 2\pi/d$. This is also natural because, in this degenerate limit, the spatial period of the structure Δz becomes d, so the general relation $\Delta k = 2\pi/\Delta z$ is still observed.

In the case $0 < |m' - m| << m$, the formation of a narrow frequency gap may be interpreted as a result of partial backscattering of the traveling waves with wavelength $2\pi/k \approx 4d$ on the weak "standing wave" (with period $2d$) of the particle mass alternation. An analysis in these terms (which may be recommended to the reader as a very good additional exercise[24]) leads, in particular, to the standard

[23] This gap between two bands of possible frequencies is frequently (especially in the physics of semiconductors) called the *bandgap*.
[24] See also QM Sec. 2.7.

level anticrossing diagram (cf. Fig. 6.2 in the lecture notes), which is clearly visible on the right bottom panel of the figure above.



Problem 6.17. Analyze the traveling wave's reflection from a "point inhomogeneity": a single particle with a different mass $m_0 \neq m$, in an otherwise uniform 1D chain – see the figure on the right.

Solution: Since we are considering a point (discrete) inhomogeneity, we should replace Eq. (6.50) of the lecture notes with its discrete version. For sinusoidal waves of frequency ω, we may write

$$q_j(t) = \text{Re}\begin{cases} a_-'\exp\{i(-kz_j - \omega t)\}, & \text{for } j \leq 0, \\ a_- \exp\{i(-kz_j - \omega t)\} + a_+ \exp\{i(kz_j - \omega t)\}, & \text{for } j \geq 0. \end{cases}$$

Plugging this solution into Eq. (6.24) for the regular particles (of mass m),

$$m\ddot{q}_j = \kappa_{ef}(q_{j+1} - q_j) - \kappa_{ef}(q_j - q_{j-1}), \qquad \text{for } j \neq 0,$$

we see that they are satisfied if k obeys the dispersion law (6.30),

$$\omega^2 = 2\frac{\kappa_{ef}}{m}(1 - \cos kd), \tag{*}$$

while the naturally modified similar equation for the special particle of mass m_0 (assumed to be located at $z = 0$), i.e. for $j = 0$,

$$m_0\ddot{q}_0 = \kappa_{ef}(q_1 - q_0) - \kappa_{ef}(q_0 - q_{-1}),$$

is satisfied if the complex amplitudes of the incident (a_-), transmitted (a_-'), and reflected (a_+) waves obey the following system of equations:

$$a_-' = a_- + a_+, \quad -\omega^2 m_0 a_-' = \kappa_{ef}\left[a_-\left(e^{-ikd} - 1\right) + a_+\left(e^{ikd} - 1\right)\right] - \kappa_{ef} a_-'\left(1 - e^{ikd}\right).$$

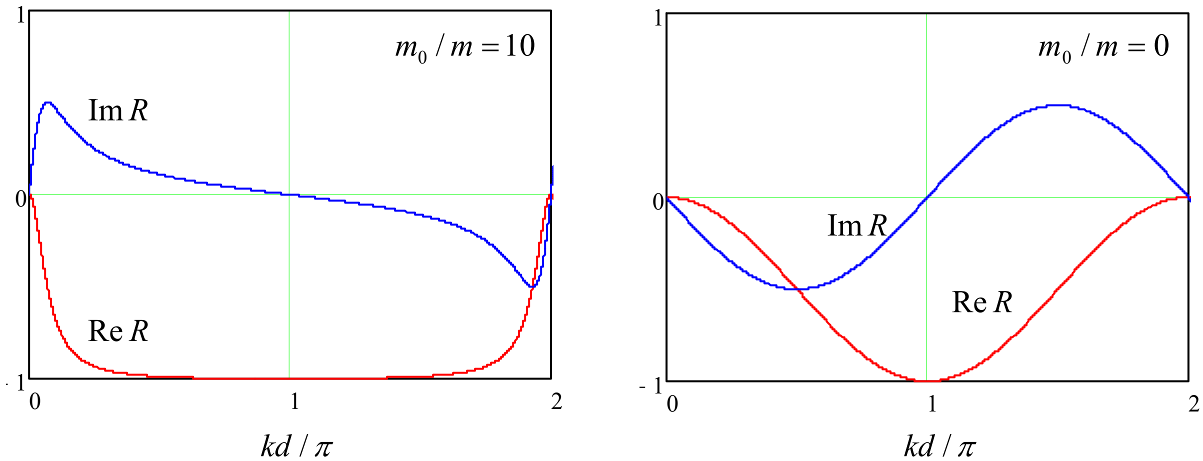
Solving this simple system of two linear equations, and plugging ω^2 from Eq. (*), we get

$$\mathcal{R} \equiv \frac{a_+}{a_-} = -\frac{m_0/m - 1}{m_0/m - 1 + i\sin kd/(1 - \cos kd)}. \tag{**}$$

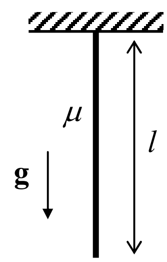
As a sanity check, at $m = m_0$ (no inhomogeneity), Eq. (*) gives no wave reflection: $\mathcal{R} = 0$ at any wavenumber k. The figure below shows plots of the real and imaginary parts of $\mathcal{R}$ as a function of the product kd, for two values of the m_0/m ratio. For $m_0 >> m$, the reflection coefficient is close to (–1) for almost any kd (i.e. any frequency), just as for the reflection from a rigid boundary. On the other hand, at $m_0 << m$, both components of the reflection coefficient are comparable.

Note, however, that for any finite m_0, both parts of $\mathcal{R}$ tend to zero at $kd \to 0$ (and also $2\pi n$ with any integer n). This is natural because in the acoustic-wave limit, the wavelength becomes much larger than d, i.e. wave is spread over many particles, and is virtually unaffected by the special mass of one of them. This averaging effect is clearly visible from the first (linear) approximation in $kd << 1$, in which Eq. (**) is reduced to

$$\mathrm{Re}\,\mathcal{R} \approx 0, \qquad \mathrm{Im}\,\mathcal{R} \approx \left(\frac{m_0}{m} - 1\right)\frac{kd}{2} \equiv \pi \frac{d}{\lambda}\left(\frac{m_0}{m} - 1\right).$$



Problem 6.18.*



(i) Explore an approximate way to analyze small waves in a continuous 1D system with parameters slowly varying along its length.

(ii) Use this method to calculate the frequencies of transverse standing waves on a freely hanging heavy rope of length l, with a constant mass μ per unit length – see the figure on the right.

(iii) For the three lowest standing wave modes, compare the results with those obtained in the solution of Problem 4 for the triple pendulum.

Hint: The reader familiar with the WKB approximation in quantum mechanics (see, e.g., QM Sec. 2.4) is welcome to adapt it for this classical application. Another possible starting point is the van der Pol approximation discussed in Sec. 5.3, which should be translated from the time domain to the space domain.

Solutions:

(i) Recalling the basics (see in Sec. 6.3 of the lecture notes), in the continuous limit $kd << 1$, waves in a ***uniform*** 1D system are described by the partial differential equation (6.40):

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right) q(z,t) = 0 . \qquad (*)$$

If we are only interested in monochromatic waves, we may look for the solution of (*) in the form

$$q(z,t) = f(z) e^{-i\omega t} .$$

With this substitution, Eq. (*) turns into the following 1D *Helmholtz equation*[25] for the function $f(z)$:

[25] See also EM Chapters 7 and 8, and QM Chapters 2 and 3.

$$\left(\frac{d^2}{dz^2}+k^2\right)f=0, \qquad \text{with } k^2\equiv\frac{\omega^2}{v^2}.$$

The fundamental solutions of this ordinary differential equation are sinusoidal waves with wave numbers $\pm k$:

$$f_\pm = a_\pm e^{\pm ikz}, \qquad \text{giving } q(z,t)=a_+e^{i(kz-\omega t)}+a_-e^{i(-kz-\omega t)}.$$

Now let us rewrite these expressions via the full wave's phases $\Psi_\pm(z) = \pm kz$:

$$f_\pm = a_\pm e^{i\Psi_\pm(z)},$$

and think about what would happen if k is not a constant, but is a function of z, due to the variation of the system's parameters, and hence of the wave velocity v:

$$k^2(z)\equiv\frac{\omega^2}{v^2(z)}.$$

It is almost evident that if $k^2(z)$ is a sufficiently slowly changing function, the above solutions should be approximately valid, with the following replacement:[26]

$$\Psi_\pm(z)\rightarrow\pm\int^z k(z')dz'.$$

Indeed, the substitution of this *WKB approximation* into the nonuniform Helmholtz equation

$$\left[\frac{d^2}{dz^2}+k^2(z)\right]f=0, \tag{**}$$

shows[27] that this solution is asymptotically valid if

$$ka \gg 1, \qquad \text{with } a\equiv\left|\frac{k}{dk/dz}\right|.$$

Physically, a is the spatial scale of relatively significant changes in the system's parameters, and the above condition means that the WKB approximation is valid for a wave with many wavelengths $\lambda = 2\pi/k$ on such an interval – in other words, if the relative change of parameters on one wavelength is small.

(ii) For a uniform, horizontally stretched rope, the transverse wave velocity is described by Eq. (6.43) of the lecture notes,

$$v=\left(\frac{\mathscr{T}}{\mu}\right)^{1/2},$$

[26] The lower limits here are not important, because the constants resulting from their choice may be readily incorporated into the phases of the complex amplitudes $a_\pm$.

[27] See also QM Sec. 3.2. Note that the accuracy of the WKB approximation may be improved by the simultaneous adjustment of the traveling wave's amplitude to conserve its power (in quantum mechanics, the probability current); for our current task, this detail is not important, except that it is crudely reflected in the mode sketches in the figure below.

where $\mathscr{T}$ is the rope's tension, and μ is its mass per unit length. In a vertically hanging rope, the tension grows toward the suspension point:

$$\mathscr{T} = \mathscr{T}(z) = \mu g z$$

(where z is the axis directed up, with the origin at the rope's end), so[28]

$$k(z) = \frac{\omega}{v(z)} = \omega\left[\frac{\mu}{\mathscr{T}(z)}\right]^{1/2} = \frac{\omega}{(gz)^{1/2}}. \qquad (***)$$

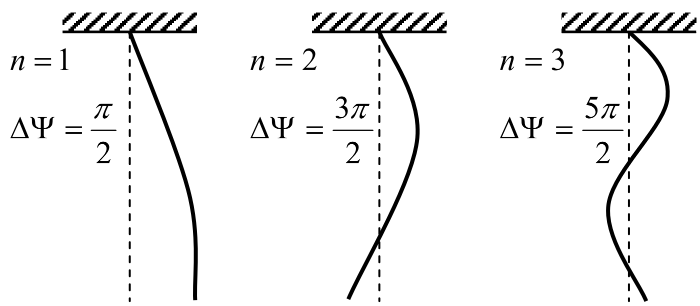
Hence the wave phase change along the rope's length is

$$\Delta\Psi = \int_0^l k(z)dz = \frac{\omega}{g^{1/2}}\int_0^l \frac{dz}{z^{1/2}} = \frac{\omega}{g^{1/2}}2l^{1/2} \equiv 2\omega\left(\frac{l}{g}\right)^{1/2}. \qquad (****)$$

For our problem, we should use the "rigid" boundary condition ($f = 0$) at the top end of the rope, and the "open" condition ($df/dz = 0$) at its lower end. According to Eq, (6.72) of the lecture notes, for such a boundary condition combination, the standing wave modes correspond to the following phase difference values:

$$\Delta\Psi = \pi\left(n - \frac{1}{2}\right), \quad \text{with } n = 1, 2, 3,..., \qquad \text{i.e. } \Delta\Psi = \frac{\pi}{2}, \ \frac{3\pi}{2}, \ \frac{5\pi}{2},...$$

(The first three modes are crudely sketched in the figure on the right.) Combining this expression with Eq. (****), we get the following result for mode frequencies:



$$\omega_n = \frac{\pi}{2}\left(n - \frac{1}{2}\right)\left(\frac{g}{l}\right)^{1/2};$$

it is exact for $n >> 1$.[29]

(iii) For the issue in question, the triple pendulum discussed in Problem 4 may be crudely modeled as a heavy rope of length $3l$ (with $\mu = m/l$, but this parameter does not participate in the final results). For such a model, the result of Task (ii) of our current problem becomes

[28] Strictly speaking, the transverse waves on a rope with variable tension $\mathscr{T}(z)$ are described not by the equation that would follow from Eqs. (**) and (***),

$$\mathscr{T}(z)\frac{d^2 f}{dz^2} + \mu\omega^2 f = 0,$$

but by a different equation that follows, in the continuous limit, from Eq. (6.24) of the lecture notes:

$$\frac{d}{dz}\left(\mathscr{T}(z)\frac{df}{dz}\right) + \mu\omega^2 f = 0.$$

However, the WKB approximations for these two equations coincide.

[29] Just for the reader's reference: this problem allows the *exact* solution in terms of the Bessel functions, in particular, giving $\omega_n = (\xi_{0n}/2)(g/l)^{1/2}$, where ξ_{0n} is the n^{th} root of the Bessel function $J_0(\xi)$; these roots are listed, for example, in the top row of EM Table 2.1. Let me refer the interested reader to a recent, very clearly written paper by Y. Verbin, *Eur. J. Phys*. **36**, 015005 (2015), and references therein.

$$\omega_n = \frac{\pi}{2\sqrt{3}}\left(n - \frac{1}{2}\right)\Omega, \qquad \text{where } \Omega \equiv \left(\frac{g}{l}\right)^{1/2};$$

for the lowest three modes this formula yields

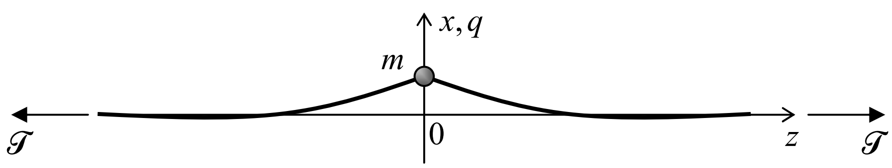
$$\omega_1 = \frac{\pi}{4\sqrt{3}}\Omega \approx 0.45\Omega, \qquad \omega_2 = \frac{3\pi}{4\sqrt{3}}\Omega \approx 1.36\Omega, \qquad \omega_3 = \frac{5\pi}{4\sqrt{3}}\Omega \approx 2.27\Omega.$$

It is very impressive how close are these numbers[30] to the values calculated in the solution of Problem 4:

$$\omega_1 \approx 0.6448\Omega, \qquad \omega_2 \approx 1.5147\Omega, \qquad \omega_3 \approx 2.508\Omega.,$$

given the crudeness of the distributed model and of the WKB approximation for our current case. Indeed, for the lowest modes, the validity condition $ka >> 1$, in our case taking the form $kl >> 1$, is rather poorly satisfied.

Problem 6.19. A particle of mass m is attached to an infinite string, of mass μ per unit length, stretched with tension $\mathscr{T}$. The particle is confined to move along the x-axis perpendicular to the string (see the figure below), in an additional smooth potential $U(x)$ with a minimum at $x = 0$. Assuming that the waves on the string are excited only by the motion of the particle (rather than any external source), reduce the system of equations describing the system to an ordinary differential equation for small oscillations $x(t)$, and calculate their Q-factor of due to the drag caused by the string.



Solution: For small deviations $q(z, t)$ of the string from the equilibrium, with $\partial q/\partial z \to 0$, the x-component of the net force exerted on the particle by two parts of the string is

$$F_x \approx \mathscr{T}\left(\frac{\partial q}{\partial z}\Big|_{z=+0} - \frac{\partial q}{\partial z}\Big|_{z=-0}\right),$$

so the equation of motion of its vertical coordinate $x(t)$ is

$$m\ddot{x} = -\frac{\partial U(x)}{\partial x} + \mathscr{T}\left(\frac{\partial q}{\partial z}\Big|_{z=+0} - \frac{\partial q}{\partial z}\Big|_{z=-0}\right). \qquad (*)$$

Very similarly, as was discussed at the beginning of Sec. 6.4 of the lecture notes, the x-component of the force exerted on a small string's fragment of length dz by the parts adjacent to it is

$$dF_x \approx \mathscr{T}\left(\frac{\partial q}{\partial z}\Big|_{z+dz/2} - \frac{\partial q}{\partial z}\Big|_{z-dz/2}\right) \approx \mathscr{T}\left[\frac{\partial^2 q}{\partial^2 z}\left(+\frac{dz}{2}\right) - \frac{\partial^2 q}{\partial^2 z}\left(-\frac{dz}{2}\right)\right] \equiv \mathscr{T}\frac{\partial^2 q}{\partial^2 z}dz.$$

[30] The reader is invited to check that the mode patterns are very similar as well.

Since the mass of this fragment is μdz, the 2nd Newton's law gives the following equation of the string's motion:

$$\mu \frac{\partial^2 q}{\partial t^2} = \mathscr{T} \frac{\partial^2 q}{\partial^2 z}, \qquad (**)$$

which is just a particular form of Eq. (6.40). Equations (*) and (**), together with the boundary condition (evident from the figure in the assignment),

$$q(0,t) = x(t),$$

fully describe the system's dynamics, provided that initial conditions have been specified.

Next, as was discussed in Sec. 6.4 of the lecture notes, the general solution of the wave equation (**) is given by Eq. (6.41)

$$q(x,t) = f_{\rightarrow}\left(t - \frac{z}{v}\right) + f_{\leftarrow}\left(t + \frac{z}{v}\right),$$

Where, in agreement with Eq. (6.43),

$$v \equiv \left(\mathscr{T} / \mu\right)^{1/2}$$

is the wave velocity, and $f_{\rightarrow}$ and $f_{\leftarrow}$ are certain functions of a single argument, which are determined by the initial and boundary conditions. If the waves on the string are excited only by the motion of the particle (rather than any external sources, in particular, any waves arriving from afar), the wave on its right part ($z > 0$) may travel only to the right, and vice versa:

$$q(z,t) = \begin{cases} f_{\rightarrow}(t - z/v), & \text{at } z \geq 0, \\ f_{\leftarrow}(t + z/v), & \text{at } z \leq 0. \end{cases}$$

Moreover, according to our boundary condition, at $z = 0$ these functions have to be equal to each other and to the particle's coordinate $x(t)$, so

$$q(z,t) = \begin{cases} x(t - z/v), & \text{at } z \geq 0, \\ x(t + z/v), & \text{at } z \leq 0, \end{cases} \qquad \text{and hence} \quad \frac{\partial q}{\partial z}(z,t) = \begin{cases} (-1/v)\dot{x}(t - z/v), & \text{at } z \geq 0, \\ (+1/v)\dot{x}(t + z/v), & \text{at } z \leq 0. \end{cases}$$

Plugging these expressions, taken at $z = \pm 0$, into Eq. (*), we get the following ordinary differential equation for the particle:

$$m\ddot{x} = -\frac{\partial U(x)}{\partial x} - 2Z\dot{x}, \qquad \text{with } Z \equiv \frac{\mathscr{T}}{v} = \left(\mu \mathscr{T}\right)^{1/2}. \qquad (***)$$

The constant Z is just the wave impedance of the string – see Eqs. (6.47)-(6.48) of the lecture notes. As Eq. (***) shows, in our case, $2Z$ plays the role of the drag coefficient η defined by Eq. (5.5): $F_u = -\eta u$, where $u \equiv \dot{x}$ is the particle's velocity.

For small oscillations near the equilibrium, the smooth potential $U(x)$ may be approximated with a quadratic parabola:

$$U(x) \approx \frac{\kappa}{2} x^2, \quad \text{where } \kappa \equiv \frac{d^2 U}{dx^2}\Big|_{x=0},$$

so Eq. (***), after the division of all its terms by m, takes the form

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = 0, \qquad \text{with } \omega^2 = \frac{\kappa}{m},$$

where $\delta \equiv \eta/2m = Z/m$ is the damping coefficient, so the Q-factor defined by Eqs. (5.11)-(5.12) is

$$Q \equiv \frac{\omega}{2\delta} = \frac{m\omega}{2Z}.$$

Note that this is a good example of an "open" physical system whose dynamics may be interpreted in two different ways. On one hand, Eqs. (*) and (**) are Hamiltonian (dissipation-free), so the full energy of the system, with strings extending to infinity, is conserved for any finite t. On the other hand, the outgoing waves $f_{\rightarrow}$ and $f_{\leftarrow}$, propagating from the particle, carry out its energy to infinity, so the subsystem including the particle plus two adjacent segments of the string of any finite length eventually loses energy and hence is dissipative. This duality gives a wonderful (and broadly used) opportunity to study not only classical but also quantum dynamics of dissipative systems while using the reliable theoretical methods developed for Hamiltonian systems.

Problem 6.20.* Use the van der Pol method to analyze the mutual phase locking of two weakly coupled self-oscillators with the dissipative nonlinearity, for the cases of:

(i) the direct coordinate coupling described by Eq. (6.5) of the lecture notes, and

(ii) a bilinear but otherwise arbitrary coupling of two similar oscillators.

Hint: In Task (ii), describe the coupling by an arbitrary linear operator, and express the result via its Fourier image.

Solutions:

(i) As it follows from the analysis of phase locking in Sec. 5.4 of the lecture notes, the effect is insensitive to the exact model of the dissipative nonlinearity, provided that the locking range is not too large. Assuming that, we can use, for example, the model described by Eq. (5.62) for each oscillator, so Eqs. (6.5) turn into

$$\begin{aligned} m_1\left(\ddot{q}_1 + 2\delta_1\dot{q}_1 + \Omega_1^2 q_1 + \beta_1\dot{q}_1^3\right) &= \kappa q_2, \\ m_2\left(\ddot{q}_2 + 2\delta_2\dot{q}_2 + \Omega_2^2 q_2 + \beta_2\dot{q}_2^3\right) &= \kappa q_1, \end{aligned} \qquad (*)$$

where $\Omega_1 \approx \Omega_2$; $m_{1,2}$, $\beta_{1,2} > 0$; and $\delta_{1,2} < 0$. Now assuming that the parameters δ, β, κ, and the detuning $\xi \equiv \Omega_1 - \Omega_2$ are sufficiently small, we may look for the solution of Eqs. (*) in the standard form (5.41):

$$q_{1,2} = A_{1,2}(t)\cos\Psi_{1,2}, \qquad \Psi_{1,2} \equiv \omega t - \varphi_{1,2}(t),$$

where ω is some (yet unknown) frequency, close to Ω_1 and Ω_2. Reviewing the discussion of the van der Pol approximation's validity (Sec. 5.3) for this case, we may readily conclude that the reduced equations (5.57) are still valid for each frequency component of the oscillations, provided that the right-hand side averaging in these equations is carried over a sufficiently large number of the corresponding periods $\Delta\Psi_{1,2} = 2\pi$, rather than just one of them, so the impact of combinational frequencies incommensurate with $\omega_{1,2}$ becomes negligible. Rewriting each of Eqs. (*) in the standard form (5.38), we get from Eqs. (5.57) the following reduced equations:

$$\dot{A}_1 = -\delta_1(A_1)\,A_1 + \frac{\kappa A_2}{2m_1\omega}\sin\varphi, \qquad \dot{\varphi}_1 = \xi_1 + \frac{\kappa A_2}{2m_1\omega A_1}\cos\varphi,$$
$$\dot{A}_2 = -\delta_2(A_2)\,A_2 - \frac{\kappa A_1}{2m_2\omega}\sin\varphi, \qquad \dot{\varphi}_2 = \xi_2 + \frac{\kappa A_1}{2m_2\omega A_2}\cos\varphi,$$

where $\delta_{1,2}(A_{1,2}) \equiv \delta_{1,2} + (3/8)\beta_{1,2}\omega^2 A_{1,2}^{\ 2}$ are nonlinear damping functions (see Eq. (5.63a) of the lecture notes), $\varphi \equiv \varphi_1 - \varphi_2$ is the phase difference between the two oscillators, while $\xi_1 \equiv \omega - \Omega_1$ and $\xi_2 \equiv \omega - \Omega_2$ are partial detunings whose difference is a known parameter: $\xi \equiv \xi_1 - \xi_2 \equiv \Omega_2 - \Omega_1$. Since the right-hand sides of the reduced equations depend only on the phase difference φ, the next good move is to eliminate the partial oscillation phases altogether by subtracting the equations for their time derivatives. The result may be represented in the usual form (5.68):

$$\dot{\varphi} = \xi + \Delta\cos\varphi, \qquad (**)$$

where in our current case

$$\Delta \equiv \Delta_1 - \Delta_2, \quad \text{with } \Delta_{1,2} \equiv \frac{\kappa}{2\omega}\frac{A_{2,1}}{m_{1.2}A_{1,2}}.$$

Now using the same argumentation as in the analysis of phase locking by an external signal in Sec. 5.4 of the lecture notes, in the weak coupling limit ($|\Delta| << |\delta_{1,2}|$), we may consider $\Delta_{1,2}$ constant, with the amplitudes $A_{1,2}$ equal to their stationary values in the absence of coupling. Then Eq. (**) immediately shows that the phase φ has a stable stationary solution at $|\xi| < |\Delta|$. Since the constancy of φ means that the oscillators have exactly the same frequency, i.e. are phase-locked, this means that the mutual phase locking range is

$$-|\Delta| \le \Omega_1 - \Omega_2 \le +|\Delta|.$$

The common oscillation frequency ω_{PL} within this range may be found from any of the two equations for the partial phases $\varphi_{1,2}$ and the stationary solution of Eq. (**), $\cos\varphi = -\xi/\Delta$:

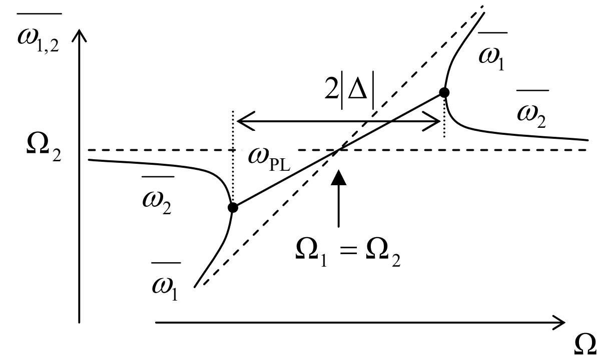
$$\omega_{PL} \equiv \dot{\Psi}_{1,2} = \omega - \dot{\varphi}_{1,2} = \omega - \left(\xi_{1,2} + \Delta_{1,2}\cos\varphi\right) = \omega - \left(\omega - \Omega_{1,2} - \frac{\Delta_{1,2}}{\Delta}\xi\right) = \Omega_{1,2} + \Delta_{1,2}\frac{\Omega_2 - \Omega_1}{\Delta_1 - \Delta_2}.$$

Note that this result does not depend on either the used oscillator index or on the assumed oscillation frequency ω, showing again the remarkable ability of the van der Pol method to recover the genuine frequency of oscillations – see the discussion of this issue in Sec. 5.3 of the lecture notes.

Another feature of the result for ω_{PL} is that within the mutual locking range, it is a linear function of any of oscillator eigenfrequencies $\Omega_{1,2}$. This conclusion is not valid outside of the locking range, where the average frequencies of the oscillations,

$$\overline{\omega_{1,2}} \equiv \overline{\Psi_{1,2}} = \omega - \overline{\dot{\varphi}_{1,2}} = \omega + \Delta_{1,2}\overline{\cos\varphi},$$

are different, nonlinear functions of the frequencies $\Omega_{1,2}$, and may be found from the time-dependent solution of Eq. (**) at $|\xi| > |\Delta|$, just as this was done in the model solution of Problem 5.12. We do not need that detailed solution in the case of large detuning, $|\xi| >> |\Delta|$, when, according to Eq. (**), the time evolution of the phase φ is virtually linear, and the time average of $\cos\varphi$ vanishes, so the average frequency of each oscillator tends to its own frequency. The resulting behavior of the average frequencies of the oscillators is shown (schematically) in the figure below.



Note a spectacular difference between this result and the anticrossing diagram (Fig. 6.2) for the apparently similar system described by Eqs. (6.5) of the lecture notes. Indeed, as the figure on the right shows, as a result of the self-oscillator interaction, their oscillation frequencies tend to stick together when the partial frequencies are close, while the anticrossing diagram describes the opposite trend of "frequency repulsion". This difference is due to the "only" substantial difference between these models, namely the fact that in the nonlinear system described by Eqs. (*), the oscillation amplitudes are stabilized by the dissipation's nonlinearity, so at weak coupling they are virtually not affected by the evolution of the oscillation phases. Indeed, it is straightforward to check that at $\delta_1(A_1) \equiv \delta_2(A_2) \equiv 0$, the reduced equations spelled out above describe the same anticrossing as the direct, exact solution of Eqs. (6.5).

Another important observation is that Δ vanishes if two oscillators are identical, so $\Delta_1 = \Delta_2$. As the analysis below will show, this is an artifact of the direct coordinate coupling assumed in Eqs. (*).

(ii) For an arbitrary bilinear coupling (with the interaction energy being an arbitrary linear operator of the product q_1q_2), Eqs. (*) may be generalized as follows:

$$m_1\left(\ddot{q}_1 + 2\delta_1\dot{q}_1 + \Omega_1^2 q_1 + \beta_1 q_1^3\right) = F\{q_2\},$$
$$m_2\left(\ddot{q}_2 + 2\delta_2\dot{q}_2 + \Omega_2^2 q_2 + \beta_2 q_2^3\right) = F\{q_1\},$$

where both (generalized) interaction forces F are described by the same linear operator K acting on the functions $q_{1,2}(t)\}$. As we know from the discussion of the Green's function in Sec. 5.1(ii) of the lecture notes, the action of such an operator may be expressed by a causal integral similar to Eq. (5.27):

$$F(t) = \int_0^\infty q(t-\tau)\mathrm{K}(\tau)d\tau,$$

where the kernel $\mathrm{K}(\tau)$ may be described by its Fourier image $\kappa(\omega)$ defined similarly to Eq. (5.28):

$$\mathrm{K}(\tau) \equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty}\kappa(\omega)e^{-i\omega\tau}d\omega, \qquad \text{so } \kappa(\omega) = \int_0^\infty \mathrm{K}(\tau)e^{i\omega\tau}d\tau.$$

In terms of this function, the interaction force F caused by some quasi-sinusoidal oscillations $q = A\cos\Psi$, with $\Psi = \omega t - \varphi$, is

$$\begin{aligned}F(t) &= \int_0^\infty A\cos[\omega(t-\tau)-\varphi]\mathrm{K}(\tau)d\tau \equiv \frac{A}{2}\int_0^\infty\left[e^{-i(\omega t-\varphi)}e^{i\omega\tau} + e^{i(\omega t-\varphi)}e^{-i\omega\tau}\right]\mathrm{K}(\tau)d\tau\\ &= \frac{A}{2}\left[e^{-i(\omega t-\varphi)}\kappa(\omega) + e^{i(\omega t-\varphi)}\kappa^*(\omega)\right] \equiv \frac{A}{2}\left[(\cos\Psi - i\sin\Psi)\kappa(\omega) + (\cos\Psi + i\sin\Psi)\kappa^*(\omega)\right]\\ &\equiv A\cos\Psi\,\mathrm{Re}\,\kappa(\omega) + A\sin\Psi\,\mathrm{Im}\,\kappa(\omega),\end{aligned}$$

so the average value participating in the second of Eqs. (5.57a) for the variable q_1 is

$$\overline{F\{q_2^{(0)}\}\cos\Psi_1} \equiv \overline{F\{A_2\cos\Psi_2\}\cos\Psi_1} \equiv A_2\overline{\cos\Psi_2\cos\Psi_1}\,\mathrm{Re}\,\kappa(\omega) + A_2\overline{\sin\Psi_2\cos\Psi_1}\,\mathrm{Im}\,\kappa(\omega)$$
$$= \frac{A_2}{2}\left[\cos(\Psi_1-\Psi_2)\mathrm{Re}\,\kappa(\omega) - \sin(\Psi_1-\Psi_2)\mathrm{Im}\,\kappa(\omega)\right] \equiv \frac{A_2}{2}\mathrm{Re}\left[e^{i\varphi}\kappa(\omega)\right],$$

and absolutely similarly:

$$\overline{F\{q_1^{(0)}\}\cos\Psi_2} = \frac{A_1}{2}\left[\cos(\Psi_2-\Psi_1)\mathrm{Re}\,\kappa(\omega) - \sin(\Psi_2-\Psi_1)\mathrm{Im}\,\kappa(\omega)\right] \equiv \frac{A_1}{2}\mathrm{Re}\left[e^{-i\varphi}\kappa(\omega)\right],$$

where, as before, $\varphi \equiv \Psi_1 - \Psi_2 = \varphi_1 - \varphi_2$ is the oscillator phase difference.

As a result, the reduced equations (5.57a) for oscillator phases yield the following generalization of Eq. (**):[31]

$$\dot{\varphi} = \xi + \frac{A_2}{2m_1\omega A_1}\mathrm{Re}\left[e^{i\varphi}\kappa(\omega)\right] - \frac{A_1}{2m_2\omega A_2}\mathrm{Re}\left[e^{-i\varphi}\kappa(\omega)\right].$$

As a sanity check, in the case of a purely real function $\kappa(\omega)$ – say, $\kappa(\omega) = \kappa = \text{const}$, as in Task (i), this equation is reduced to Eq. (**). On the other hand, for an arbitrary complex function $\kappa(\omega)$ but similar oscillators (with $A_1 = A_2$, $m_1 = m_2 \equiv m$, and $\xi_1 = \xi_2$, so $\xi = 0$)[32] it yields

$$\dot{\varphi} = \frac{1}{2m\omega}\mathrm{Re}\left[\left(e^{i\varphi} - e^{-i\varphi}\right)\kappa(\omega)\right] \equiv \xi + \frac{1}{2m\omega}\mathrm{Re}\left[2i\sin\varphi\,\kappa(\omega)\right] = -\Delta\sin\varphi, \quad \text{with } \Delta \equiv \frac{\mathrm{Im}\,\kappa(\omega)}{m\omega}. \quad (***)$$

If $\mathrm{Im}\,\kappa(\omega) \neq 0$, Eq. (***) evidently has two fixed points: $\varphi = 0$ and $\varphi = \pi$ (and of course all their physically indistinguishable cousins different from them by any phase shift multiple of 2π); however, as one may readily verify by the usual linearization of this equation, only one of them is stable:

$$\varphi = \begin{cases} 0 & \text{is stable if } \mathrm{Im}\,\kappa(\omega) > 0, \\ \pi & \text{is stable if } \mathrm{Im}\,\kappa(\omega) < 0. \end{cases}$$

Hence, depending on the type of coupling, the oscillators may phase-lock either *in-phase* or in *anti-phase*. This fact takes a very interesting form in the case of mutual phase locking in systems of $N >> 1$ similar oscillators – see the next problem.

<u>Problem 6.21</u>.* Extend Task (ii) of the previous problem to the mutual phase locking of N similar self-oscillators. In particular, explore the in-phase mode's stability for the case of so-called *global coupling* via a single force F contributed equally by all oscillators.

Solution: Naturally extending the model used in the previous problem, we may start with the system of N similar equations:

[31] Strictly speaking, this equation is valid when the characteristic bandwidth $\Delta\omega$ of the function $\kappa(\omega)$ is much larger than the locking range $|\Delta|$ (explain why!), imposing one more condition of the range smallness.

[32] In a more realistic case when the oscillators, otherwise similar, have slightly different partial frequencies Ω, so $\xi \equiv \xi_1 - \xi_2 = \Omega_2 - \Omega_1 \neq 0$, the resulting equation shows that the phase locking range width is $2|\Delta|$, where Δ is given by Eq. (***).

$$m_k\left(\ddot{q}_k + 2\delta_k \dot{q}_k + \Omega_k^2 q_k + \beta_k \dot{q}_k^3\right) = \sum_{k'=1}^{N} F_{kk'}\{q_{k'}\},$$

where the linear operators giving the interaction forces $F_{kk'}\{q_{k'}\}$ may be different for each pair of oscillators.[33] Applying the van der Pol approximation to this system in the usual way, we get the following system of the reduced equations for oscillator phases φ_k:

$$\dot{\varphi}_k = \xi_k + \frac{1}{2m_k \omega A_k} \operatorname{Re}\left[\sum_{k'=1}^{N} A_{k'} \exp\{i(\varphi_k - \varphi_{k'})\}\kappa_{kk'}(\omega)\right], \qquad (*)$$

where the complex interaction functions $\kappa_{kk'}(\omega)$ are introduced exactly as in the previous problem:

$$\kappa_{kk'}(\omega) \equiv \int_0^\infty \mathrm{K}_{kk'}(\tau) e^{i\omega\tau} d\tau, \qquad \text{where } F_{kk'}(t) = \int_0^\infty q_{k'}(t-\tau)\mathrm{K}_{kk'}(\tau) d\tau .$$

A general analysis of Eq. (*) shows[34] that the mutual self-locking is enhanced if the coupling functions depend only slowly on the "interaction distance" $|k - k'|$. The ultimate limit of this trend is the global coupling where these functions do not decay at all: $\kappa_{kk'}(\omega) = \kappa(\omega)$, i.e. all right-hand sides of the equations of motion are equal and may be rewritten as

$$F\left\{\sum_{k'=1}^{N} q_{k'}\right\},$$

i.e. describe a common force contributed equally by all oscillators, so all the interaction operators $\mathrm{K}_{kk'}$ are equal and may be stripped of their indices: $\mathrm{K}_{kk'}(\tau) \equiv \mathrm{K}(\tau)$, and $\kappa_{kk'}(\omega) \equiv \kappa(\omega)$. If, in addition, all oscillators are exactly similar, Eq. (*) reduces to

$$\dot{\varphi}_k = \frac{1}{2m\omega} \operatorname{Re}\left[\kappa(\omega)\sum_{k'=1}^{N} \exp\{i(\varphi_k - \varphi_{k'})\}\right]. \qquad (**)$$

Evidently, this system of equations has fixed points of two kinds: the *in-phase* solution with all phases φ_k equal to the same (arbitrary) phase φ_0, and the so-called *run-away* solutions with

$$\sum_{k=1}^{N} \exp\{i\varphi_k\} = 0 .$$

The simplest example of the latter solutions is the one with all phases separated by equal intervals:

$$\varphi_k = \varphi_0 + \frac{2\pi}{N} k ,$$

where the common phase φ_0 and the oscillator numbering order are arbitrary.[35]

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[33] Many (but not all!) linear systems feature the so-called *reciprocity relations* (see, e.g., EM Problems 1.18, 4.3, 7.10, and 8.2), in our current context giving $F_{kk'} = F_{k'k}$.
[34] See, e.g., Sec. 6 in A. Jain *et al.*, *Phys. Repts*. **109**, 309 (1984).
[35] If the number N is factorable, $N = N_1N_2$, another possible run-away solution is the system of N_1 groups, with N_2 equal phases in each, separated by equal intervals $2\pi/N_1$. In particular, if N is even (i.e. if $N = 2M$ with an integer M), such clustering may take the form similar to the antiphase solution studied in the previous problem, with M phases equal to some φ_0, and M other phases, to $(\varphi_0 + \pi)$.

The stability of these fixed points may be explored, as usual, by linearizing Eq. (**) with respect to small phase deviations $\tilde{\varphi}_k$:

$$\dot{\tilde{\varphi}}_k = \frac{1}{2m\omega}\operatorname{Re}\left[\kappa(\omega)\sum_{k'=1}^{N}\exp\{i(\varphi_k - \varphi_{k'})\}i(\tilde{\varphi}_k - \tilde{\varphi}_{k'})\right] \equiv -\frac{1}{2m\omega}\operatorname{Im}\left[\kappa(\omega)\sum_{k'=1}^{N}\exp\{i(\varphi_k - \varphi_{k'})\}(\tilde{\varphi}_k - \tilde{\varphi}_{k'})\right],$$

(where the phases without the tilde sign denote the fixed-point solutions, i.e. constants), and then looking for the solution of this linear system of equations in the form $\tilde{\varphi}_k = c_k e^{\lambda t}$, so the system is reduced to that of N homogeneous linear algebraic equations:

$$\lambda c_k = -\frac{1}{2m\omega}\operatorname{Im}\left[\kappa(\omega)\sum_{k'=1}^{N}\exp\{i(\varphi_k - \varphi_{k'})\}(c_k - c_{k'})\right].$$

For the (practically, most important)[36] in-phase solution $\varphi_k = \varphi_0$, these equations are reduced to

$$\lambda c_k = -\frac{1}{2m\omega}\operatorname{Im}\left[\kappa(\omega)\sum_{k'=1}^{N}(c_k - c_{k'})\right]. \qquad (***)$$

Summing up all N of these equations, and canceling similar sums on the left-hand and right-hand sides, we get simply

$$\lambda\sum_{k'=1}^{N}c_{k'} = 0.$$

This means that one root of the characteristic equation is $\lambda = 0$; it corresponds to the orbital ("indifferent") stability of the common phase φ_0, which does not affect phase locking. Of other roots, one more is also evident; it corresponds to the mode in which the sum of all distribution coefficients c_k equals zero; with this condition, Eqs. (***) yields

$$\lambda = -\frac{N}{2m\omega}\operatorname{Im}\kappa(\omega).$$

If $\operatorname{Im}\kappa(\omega)$ is negative, this root is real and positive, so the in-phase mode is unstable. Hence, just as in the case of two oscillators, the in-phase mode may be stable only if $\operatorname{Im}\kappa(\omega)$ is positive.[37]

Note that mutual phase locking may take place in self-oscillating systems of any (sometimes very counter-intuitive) origin.[38]

Problem 6.22.* Find the condition of non-degenerate parametric excitation in a system of two coupled oscillators described by Eqs. (6.5), but with a time-dependent coupling: $\kappa \to \kappa(1 + \mu\cos\omega_p t)$, with $\omega_p \approx \Omega_1 + \Omega_2$.

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[36] The in-phase mode may be practically used for getting high-power, high-frequency, narrowband electromagnetic radiation from large systems of low-power oscillators, for example, Josephson junctions – see, e.g., the 1984 review by A. Jain *et al.*, which was cited above; for more recent (rather impressive) experimental results see U. Welp *et al.*, *Nature Photonics* **7**, 702 (2013).

[37] Additional exercises for the reader: prove that there are no other roots λ in the in-phase mode, and analyze the stability of various run-away solutions.

[38] See, for example, Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence*, Dover 2003.

Hint: Use the van der Pol method, assuming the modulation depth μ, the static coupling coefficient κ, and the detuning $\xi \equiv \omega_p - (\Omega_1 + \Omega_2)$ are all sufficiently small.

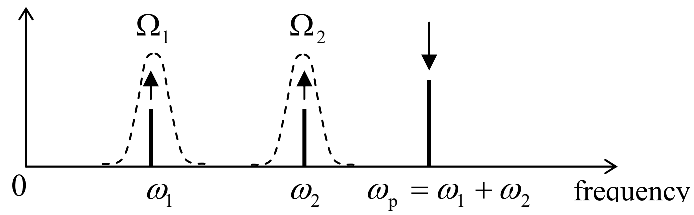
Solution: As follows from the results of Sec. 6.1 of the lecture notes, at $\kappa/m << |\Omega_2 - \Omega_1|$, the constant part of coupling does not affect the own frequencies of the system substantially, so we may neglect it. As a result, the modified Eqs. (6.5) are reduced to

$$m_1\left(\ddot{q}_1 + \Omega_1^2 q_1\right) = \mu\kappa q_2 \cos\omega_p t,$$
$$m_2\left(\ddot{q}_2 + \Omega_2^2 q_2\right) = \mu\kappa q_1 \cos\omega_p t.$$

If the modulation coefficient μ is sufficiently small, the oscillations of each of the coupled oscillators are nearly sinusoidal, so we may look for the approximate solution of this system of differential equations in the standard quasi-sinusoidal form:

$$q_{1,2} = A_{1,2}(t)\cos\Psi_{1,2}, \quad \Psi_{1,2} \equiv \omega_{1,2}t - \varphi_{1,2}(t), \quad \omega_1 + \omega_2 = \omega_p \approx \Omega_1 + \Omega_2,$$

were the speed of change of the functions $A_{1,2}$ and $\varphi_{1,2}$ in time is low – see Eq. (5.41) and its discussion, and also Fig. 6.12a, reproduced on the right. Just as in the two previous problems, the reduced equations (4.57) are still valid for each frequency component of the oscillations, provided that the right-hand side averaging in these equations is carried over a sufficiently large number of the corresponding periods $\Delta\Psi_{1,2} = 2\pi$, rather than just one of them. Concerning the particular form of the equations, from Sec. 5.5 we already know that their $\{A, \varphi\}$ form given by Eq. (5.57a) makes the analysis of the trivial fixed point ($A = A_0 = 0$) awkward. We could avoid this difficulty by using four equations Eqs. (5.57c) for $u_{1,2}$ and $v_{1,2}$, which would give us a solvable quadratic characteristic equation for λ^2. However, since the initial equations of our model are linear, it is even easier to use for this purpose Eq. (5.57b) for the complex amplitudes of the two modes. Simple averaging on their right-hand sides yields:



$$\dot{a}_{1,2} = i\xi_{1,2}a_{1,2} + i\zeta_{1,2}a_{2,1}^*, \qquad \text{with } \xi_{1,2} \equiv \omega_{1,2} - \Omega_{1,2}, \quad \zeta_{1,2} \equiv \frac{\mu\omega_{1,2}\kappa}{4m_{1,2}}.$$

We see that a combination of one of these equations (say, for $\dot{a}_1$) with the complex conjugate of the counterpart equation forms a full system. Looking for its solution in the standard form (cf. Eq. (5.80) of the lecture notes),

$$a_1(t) = c_1 e^{\lambda t}, \quad a_2^*(t) = c_2^* e^{\lambda t},$$

we get a system of two homogeneous linear equations with the following characteristic equation of self-consistency:

$$\begin{vmatrix} i\xi_1 - \lambda & i\zeta_1 \\ -i\zeta_2 & -i\xi_2 - \lambda \end{vmatrix} \equiv \lambda^2 - i\lambda\left(\xi_1 - \xi_2\right) + \xi_1\xi_2 - \zeta_1\zeta_2 = 0\,.$$

A straightforward analysis of its roots yields the following condition of instability of the trivial fixed point $a_1 = a_2^* = 0$, i.e., of the non-degenerate parametric excitation:

$$\left(\zeta_1\zeta_2\right)^{1/2} \equiv \frac{\mu\kappa}{4}\left(\frac{\omega_1\omega_2}{m_1m_2}\right)^{1/2} > \frac{|\xi_1+\xi_2|}{2}, \qquad (*)$$

which should be compared with the degenerate excitation condition

$$\frac{\mu\omega}{4} > |\xi|$$

that follows from Eq. (5.84) at $\delta = 0$. Taking into account the difference in notation, these conditions coincide if the two oscillators are similar.

The most interesting new feature of Eq. (*) is that it includes only the sum of the two detunings:

$$\xi \equiv \xi_1 + \xi_2 \equiv \left(\omega_1 - \Omega_1\right) + \left(\omega_2 - \Omega_1\right) = \omega_\text{p} - (\Omega_1 + \Omega_2)\,.$$

The physics of this result may be better revealed by writing the reduced equations in their alternative form (5.57a):

$$\begin{aligned} \dot{A}_{1,2} &= A_{2,1}\zeta_{1,2}\sin\varphi, \\ A_{1,2}\dot{\varphi}_{1,2} &= A_{1,2}\xi_{1,2} + A_{2,1}\zeta_{1,2}\cos\varphi, \quad \text{where } \varphi \equiv \varphi_1 + \varphi_2, \end{aligned} \qquad (**)$$

even if it is less convenient for the derivation of Eq. (*). We see that the right-hand sides depend on oscillation phases in only one combination, φ, so the non-degenerate excitation is not affected by slow drift of the partial phases $\varphi_{1,2}$ provided that φ is kept constant. (Such invariance reflects the insensitivity of this effect to small variations of the excitation frequencies ω_1 and ω_2 if their sum is exactly fixed, $\omega_1 + \omega_2 = \omega_\text{p}$.) If we want to obtain a closed system of the reduced equations in the $\{A, \varphi\}$ form, we need to sum up the second Eqs. (**) for the two phases, getting

$$\dot{\varphi} = \left(\xi_1 + \xi_2\right) + \frac{\mu\kappa}{4}\left(\frac{\omega_1}{m_1}\frac{A_2}{A_1} + \frac{\omega_2}{m_2}\frac{A_1}{A_2}\right)\cos\varphi,$$

so, indeed, only the sum of ξ_1 and ξ_2 is important for the non-degenerate excitation.

<u>Problem 6.23</u>. Show that the cubic nonlinearity of the type αq^3 indeed enables the parametric interaction ("four-wave mixing") of oscillations with incommensurate frequencies related by Eq. (6.92a) of the lecture notes.

Solution: Let our process $q(t)$ be a sum of intensive sinusoidal oscillations of frequency ω and certain small oscillations $\tilde{q}(t)$:

$$q(t) = A\cos\Psi + \tilde{q}(t), \qquad \text{with } \Psi = \omega t - \varphi, \quad \text{and } |\tilde{q}| << A.$$

Plugging this expression into the given nonlinear term, and expanding the result into the Taylor series in small oscillations, we get

$$q^3 = \left[A\cos\Psi + \tilde{q}(t)\right]^3 = A^3\cos^3\Psi + 3A^2\cos^2\Psi\,\tilde{q}(t) + \ldots.$$

Let us spell out the second term, proportional to the weak oscillations' magnitude:

$$3A^2\cos^2\Psi\,\tilde{q}(t) = \frac{3}{2}A^2(1+\cos 2\Psi)\tilde{q}(t) \equiv \frac{3}{2}A^2\tilde{q}(t) + \frac{3}{2}A^2\cos(2\omega t - 2\varphi)\tilde{q}(t).$$

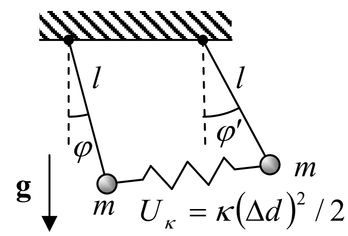
The last term of this expression has a structure similar to Eq. (5.109) of the lecture notes and describes a periodic modulation of the system's parameter with the frequency $\omega_p = 2\omega$. Now repeating the argumentation of Sec. 6.7 of the lecture notes (see Eqs. (6.89) and their discussion), we may conclude that the parametric energy exchange between the strong and weak oscillations is indeed possible at the conditions (6.90), which now take the following form:

$$2\omega = \omega_1 \pm \omega_2 ,$$

where ω_1 and ω_2 are (possibly, incommensurate) frequencies of components of the function $\tilde{q}(t)$. As was discussed in Sec. 6.7, and proved in the solution of the previous problem, the parametric excitation of oscillations ω_1 and ω_2 is possible (at a sufficiently deep parameter modulation, i.e., in our case at a sufficiently large product αA^2), for this frequency relation with the upper sign.

Problem 6.24. In the first nonvanishing approximation in small oscillation amplitudes, calculate their effect on the frequencies of the double-pendulum system that was the subject of Problem 1.

Solution: In order to describe nonlinear effects, we need to derive the equations of motion of this system (see the figure on the right) without the linear approximation made in the solutions of the listed problems. In terms of the angular deviations φ and φ' of the pendula from their equilibrium (vertical) positions, the kinetic energy of the system remains the same:



$$T = \frac{ml^2}{2}\left(\dot{\varphi}^2 + \dot{\varphi}'^2\right),$$

but its potential energy should now be calculated for arbitrary displacements of the pendula:

$$\begin{aligned} U = U_\kappa + U_g &= \frac{\kappa}{2}(\Delta d)^2 + mg(y + y') = \frac{\kappa}{2}\left[(\Delta x)^2 + (\Delta y)^2\right] + mg(\Delta y + \Delta y') \\ &= \frac{\kappa}{2} l^2 \left[(\sin\varphi - \sin\varphi')^2 + (\cos\varphi - \cos\varphi')^2\right] - mgl(\cos\varphi + \cos\varphi') \\ &\equiv -\kappa l^2 \cos(\varphi - \varphi') - mgl(\cos\varphi + \cos\varphi') + \text{const.} \end{aligned}$$

(As a reminder, in this model, the spring is not stretched at $\varphi = \varphi' = 0$, and hence at any $\varphi = \varphi'$.)

Now constructing the Lagrangian function $L \equiv T - U$, and writing the Lagrangian equations for the generalized coordinates φ and φ', we get the following equations of motion:

$$\ddot{\varphi} + \Omega^2 \sin\varphi + \omega_0^2 \sin(\varphi - \varphi') = 0, \qquad \ddot{\varphi}' + \Omega^2 \sin\varphi' - \omega_0^2 \sin(\varphi - \varphi') = 0, \qquad (*)$$

where, as in the model solution of Problem 1, $\Omega^2 \equiv g/l$ and $\omega_0^2 \equiv \kappa/m$. As a sanity check, in the linear approximation in $\varphi, \varphi' \to 0$, these equations are reduced to the ones derived in the solutions of Problems 1.8 and 2.9, and discussed in the solution of Problem 6.1. As was shown in the last solution, in the "soft" mode of small oscillations of the system, with frequency $\omega_- = \Omega$, the coordinates φ and φ'

oscillate in phase: $\varphi(t) - \varphi'(t) = 0$, while in the "hard" mode, with frequency $\omega_+ = (\Omega^2 + 2\omega_0^2)^{1/2}$, they oscillate in antiphase: $\varphi(t) + \varphi'(t) = 0$. Hence, if we transfer to new variables, defined, for example,[39] as

$$\varphi_+ \equiv \varphi - \varphi', \quad \varphi_- \equiv \frac{\varphi + \varphi'}{2}, \quad \text{so that} \quad \varphi = \varphi_- + \frac{\varphi_+}{2}, \quad \varphi' = \varphi_- - \frac{\varphi_+}{2},$$

then in each of the small oscillation modes, only one of $\varphi_\pm$ is different from zero. In these variables, Eqs. (*) become

$$\ddot{\varphi}_+ + 2\Omega^2 \sin\frac{\varphi_+}{2}\cos\varphi_- + 2\omega_0^2 \sin\varphi_+ = 0, \qquad \ddot{\varphi}_- + \Omega^2 \cos\frac{\varphi_+}{2}\sin\varphi_- = 0.$$

This (within our model, exact) system of coupled nonlinear equations defies general analytical solution, so let us simplify it by using the given condition $\varphi_\pm \to 0$. Expanding all nonlinear terms of the equations into the Taylor series in these small variables, and keeping only the first nonvanishing nonlinear terms, we get

$$\ddot{\varphi}_+ + \Omega^2\left(\varphi_+ - \frac{\varphi_+\varphi_-^2}{2} - \frac{\varphi_+^3}{24}\right) + 2\omega_0^2\left(\varphi_+ - \frac{\varphi_+^3}{6}\right)\sin\varphi_+ = 0, \quad \ddot{\varphi}_- + \Omega^2\left(\varphi_- - \frac{\varphi_-\varphi_+^2}{8} - \frac{\varphi_-^2}{6}\right) = 0. \quad (**)$$

Now we may apply to this system the harmonic balance method discussed in Sec. 6.2 of the lecture notes, by looking for its solution in the form

$$\varphi_\pm(t) = A_\pm \cos\Psi_\pm + \text{other harmonics}, \qquad \text{where } \Psi_\pm = \omega_\pm t - \theta_\pm,$$

and neglecting the small components with other frequencies in the already small nonlinear terms $O(\varphi_\pm^3)$. Then these nonlinear terms become

$$\varphi_\pm^3 = A_\pm^3 \cos^3\Psi_\pm \equiv A_\pm^3\left(\frac{3}{4}\cos\Psi_\pm + \frac{1}{4}\cos 3\Psi_\pm\right),$$
$$\varphi_\pm\varphi_\mp^2 = A_\pm A_\mp^2 \cos\Psi_\pm \cos^2\Psi_\mp \equiv A_\pm A_\mp^2\left(\frac{1}{2}\cos\Psi_\pm + \frac{1}{4}\cos\Psi_\pm \cos 2\Psi_\mp\right).$$

The last terms of these expressions have only components with frequencies $3\omega_\pm$ and $2\omega_\mp \pm \omega_\pm$. Assuming that none of them comes very close to our basic frequencies, $\omega_\pm$,[40] these terms do not participate in the balance of the basic frequency terms in Eqs. (**):

$$A_+ \cos\Psi_+\left[-\omega_+^2 + \Omega^2\left(1 - \frac{A_-^2}{4} - \frac{A_+^2}{32}\right) + 2\omega_0^2\left(1 - \frac{A_+^2}{8}\right)\right] = 0, \quad A_- \cos\Psi_-\left[-\omega_-^2 + \Omega^2\left(1 - \frac{A_+^2}{16} - \frac{A_-^2}{8}\right)\right] = 0.$$

In order for these relations to be valid at all times, i.e. for any $\Psi_\pm$, the expressions inside both square brackets should equal zero, giving us the following final result for oscillation frequencies:

$$\omega_+^2 = \Omega^2\left(1 - \frac{A_-^2}{4} - \frac{A_+^2}{32}\right) + 2\omega_0^2\left(1 - \frac{A_+^2}{8}\right), \qquad \omega_-^2 = \Omega^2\left(1 - \frac{A_+^2}{16} - \frac{A_-^2}{8}\right).$$

---

[39] The common coefficients in these definitions are arbitrary and are selected here in a way to make the forthcoming calculations as compact as possible.

[40] Each of such coincidences requires a separate analysis, but note that they are impossible in the most interesting limit of weak coupling $\omega_0 << \Omega$ when the mode frequencies $\omega_\pm$ are relatively close to each other.

In the limit $A_\pm \to 0$, the frequencies tend to the values found in Problem 1, but an increase of oscillation amplitudes leads to a reduction of both frequencies, just as in a single pendulum – see, e.g., Fig. 5.4 and its discussion in Sec. 5.2 of the lecture notes.

Note that this system would be more phenomena-rich if the pendula nonlinearities did not have the sign antisymmetry[41] – see the discussion in Sec. 6.7 of the lecture notes and the model solution of Problem 5.8. Unfortunately, a quantitative discussion of these interesting (and practically useful) effects is beyond the framework of this course.

Problem 6.25. Calculate the velocity of small transverse waves propagating on a thin, planar, elastic membrane, with a constant mass $\mathscr{m}$ per unit area, pre-stretched with force τ per unit width.

Solution: In order to re-use the analysis in Sec. 6.3 of the lecture notes, just before Eq. (6.24), it is convenient to model the membrane as a square lattice ("mesh") of particles of mass m, connected, in both directions, with strings stretched with an equal force $\mathscr{T}$ each. Let the mesh side (i.e. the distance between the adjacent particles) be d. Then, as it was argued in Sec. 6.3, the transverse force F, trying to return a certain particle to its equilibrium position on the initial plane of the membrane, has the following contribution from the tension in one longitudinal direction: $\mathscr{T}[(q_{j+1} - q_j) - (q_j - q_{j-1})]/d$, where the index j numbers the particles in this direction. The contribution to this force from the tension in the perpendicular longitudinal direction is evidently similar, with j replaced by some index numbering the particles in that direction.

As was discussed at the beginning of Sec. 6.4 of the lecture notes, at the transfer to the continuum (possible in the limit $d \to 0$), each of these sums may be replaced by the corresponding second derivative, for example,

$$\left(q_{j+1} - q_j\right) - \left(q_j - q_{j-1}\right) \to \frac{\partial q}{\partial z}\Big|_{z=z_+} d - \frac{\partial q}{\partial z}\Big|_{z=z_-} d \equiv \left(\frac{\partial q}{\partial z}\Big|_{z=z_+} - \frac{\partial q}{\partial z}\Big|_{z=z_-} \right) d \to \frac{\partial^2 q}{\partial z^2}\Big|_{z=z_j} d^2,$$

where z is the longitudinal coordinate in this direction, and $z_\pm \equiv (z_{j\pm 1} + z_j)/2$. Adding a similar contribution from the other coordinate (say, y), we get the net transverse force

$$F = \mathscr{T} d \left(\frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 q}{\partial z^2} \right) \equiv \mathscr{T} d \nabla_2^2 q,$$

so the 2nd Newton law in the transverse direction is

$$m \frac{\partial^2 q}{\partial t^2} = \mathscr{T} d \nabla_2^2 q.$$

Now noticing that in this model, the membrane's mass $\mathscr{m}$ per unit area is m/d^2, and its tension τ per unit width is $\mathscr{T}/d$, we get the 2D wave equation (6.91),

$$\frac{1}{v_t^2} \frac{\partial^2 q}{\partial t^2} - \nabla_2^2 q = 0,$$

with a very simple expression for the transverse wave velocity:

[41] That could be the case, e.g., if the spring was pre-stretched, so the equilibrium corresponded to $\varphi, \varphi' \neq 0$.

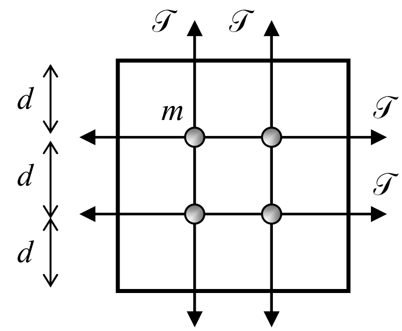
$$v_t = \left(\frac{\tau}{m}\right)^{1/2}.$$

This result, with $\tau = 2\gamma$, is also valid for a stable thin film of a fluid (e.g., of a soap solution) with the surface tension coefficient γ – see, e.g., Sec. 8.2 of the lecture notes.

Problem 6.26. A membrane discussed in the previous problem is stretched on a thin but firm plane frame of area $a\times a$.

(i) Calculate the frequency spectrum of small transverse standing waves in the system; sketch a few lowest wave modes.

(ii) Compare the results with those for a discrete-point analog of this system, with four particles of equal masses m, connected with light flexible strings that are stretched, with equal tensions $\mathscr{T}$, on a similar frame – see the figure on the right.



Hint: The frames do not allow the membrane edges/string ends to deviate from their planes.

Solutions:

(i) The 2D wave equation describing the membrane's transverse motion was derived in the solution of the previous problem:

$$\frac{1}{v_t^2}\frac{\partial^2 q}{\partial t^2} = \nabla_2^2 q, \qquad \text{with } v_t \equiv \left(\frac{\tau}{m}\right)^{1/2}.$$

(The constant v_t has the physical sense of the velocity of dispersion-free propagation of transverse waves on the membrane.) In our case, this equation has to be solved with the boundary condition $q|_{\text{on frame}} = 0$. With the x- and y-axes directed along the edges of the frame, with the origins in one of its corners, this boundary condition becomes

$$q = 0, \quad \text{at } \begin{cases} 0 \le x \le a, & \text{for } y = 0, a, \\ 0 \le y \le a, & \text{for } x = 0, a. \end{cases} \qquad (*)$$

This boundary problem may be readily solved by using the variable separation method, i.e. by looking for the solution in the form

$$q(x,y,t) = \sum_{n,m} c_{n,m} q_{n,m}, \qquad \text{with } q_{n,m} = X_n(x) Y_m(y) T_{n,m}(t),$$

where $c_{n,m}$ are constant coefficients determined by initial conditions. Plugging the partial solution $q_{n,m}$ into our wave equation, and dividing both sides by $X_n Y_m T_{n,m}$, we get

$$\frac{1}{v_t^2}\frac{1}{T_{n,m}}\frac{d^2 T_{n,m}}{dt^2} = \frac{1}{X_n}\frac{d^2 X_n}{dx^2} + \frac{1}{Y_m}\frac{d^2 Y_m}{dy^2}. \qquad (**)$$

This equality may hold for all x, y, and t only if each of its three terms is a constant. Calling these constants, respectively, k^2, k_x^2, and k_y^2, so Eq. (**) becomes

$$k^2 = k_x^2 + k_y^2,$$

we get similar linear ordinary differential equations for the functions $T_{n,m}$, X_n, and Y_m, whose solutions are sinusoidal functions of, respectively, ωt, $k_x x$, and $k_y y$, where

$$\omega = kv_\mathrm{t}.$$

Now requiring all functions X_n and Y_m to satisfy the boundary conditions following from Eq. (*):

$$X_n(0) = X_n(a) = 0, \qquad Y_m(0) = Y_m(a) = 0,$$

we get the following spectra of the corresponding wave numbers:

$$k_x = \frac{\pi}{a} n, \quad k_y = \frac{\pi}{a} m, \qquad \text{with } n, m = 1, 2, ...,$$

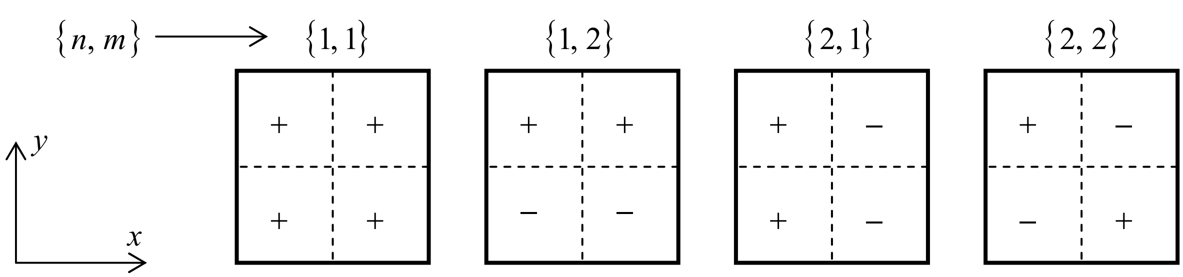
so the partial solutions of the problem (each describing a specific spatial mode, in our case a specific standing wave on the membrane) are

$$q_{n,m} = \sin\frac{\pi x n}{a} \sin\frac{\pi y m}{a} \cos(\omega_{n,m} t + \text{const}),$$

where $\omega_{n,m}$ is the mode's frequency. The spectrum of these frequencies follows from the above relations:

$$\omega_{n,m} = kv = \left(k_x^2 + k_y^2\right)^{1/2} v = \omega_0 \left(n^2 + m^2\right)^{1/2}, \qquad \text{where } \omega_0 \equiv \frac{\pi}{a} v.$$

The figure below shows the first four modes, in the order of increasing frequencies, by the signs of the functions $q_{n,m}$ in each of the four quadrants, at a certain time instant. Note that the modes {1, 2} and {2, 1} have the same frequency $\omega_{1,2} = \omega_{2,1} = \sqrt{5}\omega_0$, but differ by their orientation on the membrane.



(ii) The equation describing the motion of particles in the discrete version of the system may be derived similarly to that of the membrane. This analogy is more complete if we start not with the small, four-particle system specified in the assignment, but with a large system of similar particles and strings, with the period d in each of the directions x, y. As was shown in Sec. 6.3 of the lecture notes, for small transverse displacements (with $|q| << d$), the net force provided by the x-oriented string is

$$F_z = \mathscr{T} \frac{q_{i+1,j} + q_{i-1,j} - 2q_{i,j}}{d},$$

where the indices number the particle's Cartesian coordinates (in the units of d). Now adding a similar contribution from the y-oriented string, we get the full transverse force exerted on the particle:

$$\left(F_z\right)_{i,j} = \mathscr{T} \frac{q_{i+1,j} + q_{i-1,j} + q_{i,j+1} + q_{i,j+1} - 4q_{i,j}}{d},$$

so the 2nd Newton's law for it yields

$$m\frac{d^2 q_{i,j}}{dt^2} = \frac{\mathscr{T}}{d}\left(q_{i-1,j} + q_{i+1,j} + q_{i,j-1} + q_{i,j+1} - 4q_{i,j}\right). \qquad (***)$$

This equation has to be solved with the same boundary conditions $z(t)|_{\text{on frame}} = 0$ as for the membrane.

Solving a large system of such mutually-coupled ordinary partial equations analytically may be hard. However, the small size and high symmetry of our system (of just four particles) make such a solution very easy. Indeed, we may consider the points where the strings contact the frame as additional particles, but with $q = 0$, so they give zero contributions to the right-hand side of Eqs. (***) written for the adjacent internal particles. For example, for the top-left particle (assigning it the numbers $i = 1$, $j = 1$, with the natural sequential numbering of other particles) we may write

$$m\frac{d^2 q_{1.1}}{dt^2} = \frac{\mathscr{T}}{d}\left(0 + q_{2,1} + 0 + q_{1,2} - 4q_{1,1}\right), \qquad (****)$$

with similar simplifications for the other three particles. Plugging into the resulting system of four linear ordinary differential equations the oscillatory solution $q_{i,j} = c_{i,j}\exp\{-i\omega t\}$, we get a system of four algebraic homogeneous linear equations for four distribution coefficients $c_{i,j}$, whose condition of consistency is

$$\begin{vmatrix} -4+\lambda & 1 & 1 & 0 \\ 1 & -4+\lambda & 0 & 1 \\ 1 & 0 & -4+\lambda & 1 \\ 0 & 1 & 1 & -4+\lambda \end{vmatrix} = 0, \qquad \text{where } \lambda \equiv \frac{m\omega^2}{\mathscr{T}/d}.$$

However, instead of trying to solve this equation and then calculate the corresponding distribution coefficients, in our highly symmetric case, it is easier to assume that the four different oscillation modes of the discrete-particle system[42] have the same symmetry as the lowest modes of the membrane oscillations, sketched above. This means the following relations between the particle oscillations:

mode {1, 1}: $q_{11} = q_{12} = q_{21} = q_{22}$,

mode {1, 2}: $q_{11} = -q_{12} = q_{21} = -q_{22}$,

mode {2, 1}: $q_{11} = q_{12} = -q_{21} = -q_{22}$,

mode {2, 2}: $q_{11} = -q_{12} = -q_{21} = q_{22}$.

Due to this symmetry, the corresponding oscillation frequencies may be calculated using just one equation, for example, Eq. (****) for $q_{1,1}$. For example, with the substitution of q_{12} and q_{21} for the {1, 1} mode, we get simply

$$m\frac{d^2 q_{1.1}}{dt^2} = \frac{\mathscr{T}}{d}\left(0 + q_{1,1} + 0 + q_{1,1} - 4q_{1,1}\right), \qquad \text{i.e. } m\frac{d^2 q_{1.1}}{dt^2} + 2\frac{\mathscr{T}}{d}q_{1,1} = 0.$$

This is just the usual equation of a linear oscillator with the frequency

[42] As a reminder, the general theory of oscillations in a coupled linear system of N particles (discussed in Sec. 6.2 of the lecture notes) tells us that it may have only N normal modes.

$$\omega_{11} = \sqrt{2}\omega_0, \qquad \text{where } \omega_0 \equiv \left(\frac{\mathscr{T}}{md}\right)^{1/2}.$$

Very similarly, for the {1, 2} mode, the same Eq. (****) reduces to

$$m\frac{d^2 q_{1.1}}{dt^2} = \frac{\mathscr{T}}{d}\left(0 + q_{1,1} + 0 - q_{1,1} - 4q_{1,1}\right), \qquad \text{i.e. } m\frac{d^2 q_{1.1}}{dt^2} + 4\frac{\mathscr{T}}{d}q_{1,1} = 0.$$

The similar calculation for the mode {2, 1} yields the same final equation for q_{11}, so the frequencies of these two modes are equal (just as they are for the membrane):

$$\omega_{12} = \omega_{21} = 2\omega_0.$$

Finally, for the {2, 2} mode we get

$$m\frac{d^2 q_{1.1}}{dt^2} = \frac{\mathscr{T}}{d}\left(0 - q_{1,1} + 0 - q_{1,1} - 4q_{1,1}\right), \qquad \text{i.e. } m\frac{d^2 q_{1.1}}{dt^2} + 6\frac{\mathscr{T}}{d}q_{1,1} = 0,$$

giving the highest frequency of the spectrum:

$$\omega_{22} = \sqrt{6}\omega_0.$$

(As a sanity check, each of the values $\lambda = \{2, 4, 6\}$ corresponding to these frequencies, indeed turns the system's determinant to zero.)

To compare the results for the continuous system (the membrane) and the system of the discrete particles, we may look at the ratios of their frequencies:

$$\omega_{11}/\omega_{12}/\omega_{21}/\omega_{22} = \left(\sqrt{2}/\sqrt{5}/\sqrt{5}/\sqrt{8}\right)_{\text{continuous}} \quad \text{vs} \quad \left(\sqrt{2}/\sqrt{4}/\sqrt{4}/\sqrt{6}\right)_{\text{discrete}}.$$

So the ratios are reasonably close, with a somewhat larger deviation for the highest mode {2, 2}. Moreover, comparing the systems, we may assume that they should have the closest properties at the following relations of their parameters:

$$a = 3d, \quad \tau = \frac{\mathscr{T}}{d}, \quad \mathscr{m} = \frac{m}{d^2}.$$

Plugging these relations into the corresponding definitions of ω_0, we get

$$\omega_0 = \frac{\pi}{3}\left(\frac{\mathscr{T}}{md}\right)^{1/2} \approx 1.047\left(\frac{\mathscr{T}}{md}\right)^{1/2}_{\text{cntinuous}} \quad \text{vs } 1.000\left(\frac{\mathscr{T}}{md}\right)^{1/2}_{\text{discrete}}.$$

So, the difference between the values of this frequency scale (and hence of the fundamental frequencies $\omega_{1,1} = \sqrt{2}\omega_0$) in the two systems is below 5%.

Chapter 7. Deformations and Elasticity

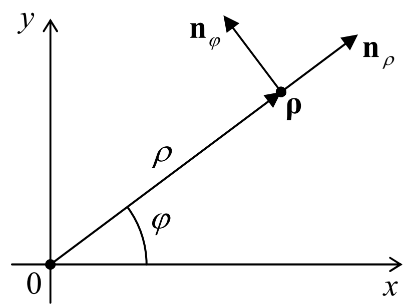
Problem 7.1. Derive Eqs. (7.16) of the lecture notes.

Hint: Besides basic calculus and the definition of the cylindrical coordinates, you may like to use Eq. (4.7) with $d\boldsymbol{\varphi} = (d\varphi)\mathbf{n}_z$.

Solution:[43] According to the definition of the cylindrical coordinates $\{\rho, \varphi, z\}$, the z-coordinate is essentially a Cartesian one, so the last of Eqs. (7.16), $s_{zz} = \partial q_z/\partial z$, just repeats Eq. (7.9b) for $r_{j'} = r_j = z$. So, let us derive just the first two of these formulas,

$$s_{\rho\rho} = \frac{\partial q_\rho}{\partial \rho}, \quad s_{\varphi\varphi} = \frac{1}{\rho}\left(q_\rho + \frac{\partial q_\varphi}{\partial \varphi}\right),$$

for clarity taking $q_z = 0$ and $z = \text{const}$, i.e. essentially using the polar coordinates $\{\rho, \varphi\}$ – see the figure on the right. In these coordinates, the radius vector of a geometric point is a two-dimensional vector:



$$\boldsymbol{\rho} = \rho \mathbf{n}_\rho . \qquad (*)$$

This means that in the local orthogonal 2D coordinates with their axes directed along the unit vectors $\mathbf{n}_\rho \equiv \boldsymbol{\rho}/\rho$ and $\mathbf{n}_\varphi \perp \mathbf{n}_\rho$ (see the figure again), the vector $\boldsymbol{\rho}$ has just one scalar component. However, an infinitesimal but otherwise arbitrary change $d\boldsymbol{\rho}$ of this vector generally has both components. Indeed, calculating this differential, on the right-hand side of Eq. (*), we need to differentiate not only the scalar ρ, but also the unit vector $\mathbf{n}_\rho$, because a shift $d\boldsymbol{\rho}$ generally changes the angle φ, and hence the direction of $\mathbf{n}_\rho$:

$$d\boldsymbol{\rho} = (d\rho)\mathbf{n}_\rho + \rho\, d\mathbf{n}_\rho . \qquad (**)$$

The last differential may be calculated by either differentiating the basic formulas relating the polar coordinates and the Cartesian coordinates $\{x, y\}$:

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi,$$

or even simpler, by considering $\mathbf{n}_\rho$, $\mathbf{n}_\varphi$, and $\mathbf{n}_z = \mathbf{n}_\rho \times \mathbf{n}_\varphi$ a usual set of three mutually perpendicular unit vectors $\mathbf{n}_j$, $\mathbf{n}_{j'}$, and $\mathbf{n}_{j''}$, and applying to them Eqs. (4.7) with $d\boldsymbol{\varphi} = (d\varphi)\,\mathbf{n}_z$:

$$d\mathbf{n}_\rho = d\boldsymbol{\varphi}\times\mathbf{n}_\rho = (d\varphi)\mathbf{n}_z\times\mathbf{n}_\rho = (d\varphi)\mathbf{n}_\varphi, \qquad d\mathbf{n}_\varphi = d\boldsymbol{\varphi}\times\mathbf{n}_\varphi = (d\varphi)\mathbf{n}_z\times\mathbf{n}_\varphi = -(d\varphi)\mathbf{n}_\rho . \qquad (***)$$

With the first of these relations, Eq. (**) becomes

$$d\boldsymbol{\rho} = (d\rho)\mathbf{n}_\rho + (\rho\, d\varphi)\mathbf{n}_\varphi .$$

[43] I apologize to the readers for whom this solution is too detailed/slow-paced, but when I taught this material at Stony Brook, I ran into some quite strong students who could not solve this problem.

This (perhaps, obvious) relation shows that in the polar coordinates, the role of the Cartesian coordinate differentials dr_j participating in Eq. (7.9b) is played by $d\rho$ and $\rho\, d\varphi$. Hence the diagonal elements of the strain tensor we are looking for may be expressed as[44]

$$s_{\rho\rho} = \left(\frac{\partial \mathbf{q}}{\partial \rho}\right)_\rho \equiv \frac{\partial \mathbf{q}}{\partial \rho}\cdot \mathbf{n}_\rho, \quad s_{\varphi\varphi} = \left(\frac{\partial \mathbf{q}}{\rho\,\partial \varphi}\right)_\varphi \equiv \frac{1}{\rho}\frac{\partial \mathbf{q}}{\partial \varphi}\cdot \mathbf{n}_\varphi. \qquad (****)$$

What remains is to express the involved derivatives via the scalar components of the deformation vector $\mathbf{q} = q_\rho \mathbf{n}_\rho + q_\varphi \mathbf{n}_\varphi$, in the same polar coordinates. With the two relations (***) ready, this is easy:

$$\frac{\partial \mathbf{q}}{\partial \rho} = \frac{\partial q_\rho}{\partial \rho}\mathbf{n}_\rho + q_\rho \frac{\partial \mathbf{n}_\rho}{\partial \rho} + \frac{\partial q_\varphi}{\partial \rho}\mathbf{n}_\varphi + q_\varphi \frac{\partial \mathbf{n}_\varphi}{\partial \rho} = \frac{\partial q_\rho}{\partial \rho}\mathbf{n}_\rho + 0 + \frac{\partial q_\varphi}{\partial \rho}\mathbf{n}_\varphi + 0,$$

$$\frac{\partial \mathbf{q}}{\partial \varphi} = \frac{\partial q_\rho}{\partial \varphi}\mathbf{n}_\rho + q_\rho \frac{\partial \mathbf{n}_\rho}{\partial \varphi} + \frac{\partial q_\varphi}{\partial \varphi}\mathbf{n}_\varphi + q_\varphi \frac{\partial \mathbf{n}_\varphi}{\partial \varphi} = \frac{\partial q_\rho}{\partial \varphi}\mathbf{n}_\rho + q_\rho \mathbf{n}_\varphi + \frac{\partial q_\varphi}{\partial \varphi}\mathbf{n}_\varphi - q_\varphi \mathbf{n}_\rho,$$

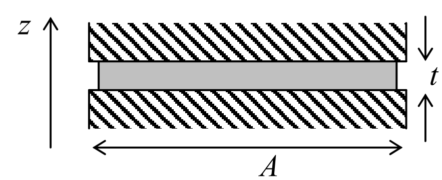
so

$$\frac{\partial \mathbf{q}}{\partial \rho}\cdot \mathbf{n}_\rho = \frac{\partial q_\rho}{\partial \rho}, \qquad \frac{\partial \mathbf{q}}{\partial \varphi}\cdot \mathbf{n}_\varphi = q_\rho + \frac{\partial q_\varphi}{\partial \varphi}.$$

Plugged into Eq. (****), these expressions give the formulas we needed to prove.

Note that we have done more than what we were asked for: the obtained expressions for $\partial\mathbf{q}/\partial\rho$ and $\partial\mathbf{q}/\partial\varphi$ contain all the terms necessary to spell out the off-diagonal elements of the strain tensor as well. Moreover, the same expressions, with $\mathbf{q}$ replaced with an arbitrary vector function $\mathbf{f}$, enable a one-step proof of MA Eq. (10.4) – a useful exercise for those who want to brush up on their vector calculus. Finally, the derivation of Eqs. (7.17) for the strain tensor in the spherical coordinates may be carried out in absolutely the same way, though it gives somewhat bulkier formulas.

Problem 7.2. A uniform thin sheet of an isotropic elastic material, of thickness t and area $A >> t^2$, is compressed by two plane, parallel, broad, rigid surfaces – see the figure on the right. Assuming that there is no slippage between the sheet and the surfaces, calculate the relative compression ($-\Delta t/t$) as a function of the compressing force. Compare the result with that for the tensile stress calculated in Sec. 7.3 of the lecture notes.



Solution: Let the z-axis be directed across the sheet – see the figure above. Due to the system's geometry and the no-slippage condition, the sheet cannot shrink substantially in either of the transverse directions (x and y), so we may take $s_{xx} = s_{yy} = 0$ and Tr (s) = $s_{zz} = \Delta t/t$. Plugging these relations into Hooke's law in the form of Eq. (7.49a), we get the following expression for the applied force $F = -\sigma_{zz}A$:

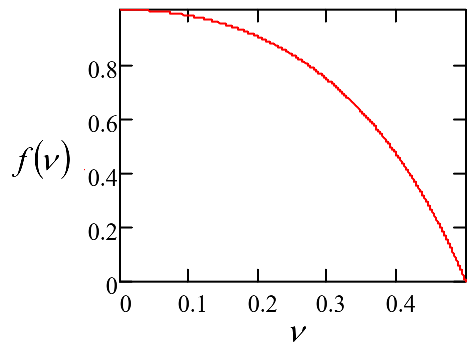
$$F = -\frac{E}{1+\nu}\left(1 + \frac{\nu}{1-2\nu}\right)s_{zz}A = -\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\frac{\Delta t}{t}A,$$

[44] Note that using the component index in these relations for the numerators only rather than for the whole fractions would be an error, giving a wrong result for $s_{\varphi\varphi}$ – exactly this point was difficult for some of my Stony Brook students, perhaps because in the Cartesian coordinates, this is permissible.

so the relative compression

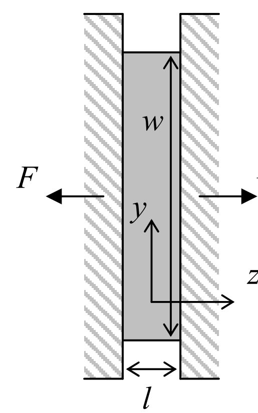
$$-\frac{\Delta t}{t}=\frac{1}{E}\frac{F}{A}f(\nu), \qquad \text{with } f(\nu)\equiv\frac{(1+\nu)(1-2\nu)}{(1-\nu)}. \qquad (*)$$

This result is substantially different from Eq. (7.45) describing the longitudinal deformation at the tensile stress experiment shown in Fig. 7.6 of the lecture notes. Namely, since in the realistic range of the Poisson's ratio values, $0 < \nu < ½$, the factor $f(\nu)$ in Eq. (*) is always less than 1 (see its plot in the figure on the right), the longitudinal deformation is always *smaller* than that at the tensile stress, at the same applied force per unit area. Physically, the difference is due to the impossibility for the sheet to swell/shrink in lateral dimensions, because of the no-slippage condition. This is why only this situation, rather than the tensile stress, deserves the name of a purely one-dimensional deformation.



Problem 7.3. Two opposite edges of a thin but wide sheet of an isotropic elastic material are clamped in two rigid, plane, parallel walls that are pulled apart with force F, along the sheet's length l. Find the relative extension $\Delta l/l$ of the sheet in the direction of the force and its relative compression $\Delta t/t$ in the perpendicular direction, and compare the results with Eqs. (7.45)-(7.46) of the lecture notes for the tensile stress and with the solution of the previous problem.

Solution: Introducing the Cartesian coordinates as shown in the figure on the right, for our case $t << l << w$, we may immediately write $s_{yy} = 0$ (because the sheet cannot shrink along the y-axis, due to its large width), $\sigma_{zz} = F/A$ (describing the uniform distribution of the applied force over the sheet's cross-section of area $A = wt$), and $\sigma_{xx} = 0$ (no forces in the x-direction, because of free horizontal surfaces and small thickness of the sheet). Plugging these relations into Hooke's law in the form of Eqs. (7.49a) written for the two remaining diagonal elements of the stress tensor, we get a system of two linear equations for s_{xx} and s_{zz}:



$$\frac{E}{1+\nu}\left[s_{zz}+\frac{\nu}{1-2\nu}(s_{xx}+s_{zz})\right]=\frac{F}{A},$$

$$\frac{E}{1+\nu}\left[s_{xx}+\frac{\nu}{1-2\nu}(s_{xx}+s_{zz})\right]=0.$$

Its solution gives us the final answers:

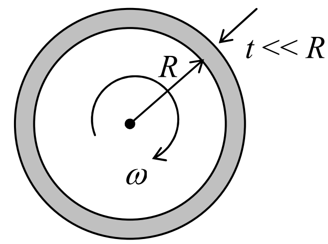
$$\frac{\Delta l}{l}\equiv s_{zz}=\frac{1-\nu^2}{E}\frac{F}{A}, \qquad \frac{\Delta t}{t}\equiv s_{xx}=-\frac{\nu(1+\nu)}{E}\frac{F}{A}\equiv-\frac{\nu}{1-\nu}s_{zz}.$$

These results are different from those for both the standard tensile stress experiment discussed in Sec. 7.3, and the purely 1D deformation discussed in Problem 1. Namely, the longitudinal extension in our current problem is intermediate: *smaller* than that at the tensile stress, but *larger* than at the purely

1D deformation (for the same applied force per unit area). The differences are due to the possibility of the sheet's shrinkage along just one of its transverse dimensions (t). Note also that the relative transverse shrinkage,

$$-\frac{s_{xx}}{s_{zz}} = \frac{\nu}{1-\nu},$$

is larger than that at the tensile stress – cf. Eq. (7.46). Physically, this increase is a partial compensation for the impossibility of shrinking in the second transverse direction.



Problem 7.4. Calculate the radial extension ΔR of a thin, long, round cylindrical pipe due to its rotation with a constant angular velocity ω about its symmetry axis (see the figure on the right), in terms of the elastic moduli E and ν.

Solution: The problem may be solved in two ways. In the first, more general approach, valid for an arbitrary pipe thickness, we may use the results of Sec. 4.6 of the lecture notes to rewrite the static equilibrium equation (7.53) in the non-inertial reference frame rotating with the pipe, by adding the bulk-distributed centrifugal inertial "forces" (4.93), $\mathbf{f}_c = -\rho\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \equiv \rho\omega^2\boldsymbol{\rho}$, to the real bulk forces $\mathbf{f}$ (which are assumed to be negligible in our current case):

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\nabla(\nabla\cdot\mathbf{q}) - \frac{E}{2(1+\nu)(1-2\nu)}\nabla\times(\nabla\times\mathbf{q}) + \rho\omega^2\boldsymbol{\rho} = 0. \qquad (*)$$

(Please mind the fonts: here ρ is the pipe material's density, while $\boldsymbol{\rho}$ is the 2D radius vector in the pipe's cross-section.) Because of the evident axial symmetry of the problem, $\mathbf{q} = \mathbf{n}_\rho q(\mathrm{\rho})$, where $\mathrm{\rho} \equiv |\boldsymbol{\rho}|$, the double cross-product in Eq. (*) vanishes, while the remaining terms have only one (radial) component. As a result, in our case, the mathematical identities MA Eqs. (10.4) and (10.2) are reduced to, respectively,

$$f \equiv \nabla\cdot\mathbf{q} = \frac{1}{\mathrm{\rho}}\frac{d}{d\mathrm{\rho}}(\mathrm{\rho} q), \quad \text{and} \quad \nabla(\nabla\cdot\mathbf{q}) \equiv \nabla f = \mathbf{n}_\rho\frac{df}{d\mathrm{\rho}} = \mathbf{n}_\rho\frac{d}{d\mathrm{\rho}}\left[\frac{1}{\mathrm{\rho}}\frac{d}{d\mathrm{\rho}}(\mathrm{\rho} q)\right],$$

so Eq. (*) yields the following ordinary differential equation:

$$\frac{d}{d\mathrm{\rho}}\left[\frac{1}{\mathrm{\rho}}\frac{d}{d\mathrm{\rho}}(\mathrm{\rho} q)\right] = -C\mathrm{\rho}, \quad \text{where } C \equiv \rho\omega^2\frac{(1+\nu)(1-2\nu)}{E(1-\nu)}.$$

It is straightforward to integrate this equation over ρ twice, just as it was done for the spherical shell problem in Sec. 7.4 of the lecture notes. The result is

$$q(\mathrm{\rho}) = -\frac{C}{8}\mathrm{\rho}^3 + a\mathrm{\rho} + \frac{b}{\mathrm{\rho}}, \qquad (**)$$

where a and b are the integration constants. Now this result and Eqs. (7.16) of the lecture notes may be readily combined to calculate diagonal elements of the strain tensor:

$$s_{\rho\rho} = \frac{dq}{d\mathrm{\rho}} = -\frac{3C}{8}\mathrm{\rho}^2 + a - \frac{b}{\mathrm{\rho}^2}, \quad s_{\varphi\varphi} = \frac{q}{\mathrm{\rho}} = -\frac{C}{8}\mathrm{\rho}^2 + a + \frac{b}{\mathrm{\rho}^2}, \quad s_{zz} = 0, \quad \text{so } \mathrm{Tr}(\mathrm{s}) = -\frac{C}{2}\mathrm{\rho}^2 + 2a.$$

Applying Hooke's law in the form given by Eq. (7.49a) of the lecture notes, we get the following radial diagonal element of the stress tensor:

$$\sigma_{\rho\rho} = \frac{E}{1+\nu}\left[\left(-\frac{3C}{8}\rho^2 + a - \frac{b}{\rho^2}\right) + \frac{\nu}{1-2\nu}\left(-\frac{C}{2}\rho^2 + 2a\right)\right].$$

Per Eq. (7.19), at both the inner ($\rho = R$) and outer ($\rho = R + t$) surfaces, $\sigma_{\rho\rho}$ should equal $-\mathcal{P}$. Our task is to calculate the deformation due to the rotation alone, so at these boundary values of ρ, we may take $\sigma_{\rho\rho} = 0$. These two boundary conditions give us a simple system of two linear equations for the constants a and b. Solving it, we get

$$a = \frac{C}{8}(3-2\nu)\left[(R+t)^2 + R^2\right], \qquad b = \frac{C}{8}\frac{3-2\nu}{1-2\nu}(R+t)^2 R^2.$$

This general result is valid for arbitrary R and t. For our simple case $t \ll R$, it reduces to

$$a = \frac{C}{8}(3-2\nu)2R^2, \qquad b = \frac{C}{8}\frac{3-2\nu}{1-2\nu}R^4.$$

Plugging these values into Eq. (**), we finally find

$$\Delta R \equiv q(R) = \frac{C}{8}R^3\left[(-1) + 2(3-2\nu) + \frac{3-2\nu}{1-2\nu}\right] \equiv \rho\omega^2 R^3 \frac{1-\nu^2}{E}. \qquad (***)$$

Curiously, the deformation does not vanish in the limit $t \to 0$, because both the centrifugal "force" f_c and the material's resistance to the force decrease in the same proportion to the pipe's thickness.

The second, much shorter way to solve this problem is to reuse the results of Problem 2. Indeed, the centrifugal "force" is equivalent to the net outward pressure $\mathcal{P}_{ef} \equiv F/A = (m/A)\omega^2 R = \rho t\omega^2 R$. Applying the arguments given in the lecture notes to derive Eq. (7.63), we can say that the pressure is equivalent to the effective force (per unit length of the pipe)

$$\frac{F}{l} = R\mathcal{P}_{ef} = \rho t\omega^2 R^2,$$

applied tangentially to each wall of the pipe. But as it was shown in the solution of Problem 3, such a force, applied to a thin sheet that cannot shrink along its widest transverse dimensions, gives the following relative extension:

$$\frac{\Delta l}{l} = \frac{1-\nu^2}{E}\frac{F}{lt},$$

where l is its length and $t \ll l$ is thickness. For a round pipe, this relative extension is equal to $\Delta R/R$, so, combining the last two formulas, we return to Eq. (***).

Problem 7.5.* A static force **F** is exerted on an inner point of a uniform and isotropic elastic body. Calculate the spatial distribution of the deformation created by the force, assuming that far from the point of its application and the points we are interested in, the body's position is kept fixed.

Solution: In a coordinate frame with its center at the force application point, it may be described by the following force density **f** participating in Eq. (7.52) of the lecture notes:

$$\mathbf{f} = \mathbf{F}\delta(\mathbf{r}),$$

where $\delta(\mathbf{r})$ is the 3D delta-function.[45] Let us look for the solution of that equation,

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\nabla(\nabla\cdot\mathbf{q}) - \frac{E}{2(1+\nu)}\nabla\times(\nabla\times\mathbf{q}) = -\mathbf{F}\delta(\mathbf{r}),$$

in the form $\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2$, where $\mathbf{q}_1$ is the solution of the following Poisson equation:

$$\nabla^2\mathbf{q}_1 = -\frac{2(1+\nu)}{E}\mathbf{F}\delta(\mathbf{r}). \qquad (*)$$

so $\mathbf{q}_2$ satisfies the following equation:

$$\nabla(\nabla\cdot\mathbf{q}_2) + (1-2\nu)\nabla^2\mathbf{q}_2 = -\nabla(\nabla\cdot\mathbf{q}_1). \qquad (**)$$

Let us start with Eq. (*). Aligning one of the coordinate axes with the direction of the vector $\mathbf{F}$, we see that the Laplace equations for other Cartesian components q_{1j} have zero right-hand sides (i.e. are the *Laplace equations*: $\nabla^2 q_{1j} = 0$). Since, according to the problem's assignment, we may neglect the deformations at $r \to \infty$, these Laplace equations have trivial solutions $q_{1j} = 0$. Hence we may take $\mathbf{q}_1 = q_1\mathbf{n}_F$, where $\mathbf{n}_F \equiv \mathbf{F}/F$ and q_1 is a scalar function satisfying the spherically symmetric equation

$$\nabla^2 q_1 = -\frac{2(1+\nu)}{E}F\delta(\mathbf{r}),$$

with spherically symmetric boundary conditions: $q_1 \to \infty$ at $r \to \infty$. Hence its solution should be spherically symmetric as well: $q_1 = q_1(r)$, and for such a function, the Laplace operator may be simplified,[46] reducing this equation to

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dq_1}{dr}\right) = -\frac{2(1+\nu)}{E}F\delta(\mathbf{r}).$$

Since the high-hand side equals zero for any $r \neq 0$, we may readily integrate this equation twice, getting

$$q_1 = \frac{C}{r}, \qquad \text{for } r > 0.$$

where C_1 is a constant. This constant may be found from the requirement that the 3D integrals of both sides of the equation over a closed volume containing the point $r = 0$, for example, a sphere of some radius $R > 0$, gives the same result:[47]

$$4\pi\int_0^R r^2 dr\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dq_1}{dr}\right) \equiv 4\pi\int_{r=0}^{r=R} d\left(r^2\frac{dq_1}{dr}\right) \equiv 4\pi\left(r^2\frac{dq_1}{dr}\right)_{r=R} \equiv -4\pi C = -\frac{2(1+\nu)}{E}F,$$

finally obtaining

$$\mathbf{q}_1 = \frac{(1+\nu)}{2\pi Er}\mathbf{F}. \qquad (***)$$

[45] See, e.g., MA Eqs. (14.5) and (14.6).

[46] See, e.g., MA Eq. (10.9) with $\partial/\partial\theta = \partial/\partial\varphi = 0$.

[47] This is a fast (and admittedly, not very rigorous :-) derivation of the mathematical equality behind the well-known Gauss law of electrostatics – see, e.g., EM Sec. 1.2.

Now proceeding to Eq. (**), let us take the curl of both sides. Since the curl of any gradient equals zero,[48] we get $\nabla\times(\nabla^2\mathbf{q}_2) \equiv \nabla^2(\nabla\times\mathbf{q}_2) = 0$. But at $r \to \infty$, we should have $\mathbf{q} \to 0$. Since, according to Eq. (***), $\mathbf{q}_1 \to 0$ in this limit as well, so both $\mathbf{q}_2$ and hence its curl should also vanish. Hence, just as for the non-aligned components q_{1j}, we may conclude that $\nabla\times\mathbf{q}_2 = 0$ everywhere. But such a zero-curl ("vortex-free") vector field may be always represented as a gradient of some scalar function: $\mathbf{q}_2 = \nabla\phi$. With this substitution, Eq. (**) becomes

$$\nabla\left(\nabla\cdot\nabla\phi\right)+\left(1-2\nu\right)\nabla^2\nabla\phi = -\nabla\left(\nabla\cdot\mathbf{q}_1\right), \quad \text{i.e.}\ \nabla\left[2\left(1-\nu\right)\nabla^2\phi+\nabla\cdot\mathbf{q}_1\right]=0\,.$$

Hence the function in the last square brackets is constant in the whole space. Since both its terms should tend to zero at $r \to \infty$, the function has to be zero everywhere, so we get

$$\nabla^2\phi = -\frac{1}{2\left(1-\nu\right)}\nabla\cdot\mathbf{q}_1 \equiv -\frac{1+\nu}{4\pi\left(1-\nu\right)E}\mathbf{F}\cdot\nabla\frac{1}{r}\cdot$$

We may rewrite this equation as

$$\nabla^2\left[\phi+\frac{1+\nu}{4\pi\left(1-\nu\right)E}\mathbf{F}\cdot\nabla\psi\right]=0,$$

where $\psi(r)$ is a spherically symmetric scalar function that satisfies the following Poisson equation:

$$\nabla^2\psi = \frac{1}{r}, \qquad \text{i.e.}\ \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right)=\frac{1}{r}, \qquad (****)$$

and argue, just as was done twice earlier in this solution, that the expression in the square brackets has to equal zero everywhere, so

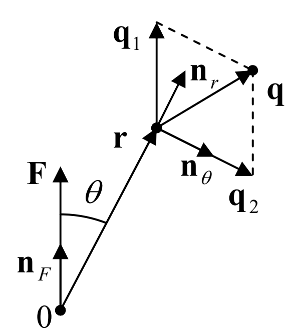
$$\phi = -\frac{1+\nu}{4\pi\left(1-\nu\right)E}\mathbf{F}\cdot\nabla\psi\,.$$

What remains is to find the function $\psi(r)$. A simple double integration of Eq. (****) yields

$$\psi = \frac{r}{2}-\frac{C_1}{r}+C_2\,,$$

where $C_{1,2}$ are the integration constants. The first of them scales an unphysical singularity at $r \to 0$, while the second one is not essential, because it gives no contribution to $\nabla\psi$. So, taking $C_1 = C_2 = 0$, we get $\nabla\psi = \mathbf{n}_r/2$ (where $\mathbf{n}_r \equiv \mathbf{r}/r$ is the unit vector directed from the origin, i.e. the force application point, toward the point $\mathbf{r}$ where the deformation is observed), so

$$\phi = -\frac{1+\nu}{8\pi\left(1-\nu\right)E}\mathbf{F}\cdot\mathbf{n}_r\,, \qquad \text{and}\ \ \mathbf{q}_2 \equiv \nabla\phi = -\frac{1+\nu}{8\pi\left(1-\nu\right)E}\nabla\left(\mathbf{F}\cdot\mathbf{n}_r\right).$$



The last gradient is most simply calculated in polar coordinates, by aligning the z-axis with the force $\mathbf{F}$. In this case, $\mathbf{F}\cdot\mathbf{n}_r = F\cos\theta$, where θ is the polar angle – see the figure on the right. Since this scalar product depends only on θ, its gradient has only one component,[49] $\mathbf{n}_\theta\,[\partial(F\cos\theta)/\partial\theta]/r = -\,\mathbf{n}_\theta\,F\sin\theta/r$. However, since in these coordinates, the vector $\mathbf{F}$ may be represented as $(\mathbf{n}_r F\cos\theta\, -\, \mathbf{n}_\theta$

[48] See, e.g., MA Eq. (11.1).

[49] See, e.g., MA Eq. (10.8), with $\partial/\partial r = \partial/\partial\varphi = 0$.

$F\sin\theta$), this simple result may be also rewritten in another form: $(\mathbf{F}\cdot\mathbf{n}_r)\mathbf{n}_r - \mathbf{F}$. Hence we may write

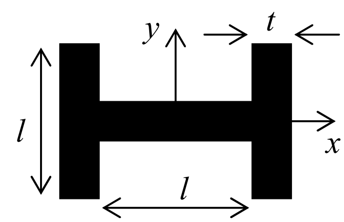
$$\mathbf{q}_2 = \frac{1+\nu}{8\pi(1-\nu)E}\frac{F\sin\theta}{r}\mathbf{n}_\theta \equiv \frac{1+\nu}{8\pi(1-\nu)E}\frac{(\mathbf{F}\cdot\mathbf{n}_r)\mathbf{n}_r - \mathbf{F}}{r}.$$

Merging this result with Eq. (***), we finally get

$$\mathbf{q} \equiv \mathbf{q}_1 + \mathbf{q}_2 = \frac{1+\nu}{8\pi(1-\nu)E}\frac{(3-4\nu)\mathbf{F} + (\mathbf{F}\cdot\mathbf{n}_r)\mathbf{n}_r}{r} \equiv \frac{1+\nu}{8\pi(1-\nu)E}\left[(3-4\nu)\mathbf{n}_F + \cos\theta\,\mathbf{n}_r\right]\frac{F}{r}.$$

Note that the scaling $q \propto 1/r$ agrees with simple physical reasoning. Indeed, in this case, according to Eq. (7.9b) of the lecture notes, the elements of the strain tensors scale as $1/r^2$. According to Hooke's law (7.32), the stress tensor elements should follow the same scaling, so the total stress force exerted on a sphere of radius r, drawn around the force application force, is independent of r – as it should be because in the static situation we have analyzed, this force has to exactly compensate the exerted force **F**.

This problem, first solved by Lord Kelvin in 1848, is important due to the mighty linear superposition principle (valid in the elastic limit): the deformation of a body, due to an *arbitrary* distributed force **f**(**r**) exerted on its interior, may be represented as a spatial integral of this function multiplied by the weight (essentially, the spatial Green's function[50]) calculated above. The next (practically, even more important) step in this field was made in 1879 by J. Boussinesq who solved a similar problem for an elastic half-space with a plane open border, with a force **F** exerted on some point of this border; this *Boussinesq solution*[51] is especially valuable for various aspects of geophysics.



Problem 7.6. A long uniform rail with the cross-section shown in the figure on the right is being bent with the same (small) torque twice: first within the xz-plane and then within the yz-plane. Assuming that $t \ll l$, find the ratio of the bending deformations in these two cases.

Solution: At thin rod bending by torque τ, the deformation q at any point is proportional to $1/R$, where R is the created curvature radius. In Sec. 7.5 of the lecture notes, it was proved that

$$\frac{1}{R} = \frac{\tau}{EI},$$

where the "moment of inertia" I is given by the integral over the cross-section's area A:

$$I = \int_A r_d^2 d^2 r$$

and r_d is the point's distance from the neutral plane. (For our symmetric geometry, the plane evidently passes through the geometric center of the cross-section, $x = y = 0$, for both bending directions.) Using the specific geometry of the rail, shown in the figure above, and the strong relation $t \ll l$, we get the following deformation ratio:

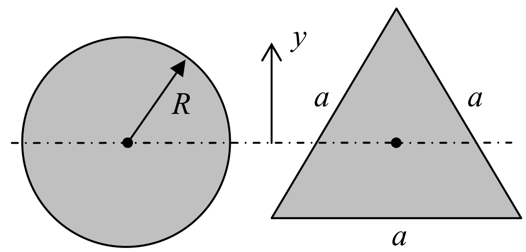
[50] Note that in our current case, the Green's function is a tensor because it relates two proportional but generally not parallel vectors **q** and **F**.

[51] It may be found, for example, in Sec. 8 of the *Theory of Elasticity* by Landau and Lifshitz – see *References*.

$$\frac{q_y}{q_x} = \frac{I_x}{I_y} \approx \frac{4t\int_0^{l/2} y^2 dy}{2t\int_0^{l/2} x^2 dx + 2l\left(\frac{l}{2}\right)^2 t} = \frac{\frac{1}{6}tl^3}{\frac{1}{12}tl^3 + \frac{1}{2}tl^3} = \frac{2}{7}.$$

We see that the rail is much more rigid when it is bent within the *xz* plane. This result explains the standard shape of the rails used on railroad tracks and in construction, though, in engineering practice, *t* is not *much* smaller than *l*.

Problem 7.7. Two thin rods of the same length and mass are made of the same isotropic and elastic material. The cross-section of one of them is a circle, while the other one is an equilateral triangle – see the figure on the right. Which of the rods is stiffer for bending along its length? Quantify the relation. Does the result depend on the bending plane's orientation?



Solution: According to the analysis in Sec. 7.5, the rod's stiffness (at a fixed applied torque) may be characterized by its curvature radius, which is, in turn, proportional to the "moment of inertia"

$$I_y = \int_A y^2 d^2 r,$$

where *y* is the distance of the point from the neutral plane that passes through the cross-section's center of mass – see the dashed line in the figure above. For a circle, we can readily get the following result:

$$I_{\text{circle}} = \int_0^R \rho d\rho \int_0^{2\pi} d\varphi\, y^2 = \int_0^R \rho d\rho \int_0^{2\pi} d\varphi (\rho \sin\varphi)^2 = \int_0^R \rho^3 d\rho \int_0^{2\pi} \sin^2\varphi\, d\varphi = \pi \int_0^R \rho^3 d\rho = \frac{\pi}{4} R^4,$$

which is of course independent of the bending direction. For an equilateral triangle with side *a*, the moment of inertia has been calculated in the solution of Problem 4.2(ii):

$$I_{\text{triangle}} = \frac{\sqrt{3}}{96} a^4,$$

and also does not depend on the direction of axis *y*, i.e. on the bending plane orientation.

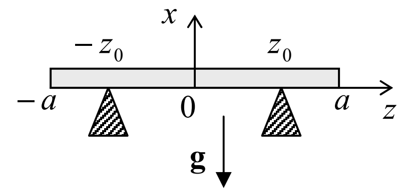
For the rods of the same mass and density, the areas of their cross-sections have to be equal:

$$\pi R^2 = \frac{\sqrt{3}}{4} a^2.$$

From here,

$$\frac{I_{\text{circle}}}{I_{\text{triangle}}} = \frac{(\pi/4) R^4}{(\sqrt{3}/96)\, a^4} = \frac{\pi}{4} \frac{96}{\sqrt{3}} \left(\frac{\sqrt{3}}{4\pi} \right)^2 \equiv \frac{9}{2\pi\sqrt{3}} \approx 0.827 < 1,$$

as could be expected, because the points of the round cross-section are, on average, closer to the neutral plane. Hence, the round cross-section gives the rod a slightly lower bending stiffness than the triangular cross-section.



Problem 7.8. A thin, uniform, initially straight elastic beam is placed on two point supports at the same height – see the figure on the right. Calculate the support placements that:

(i) ensure that the beam's ends are horizontal, and

(ii) minimize the largest deflection of the beam from the horizontal baseline.

Hint: For Task (ii), an approximate answer (with an accuracy better than 1%) is acceptable.

Solution: Due to the problem's symmetry, the optimal support placement should be also symmetric, so it is sufficient to calculate the position of just one support point – say, its distance z_0 from the beam's center 0 – see the figure above. For that, we may integrate the system of the first-order differential equations given by Eqs. (7.74)-(7.77) of the lecture notes – just as it was done for the similar system in Sec. 7.5, but now separately on each of two z-segments: $[0, z_0]$, and $[z_0, a]$, where a is the half-length of the beam. Denoting the variables on these segments with upper indices (respectively, – and +), we have to use the following boundary conditions – see Fig. 7.9b of the lecture notes:

Position	$z = 0$	$z = z_0$	$z = a$
Boundary conditions	$F_x^- = 0$		$F_x^+ = 0$
		$\tau_y^- = \tau_y^+$	$\tau_y^+ = 0$
	$\varphi^- = 0$	$\varphi^- = \varphi^+$	
		$q_x^- = q_x^+ = 0$	

With $f_x = -\rho g$, where ρ is the beam material's density, the straightforward sequential integration of Eqs. (7.74)-(7.77), using the above boundary conditions for each segment (at this stage, besides those for τ_y and φ at $z = z_0$), yields:[52]

$$F_x^- = \rho g A z, \qquad F_x^+ = \rho g A(z-a),$$

$$\tau_y^- = -\rho g A\left(\frac{z^2}{2} + c_1\right), \qquad \tau_y^+ = -\rho g A\frac{(z-a)^2}{2},$$

$$\varphi^- = -\frac{\rho g A}{EI_y}\left(\frac{z^3}{6} + c_1 z\right), \qquad \varphi^+ = -\frac{\rho g A}{EI_y}\left[\frac{(z-a)^3}{6} + c_2\right],$$

$$q_x^- = -\frac{\rho g A}{EI_y}\left(\frac{z^4 - z_0^4}{24} + c_1\frac{z^2 - z_0^2}{2}\right), \qquad q_x^+ = -\frac{\rho g A}{EI_y}\left[\frac{(z-a)^4 - (z_0-a)^4}{24} + c_2(z - z_0)\right].$$

Now we can find the two integration constants $c_{1,2}$ by using the two remaining conditions (for τ_y and φ at $z = z_0$) – see the middle column in the above table:

$$\frac{z_0^2}{2} + c_1 = \frac{(z_0-a)^2}{2}, \qquad \frac{z_0^3}{6} + c_1 z_0 = \frac{(z_0-a)^3}{6} + c_2.$$

[52] As a sanity check: the first pair of these formulas correctly describes a jump of F_x, at the point $z = z_0$, by $\Delta F_x = -\rho g A a$, due to the point support force.

By solving this system of equations (an easy job, because the first equation includes only c_1), we get

$$c_1 = \frac{a^2}{2} - az_0, \qquad c_2 = \frac{a^3}{6} - \frac{az_0^2}{2}.$$

Now we are ready to address the problem's tasks.

(i) The beam's ends being horizontal means that their slope $\varphi^+|_{z=a}$ should vanish. From the above formula for φ^+, we see that this happens if $c_2 = 0$, and the last formula for this coefficient shows that this happens if the supports are placed in the so-called *Airy points*:[53]

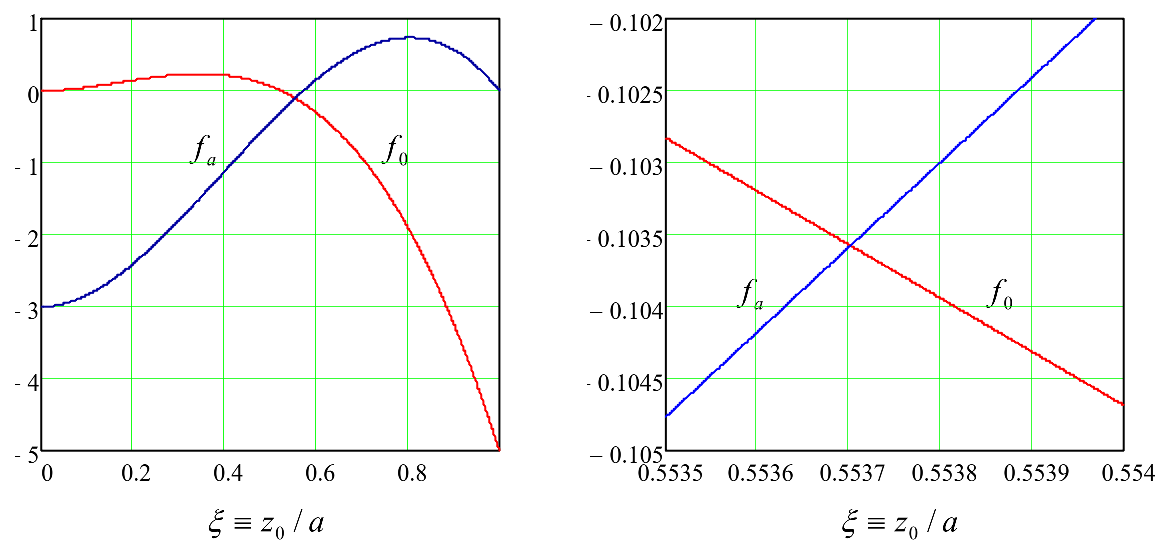
$$\frac{a^3}{6} - \frac{az_0^2}{2} = 0, \qquad \text{i.e. } z_0 = \pm\frac{1}{\sqrt{3}}a \approx 0.5773\,a.$$

(ii) Since the beam's bending is rather smooth, its deflection from the horizontal line may reach its maximum either in its middle or at its ends, so we need to spell out our result for $q_x^{\pm}(z)$ only at the points $z = 0$ and $z = a$:[54]

$$q_x^-(z=0) = -\frac{\rho g A}{EI_y}\left(-\frac{z_0^4}{24} - c_1\frac{z_0^2}{2}\right) = \frac{\rho g A a^4}{24EI_y}f_0(\xi), \qquad \text{with } f_0(\xi) \equiv \xi^4 - 12\xi^3 + 6\xi^2,$$

$$q_x^+(z=a) = -\frac{\rho g A}{EI_y}c_2(a - z_0) = \frac{\rho g A a^4}{24EI_y}f_a(\xi), \qquad \text{with } f_a(\xi) \equiv \xi^4 - 16\xi^3 + 18\xi^2 - 3,$$

where $\xi \equiv z_0/a$ is the normalized position of the support. The functions $f_0(\xi)$ and $f_a(\xi)$ are plotted in the figure below. (The right panel is a zoom-in on the region where the function plots cross.)



The plots show that the minimized largest deflection's magnitude,

[53] After G. B. Airy (of the Airy functions' fame), who calculated these points in 1845.

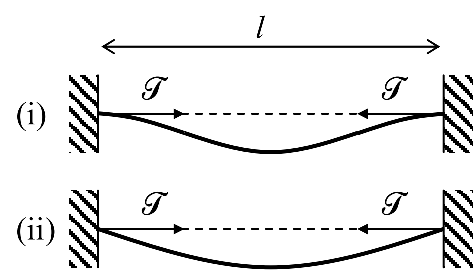
[54] A good sanity check here is that if the support is placed directly under the point of interest, the vertical deflection vanishes: $f_0(0) = f_a(1) = 0$.

$$\min_{z_0}\left[\max_z\left(\left|q_x\right|\right)\right] \approx 0.1036\,\frac{\rho g A a^4}{24 E I_y} \approx 0.00432\,\frac{\rho g A a^4}{E I_y},$$

is achieved at

$$z_0 \equiv \xi a \approx 0.5537\, a\,.$$

Note how close the "minimum-sag" support positions $\pm z_0$ are to the Airy points. Just for the reader's reference, these points are even closer to the so-called *Bessel support points* ($z_0 \approx 0.5594\, a$) providing the largest longitudinal extension of the rod's length by its weight. (Their calculation, using our intermediate results, is a good additional exercise recommended to the reader.) Historically, the Bessel points were of large importance for the legacy length standards in the form of rigid rods: since they maximize the length, small changes due to unintentional errors in z_0 are minimized. For example, the meter's legal definition accepted internationally between 1927 and 1960 prescribed placing the meter's standard on supports positioned at these points.



Problem 7.9. Calculate the largest longitudinal compression force $\mathscr{T}$ that may be withstood by a thin, straight, elastic rod without buckling (see the figure on the right) for each of the shown cases:

(i) the rod's ends are clamped, and
(ii) the rod is free to turn about the support points.

Solution:

As was discussed in Sec. 7.8 of the lecture notes, the longitudinal stretching of a rod with force $\mathscr{T}$ adds the additional term $\mathscr{T}\partial^2 q_x/\partial z^2$ to the equation of transverse waves propagating along the rod, leading to the modification of their dispersion relation (7.136) to the form (7.137):

$$\omega^2 = \frac{1}{\rho A}\left(E I_y k^4 + \mathscr{T} k^2\right).$$

Reviewing these arguments, we may see that they all hold even if the rod is compressed rather than stretched, besides that if $\mathscr{T}$ means the compression force's magnitude, the sign before it now has to be changed, giving

$$\omega^2 = \frac{1}{\rho A}\left(E I_y k^4 - \mathscr{T} k^2\right).$$

This formula shows that at a fixed wave number k, the increase of compression reduces the wave's frequency and that eventually, at

$$\mathscr{T} > \mathscr{T}_{\max} = E I_y k^2, \qquad (*)$$

ω^2 becomes negative. This means that ω becomes imaginary, i.e. the time dependence of the transversal deviations changes from the $\exp\{-i\omega t\}$ assumed at the derivation of the dispersion relation, to $\exp\{\pm\Lambda t\}$, with real $\Lambda = i\omega$. Physically this means that at $\mathscr{T} = \mathscr{T}_{\max}$, the transverse perturbation of the rod's initial straight shape, in the form of a sinusoidal[55] standing wave with wavelength $\lambda = 2\pi/k$, becomes unstable and leads to buckling.

For the geometry shown on panel (i) of the figure above, with both ends of the rod clamped (i.e. not only the deviations q_x, but also the slopes $\partial q_x/\partial z$ equal zero at $z = 0$ and $z = l$), the longest possible standing wave, and hence the lowest $\mathscr{T}_{\max}$, correspond to $\lambda = l$, i.e. $k = 2\pi/l$, so Eq. (*) yields

$$\mathscr{T}_{\max} = 4\pi^2 \frac{EI_y}{l^2}, \qquad \text{for geometry (i)}.$$

However, the geometry shown on panel (ii) clearly allows for a longer wave, with $\lambda/2 = l$, i.e. $k = \pi/l$, so Eq. (*) gives a four times lower buckling threshold:[56]

$$\mathscr{T}_{\max} = \pi^2 \frac{EI_y}{l^2}, \qquad \text{for geometry (ii)}.$$

Note that the thresholds are proportional to the "moment of inertia" I_y defined by Eq. (7.70), so the buckling occurs within the plane $[x, z]$ with the smallest I_y. Another notable feature of these results is a strong (inverse-quadratic) dependence of $\mathscr{T}_{\max}$ on l, so in very long rod-like structures (such as vertical beams of tall buildings, carrying their weight), the buckling is hard to avoid at even modest compression, and additional stabilizing elements (such as transverse beams in building frames and spikes on railway tracks[57]) are necessary to ensure their stability.

<u>Problem 7.10</u>. A thin elastic pole with a square cross-section of area $A = a\times a$ is firmly dug into the ground in the vertical position, sticking out by height $h >> a$.

(i) What largest compact mass M may be placed straight on the top of a light pole without stability loss?

(ii) In the absence of such an additional mass, how massive a uniform pole may be to retain its stability?

Hint: For Task (ii), you may use the same WKB approximation as in Problem 6.18.

Solutions:

(i) This task is almost completely similar to the two situations considered in the previous problem, besides different boundary conditions, so we may use Eq. (*) of its model solution, with the compressing tension $\mathscr{T}$ duly replaced with Mg:

[55] A thoughtful reader might have noticed that generally, both dispersion relations displayed above allow, for a given ω^2, not only positive but also negative values of k^2, and hence not only sinusoidal waves proportional to $\exp\{\pm ikz\}$ but also additional exponential terms proportional to $\exp\{\pm kz\}$. The latter solutions are, indeed, important for the analysis of standing transverse waves – see, e.g., Problem 23 below. However, in our current case $\omega^2 = 0$, these exponential terms have $k = 0$, and hence describe just the average displacement of the rod, clearly visible on the (quite realistic) sketches in the assignment.

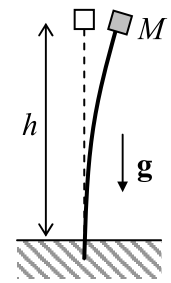
[56] This formula was first derived by L. Euler himself – as early as in 1757.

[57] In the latter case, the compression force is caused by the thermally induced stress. Such stress may be very high, necessitating small distances between the spikes in the modern gap-free ("continuously welded") rails.

$$M_{\max} = \frac{EI_y}{g} k^2 ,$$

with the "moment of inertia" I_y calculated from Eq. (7.70) of the lecture notes, for our square cross-section:

$$I_y = \int_A x^2 dxdy = a \int_{-a/2}^{+a/2} x^2 dx = \frac{a^4}{12} .$$



Let us find the lowest possible wave number k of the transversal deflection mode that arises at $M > M_{\max}$. Sketching this mode (see the figure on the right), we may note that this boundary condition at the top end of the pole allows its slope only to grow with height, because the poll's curvature (7.73), proportional to the torque, cannot change its sign. On the other hand, at its lower end, the slope has to equal zero because of the "clamping" boundary condition. Hence the lowest k corresponds to the pole's height h equal to one-quarter of the sinusoidal deviation's wavelength $\lambda = 2\pi/k$, i.e. $h = \lambda/4 = \pi/2k$, so $k = \pi/2h$, and, finally,

$$M_{\max} = \frac{E}{g} I_y k^2 = \frac{E}{g} \frac{a^4}{12} \left(\frac{\pi}{2h} \right)^2 \equiv \frac{\pi^2}{48} \frac{Ea^4}{gh^2} . \qquad (*)$$

(ii) In the absence of an additional weight, the poll's compression force $\mathscr{T}$ grows, from its top to the bottom, under its own weight. For a uniform poll with the mass $\mu = M/h$ per unit length,

$$\mathscr{T} = \mu g (h - z) = Mg \left(1 - \frac{z}{h} \right),$$

where the vertical coordinate z is referred to the ground level. Repeating the arguments made in the model solution of Problem 6.18, we see that the local wave number $k(z)$ of the WKB standing wave of the transversal displacements, in the limit $\omega \to 0$, may be found from the following natural generalization of the solution of the previous problem:

$$EI_y k^2(z) = \mathscr{T}(z) = Mg \left(1 - \frac{z}{h} \right), \qquad \text{i.e. } k(z) = \left[\frac{Mg}{EI_y} \left(1 - \frac{z}{h} \right) \right]^{1/2} .$$

Now, with the same boundary conditions as in Task (i), the stability threshold may be found from the requirement

$$\Psi(h) \equiv \int_0^h k(z) dz \equiv \left(\frac{M_{\max} g}{EI_y} \right)^{1/2} \int_0^h \left(1 - \frac{z}{h} \right)^{1/2} dz \equiv \left(\frac{M_{\max} g}{EI_y} \right)^{1/2} \frac{2}{3} h = \frac{\pi}{2},$$

finally giving a result that differs from Eq. (*) only by a numerical factor:

$$M_{\max} = \left(\frac{3\pi}{4} \right)^2 \frac{EI_y}{gh^2} \equiv \frac{3\pi^2}{64} \frac{Ea^4}{gh^2} .$$

It is only natural that the additional factor (9/4) is larger than 1, because a substantial part of the uniform pole's mass is closer to the ground than in Task (i). Note, however, that for a wave that long ($\lambda = 4h$), the WKB approximation may give an error of more than 10%.

Problem 7.11. Calculate the potential energy of a small and slowly changing but otherwise arbitrary bending deformation of a uniform, initially straight elastic rod. Can the result be used to derive the dispersion relation (7.136)?

Solution: The general expression for the potential energy density of an elastic deformation is given by Eq. (7.50) of the lecture notes:

$$U = \int_V u(\mathbf{r}) d^3 r, \quad u(\mathbf{r}) = \frac{1}{2} \sum_{j,j'=1}^{3} \sigma_{jj'} s_{j'j} . \qquad (*)$$

According to Eq. (7.64), at weak bending of a very long (and/or very thin) rod, the stress tensor is diagonal,

$$\sigma_{jj'} = \sigma_{zz} \delta_{jz} ,$$

where the z-axis is directed along the rod's axis – see Fig. 7.8 of the lecture notes. If the bending is elastic, the stress σ_{zz} is related to the strain s_{zz} by Eq. (7.42). Taking into account the definition (7.44) of the Young modulus E, this relation may be rewritten simply as $\sigma_{zz} = Es_{zz}$, so the second of Eqs. (*) is reduced to

$$u(\mathbf{r}) = \frac{1}{2E} \sigma_{zz}^2 .$$

Now we may plug into this relation the distribution of σ_{zz} given by Eq. (7.67), getting

$$u(\mathbf{r}) = \frac{1}{2E} \left(\frac{Ex}{R} \right)^2 = \frac{E}{2R^2} x^2, \qquad \text{so } U = \int dz \int_A dxdy\, u(\mathbf{r}) = \frac{E}{2} \int \frac{dz}{R^2} \int_A x^2 dxdy = \frac{EI_y}{2} \int \frac{dz}{R^2} ,$$

where $R \equiv 1/(\partial \varphi_y / \partial z)$ is the curvature radius (which may, for an arbitrary deformation, vary along the rod's length), A is the rod cross-section area, I_y is its "moment of inertia" defined by Eq. (7.70), and the integration over z is extended over the whole length of the rod. Finally, using Eq. (7.77), which is valid for small bending of an initially straight rod, we may express the energy U via the rod's deviation $q_x(z)$ from its equilibrium (straight) shape:

$$U = \frac{EI_y}{2} \int \left(\frac{\partial^2 q_x}{\partial^2 z} \right)^2 dz . \qquad (**)$$

As it follows from its derivation, this expression is valid when the deformation has an equilibrium distribution over every cross-section of the rod, but not necessarily over its length. This is why this formula may be used, in particular, for an analysis of bending waves $q_x(z, t)$ with wavelength $\lambda >> a$, where a is the largest linear dimension of the cross-section. Indeed, complementing Eq. (**) with the evident expression for the kinetic energy of the rod's bending,

$$T = \frac{\rho A}{2} \int \left(\frac{\partial q_x}{\partial t} \right)^2 dz ,$$

we may write its Lagrangian function as

$$L \equiv T - U = \int \mathscr{L}\, dz, \qquad \text{with } \mathscr{L} = \frac{\rho A}{2} \left(\frac{\partial q_x}{\partial t} \right)^2 - \frac{EI_y}{2} \left(\frac{\partial^2 q_x}{\partial^2 z} \right)^2 . \qquad (***)$$

While the first term on the right-hand side of the last expression, for what is called the *Lagrange function's density* $\mathscr{L}$, is very ordinary and was repeatedly encountered earlier in the course, the second term is not. This is understandable because here we are dealing with a distributed system, which may be viewed as a system with infinitely many degrees of freedom – say, displacements q_x in all infinitesimally close points z. Handling such distributed Lagrangian (and Hamiltonian) functions is the subject of *field theories* – either classical or quantum.[58] However, we may use the following plausible (if not completely strict) reasoning.

Since $\mathscr{L}$ is a quadratic form of q_x (or rather of its derivatives over z and t), we may expect the resulting Lagrange equation of motion for q_x to be linear. Thus its arbitrary solution may be Fourier-expanded into a linear superposition of standing waves of the type $a_k(t)\cos kz$ or $a_k(t)\sin kz$, with real $a_k(t)$ and k.[59] Plugging such a standing waveform into Eq. (***), and averaging the Lagrangian function over any *spatial* interval equal to a multiple of the half-wavelength,[60] we get

$$\mathscr{L}_{\text{ave}} = \frac{\rho A}{2}\dot{a}_k^2 - \frac{EI_y}{2}\left(-k^2 a_k\right)^2 \equiv \frac{\rho A}{2}\dot{a}_k^2 - \frac{EI_y}{2}k^4 a_k^2 .$$

Let me hope that the reader recognizes in the last expression the usual Lagrangian function (5.1) of a harmonic oscillator, in this case with the frequency

$$\omega = \left(\frac{EI_y}{\rho A}\right)^{1/2} k^2 .$$

But this is exactly the dispersion relation (7.136).

Problem 7.12. Calculate the torsional rigidity of a long uniform rod whose cross-section is an ellipse with semi-axes a and b.

Solution: Since the cross-section of the rod is not circular, its rigidity C has to be calculated using general Eqs. (7.101) of the lecture notes, with the function $\chi(x, y)$ calculated from Eq. (7.100),

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\chi = -1, \qquad (*)$$

with the boundary condition (7.99): $\chi|_{\text{boundary}}$ = const. Directing the x- and y-axes along the major semi-axes a and b of the ellipse, we may describe the boundary in the well-known canonical form

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)_{\text{boundary}} = 1. \qquad (**)$$

Due to the similarity of the functional forms in Eqs. (*) and (**), the solution of this boundary problem may be readily guessed: $\chi = c_1(x^2/a^2 + y^2/b^2) + c_2$. Now selecting the constant c_1 to satisfy Eq. (*), and the inconsequential constant c_2 to have $\chi|_{\text{boundary}} = 0$ (just for convenience), we get

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[58] In this series, these theories are discussed only in passing: in EM Sec. 9.8 and QM Sec. 9.1.

[59] Such superpositions do not limit us to standing waves alone. For example, a traveling wave $\text{Re}[a\exp\{i(kz - \omega t)\}]$, with an arbitrary complex amplitude a, may be represented as the standing wave superposition $a_c(t)\cos kz + a_s(t)\sin kz$, with phase-shifted real functions $a_c(t) = |\,a\,|\cos[\omega t - \arg(a)]$ and $a_s(t) = |\,a\,|\sin[\omega t - \arg(a)]$.

[60] Such averaging excludes from consideration the fast (with frequency 2ω) "re-pumping" of the standing wave energy between its kinetic and potential components – see, e.g., the model solution of Problem 6.9.

$$\chi = \frac{1 - x^2/a^2 - y^2/b^2}{2/a^2 + 2/b^2} \equiv \frac{a^2b^2}{2(a^2+b^2)}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right),$$

With this expression, Eq. (7.101b) gives

$$C = \frac{2\mu a^2 b^2}{a^2+b^2}\int_A \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dxdy \equiv \frac{2\mu a^3 b^3}{a^2+b^2}\int_A \left(1 - \xi^2 - \zeta^2\right) d\xi d\zeta ,$$

where, in the last form, $\xi \equiv x/a$ and $\zeta \equiv y/b$. According to Eq. (**), in these dimensionless coordinates, the ellipse is just a circle of a unit radius, so, by using the dimensionless polar coordinates defined as $\xi \equiv \rho\cos\varphi$ and $\zeta \equiv \rho\sin\varphi$ (so that $\xi^2 + \zeta^2 = \rho^2$ and $d\xi d\zeta = \rho d\rho d\varphi$), we get

$$C = \frac{2\mu a^3 b^3}{a^2+b^2} 2\pi \int_0^1 \left(1-\rho^2\right)\rho d\rho = \frac{2\mu a^3 b^3}{a^2+b^2} 2\pi \frac{1}{4} \equiv \pi\mu \frac{a^3b^3}{a^2+b^2}.$$

This simple result is very illuminating. Indeed, for the particular case $a = b = R$, we get

$$C = \frac{\pi\mu}{2} R^4 ,$$

i.e. recover Eq. (7.90) of the lecture notes (with $R_1 = 0$, $R_2 = R$) for a rod with a circular cross-section of radius R. On the other hand, in the limit of a very stretched cross-section, say with $a >> b$, we get an expression,

$$C = \pi\mu\, ab^3 \equiv \frac{\pi}{16}\mu(2a)(2b)^3 , \qquad (***)$$

which is functionally similar (and even numerically close) to Eq. (7.104), $C = \mu w t^3/3$, for the rectangular cross-section with $w >> t$.

Problem 7.13. Calculate the potential energy of a small but otherwise arbitrary torsional deformation $\varphi_z(z)$ of a uniform and straight elastic rod.

Solution: First, for rods with round cross-sections, the strain and stress tensor elements are described by the following simple relations – see Eqs. (7.86)-(7.87) of the lecture notes:

$$s_{xx} = s_{yy} = s_{zz} = 0, \quad s_{xy} = s_{yx} = 0, \quad s_{xz} = s_{zx} = -\frac{\kappa}{2} y, \quad s_{yz} = s_{zy} = \frac{\kappa}{2} x ,$$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0, \quad \sigma_{xy} = \sigma_{yx} = 0, \quad \sigma_{xz} = \sigma_{zx} = -\mu\kappa y, \quad \sigma_{yz} = \sigma_{zy} = \mu\kappa x ,$$

where the z-axis is directed along the rod's axis, the direction of the x-axis is arbitrary (due to the axial symmetry of the system), $\kappa \equiv \partial\varphi_z/\partial z$ is the torsional parameter defined by Eq. (7.83), and μ is the shear modulus of the rod's material. Plugging these expressions into the general Eq. (7.50) for the potential energy density of a deformation,

$$u(\mathbf{r}) = \frac{1}{2}\sum_{j,j'=1}^{3} \sigma_{jj'} s_{j'j} ,$$

we get

$$u(\mathbf{r}) = \frac{\mu}{2}\kappa^2\left(x^2 + y^2\right) \equiv \frac{\mu}{2}\kappa^2\rho^2 ,$$

where ρ is the point's distance from the rod's axis. Now integrating this expression over the rod's cross-section, we get the following linear density of the potential energy:

$$\frac{dU}{dz} = \int_A u(\mathbf{r})dxdy = \frac{\mu}{2}\kappa^2 I_z \equiv \frac{\mu I_z}{2}\left(\frac{\partial\varphi_z}{\partial z}\right)^2, \qquad \text{where } I_z \equiv \int_A \rho^2 dxdy .$$

The total potential energy U of the rod now may be readily expressed as the following integral:

$$U = \int \frac{dU}{dz}dz = \frac{\mu I_z}{2}\int\left(\frac{\partial\varphi_z}{\partial z}\right)^2 dz, \qquad (*)$$

over its length.

As was discussed in Sec. 7.6 of the lecture notes, for rods with non-circular cross-sections, the deformation distribution over the cross-section is more complex, but U still may be readily calculated using a different, more general approach. Let the torque $\boldsymbol{\tau} = \tau_z\mathbf{n}_z$, causing the torsion, be created by a pair of forces $\mathbf{F}$ (see Fig. 7.10 of the lecture notes) applied at a distance ρ from the axis and perpendicular to the polar vector $\boldsymbol{\rho}$; then the basic Eq. (1.34) gives $\tau_z = F\rho$. A small change $\delta\varphi_z$ of the torsion angle causes a displacement, of magnitude $\delta r = \rho\delta\varphi_z$, of the force's application point, and hence causes the elementary work $\delta\mathscr{W} = F\delta r = F\rho\delta\varphi_z = \tau_z\delta\varphi_z$ of the forces on the rod.[61] At slow torsion, all this work goes to an increment of the rod's potential energy U:

$$\delta U = \tau_z\delta\varphi_z . \qquad (**)$$

For an elastic deformation whose distribution does not change significantly on a small segment of length dz, the torque is proportional to the torsion parameter κ – see Eq. (7.84):

$$\tau_z = C\frac{d\varphi_z}{dz},$$

where C is the torsional rigidity, so the integration of Eq. (**) from $\varphi_z = 0$ to a certain nonvanishing torsion $d\varphi_z$ yields

$$dU = \int \delta U = \int \tau_z\delta\varphi_z = \frac{C}{dz}\int_0^{d\varphi_z}\varphi_z\delta\varphi_z = C\frac{d\varphi_z^2}{2dz},$$

and the potential energy density per unit length of the rod is

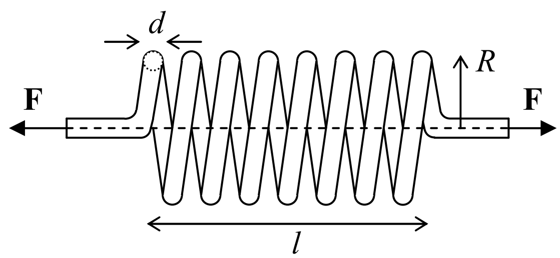
$$\frac{dU}{dz} = C\frac{d\varphi_z^2}{2dz^2} = \frac{C}{2}\left(\frac{d\varphi_z}{dz}\right)^2 .$$

This expression is valid even for time-dependent phenomena (e.g., waves) if their longitudinal length scale is much larger than the largest size of the cross-section, provided that the derivative over z is considered partial, so

$$U = \int \frac{dU}{dz}dz = \frac{C}{2}\int\left(\frac{\partial\varphi_z}{\partial z}\right)^2 dz .$$

[61] It is intuitively clear that this relation is independent of the way the torque has been created. A good additional exercise: use Eqs. (1.34), (7.1), and the "operand rotation rule" MA Eq. (7.6) to prove the following generalization of this result: $\delta\mathscr{W} = \boldsymbol{\tau}\cdot\delta\boldsymbol{\varphi}$, valid for any direction of the vectors $\boldsymbol{\tau}$ and $\delta\boldsymbol{\varphi}$.

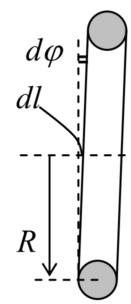
This expression is valid for rods with any cross-section; if it has a round shape, we may use Eq. (7.89), $C = \mu I_z$, and the general result is reduced to Eq. (*), providing a useful sanity check.



Problem 7.14. Calculate the spring constant $\kappa \equiv dF/dl$ of a coil made of a uniform elastic wire with a circular cross-section of diameter d, wound as a dense round spiral of $N >> 1$ turns of radius $R >> d$ – see the figure on the right.

Solution: The dominating deformation in this system is the wire's torsion, due to the torque

$$\tau = \frac{F}{R} \tag{*}$$



created by the stretching force **F** applied at the center of the coil (see the figure above), i.e. at the distance R from the wire cross-section's center. The full torsion angle $\Delta\varphi$ caused by the torque is uniformly distributed along the full length $L = (2\pi R)N$ of the wire. As the figure on the right shows, any elementary twist of the wire's cross-section by angle $d\varphi$ leads to the spring end's shift (along the coil's axis) by $dl = Rd\varphi$. Summing up such shifts over the whole wire's length, we may express the full spring's extension as

$$\Delta l = R\Delta\varphi, \tag{**}$$

so the torsion parameter, defined by Eq. (7.83) of the lecture notes, is

$$\frac{d\varphi}{dL} = \frac{\Delta\varphi}{L} = \frac{\Delta l}{LR} = \frac{\Delta l}{2\pi NR^2}.$$

According to Eqs. (7.84) and (7.90), the torque corresponding to such torsion is

$$\tau = C\frac{d\varphi}{dL} = \frac{\pi\mu(d/2)^4}{2}\frac{\Delta l}{2\pi NR^2} = \frac{\mu d^4}{64NR^2}\Delta l.$$

Comparing this expression with Eq. (*), we get the following final result for the spring constant:[62]

$$\kappa \equiv \frac{F}{\Delta l} = \frac{\mu d^4}{64NR^3}. \tag{***}$$

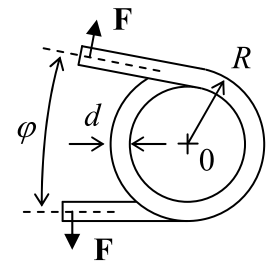
Note that for the ultimately dense winding, $l = Nd$, this result may be rewritten as

$$\kappa = \frac{\mu d^5}{8lD^3} = \frac{\mu A}{2\pi l}\left(\frac{d}{D}\right)^3 = \frac{1}{2\pi}\frac{\mu}{E}\left(\frac{d}{D}\right)^3\kappa' = \frac{1}{4\pi(1+\nu)}\left(\frac{d}{D}\right)^3\kappa' << \kappa',$$

[62] Note that this result may be also obtained using the solution of the previous problem – by expressing the potential energy U of the spring via its extension Δl, and then requiring it to be equal to $\kappa(\Delta l)^2/2$. Though this additional exercise is highly recommended to the reader, this approach does not eliminate the need for the only non-trivial step of the above solution – derivation of the kinematic relation (**).

where $D \equiv 2R$ is the coil's diameter, while $\kappa' \equiv AE/l$, according to Eq. (7.47), is the spring constant of a straight wire of the same cross-section area $A = \pi(d/2)^2$ and the same length l. Due to the large value of the elastic moduli of typical construction materials like steel (see, e.g., Table 7.1 in the lecture notes), such spiral coils with $d << D$ are virtually the only simple way to implement compact but relatively "soft" springs. As a result, they are very widely used in engineering and physical experimentation.

This solution gives a very nice illustration of how creative we should be when solving particular problems of physics. Indeed, the above "hand-waving" reasoning leads to the solution much faster than a more formal approach (say, solving the general Eq. (7.52) for the spring's geometry) would.



Problem 7.15. The coil studied in the previous problem is now used as what is sometimes called the *torsion spring* – see the figure on the right. Find the corresponding spring constant $d\tau/d\varphi$, where $\boldsymbol{\tau}$ is the torque of the external forces **F** relative to the center of the coil (point 0).

Solution: In contrast to the situation in the previous problem, the dominating deformation here is wire bending resulting in an additional, torque-induced curvature – on top of the initial, stress-free curvature $1/R$:

$$d\left(\frac{1}{R}\right) = \frac{d\varphi}{L} = \frac{d\varphi}{2\pi RN},$$

where L is the total length of the wire. Hence, using Eq. (7.69) of the lecture notes, we may write

$$\frac{d\tau}{d\varphi} = EI_y \frac{d(1/R)}{d\varphi} = \frac{EI_y}{2\pi RN}, \qquad (*)$$

where I_y is the "moment of inertia" of the wire's cross-section, given by Eq. (7.70):

$$I_y \equiv \int_A x^2 dA,$$

and x is the local axis within the coil's plane, directed perpendicular to the local wire's axis. For a wire with a round cross-section, this integral may be readily calculated – either as in the model solution of Problem 7 or by noticing that due to the axial symmetry of the cross-section, the integral of y^2 must give the same result, so

$$I_y = \frac{1}{2}\int_A \left(x^2 + y^2\right) d^2 r = \frac{1}{2} 2\pi \int_0^{d/2} r^2 r dr = \frac{\pi d^4}{64}.$$

Plugging this result into Eq. (*), we see that the spring constant,

$$\frac{d\tau}{d\varphi} = \frac{Ed^4}{128 RN},$$

depends only on Young's modulus, but not on the shear modulus. This is natural because the deformation of the "torsion" spring's material (bending) has nothing to do with the genuine torsion that was discussed in Sec. 7.6 of the lecture notes.

Problem 7.16. Use Eqs. (7.99) and (7.100) of the lecture notes to recast Eq. (7.101b) for the torsional rigidity C of a thin rod into the form given by Eq. (7.101c).

Solution: Let us first calculate the divergence of the vector $\chi\nabla\chi$, at this stage without any special assumptions about the scalar function $\chi(\mathbf{r})$:

$$\nabla\cdot(\chi\nabla\chi) = \nabla\chi\cdot\nabla\chi + \chi\,\nabla\cdot\nabla\chi \equiv (\nabla\chi)^2 + \chi\nabla^2\chi\,.$$

Now let us apply this result to the particular function $\chi(x, y)$ defined by Eqs. (7.98), which does not depend on the third Cartesian coordinate (z), and whose 2D Laplacian operator equals (–1) – see Eq. (7.100). The result is

$$\nabla_{x,y}\cdot(\chi\nabla_{x,y}\chi) = (\nabla_{x,y}\chi)^2 - \chi, \quad \text{giving} \;\; \chi = (\nabla_{x,y}\chi)^2 - \nabla_{x,y}\cdot(\chi\nabla_{x,y}\chi).$$

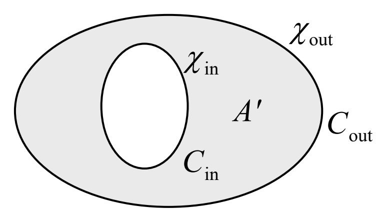
Plugging the last expression into Eq. (7.101b), and applying the 2D form of the divergence theorem[63] to the second term, we get

$$C = 4\mu\int_A (\nabla_{x,y}\chi)^2\,dxdy - \chi_{\text{border}}\left(A + \oint_{\text{border}}(\nabla_{x,y}\chi)_n\,dl\right).$$

But according to its definition, the function $\chi(x, y)$ is defined to an arbitrary constant, and according to Eq. (7.99), it does not change at the cross-section's border, so we always may take $\chi_{\text{border}} = 0$, thus arriving at Eq. (101c).

Problem 7.17.* Generalize Eq. (7.101b) of the lecture notes to the case of a thin rod with more than one cross-section's boundary. Use the result to calculate the torsional rigidity of a thin round pipe, and compare it with Eq. (7.93).

Solution: As was noted in Sec. 7.6 of the lecture notes, Eq. (7.99) does not forbid the function $\chi(x, y)$ from having different values at different boundaries of the rod if their contours at the rod's cross-section are disconnected. For example, in a hollow pipe (see the figure on the right), the values of χ_{out} and χ_{in} may be different. In this case, reviewing the integration by parts in Eq. (7.101b), we may express the torsional rigidity as



$$C = 4\mu\left[\int_{A'}\chi dxdy + \chi_{\text{in}}A_{\text{in}} - \chi_{\text{out}}A_{\text{out}}\right], \qquad (*)$$

where A_{out} and A_{in} are the areas within, respectively, the external and internal boundary contours, so the area of the material-filled part of the cross-section is $A' = A_{\text{out}} - A_{\text{in}}$. In this case, only one of the constants χ_{out} and χ_{in} may be selected arbitrarily; their difference evidently affects the results.

The relation between these constants may be obtained from the requirement for the longitudinal deformation function $\psi(x,y)$ to be unique, and hence the integral of its differential $d\psi$ along any of the borders (say, the k^{th} boundary contour C_k) to equal zero. From this condition and Eq. (7.98), we get

63 See, e.g., MA Eq. (12.2).

$$0=\oint_{C_k} d\psi \equiv \oint_{C_k}\left(\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy\right)=2\oint_{C_k}\left(\frac{\partial\chi}{\partial y}dx-\frac{\partial\chi}{\partial x}dy\right)-\oint_{C_k}(xdy-ydx).$$

Using the same argumentation as at the derivation of Eq. (7.97) from Fig. 7.11, we see that the expression under the first integral is equal to $(\partial\chi/\partial n)dl$, where n is the external normal to the k^{th} boundary. The second integral may be worked out by transferring to polar coordinates: $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, so $(xdy - ydx) = \rho\cos\varphi\, d(\rho\sin\varphi) - \rho\sin\varphi\, d(\rho\cos\varphi) = \rho^2 d\varphi$, i.e. twice the elementary area dA_k – see, e.g., Fig. 3.4 of the lecture notes and its discussion. As a result, we get the following boundary condition:

$$\oint_{C_k}\frac{\partial\chi}{\partial n}dl=-A_k, \qquad (**)$$

where A_k is the area limited by the k^{th} boundary.

Let us use Eqs. (*) and (**) to calculate the torsional rigidity C of a pipe with a constant and small wall thickness $t << a$, but otherwise with an arbitrary cross-section. In this case, $\partial\chi/\partial n$ in Eq. (**) for the outer border may be approximated as $(\chi_{\text{out}} - \chi_{\text{in}})/t$, so the left-hand side of this equality is approximately equal to $p(\chi_{\text{out}} - \chi_{\text{in}})/t$, where p is the border's perimeter, while its right-hand side is $(-A)$, where (in contrast to $A' = A_{\text{out}} - A_{\text{in}}$) $A \approx A_{\text{in}} \approx A_{\text{out}}$ is the area limited by the pipe's cross-section. (For the inner border, Eq. (**) gives the same result, because of the opposite direction of the external normal to that border.) In this approximation, the first term in Eq. (*) is negligible,[64] while the two other terms yield

$$C\approx 4\mu A(\chi_{\text{in}}-\chi_{\text{out}})\approx\frac{4\mu tA^2}{p}. \qquad (***)$$

For a round pipe of radius R, $A = \pi R^2$ and $p = 2\pi R$, so Eq. (***) is reduced to Eq. (7.91b), which was derived in Sec. 7.6 using the assumption $\psi(x, y) = 0$.

It is interesting (and practically important) that the torsional rigidity of a thin pipe may be strongly decreased by making an even a very thin cut of its wall, parallel to the pipe's length. Indeed, after such a cut, the outer and inner surfaces of the pipe become a single surface, and the function χ on them should be the same. As a result, we can repeat all the arguments leading to Eq. (7.104) to see that it is applicable to this case, with the replacement of w with $p \sim a$. The resulting torsional rigidity is of the order of μat^3, i.e. much lower than that given by Eq. (***) – which is of the order of μta^3. The physical reason for this dramatic reduction is that the cut pipe's torsion causes a large longitudinal shift $\kappa\Delta\psi$ between the cross-section's points on both sides of the cut, while in the uncut pipe, such a shift is impossible.

Problem 7.18. Prove that in a uniform isotropic medium, an arbitrary (not necessarily plane) elastic wave may be decomposed into a longitudinal wave with $\nabla\times\mathbf{q}_l = 0$ and a transverse wave with $\nabla\cdot\mathbf{q}_t = 0$, and find the equations satisfied by these functions.

[64] Indeed, taking one of the boundary values of χ for zero, we see that the term's contribution is of the order of $4\mu\chi A'$, i.e. is much smaller than the last two terms.

Solution: Per the discussion in the lecture notes, such a wave is described by Eq. (7.107) with **f**(**r**, t) = 0. Plugging into this equation the suggested decomposition $\mathbf{q} = \mathbf{q}_l(\mathbf{r}, t) + \mathbf{q}_t(\mathbf{r}, t)$, dividing all terms by ρ, and taking into account Eqs. (7.112) and (7.116), we may represent the result as

$$\frac{\partial^2}{\partial t^2}(\mathbf{q}_l + \mathbf{q}_t) = v_t^2 \nabla^2 (\mathbf{q}_l + \mathbf{q}_t) + (v_l^2 - v_t^2)\nabla(\nabla \cdot \mathbf{q}_l), \qquad (*)$$

because, by the definition of $\mathbf{q}_t$, its divergence vanishes. Let us take the divergence of both sides and use the same condition again:

$$\nabla \cdot \frac{\partial^2}{\partial t^2}\mathbf{q}_l = v_t^2 \nabla^2 (\nabla \mathbf{q}_l) + (v_l^2 - v_t^2)\nabla^2 [\nabla \cdot \mathbf{q}_l], \qquad \text{i.e. } \nabla \cdot \left(\frac{\partial^2}{\partial t^2}\mathbf{q}_l - v_l^2 \nabla^2 \mathbf{q}_l \right) = 0 .$$

By the definition of $\mathbf{q}_l$, not only the divergence but also the curl of the last parentheses have to be zero, and vector algebra tells us that if this is true for any vector in the whole space, this vector should equal zero everywhere. Hence we may write

$$\frac{\partial^2}{\partial t^2}\mathbf{q}_l - v_l^2 \nabla^2 \mathbf{q}_l = 0 .$$

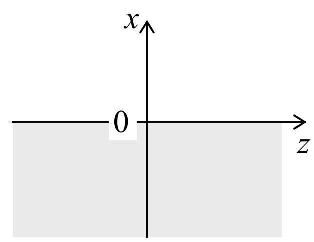
This *3D wave equation* is a very natural extension of the 1D wave equation (6.40) whose solutions were discussed in Secs. 6.4 and 6.5 of the lecture notes. It describes the longitudinal acoustic (dispersion-free) waves propagating with velocity v_l. (Note again that Eq. (7.107) is more general.)

Now, very similarly, taking the curl of both sides of Eq. (*), using the fact that $\nabla \times \mathbf{q}_l = 0$, and repeating the same arguments about a vector with vanishing divergence and curl, we may see that the transverse waves satisfy a similar 3D wave equation, but with the speed v_t.

Problem 7.19.* Use the wave equations derived in the solution of the previous problem and the semi-quantitative description of the Rayleigh surface waves given in Sec. 7.7 of the lecture notes, to calculate the structure of the waves and to derive Eq. (7.127).

Solution: Let us direct the x-axis normally to the surface we are considering, and analyze a "1D-plane" wave whose variables are independent of one of the remaining Cartesian coordinates (say, y):

$$\frac{\partial}{\partial y} = 0 .$$



Let us align the z-axis with the direction of the wave's propagation – see the figure on the right. Then, according to the description of the Rayleigh waves in Sec. 7.7, the net particle displacement vector **q** may have only two nonvanishing Cartesian components:

$$q_y = 0 .$$

In this case, the equations for the longitudinal and transverse components $\mathbf{q}_l$ and $\mathbf{q}_t$ of the Rayleigh wave, which were derived in the solution of the previous problem, are reduced to

$$\frac{\partial^2}{\partial t^2}\mathbf{q}_l - v_l^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right)\mathbf{q}_l = 0, \qquad \frac{\partial^2}{\partial t^2}\mathbf{q}_t - v_t^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right)\mathbf{q}_t = 0.$$

Let us look for their solution in the form of "monochromatic" (single-frequency) waves propagating along the z-axis with a common wave vector k:

$$\mathbf{q}_l = \text{Re}\left[\mathbf{a}_l \exp\{i(kz - \omega t)\}\right], \qquad \mathbf{q}_t = \text{Re}\left[\mathbf{a}_t \exp\{i(kz - \omega t)\}\right].$$

Plugging these solutions into the above wave equations, we get two similar 1D differential equations for the z-dependence of the wave amplitudes:

$$\frac{d^2\mathbf{a}_l}{dx^2} = \kappa_l^2 \mathbf{a}_l, \qquad \frac{d^2\mathbf{a}_t}{dx^2} = \kappa_t^2 \mathbf{a}_t, \qquad \text{where } \kappa_l^2 \equiv k^2 - \frac{\omega^2}{v_l^2}, \quad \kappa_t^2 \equiv k^2 - \frac{\omega^2}{v_t^2}. \qquad (*)$$

These equations have physically acceptable solutions $\mathbf{a}_l \propto \exp\{\kappa_l x\}$ and $\mathbf{a}_t \propto \exp\{\kappa_t x\}$, decaying into the medium's bulk (i.e. at $x \to -\infty$, see the figure above), with real and positive κ_l and κ_t, only if their squares are positive, i.e. if the common constant k^2 is larger than both ω^2/v_l^2 and ω^2/v_t^2. (This means that the Rayleigh wave's velocity $v_R \equiv \omega/k$ has to be lower than both v_l and v_t.) If so, we may spell out the above expressions for $\mathbf{q}_l$ and $\mathbf{q}_t$, by representing their Cartesian components as

$$q_{lj} = \text{Re}\left[A_{lj} \exp\{\kappa_l x + i(kz - \omega t)\}\right], \qquad \mathbf{q}_{tj} = \text{Re}\left[A_{tj} \exp\{\kappa_t x + i(kz - \omega t)\}\right], \qquad (**)$$

where j means either x or z, while A_{lj} and A_{tj} are certain complex amplitudes. The relation between these amplitudes depends on the wave's type. As was discussed in Sec. 7.7 of the lecture notes and in the previous problem, in longitudinal waves, $\nabla\times\mathbf{q}_l = 0$; in our case of just two nonvanishing components of this vector, this means $\partial q_{lz}/\partial x - \partial q_{lx}/\partial z = 0$. For the spatial dependence given by the first of Eq. (**), this gives the relation $\kappa_l A_{lz} - ikA_{lx} = 0$, so we may take, for example,

$$A_{lx} = -i\kappa_l A_l, \qquad A_{lz} = kA_l,$$

where A_l is the effective complex amplitude of the longitudinal wave. On the other hand, in transverse waves, $\nabla\cdot\mathbf{q}_t = 0$, so in our 2D case, $\partial q_{tx}/\partial x + \partial q_{tz}/\partial z = 0$, and using the second of Eqs. (**), we see that the relation between its Cartesian amplitudes is different: $\kappa_t A_{tx} + ikA_{tz} = 0$, so we may write

$$A_{tx} = -ikA_t, \qquad A_{tz} = \kappa_t A_t.$$

The relation between the effective amplitudes A_l and A_t of the composite Rayleigh wave is determined by the boundary conditions at the plane surface $x = 0$. Since the surface is free, all Cartesian components of the surface forces $d\mathbf{F}$ should vanish for all of its elementary areas dA_x. According to the definition (7.18) of the stress tensor, this means

$$\sigma_{xx} = \sigma_{yx} = \sigma_{zx} = 0, \qquad \text{for } x = 0.$$

According to Hooke's law in the form (7.49a), with $s_{yy} = 0$ (because of $q_y = 0$), this is only possible if

$$s_{yx} = 0, \quad s_{zx} = 0, \quad \text{and } s_{xx} + \frac{\nu}{1-2\nu}(s_{xx} + s_{zz}) = 0, \quad \text{i.e. } (1-\nu)s_{xx} + \nu s_{zz} = 0, \quad \text{for } x = 0. \qquad (***)$$

With the definition Eq. (7.9b) of the symmetric stress tensor, the first of these equations,

$$s_{yz} \equiv \frac{1}{2}\left(\frac{\partial q_y}{\partial z} + \frac{\partial q_z}{\partial y}\right) = 0, \qquad \text{for } x = 0,$$

is in full agreement with our initial assumption that $q_y = 0$ and $\partial/\partial y = 0$ everywhere, but does not give any additional information. So, let us spell out the second and the third of Eqs. (**), in the latter case using Eq. (7.117) to express the ratio $\nu/(1-\nu)$ as $(1 - 2v_t^2/v_l^2)$:

$$\frac{\partial q_z}{\partial x} + \frac{\partial q_x}{\partial z} = 0, \qquad v_l^2 \frac{\partial q_x}{\partial x} + \left(v_l^2 - 2v_t^2\right)\frac{\partial q_z}{\partial z} = 0, \qquad \text{for } x = 0.$$

Now decomposing both Cartesian components as above, $q_x = q_{lx} + q_{tx}$, $q_z = q_{lz} + q_{tz}$, employing the spatial dependences (**) of the component waves to spell out their derivatives, using the above expressions of their complex amplitudes via A_l and A_t, and (in the second case) plugging in the expressions for v_l^2 and v_t^2 following from Eqs. (*), we get the following system of two homogeneous linear equations:

$$2A_l k\kappa_l + A_t\left(k^2 + \kappa_t^2\right) = 0, \qquad A_l\left(k^2 + \kappa_t^2\right) + 2A_t k\kappa_t = 0. \qquad (****)$$

These equations are compatible if the determinant of the system equals zero:

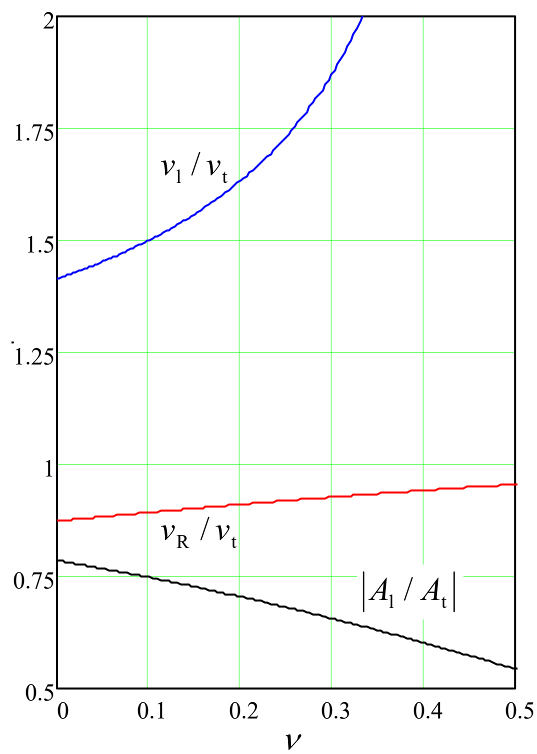
$$\begin{vmatrix} 2k\kappa_l & k^2 + \kappa_t^2 \\ k^2 + \kappa_t^2 & 2k\kappa_t \end{vmatrix} = 0, \qquad \text{i.e.} \quad \left(k^2 + \kappa_t^2\right)^2 = 4k^2\kappa_t\kappa_l .$$

Squaring both parts of this characteristic equation and plugging in Eqs. (*) for κ_l^2 and κ_t^2, we get the following relation:

$$\left(k^2 - \frac{\omega^2}{2v_t^2}\right)^4 = k^4\left(k^2 - \frac{\omega^2}{v_t^2}\right)\left(k^2 - \frac{\omega^2}{v_l^2}\right),$$

which is equivalent to Eq. (7.127) for the Rayleigh wave's phase velocity $v_R \equiv \omega/k$.

The figure on the right shows this velocity (and also v_l – please revisit Eq. (7.117) if you need) in terms of v_t, as functions of the Poisson ratio ν. One can see that in the physically realistic range $0 \leq \nu \leq \frac{1}{2}$, v_R is just slightly lower than v_t, while v_l is significantly higher than both of them.



Now the A_l/A_t ratio, i.e. the proportion of the longitudinal and transverse components in the Rayleigh wave, may be obtained by plugging the solution of the characteristic equation back into any of Eqs. (****). The result may be represented as

$$\frac{A_l}{A_t} = -\frac{\left(1 - v_R^2/v_t^2\right)^{1/2}}{1 - v_R^2/2v_t^2},$$

and is also plotted in the figure on the right. We may see that the magnitude of A_l is somewhat lower than A_t, but not by much, so for a quantitative description of the Rayleigh waves, the account of both components is indeed necessary. Let me emphasize again that per the above analysis, in any of these partial waves, the body particles oscillate in both the x- and z-directions, with $\pm\pi/2$ phase shifts between them, so the particle trajectories are elliptic, with the ellipse axes decreasing simultaneously and exponentially with the distance from the surface.

Problem 7.20.* Calculate the modes and frequencies of free radial oscillations of a sphere of radius R, made of a uniform elastic material.

Solution: The term "radial oscillations" means that in the spherical coordinates with the origin in the sphere's center, the displacement vector $\mathbf{q}(\mathbf{r}, t)$, at all points with $0 \leq r \leq R$, may have only the radial component depending only on $|r| \equiv \mathbf{r}$ and time t. According to the vector algebra,[65] the curl of such a vector equals zero. Hence, transforming Eq. (7.107) of the lecture notes exactly as this was done in statics at the transfer from Eq. (7.52) to Eq. (7.53), taking $\nabla\times\mathbf{q} = 0$ and $\mathbf{f}(\mathbf{r}, t) = 0$, and with the account of Eq. (7.112), we reduce this equation to

$$\frac{\partial^2 \mathbf{q}}{\partial t^2} = v_1^2 \nabla(\nabla \cdot \mathbf{q}).$$

Spelling out the right-hand side for our case $\mathbf{q} = \mathbf{n}_r q(r)\cos\omega t$,[66] we get the scalar differential equation

$$-\omega^2 q = v_1^2 \frac{d}{dr}\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2 q\right)\right].$$

Now comes the only creative step of this solution (which justifies the problem's star marking) – the trick that is common for analyzing spherically symmetric waves of any nature.[67] Looking for the solution of the above equation in the form $q \equiv -d\phi/dr$, with $\phi \equiv f/r$, for the function $f(r)$ we get the well-familiar 1D Helmholtz equation[68]

$$\frac{d^2 f}{dr^2} + k^2 f = 0, \qquad \text{with } k \equiv \frac{\omega}{v_1},$$

whose general solution may be represented as a linear combination of functions $\sin kr$ and $\cos kr$. However, only the former of these functions corresponds to a final (and hence physically acceptable) value of $q(0)$, so for the "displacement potential" ϕ we get the following oscillation waveform:

$$\phi \propto \frac{1}{r}\sin kr, \quad \text{i.e. } q = A\frac{d}{dr}\frac{\sin kr}{r}, \qquad (*)$$

where A is an arbitrary constant (scaling the vibration amplitude) determined by initial conditions.

The eigenvalues of the wave vector k (and hence the set of free oscillation frequencies $\omega = v_1 k$) may be found from the boundary condition on the sphere's surface. The assignment implies it to be open, so all scalar components of the elementary forces $d\mathbf{F}$ should vanish for any $d\mathbf{A} = dA\mathbf{n}_r$ at the surface. According to Eq. (7.18), this means, in particular, that

$$\sigma_{rr} = 0, \qquad \text{at } r = R.$$

[65] See, e.g., MA Eq. (10.11).
[66] See, e.g., MA Eq. (10.10) and then MA Eq. (10.8).
[67] See, e.g., EM Sec. 8.1 and QM Sec. 3.1.
[68] See, e.g., Eq. (6.69).

According to Hooke's law in the form (7.49a), this condition requires the elements of the strain tensor to be related as

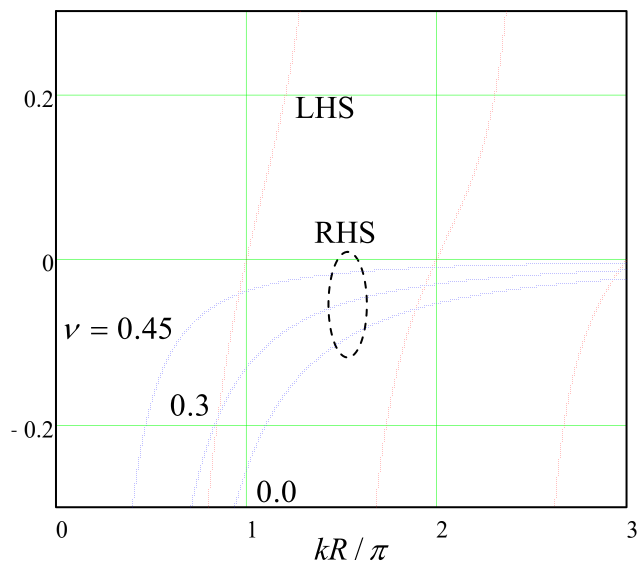
$$s_{rr} + \frac{\nu}{1-2\nu}\left(s_{rr} + s_{\theta\theta} + s_{\varphi\varphi}\right) = 0, \quad \text{i.e.} \ (1-\nu)s_{rr} + \nu\left(s_{\theta\theta} + s_{\varphi\varphi}\right) = 0, \quad \text{for} \ r = R .$$

According to Eq. (7.17), for a purely radial displacement $q_r \equiv q$ (i.e. for $q_\theta = q_\varphi = 0$) giving $s_{rr} = dq/dr$, $s_{\theta\theta} = s_{\varphi\varphi} = q/r$,[69] this means

$$(1-\nu)\frac{dq}{dr} + 2\nu\frac{q}{r} = 0, \quad \text{at} \ r = R.$$

Now calculating the left-hand side of this relation from Eq. (*), we get the following characteristic equation for the dimensionless product kR:

$$\frac{\tan kR}{kR} = \frac{1}{1-[(1-\nu)/2(1-2\nu)](kR)^2} .$$



The figure on the right shows the plots of both sides of this equation, with the right-hand side plotted for several values of the Poisson ratio ν (blue lines). The plots show that for either $kR \to \infty$ or $\nu \to$ ½ (or both), the eigenvalues of k are well described by the following simple formula:

$$k = k_n \equiv \frac{\pi}{R}n, \quad \text{i.e.} \ \omega_n = \frac{\pi v_l}{R}n, \quad \text{with} \ n = 1, 2, 3, ...,$$

meaning that n half-waves π/k of the longitudinal standing waves fit the sphere's radius R. However, for smaller values of ν, the lower oscillation modes correspond to somewhat smaller k (and hence of $\omega = v_l k$) because of the 3D geometry of the sphere. (For most common materials, with $\nu \sim 0.3$, this reduction does not exceed 20%.)

Note that besides such purely radial oscillation modes, an elastic sphere also has resonances involving angle-dependent displacements. However, their analysis involves special functions called *spherical harmonics*; in this series, their discussion is postponed until the EM and QM parts.

<u>Problem 7.21</u>. A long steel wire has a circular cross-section with a 3-mm diameter and is pre-stretched with a constant force of 100 N. Which of the longitudinal and transverse waves with a frequency of 1 kHz has the largest group velocity in the wire? Accept the following parameters for the steel (see Table 7.1 in the lecture notes): E = 170 GPa, ν = 0.30, ρ = 7.8 g/cm^3.

Solution:

(i) <u>Longitudinal waves</u>. Making a guess that the wavelength $\lambda = 2\pi/k$ is much larger than the wire's diameter $D = 3\times10^{-3}$ m (so we are dealing with the tensile waves), we may use Eq. (7.133) of the lecture notes:

[69] Please note that neglecting the non-radial components $s_{\theta\theta}$ and $s_{\varphi\varphi}$ of the strain tensor would give an error.

$$v = (E/\rho)^{1/2}.$$

For our parameters, $v \approx 4.7$ km/s, corresponding to the wavelength $\lambda = v/f \approx 4.7$ m, so our guess $\lambda >> D$ and hence the above result for v are indeed valid. Since such tensile waves are dispersion-free, the calculated value of v is valid for both the phase and group velocities.

(ii) Torsional waves. Again assuming that the relation $\lambda >> D$ is valid for these waves, we may use Eq. (7.141) of the lecture notes for v. Since the rod has an axially-symmetric cross-section, its torsional rigidity may be calculated using Eq. (7.89). As was discussed in Sec. 7.8 of the lecture notes, the resulting velocity coincides with that of the bulk transverse waves – see Eq. (7.116):

$$v = \left(\frac{\mu}{\rho}\right)^{1/2} = \left[\frac{E}{2(1+\nu)\rho}\right]^{1/2}.$$

For our parameters, this velocity is close to 2.9 km/s, i.e. lower than that of the longitudinal waves; the assumption $\lambda >> D$ is again valid. These waves are dispersion-free as well, so the above result is valid for the group velocity as well as for the phase velocity.

(iii) Bending waves. Generally, for bending waves in a rod with a background constant tension, we have to use Eq. (7.137),

$$\omega^2 = \frac{1}{\rho A}\left(EI_y k^4 + \mathscr{T}k^2\right), \qquad (*)$$

which yields a quadratic equation for k^2. However, the specified background tension $\mathscr{T}$ of the wire is so low that we may guess that the corresponding second term on the right-hand side of this formula is negligible is comparison with the first one. (This guess is to be justified *a posteriori*.) By ignoring that term, we get

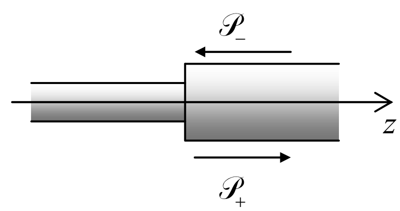
$$\omega = \left(\frac{EI_y}{\rho A}\right)^{1/2} k^2, \qquad k = \left(\frac{EI_y}{\rho A}\right)^{-1/4} \omega^{1/2}, \qquad v_{\text{gr}} \equiv \frac{d\omega}{dk} = 2\left(\frac{EI_y}{\rho A}\right)^{1/2} k = 2\left(\frac{EI_y}{\rho A}\right)^{1/4} \omega^{1/2},$$

where I_y is the "moment of inertia" integral (7.70). For a round cross-section, the integral may be easily worked in several ways; perhaps the most elegant one (already used in the solution of Problem 15) is to notice that due to the axial symmetry of the cross-section, the integral of y^2 must give the same result, so

$$I_y = I_x = \frac{1}{2}\int_A \left(x^2 + y^2\right) d^2r = \frac{1}{2}\int_0^{D/2} r^2\, 2\pi r\, dr = \frac{\pi D^4}{64}.$$

For our parameters, the resulting group velocity turns out to be close to 0.30 km/s (i.e. is much lower than that of both the tensile and torsional waves), and $k \approx 42$ m^{-1}, so the wavelength $\lambda = 2\pi/k \approx$ 0.15 m is comfortably larger than $D = 0.003$ m, confirming the validity of our analysis. Finally, plugging the calculated value of k into Eq. (*), we see that our other guess was also correct: the second term in parentheses of Eq. (*) is an order of magnitude smaller than the first one and hence is negligible for our semi-quantitative purposes.

Comparing all the results, we see that the longitudinal (actually, tensile) waves have the largest group velocity. This is very typical for all elastic waves.



Problem 7.22. Define and calculate the wave impedances for (i) tensile and (ii) torsional waves in a thin rod, that are appropriate in the long-wave limit. Use the results to calculate the fraction of each wave's power $\mathscr{P}$ reflected from a firm connection of a long rod with a round cross-section to a similar rod with a twice smaller diameter – see the figure on the right.

Solution: As follows from the discussions in Secs. 6.4 and 7.7 of the lecture notes, the wave impedance Z is the ratio of the appropriate cross-section-global force variable to the displacement velocity. The reason for such a "global" definition is that the strain and stress distributions over the rod's cross-section, which may be quite elaborate near a sharp interface, relax to their quasi-stationary forms (calculated in Secs. 7.3 and 7.6) on a relatively small length scale $\Delta z \sim D << \lambda \equiv 2\pi/k$, which may be disregarded at the analysis of long waves' propagation.[70] Such definitions, in particular, enable one to repeat all the arguments leading to Eq. (6.55)-(6.56), and use the latter of these results to calculate the required power ratio:

$$\frac{\mathscr{P}_+}{\mathscr{P}_-} = \left(\frac{Z - Z'}{Z + Z'}\right)^2 . \qquad (*)$$

For the tensile (longitudinal) waves in thin rods, the appropriate force variable is just the total *force*,

$$F_z = \int_A dF_z = \int_A \sigma_{zz} dxdy ,$$

while for the torsional (transverse) waves in such rods, it is the total *torque*,

$$\tau_z = \int_A (\mathbf{r} \times d\mathbf{F})_z = \int_A (xdF_y - ydF_x) = \int_A (x\sigma_{yz} - y\sigma_{xz}) dxdy .$$

As a result, the modification of the impedance definition (6.47), appropriate for the tensile waves, is

$$Z_{\text{tensile}} \equiv \frac{F_z}{\partial q_z / \partial t} = \frac{A\sigma_{zz}}{\partial q_z / \partial t} = AE\frac{\partial q_z / dz}{\partial q_z / \partial t} = AE\frac{k}{\omega} = \frac{AE}{v} = A(E\rho)^{1/2} ,$$

where A is the cross-section area, and Eqs. (7.130) and (7.133) have been used. For a rod with a round cross-section of diameter D, this impedance is proportional to D^2, so $Z'/Z = 1/4$, and hence Eq. (*) yields

$$\left(\frac{\mathscr{P}_+}{\mathscr{P}_-}\right)_{\text{tensile}} = \left(\frac{Z - Z/4}{Z + Z/4}\right)^2 = \left(\frac{3}{5}\right)^2 = 36\% .$$

Similarly (but not identically!), for the torsional waves

$$Z_{\text{torsional}} \equiv \frac{\tau_z}{\partial \varphi / \partial t} = C\frac{\partial \varphi / \partial z}{\partial \varphi / \partial t} = C\frac{k}{\omega} = \frac{C}{v} ,$$

[70] Note that such simple definitions of global variables are not valid for *all* elastic waves, because the strain and strain redistribution over the cross-section may give rise to other wave modes (of the same frequency). However, in thin rods, all other modes have wavelengths of the order of the diameter D of the rod's cross-section (or even smaller), so their lowest ("threshold") frequency is of the order of D/v_t or D/v_l, and hence is much higher than the frequency of the long waves we are considering here.

where C is the torsional rigidity (7.84), and, by using Eqs. (7.89) and (7.141), we get

$$Z_{\text{torsional}} = (\mu\rho)^{1/2} I_z = (\mu\rho)^{1/2} \int_A (x^2 + y^2) dxdy \propto D^4 .$$

Accordingly, for these waves, the impedance ratio for our case is higher, $Z'/Z = 1/2^4 \equiv 1/16$, so per Eq. (*), the wave reflection is much closer to the full one:

$$\left(\frac{\mathscr{P}_+}{\mathscr{P}_-}\right)_{\text{torsional}} = \left(\frac{Z - Z/16}{Z + Z/16}\right)^2 = \left(\frac{15}{17}\right)^2 \approx 78\% .$$

Problem 7.23. Calculate the fundamental frequency of small transverse standing waves on a free uniform thin rod, and the position of displacement nodes in this mode.

Hint: A numerical solution of the final transcendental equation is acceptable.

Solution: The dispersion relation of such waves, given by Eq. (7.136) of the lecture notes, allows for two signs of k^2 for each (positive or negative) frequency ω:

$$k^2 = \pm\left(\frac{\rho A}{EI_y}\right)^{1/2} \omega ,$$

with the positive sign resulting in real values $\pm k$ of the wave number, and the negative sign in its imaginary values $\pm ik$, where k is real and positive:

$$k \equiv \left(\frac{\rho A}{EI_y}\right)^{1/4} \omega^{1/2} .$$

This fact implies that, somewhat counter-intuitively, standing waves on the rod may have not only the usual terms proportional to $\exp\{\pm ikz\}$ but also the terms proportional to $\exp\{\pm i\,(\pm ik)z\} \equiv \exp\{\mp kz\}$. For x-polarized standing waves of a uniform rod positioned along the z-axis, described by Eq. (7.135), this means that the general single-frequency ("monochromatic") solution $q_x(z, t)$ of this equation is proportional to the following spatial factor:

$$q_x(z) = a_+ e^{+ikz} + a_- e^{-ikz} + b_+ e^{+kz} + b_- e^{-kz}, \qquad \text{real if } a_- = a_+^*, \quad b_- = b_+^* . \qquad (*)$$

The constant coefficients $a_\pm$ and $b_\pm$ may be found from the boundary conditions – in our current case of a free rod, the conditions that the transverse forces $F_\pm$ and the bending torques $\tau_\pm$ exerted on both ends, equal zero. According to Eqs. (7.75)-(7.77) of the lecture notes, for a rod of length $2a$, centered at $z = 0$, this means that

$$\frac{d^3 q_x}{dz^3}\bigg|_{z=\pm a} = 0, \qquad \frac{d^2 q_x}{dz^2}\bigg|_{z=\pm a} = 0.$$

Applying these conditions to the solution (*), we see that it should be a symmetric function of z,

$$q_x(z) = A\cos kz + B\cosh kz , \qquad (**)$$

with the coefficients A and B satisfying two homogeneous linear equations:[71]

$$A\sin ka + B\sinh ka = 0, \qquad -A\cos ka + B\cosh ka = 0. \qquad (***)$$

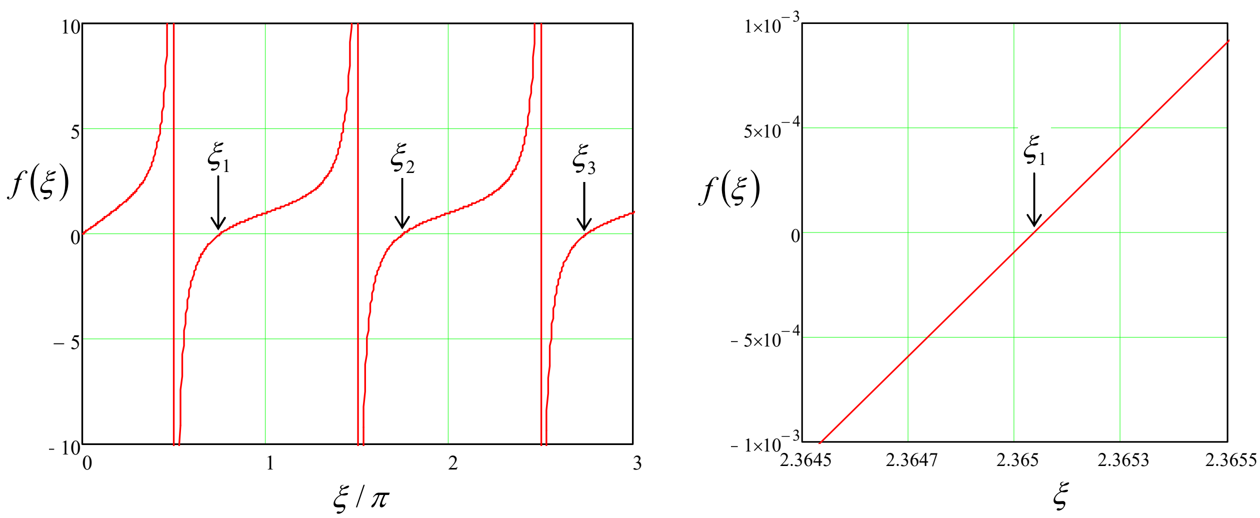
They are compatible only if the determinant of this system equals zero:

$$\begin{vmatrix} \sin ka & \sinh ka \\ -\cos ka & \cosh ka \end{vmatrix} = 0,$$

i.e. if the product ka coincides with one of the roots of the (beautiful!) transcendental equation

$$f(\xi) \equiv \tan\xi + \tanh\xi = 0.$$

The left panel of the figure below shows the "global" plot of this function $f(\xi)$, while its right panel zooms on a close vicinity of its first root ξ_1 corresponding to the lowest value of k (and hence of ω), i.e. to the fundamental standing wave mode.[72]



They show that with the accuracy sufficient for most purposes, $\xi_1 \approx 2.36505$, i.e.

$$k_1 = \frac{\xi_1}{a} \approx \frac{2.36505}{a}, \qquad \omega_1 = \left(\frac{EI_y}{\rho A}\right)^{1/2} k_1^2 = \left(\frac{EI_y}{\rho A}\right)^{1/2} \frac{\xi_1^2}{a^2} \approx 5.5934\left(\frac{EI_y}{\rho A a^4}\right)^{1/2}.$$

It is instructive to compare this result for k with that for the usual sinusoidal standing waves described by the standard 1D wave equation (6.40) with "open" boundary conditions $dq/dz|_{z=\pm a} = 0$ – valid, for example, for longitudinal elastic waves on the same free rod. In that case, the displacements are proportional to $\sin kx$, with the lowest value $ka = \pi/2$, i.e. $k = \pi/2a \approx 1.57/a$ – a very significant (~30%) difference from our k_1.

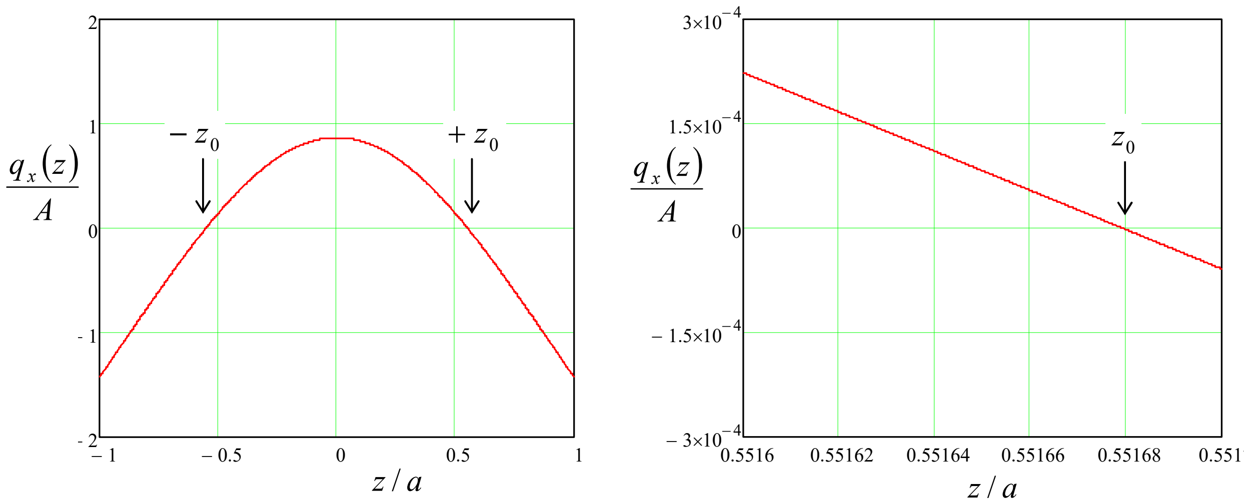
[71] It is straightforward to check that the first of these equations guarantees the absence of oscillations of the rod's center of mass, providing a good sanity check of our calculations for a free rod.

[72] Note that since with the growth of ξ, the function $\tanh\xi$ rapidly approaches 1, the higher roots ξ_n of the function $f(\xi)$ are very close to their asymptotic values $\pi(n - 1/4)$, and even for the fundamental mode ($n = 1$) we are discussing, this expression gives an error below 1%.

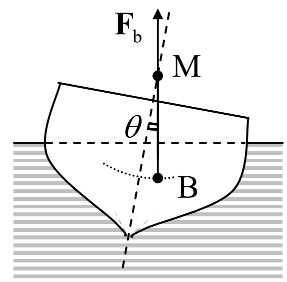
To complete our assignment, we need to plug the calculated value k_1 into Eq. (**), and use it to find the points $\pm z_0$ where $q_x(z) = 0$, using any of Eqs. (***) to find the A/B ratio:

$$q_x(z) = A\left(\cos k_1 z - \frac{\sin k_1 a}{\sinh k_1 a}\cosh k_1 z\right) \equiv A\left(\cos k_1 z + \frac{\cos k_1 a}{\cosh k_1 a}\cosh k_1 z\right).$$

The left panel of the figure below shows the general view of this function, i.e. of the fundamental mode's waveform, while its right panel zooms on a close vicinity of its positive root z_0. They show that $\pm z_0 \approx \pm 0.55168\ a$; note how close are these values to the "minimum-sag" support positions ($\pm 0.5537\ a$) calculated in the solution of Problem 8, as well as to the Bessel support points ($\pm 0.5594\ a$) mentioned in the same model solution. The physical importance of the z_0 calculated in our current solution is that two point supports of a horizontal rod, placed at points $\pm z_0$, would cause no disturbance of this fundamental mode of the transverse oscillations.



Chapter 8. Fluid Dynamics



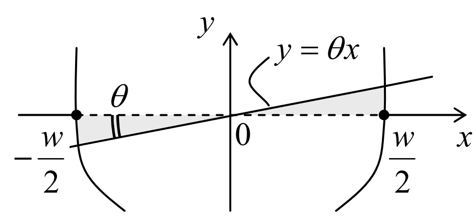
Problem 8.1. For a mirror-symmetric but otherwise arbitrary shape of a ship's hull, derive an explicit expression for the height of its metacenter M – see Fig. 8.3 of the lecture notes, partly reproduced on the right. Spell out this expression for a rectangular hull.

Solution: As was discussed in Sec. 8.1 of the lecture notes, an important point of this problem is the buoyancy center B: the center of mass of the submerged part of the hull, filled with any uniform material. According to the c.o.m.'s definition (4.13), we may find the buoyancy center's radius vector as

$$\mathbf{R}_B = \frac{1}{V}\int_{\substack{\text{submerged}\\ \text{part}}} \mathbf{r}\, d^3r, \qquad \text{where } V = \int_{\substack{\text{submerged}\\ \text{part}}} d^3r . \tag{*}$$

We may expect that in our case, due to the mirror symmetry of the hull, its tilt by a small angle θ results only in a proportional horizontal shift of that point, while its vertical shift is of the order in θ^2 or higher – a reasonable guess that still has to be verified. As the figure above shows, in this case, the metacenter's height over B is just[73]

$$\text{MB} = \lim_{\theta\to 0} \frac{\Delta x_B}{\theta} . \tag{**}$$



Let us redraw the critical part of the above figure, the waterline's vicinity, in the reference frame bound to the ship, with the x-axis positioned at the waterline in the absence of the tilt, and the y-axis residing in the hull's symmetry plane – see the figure on the right. Since the tilt does not change the contributions to the integrals (*) from the parts that were and still are submerged, their changes are restricted to the two triangular volumes shaded in the figure – the right one getting under the waterline and the left one getting above it. Since the areas of these triangles are, to the first order in small θ, equal to each other,[74] the vertical component of $\mathbf{R}_B$ does not change, while its horizontal component changes by

$$\Delta x_B = \frac{1}{V}\int_{\substack{\text{waterline}\\ \text{area}}} x\,dx\,dz \left| \int_0^{\theta x} dy \right| = \frac{\theta}{V}\int_{\substack{\text{waterline}\\ \text{area}}} x^2 dx dz , \tag{***}$$

so Eq. (**) yields a tilt-independent value

$$\text{MB} = \frac{I_z}{V}, \quad \text{where } I_z \equiv \int_{\substack{\text{waterline}\\ \text{area}}} x^2 dx dz .$$

[73] Note that the popular term *metacentric height* refers not to this MB but to the height of M over the *full ship's* center of mass C – see Fig. 8.3. As was discussed in Sec. 8.1, it is the metacentric height that determines the floatation stability.

[74] If the hull's sides are nearly horizontal (| dw/dy | >> 1) at the waterline, this is true only for very small tilts.

(Just as in the theory of deformations,[75] the last integral is commonly called the *moment of inertia* of the waterline area of the ship for its rotation about the hull's longitudinal axis z.) This means that in the small-angle approximation, point M does not move with the tilt, thus justifying the very notion of the metacenter.

If, in the absence of the tilt, the submerged part of the hull has a rectangular shape: $V = w\times h\times l$, i.e. its sides are strictly vertical ($dw/dy = 0$), the above result may be not only spelled out but even generalized for an arbitrary tilt θ. Indeed, in this case, the line $y = \theta x$ in the figure above becomes $y = x\tan\theta$, and we readily get

$$\Delta x_\text{B} = \frac{w^2}{12h}\tan\theta, \qquad \Delta y_\text{B} = \frac{w^2}{24h}\tan^2\theta.$$

In the limit $\theta \to 0$, the first of these expressions agrees with Eq. (***), while Δy_B is indeed quadratic in this small parameter:

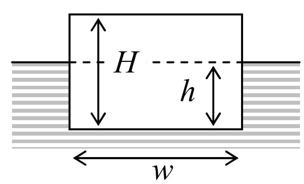
$$\Delta x_\text{B} \to R\theta, \qquad \Delta y_\text{B} \to R\frac{\theta^2}{2}, \qquad \text{where } R \equiv \frac{w^2}{12h} \equiv \frac{w^3 l/12}{whl} = \frac{I_z}{V}.$$

Comparing these results with the Taylor expansion of the Cartesian coordinates of a point near the bottom of a circle of radius R:

$$\Delta x_\text{R} = R\sin\theta \to R\theta, \quad \Delta y_R = R(1-\cos\theta) \to R\frac{\theta^2}{2}, \quad \text{at } \theta \to 0,$$

we see that at small tilts θ, the buoyancy center B moves along a circle of the radius R around the metacenter – see the dotted line in the first figure above. Since point B is at the height $h/2$ above the body's bottom, the metacenter M is located at the height ($h/2 + w^2/12h$) above the same level.

If the body with the same simple geometry and full dimensions $w\times H\times l$ (see the figure on the right) is also uniform, its center of mass C is located at height $H/2$ above the bottom, and the above result for M means that the floating position is stable with respect to rotation about the z-axis (directed along the length l) if



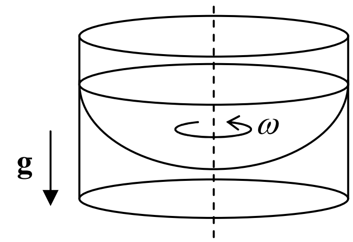
$$\text{MC} = \left(\frac{h}{2} + \frac{w^2}{12h}\right) - \frac{H}{2} > 0\,.$$

This formula shows that at the given dimensions, there is a value of h, equal to $w/\sqrt{6}$, at which the distance MC is smallest:

$$(\text{MC})_\text{min} = \frac{w}{\sqrt{6}} - \frac{H}{2},$$

i.e. the floating position is the least stable. This means, in particular, that if $w > (3/2)^{1/2}H \approx 1.22\,H$, then MC > 0 at any h. This means that any rectangular floating body with l, $w > (3/2)^{1/2}H$ is rotationally stable at any $h < H$, i.e. at any mass that still enables floatation.

[75] See, e.g., Eqs. (7.72) and (7.89) and their discussion in Chapter 7.



Problem 8.2. Neglecting surface tension, find the stationary shape of the open surface of an incompressible heavy fluid's surface in a container rotated about a vertical axis with a constant angular velocity ω – see the figure on the right.

Solution: In order to spell out the condition (8.4) of the fluid's equilibrium in the non-inertial reference frame rotating with the container, we have to add, to the real bulk–distributed gravity forces with the volumic density $\rho\mathbf{g} = -\mathbf{n}_z\rho g$, the centrifugal "forces" (4.93), with the volumic density $\mathbf{n}_\rho\rho\omega^2\mathrm{\rho}$, where $\mathrm{\rho} \equiv (x^2 + y^2)^{1/2}$ is the distance from the rotation axis (taken for axis z), while ρ is the fluid density. The result is

$$-\nabla\mathcal{P} - \mathbf{n}_z\rho g + \mathbf{n}_\rho\rho\omega^2\mathrm{\rho} = 0, \qquad (*)$$

so the radial component of this equation,

$$-\frac{\partial\mathcal{P}}{\partial\rho} + \rho\omega^2\mathrm{\rho} = 0,$$

may be immediately integrated over $\mathrm{\rho}$ to yield

$$\mathcal{P} = \frac{1}{2}\rho\omega^2\mathrm{\rho}^2 + C, \qquad (**)$$

where the integration "constant" C may be a function of z – the function still to be determined. Plugging this solution into the z-component of Eq. (*),

$$-\frac{\partial\mathcal{P}}{\partial z} - \rho g = 0,$$

we get a very simple differential equation $-dC/dz - \rho g = 0$, with the evident solution, $C = -\rho gz + \text{const}$. Thus, Eq. (**) becomes

$$\mathcal{P} = \frac{1}{2}\rho\omega^2\mathrm{\rho}^2 - \rho gz + \text{const}.$$

Now, neglecting the surface tension, we may require the pressure at the surface to be equal to the constant external pressure (say, $\mathcal{P}_0$), getting the following result:

$$\left(\frac{1}{2}\rho\omega^2\mathrm{\rho}^2 - \rho gz\right)_{\text{surface}} = \mathcal{P}_0 = \text{const},$$

showing that the surface has a parabolic shape,

$$z\big|_{\text{surface}} = \frac{1}{2g}\omega^2\mathrm{\rho}^2 + \text{const},$$

independent of the fluid's density and the container's shape.

As a parenthetic remark, this shape is perfect for light focusing, with no so-called *spherical aberrations*[76] typical for the spherical surfaces of traditional refracting and reflecting lenses. This fact is being used for making large telescope mirrors by rotating dishes with liquid mercury. (To the best of my knowledge, the largest one ever built was the Large Zenith Telescope near Vancouver, Canada, with a

[76] See, e.g., Chapter V in M. Born *et al.*, *Principles of Optics*, 7th ed., Cambridge U. Press, 1999.

mirror diameter of 6 meters, rotating at ~6.7 revolutions per minute, i.e. with $\omega \approx 0.7\ \text{s}^{-1}$, so its focal distance $f = g/2\omega^2$ was close to 10 m.[77]) The reader is invited to contemplate the challenges faced by such telescopes. (Hints: telescope aiming at a target, surface waves, harmful mercury vapors, etc.)

Note also another possible approach to the solution of this problem: use the variational calculus to minimize the effective potential energy (4.96b) of the fluid in the rotating, non-inertial reference frame:

$$U_{\text{ef}} = \int_{z\leq z_{\text{surface}}} \left(\rho g z - \frac{1}{2}\rho\mathrm{\rho}^2\omega^2\right) d^3r = 2\pi\rho \int_0^R \left(\frac{1}{2} g z^2\big|_{\text{surface}} - \frac{1}{2}\mathrm{\rho}^2\omega^2 z\big|_{\text{surface}}\right)\mathrm{\rho}\, d\mathrm{\rho}$$

(where the first term in the parentheses describes the potential energy of the fluid in the gravity field, while the second one is the effective potential energy of the centrifugal "force"), upon the condition of constant fluid volume

$$V = \int_{z\leq z_{\text{surface}}} d^3r = 2\pi \int_0^R z\big|_{\text{surface}} \mathrm{\rho}\, d\mathrm{\rho}\,.$$

Such a solution is left for the reader's exercise. (If your variational calculus skills are a bit rusty, the solution of Problem 4 below may serve as a reminder.)

Problem 8.3. In the first order in the so-called *flattening* $f \equiv (R_e - R_p)/R_p << 1$ of the Earth (where R_e and R_p are, respectively, its equatorial and polar radii), calculate it within a simple model in that our planet is a uniformly-rotating nearly-spherical fluid ball, whose gravity field is dominated by a relatively small spherical core. Compare your result with the experimental value of f, and discuss the difference.

Hint: You may use experimental values $R_e \approx 6{,}378$ km, $R_p \approx 6{,}357$ km, and $g \approx 9.807\ \text{m/s}^2$.

Solution: In our simple model, in the 0th approximation in $f << 1$, the Earth is an exact sphere rotating about its polar axis with the angular velocity $\omega_E \approx 2\pi/(24\times60\times60)\ \text{s}^{-1} \approx 0.727\times10^{-6}\ \text{s}^{-1}$. According to Eqs. (1.15) and (3.49) of the lecture notes, the gravitational potential energy of a small element dm of the fluid at its surface, in the field of the Earth's compact core of mass M_E, is

$$dU_g(\mathbf{r}) = -\frac{GM_E dm}{r}, \qquad \text{at } r \approx R_E \approx \frac{R_e + R_p}{2}.$$

In a non-inertial reference frame rotating with the Earth (which is natural for our analysis because, in it, the fluid is in static equilibrium), this dU_g should be supplemented with the centrifugal potential energy dU_{cf} (4.96b) due to the inertial "force" (4.93), so the full effective potential energy of the element is

$$dU_{\text{ef}}(\mathbf{r}) \equiv dU_g + dU_{\text{cf}} = -\frac{GM_E dm}{r} - \frac{1}{2}(\boldsymbol{\omega}_E \times \mathbf{r})^2 dm \equiv -\left(\frac{GM_E}{r} - \frac{\omega_E^2 r^2}{2}\sin^2\theta\right)dm, \qquad (*)$$

where θ is the polar angle ("colatitude") of the point's location on the planet.

[77] See https://en.wikipedia.org/wiki/Large_Zenith_Telescope. A somewhat smaller but apparently more practical International Liquid Mirror Telescope (ILMT) was recently started at the 2,450-meter Indian Observatory on Mount Devastal in the Himalayas – see, e.g., https://www.science.org/content/article/liquid-mirror-telescope-opens-india.

According to Eq. (8.4), in the absence of a substantial horizontal gradient of atmospheric pressure, the effective force $d\mathbf{F}_{ef} = -\nabla(dU_{ef})$ exerted on the mass dm should have no tangential component, i.e. U_{ef} has to be constant on the fluid's surface. Per Eq. (*), this condition gives the following equation for the latitude-dependent Earth's radius R:

$$-\frac{GM_E}{R} - \frac{\omega_E^2 R^2}{2}\sin^2\theta = \text{const.}$$

Due to the relative smallness of the second term (resulting in the smallness of the Earth flattening), this equation may be simplified by expanding the first term into the Taylor series in the small deviation $\Delta R(\theta) \equiv R - R_E$ of R from the average Earth's radius R_E, and then keeping only two leading terms:

$$-\frac{GM_E}{R_E} + \frac{GM_E}{R_E^2}\Delta R(\theta) - \frac{\omega_E^2 R_E^2}{2}\sin^2\theta = \text{const.}$$

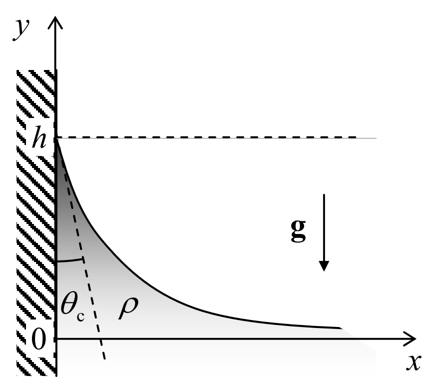
From here,

$$\Delta R(\theta) = \frac{\omega_E^2 R_E^4}{2GM_E}\sin^2\theta + \text{const}, \qquad \text{so } f \equiv \frac{R_e - R_p}{R_p} \approx \frac{\Delta R(\pi/2) - \Delta R(0)}{R_E} = \frac{\omega_E^2 R_E^3}{2GM_E}.$$

Now recalling that the average gravity g on the Earth's surface is GM_E/R_E^2, we may finally write

$$f \approx \frac{\omega_E^2 R_E}{2g}.$$

With the given data and the above value of ω_E, this simple model yields $f \approx 1.72\times10^{-3}$, while the experimental value is somewhat higher, $f_{exp} \approx 3.30\times10^{-3}$. The difference is mostly due to the fact that in reality, the Earth's gravity is provided by all its parts rather than a small core. (A similar model but with a constant mass density gives $f \approx 4.37\times10^{-3}$, on the opposite side of the actual value.)



Problem 8.4.* Use two different approaches to calculate the stationary shape of the surface of an incompressible liquid of density ρ near a vertical plane wall, in a uniform gravity field – see the figure on the right. In particular, find the height h of the liquid's rise at the wall surface as a function of the contact angle θ_c.

Solutions: meniscus near flat wall

Approach 1. Let us introduce the Cartesian coordinates as shown in the figure on the right, with $y = 0$ corresponding to the unperturbed liquid level, i.e. to the height of its surface at $x \to \infty$. The cylindrical geometry of the system ($\partial/\partial z = 0$) means that one of the principal radii of the surface curvature (e.g., R_2) is infinite, while R_1 is expressed by the well-known formula for the curvature of a smooth in-plane line $y(x)$:[78]

$$\frac{1}{R_1} = \frac{d^2y}{dx^2}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-3/2}.$$

[78] See, e.g., MA Eq. (4.3).

Plugging these expressions, as well as the Pascal equation (8.6) for the pressure difference at both sides of the surface,

$$\mathcal{P}_1 - \mathcal{P}_2 = \rho g y,$$

into the Young-Laplace formula (8.13), we get

$$\rho g y = \gamma \frac{d^2 y}{dx^2}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-3/2}.$$

Using the definition of the capillary constant a_c given by Eq. (8.14), this nonlinear differential equation for the surface's shape $y(x)$ may be rewritten in a more compact form:

$$y = \frac{a_c^2}{2}\frac{d^2 y}{dx^2}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-3/2}. \qquad (*)$$

For the first integration of this second-order equation, let us denote $dy/dx \equiv \xi$, so we may rewrite $d^2y/dx^2 \equiv d(dy/dx)/dx$ as $d\xi/dx = (d\xi/dy)(dy/dx) = (d\xi/dy)\xi$. Then Eq. (*) becomes

$$y = \frac{a_c^2}{2}\frac{d\xi}{dy}\xi\left(1+\xi^2\right)^{-3/2}, \quad \text{i.e.} \quad y dy = \frac{a_c^2}{2}\frac{\xi d\xi}{\left(1+\xi^2\right)^{3/2}}.$$

Now integrating both sides of the last equation, we get

$$\frac{y^2}{2} = \frac{a_c^2}{2}\int\frac{\xi d\xi}{\left(1+\xi^2\right)^{3/2}} \equiv \frac{a_c^2}{2}\int\frac{d\left(1+\xi^2\right)/2}{\left(1+\xi^2\right)^{3/2}} = -\frac{a_c^2}{2}\frac{1}{\left(1+\xi^2\right)^{1/2}} + C.$$

The integration constant C may be found from the condition that at large distances from the wall, where $y \to 0$, the derivative $\xi \equiv dy/dx$ should approach 0 as well – see the figure above. This condition yields $C = a_c^2/2$, so after the division of all terms by $a_c^2/2$, this first integral may be rewritten as

$$\frac{1}{\left(1+\xi^2\right)^{1/2}} = 1 - \frac{y^2}{a_c^2}. \qquad (**)$$

This formula is sufficient for the calculation of the maximum height h of the liquid's meniscus. Indeed, as the figure above shows, at the wall, where $y \to h$, the derivative $\xi \equiv dy/dx$ has to equal (–cotanθ_c), so Eq. (**) yields

$$h = a_c\left[1 - \frac{1}{\left(1+\text{cotan}^2\theta_c\right)^{1/2}}\right]^{1/2} \equiv a_c\left(1-\sin\theta_c\right)^{1/2}. \qquad (***)$$

Thus, for a good hydrophilic surface with $\theta_c << 1$, we get $h \approx a_c$, giving a simple way to measure the capillary height (8.14), and hence (at known ρ and g) the surface tension constant γ of a liquid.[79]

[79] An alternative, more popular method of measuring γ is to use the Jurin rule (8.16) for the liquid's rise in a capillary tube with a known small internal radius $a << a_c$.

Now to find the surface's shape, we may solve Eq. (**) for $1/\xi \equiv dx/dy$, selecting the proper sign at the square root to ensure that $dy/dx < 0$ – see the figure above. The result is

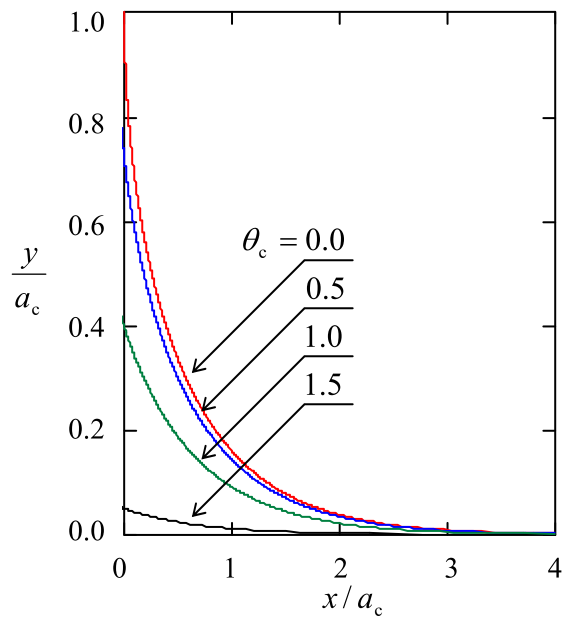
$$\frac{dx}{dy} = -\frac{a_c^2 - y^2}{y\left(2a_c^2 - y^2\right)^{1/2}}, \quad \text{i.e.}\ dx = -\frac{a_c^2 - y^2}{y\left(2a_c^2 - y^2\right)^{1/2}}dy.$$

Integrating both parts of the latter equation, we get

$$x = \frac{a_c}{\sqrt{2}}\cosh^{-1}\frac{\sqrt{2}a_c}{y} - \left(2a_c^2 - y^2\right)^{1/2} + x_0,$$

where the constant x_0 has to be selected so that at $x \to 0$, $y \to h$, where h is given by Eq. (***).

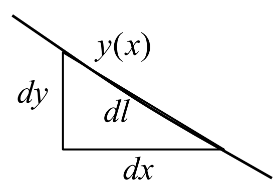
This result is a bit bulky but conceptually very simple: the meniscus' shape (in the units of a_c) is universal, and does not depend on the contact angle θ_c; this constant affects only the horizontal shift x_0 of the profile relative to the wall – see the figure on the right.



Approach 2. An alternative way to solve (or rather to begin solving) this problem, without appealing to the Young-Laplace formula, is to notice that the stationary shape $y(x)$ of the meniscus should minimize the total potential energy (per unit width) of the system,

$$\frac{U}{W} = \rho g\int_0^\infty dx\int_0^{y(x)} y'dy' + \gamma l = \rho g\int_0^\infty \frac{y^2}{2}dx + \gamma l,$$

where the first term is the gravitational energy (referred to its value for the undisturbed surface level $y = 0$), and the second term, in which $l \equiv A/W$ is the length of the line $y(x)$, is the surface energy (8.10). From the elementary geometry (see the figure on the right),



$$dl = \left(dx^2 + dy^2\right)^{1/2} = dx\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2},$$

so we have to minimize the integral

$$\frac{U}{W} = \int_0^\infty \left\{\rho g\frac{y^2}{2} + \gamma\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}\right\}dx,$$

under the following boundary conditions:

$$\frac{dy}{dx} = -\text{cotan}\,\theta_c, \text{ for } x = 0; \qquad y \to 0, \text{ for } x \to \infty. \qquad (****)$$

The necessary condition of the minimum of U that its variation due to an arbitrary small variation $\delta y(x)$ of the surface profile vanishes:

$$\frac{\delta U}{W} \equiv \delta \int_0^\infty \left\{ \rho g \frac{y^2}{2} + \gamma \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{1/2} \right\} dx \equiv \int_0^\infty \rho g y \delta y dx + \gamma \int_0^\infty \delta \left[\left(1 + \xi^2 \right)^{1/2} \right] dx = 0 ,$$

where, as in Approach 1, $\xi \equiv dy/dx$. As was discussed in Sec. 2.1 of the lecture notes (see, in particular, Fig. 2.3 and its discussion), the usual (in this case, dx) and variational differentials may be swapped at will, so the expression under the second integral may be recast as

$$\delta \left[\left(1 + \xi^2 \right)^{1/2} \right] dx = \frac{\partial}{\partial \xi} \left[\left(1 + \xi^2 \right)^{1/2} \right] \delta \xi \, dx \equiv \frac{\xi}{\left(1 + \xi^2 \right)^{1/2}} \delta \left(\frac{dy}{dx} \right) dx = \frac{\xi}{\left(1 + \xi^2 \right)^{1/2}} \frac{d(\delta y)}{dx} dx = \frac{\xi}{\left(1 + \xi^2 \right)^{1/2}} d(\delta y).$$

Now we may take that integral by parts:

$$I \equiv \int_{x=0}^{x=\infty} \frac{\xi}{\left(1 + \xi^2 \right)^{1/2}} d(\delta y) = \left[\frac{\xi}{\left(1 + \xi^2 \right)^{1/2}} \delta y \right]_{x=0}^{x=\infty} - \int_{x=0}^{x=\infty} \delta y d \left[\frac{\xi}{\left(1 + \xi^2 \right)^{1/2}} \right].$$

Due to the boundary conditions (****), which require ξ to be constant at the lower limit and δy to vanish at the upper limit, the first term is constant, so ignoring this inconsequential constant, we get

$$I = -\int_{x=0}^{x=\infty} \delta y d \left[\frac{\xi}{\left(1 + \xi^2 \right)^{1/2}} \right] = \int_{x=0}^{x=\infty} \delta y \frac{1}{\left(1 + \xi^2 \right)^{3/2}} d\xi = \int_0^\infty \delta y \frac{1}{\left(1 + \xi^2 \right)^{3/2}} \frac{d\xi}{dx} dx = \int_0^\infty \delta y \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{-3/2} \frac{d^2 y}{dx^2} dx ,$$

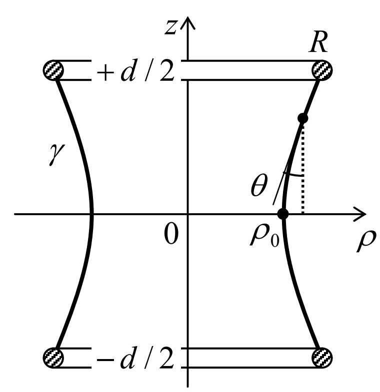
As a result, the zero-variation condition takes the form

$$\int_0^\infty \delta y \left\{ \rho g y + \gamma \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{-3/2} \frac{d^2 y}{dx^2} \right\} dx = 0 .$$

This equality should be valid for an arbitrary (though small) variation $\delta y(x)$, which is possible only if the expression in the curly brackets vanishes at each point. But this condition,

$$\rho g y + \gamma \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{-3/2} \frac{d^2 y}{dx^2} = 0 ,$$

is exactly Eq. (*) obtained in the first approach, so from this point on we may proceed similarly. The reader may see that for this problem, the variational approach leads to longer calculations, but for other problems, it may be the only way toward the solution.



Problem 8.5.* A soap film with surface tension γ is stretched between two similar, coaxial, thin, round rings of radius R – see the figure on the right. Neglecting gravity, calculate the equilibrium shape of the film, and the external force needed for keeping the rings at distance d.

Solution: Just as in Approach 2 in the solution of the previous problem, one way to get a differential equation for the (evidently, axial-symmetric) shape $\rho(z)$ of the film, where $\rho = (x^2 + y^2)^{1/2}$ is the distance from system's axis (see the figure), is to require that the stationary shape minimizes the potential energy U of the system. At negligible gravity effects, U is just twice the energy (8.10) of each surface of the film, i.e. $U = 2\gamma A$, so the requirement is reduced to that for the film's area,

$$A = 2\pi \int_{-d2}^{+d/2} \rho dl = 2\pi \int_{-d2}^{+d/2} \rho \frac{dz}{\cos\theta} = 2\pi \int_{-d2}^{+d/2} \rho \left[1 + \left(\frac{d\rho}{dz}\right)^2\right]^{1/2} dz\ ,$$

to take the smallest possible value at a fixed distance d. (The last step in the above formula uses the geometric relation $d\rho/dz = \tan\theta$ – see the figure above.)

While such calculation by using the variational method is a very useful exercise, highly recommended to the reader, a much faster solution of this particular problem may be carried out just by noticing that in the equilibrium, the total force between two parts of the film, mentally separated by any horizontal plane $z = \text{const}$,

$$|\,F\,| = |\,F_z| = 2\gamma \times 2\pi\rho\cos\theta \equiv 4\pi\gamma \frac{\rho}{\left[1 + \tan^2\theta\right]^{1/2}} = 4\pi\gamma \frac{\rho}{\left[1 + (d\rho/dz)^2\right]^{1/2}}\ ,$$

has to be independent of z. (Indeed, otherwise, the vertical component of the total force acting on the film's fragment between two horizontal planes with some z_1 and z_2, would not vanish, as it is necessary for the film's equilibrium.) Moreover, this constant should be equal to its value $4\pi\gamma\rho_0$ at the film's "neck" (in the figure above, at $z = 0$), where $d\rho/dz = 0$ because of the system's mirror symmetry with respect to this plane. This condition yields the following differential equation for the function $\rho(z)$:

$$\frac{\rho}{\left[1 + (d\rho/dz)^2\right]^{1/2}} = \rho_0\ . \qquad (*)$$

Let us first solve this equation for $d\rho/dz$, and then separate the variables ρ and z:

$$\frac{d\rho}{dz} = \pm\left(\frac{\rho^2}{\rho_0^2} - 1\right)^{1/2}, \qquad \text{so} \quad \frac{d\rho}{\left(\rho^2/\rho_0^2 - 1\right)^{1/2}} = \pm dz\ .$$

Now we may integrate both parts from $z = 0$ to an arbitrary z – for example, on the interval $[0, +d/2]$ where $d\rho/dz \geq 0$:

$$\int_0^{\rho} \frac{d\rho'}{\left(\rho'^2/\rho_0^2 - 1\right)^{1/2}} = \int_0^z dz', \qquad \text{i.e.} \quad \int_0^{\rho/\rho_0} \frac{d\xi}{\left(\xi^2 - 1\right)^{1/2}} = \frac{z}{\rho_0}, \quad \text{where } \xi \equiv \frac{\rho}{\rho_0}.$$

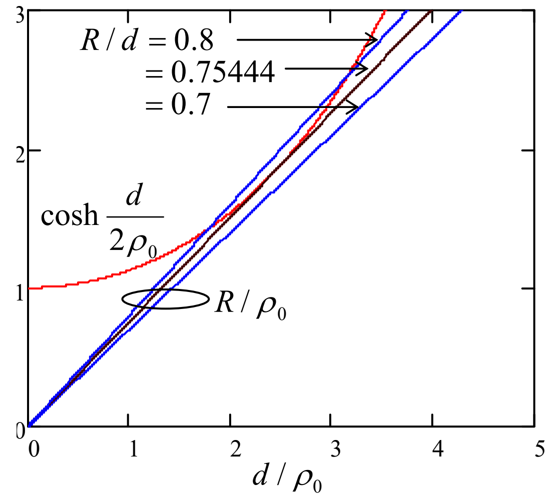
The integral on the left-hand side may be worked out using the variable substitution $\xi \equiv \cosh\alpha$, because then both the numerator, $d\xi = \sinh\alpha\, d\alpha$, and the denominator, $(\xi^2 - 1)^{1/2} = \sinh\alpha$, are proportional to the same function of α, which cancels. As a result, the integral is just $\int d\alpha = \alpha$, and the equation yields simply $\alpha = z/\rho_0$, i.e.

$$\frac{\rho(z)}{\rho_0} \equiv \xi = \cosh\alpha \equiv \cosh\frac{z}{\rho_0}\ .$$

This is the equilibrium shape of the film. What remains is just to find the constant ρ_0 from the boundary condition imposed by the rings supporting the film: $\rho(\pm d/2) = R$. This condition yields a transcendental equation for ρ_0:

$$\frac{R}{\rho_0} = \cosh \frac{d}{2\rho_0}, \tag{**}$$

which has a solution[80] only if the ratio R/d is larger than a certain dimensionless constant, approximately equal to 0.75444 – see the figure on the right. (It shows both sides of Eq. (**) as functions of the ratio d/ρ_0, for three close values of the parameter R/d.)



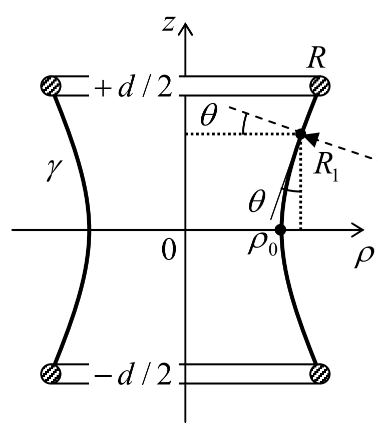
As the solution shows, the equilibrium shape of the film is independent of its surface tension coefficient γ. This coefficient, however, scales the total (vertical) force acting on each supporting ring. As was argued above, the force has to be equal to the total tension force at the film's neck:

$$|F| = 4\pi\gamma\rho_0 .$$

Note that since the neck's radius ρ_0, and hence the force, *decrease* as the rings are pulled apart (i.e. as d is increased at fixed R), this arrangement cannot be used for a stable suspension of the lower ring on the top one in a uniform gravity field.

In addition to this solution, let me show how Eq. (*) may be obtained in a longer but more regular way. Applying the Young-Laplace formula (8.13) to two points very close to some point of the film, but on its opposite sides, we see that since both points are outside of the film, i.e. $\mathcal{P}_1 - \mathcal{P}_2 = 0$, the principal radii of the film's curvature should satisfy a very simple relation:

$$\frac{1}{R_1} = -\frac{1}{R_2}, \quad \text{i.e.} \left|\frac{1}{R_1}\right| = \left|\frac{1}{R_2}\right|. \tag{***}$$



(The negative sign in the first of these formulas means that the corresponding curvature centers should be located on the opposite sides of the film – as they indeed are if the film's shape is as sketched in the first figure above, reproduced on the right with a few additions, with the neck's radius $\rho_0 \equiv \rho(0)$ smaller than the rings' radius R.)

For our axial-symmetric geometry, one of the principal radii (say, R_1) has to reside in the plane passing through the axis, i.e. in the plane of the drawing of the figure on the right, and hence R_1 is just the radius of curvature of the planar curve $\rho(z)$:[81]

$$\frac{1}{R_1} = \frac{d^2\rho/dz^2}{\left[1+(d\rho/dz)^2\right]^{3/2}}.$$

The second principal radius, R_2, has to lie in the perpendicular plane; the dashed line in the last figure above shows the common line of this plane and that of the drawing. Since the angle between this

[80] Actually, as the figure on the right shows, at the ratios R/d above the critical value, Eq. (**) has two solutions, but only one of them, with a larger value of ρ_0, is stable.

[81] See, e.g., MA Eq. (4.3). Note that this formula was already used in the model solution of Problem 3.

line and the line z = const equals $\theta \equiv \tan^{-1}(d\rho/dz)$, the principal curvature $1/R_2$ is $\cos\theta$ times that of the cross-section z = const, which is evidently $1/\rho$:[82]

$$\left|\frac{1}{R_2}\right| = \frac{\cos\theta}{\rho} \equiv \frac{1}{\rho\left[1+(d\rho/dz)^2\right]^{1/2}}.$$

As a result, Eq. (***) yields the following equation for the function $\rho(z)$:[83]

$$\frac{d^2\rho/dz^2}{\left[1+(d\rho/dz)^2\right]^{3/2}} = \frac{1}{\left[1+(d\rho/dz)^2\right]^{1/2}}, \qquad \text{i.e.} \quad \frac{d^2\rho}{dz^2}\rho = 1+\left(\frac{d\rho}{dz}\right)^2.$$

It is straightforward to find the first integral of this second-order differential equation. For this, we may represent the second derivative as[84]

$$\frac{d^2\rho}{dz^2} \equiv \frac{d}{dz}\left(\frac{d\rho}{dz}\right) = \frac{d}{d\rho}\left(\frac{d\rho}{dz}\right)\frac{d\rho}{dz} = \frac{1}{2}\frac{d}{d\rho}\left(\frac{d\rho}{dz}\right)^2,$$

so the equation becomes

$$\frac{1}{2}\frac{d}{d\rho}\left(\frac{d\rho}{dz}\right)^2\rho = 1+\left(\frac{d\rho}{dz}\right)^2, \qquad \text{i.e.} \quad \frac{1}{2}\frac{d\left[1+(d\rho/dz)^2\right]}{1+(d\rho/dz)^2} = \frac{d\rho}{\rho}.$$

Now both sides may be readily integrated from ρ_0 to an arbitrary $\rho \geq \rho_0$, giving

$$\frac{1}{2}\ln\left[1+\left(\frac{d\rho}{dz}\right)^2\right] = \ln\rho - \ln\rho_0, \qquad \text{i.e.} \quad \left[1+\left(\frac{d\rho}{dz}\right)^2\right]^{1/2} = \frac{\rho}{\rho_0},$$

i.e. exactly Eq. (*). Such a more general approach may be invaluable in less symmetric situations.

Problem 8.6. A solid sphere of radius R has been placed into a vorticity-free steady flow, with velocity v_0, of an ideal incompressible fluid. Find the spatial distribution of the fluid's velocity and pressure, and in particular their extreme values. Compare the results with those obtained in Sec. 8.4 of the lecture notes for a round cylinder.

Solution: Just as in a similar problem for a round cylinder (see Sec. 8.4 of the lecture notes), we should solve the Laplace equation $\nabla^2\phi = 0$ for the velocity's potential, with similar boundary conditions:

$$\phi\big|_{r\to\infty} = -v_0 r\cos\theta, \qquad \frac{\partial\phi}{\partial r}\Big|_{r=R} = 0.$$

Now, however, we should use the spherical rather than cylindrical coordinates, with the polar axis directed along the unperturbed fluid velocity, to better exploit the problem's symmetry. In this case, the

[82] This and other useful formulas for such axially-symmetric "surfaces of revolution" may be found in literature – see, e.g., a good online collection at http://mathworld.wolfram.com/SurfaceofRevolution.html.
[83] This is exactly the equation the variational approach would give us.
[84] As a reminder, such transformation (in various forms) was used in this course repeatedly – the first time, at the derivation of Eq. (1.20).

standard variable separation procedure gives the following general axial-symmetric solution of the Laplace equation:[85]

$$\phi(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta),$$

where $P_l(x)$ are the Legendre polynomials. Since the boundary condition at infinity includes only one of these polynomials, $\cos\theta = P_1(\cos\theta)$, only one term (with $l = 1$) of the above sum may have a non-zero coefficient, with the matching value of A_1:

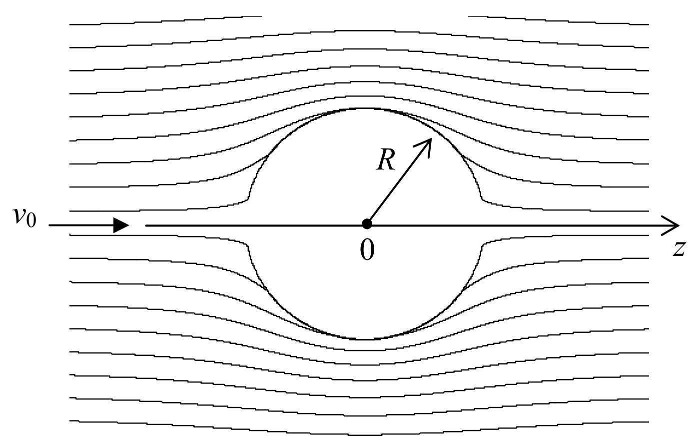
$$\phi(r,\theta)=\left(-v_0 r+\frac{B_1}{r^2}\right)\cos\theta .$$

In order the find the remaining coefficient B_1, we have to use the second boundary condition:

$$\frac{\partial\phi}{\partial r}\Big|_{r=R}\equiv\left(-v_0-\frac{2B_1}{R^3}\right)\cos\theta=0,$$

giving $B_1 = -v_0R^3/2$, i.e., finally,

$$\phi(r,\theta)=-v_0\left(r+\frac{R^3}{2r^2}\right)\cos\theta .$$



The Cartesian components of the fluid's velocity may be found by direct differentiation of the potential: $v_j = -\partial\phi/\partial r_j$. (The figure on the right shows the corresponding streamlines.) In particular, the maximum magnitude of the velocity is achieved on the sphere surface's "equator" ($\theta = \pi/2$):

$$v_{\max}=-\frac{\partial\phi}{\partial z}\bigg|_{\substack{r=R\\ z=0}}=-\frac{1}{R}\frac{\partial\phi}{\partial\theta}\bigg|_{\substack{\theta=\pi/2\\ r=R}}=\frac{3}{2}v_0 . \qquad (*)$$

Note that in the solution of a similar problem for a cylinder (Sec. 8.4) the coefficient in the corresponding expression was 2 rather than 3/2.

In order to find the pressure distribution, we may use Eq. (8.27) with u = const:

$$\mathcal{P}=\mathcal{P}_0-\frac{\rho}{2}v^2,$$

where $\mathcal{P}_0$ is a constant having the physical sense of the pressure's maximum, which is achieved at the sphere's "poles" ($\theta = 0, \pi$), where $v = 0$. (Note that with this reference value, the base value of pressure at infinity is not $\mathcal{P}_0$, but rather $\mathcal{P}_\infty = \mathcal{P}_0 - \rho v_0^2/2$.) Combining Eqs. (*) and (**), we see that the minimum pressure, achieved at the sphere's equator, is

$$\mathcal{P}_{\min}=\mathcal{P}_0-\frac{9}{8}\rho v_0^2=\mathcal{P}_\infty-\frac{5}{8}\rho v_0^2 .$$

[85] See, e.g., EM Sec. 2.8, in particular Eq. (2.172).

Problem 8.7.* Solve the same problem for a long and thin solid strip of width $2w$, with its plane normal to the unperturbed fluid flow.

Hint: You may like to use the so-called *elliptic coordinates* $\{\mu, \eta\}$ defined by their relations with the Cartesian coordinates $\{x, y\}$:

$$x = C\cosh\mu\cos\nu, \quad y = C\sinh\mu\sin\nu, \qquad \text{with } 0 \le \mu < \infty, \; -\pi \le \nu < +\pi,$$

where C is a constant; in these coordinates,

$$\nabla^2 = \frac{1}{C^2(\cosh^2\mu - \cos^2\nu)}\left(\frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2}\right).$$

Solution: The elliptic coordinates are an example of *curvilinear orthogonal coordinates* – of which the polar and spherical coordinates are other examples. Such coordinates are especially convenient for considering objects on whose surfaces some of these coordinates are constant.[86] In our current case, let us direct the y-axis along the initial fluid flow, i.e. normal to the strip's plane, and the x-axis across the strip, so the z-axis is aligned with the strip's length. (Obviously, with this choice, our solution should be z-independent.) As the red lines in the figure on the right show, in this case, with the natural choice $C = w$, the lines of constant μ tend to the strip's surface at $\mu \to 0$, while approaching round circles at $\mu \to \infty$. In the latter limit, the lines of constant ν (shown blue) become straight radial lines at angles ν to the x-axis, while at $\mu \to 0$ these lines run into the strip's cross-section (shown with the bold red line) at

$$x_0 \equiv x\big|_{\mu=0} = w\cos\nu. \qquad (*)$$

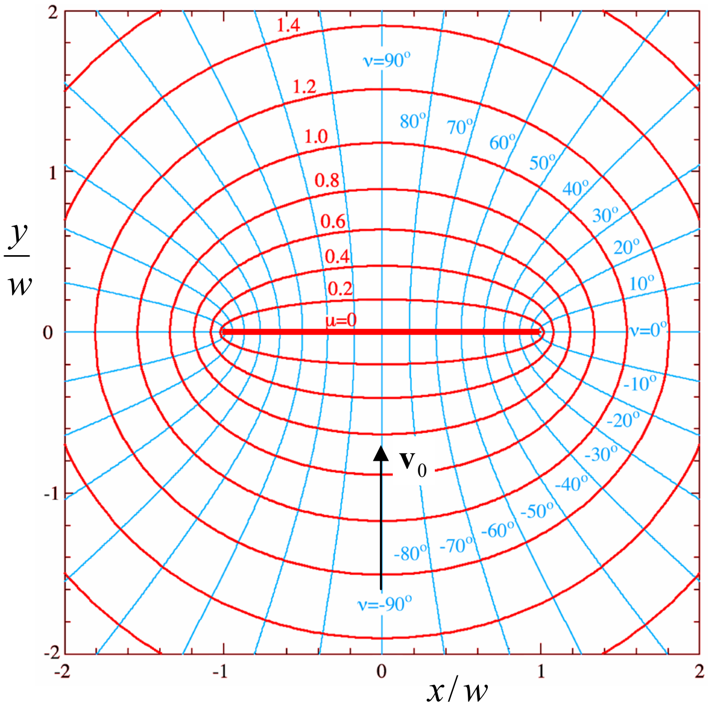


Figure adapted from https://en.wikipedia.org/ as public-domain material.

The main value of these coordinates is that (as the formula given in the *Hint* shows), the Laplace equation (8.29) looks very simple in them, just like in the Cartesian coordinates:

$$\frac{\partial^2\phi}{\partial\mu^2} + \frac{\partial^2\phi}{\partial\nu^2} = 0. \qquad (**)$$

Because of that, some boundary problems of this geometry may be solved with the corresponding potential being a function of just one coordinate, either μ or ν. Unfortunately, in our case, this is not true because due to the boundary condition (8.31), which, with our coordinate choice, becomes

[86] Several examples of such applications are given in the EM part of this series – see, especially, EM Sec. 2.4.

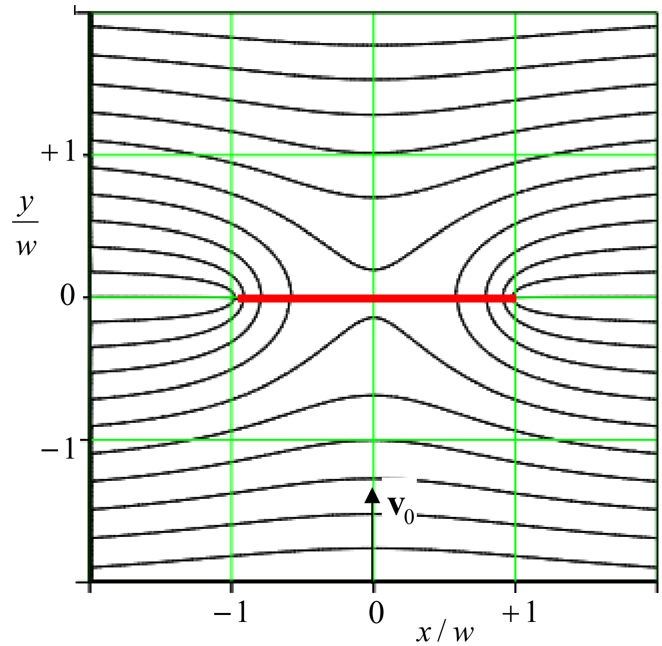
$$\frac{\partial \phi}{\partial y} \to -v_0, \qquad \text{at } \frac{x^2 + y^2}{w^2} \to \infty, \qquad (***)$$

the equipotential lines far from the strip should be parallel to the x-axis, and the figure above shows this is not the case for either μ or ν, and hence for any function of one of these arguments.

However, a thoughtful look at the coordinates' definition shows how simply this issue may be addressed: it is sufficient to swap the operands of these two products, i.e. to consider a new function

$$\phi = a \cosh \mu \sin \nu, \qquad (****)$$

where a is a constant. An elementary differentiation shows that this function satisfies the Laplace equation (**), and as the top figure on the right shows, its equipotential lines do satisfy Eq. (***), provided that $a = -v_0 w$. Indeed, at $\mu \to \infty$, $\cosh \mu \to \sinh \mu \to e^{\mu}/2$, so



$$y \to \frac{w}{2} e^{\mu} \sin \nu, \quad \phi \to -v_0 \frac{w}{2} e^{\mu} \sin \nu = -v_0 y.$$

Finally, this function also satisfies Eq. (8.30) on both surfaces of the strip: since at $\mu \to 0$, we have $y \to w\mu \sin \nu$ and $\phi \to -v_0 w \sin \nu$, then

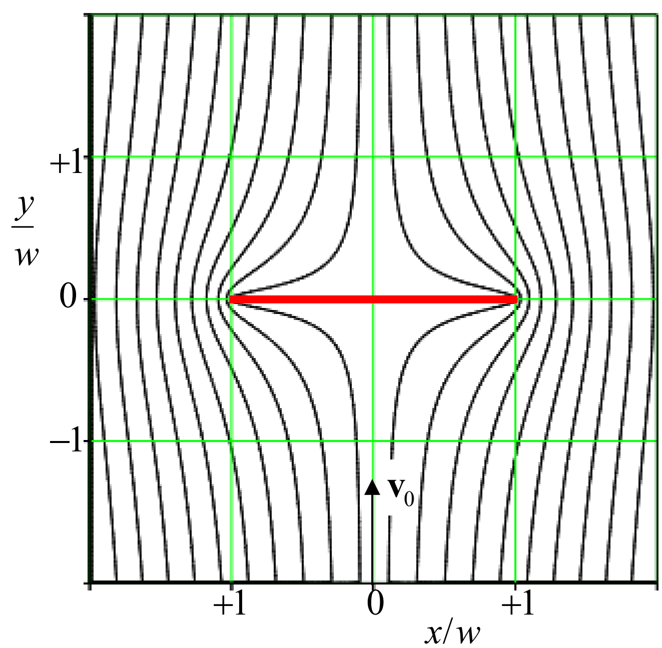
$$\left.\frac{\partial \phi}{\partial y}\right|_{\substack{y=0 \\ |x| \leq w}} = \left.\frac{\partial \phi}{\partial y}\right|_{\mu=0} = 0.$$

So, Eq. (****) with $a = -v_0 w$ is the solution of our boundary problem. (The solutions of such Laplace problems are unique.)

The bottom figure on the right shows the corresponding fluid streamlines. (As it follows from Eq. (****), they coincide with the constant-value lines of another product of the same basic functions of μ and ν:

$$\psi \propto \sinh \mu \cosh \nu;$$

actually, ψ and ϕ are the real and imaginary parts of the same complex function $\sinh(\mu + i\nu)$, while x and y are those of a sibling function, $\cosh(\mu + i\nu)$.)

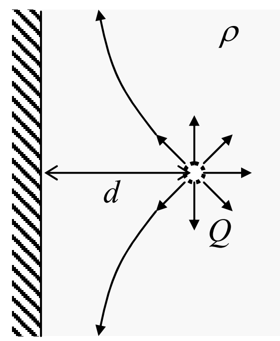


Now it is easy to calculate the fluid velocity at the strip's surfaces, i.e. at $\mu, y \to 0$ and $|x| < w$. Indeed, in this limit, x follows Eq. (*), so the only nonvanishing (tangential) component of the velocity is

$$v_x = -\frac{\partial \phi}{\partial x}\bigg|_{\substack{y=0\\ |x|\leq w}} = -\frac{\partial \phi}{\partial x}\bigg|_{\mu=0} = v_0 \frac{d(\sin\nu)}{d(\cos\nu)} = -v_0 \frac{\cos\nu}{\sin\nu} = \mp v_0 \frac{x}{\left[w^2 - x^2\right]^{1/2}},$$

with the upper (negative) sign on the top surface, $y = +0$. This formula shows that the velocity vanishes at the "stagnation points" $x = 0$, $y = \pm 0$, and diverges at the strip's edges $x = \pm w$, $y = 0$, where the surface's curvature is formally infinite.[87] The last feature is typical for the potential flow of incompressible fluids around sharp points of solid surfaces.

A remark for the reader already familiar with the basics of turbulence (e.g., by reading Secs. 8.5-8.6 of the lecture notes): looking at the Reynolds parameter definition (8.74), it is natural to guess that at the points where $\mathbf{v} \to \infty$, turbulence should start at very low initial velocities v_0, i.e. even if the formal evaluation of this parameter using v_0 for velocity and w for the object size gives a number much smaller than 1. This turns out to be true; indeed, for all thin flat objects normal to the fluid flow direction, the drag coefficient C_d defined by Eq. (8.73) remains nearly constant (close to 1) in an extremely very broad range of velocities – see, e.g., the data for a flat disk in Fig. 8.15.



Problem 8.8. A small source, located at distance d from a plane wall of a container filled with an ideal incompressible fluid of density ρ, injects additional fluid isotropically, with a time-independent mass current ("discharge") $Q \equiv dM/dt$ – see the figure on the right. Calculate the fluid's velocity distribution and its pressure on the wall, created by the flow.

Hint: Recall the charge image method in electrostatics,[88] and contemplate its possible analog.

Solution: Since an isotropic source does not create vorticity, we may assume that in the whole stationary velocity distribution, there is no vorticity as well: $\nabla\times\mathbf{v} = 0$,[89] so we may use Eq. (8.24) of the lecture notes, $\mathbf{v}(\mathbf{r}) = -\nabla\phi$, where the spatial distribution of the velocity's potential ϕ satisfies the Laplace equation (8.29),

$$\nabla^2\phi = 0, \qquad (*)$$

everywhere inside the fluid, besides the source point. If the isotropic source were inside an infinite fluid, the potential distribution would be spherically symmetric with respect to the source position $\mathbf{r}'$:

$$\phi = \phi(R), \qquad \text{where } R = |\mathbf{R}|, \quad \mathbf{R} \equiv \mathbf{r} - \mathbf{r}'.$$

For the stationary flow we are interested in, the function $\phi(R)$ might then be readily found from the continuity equation (8.20), with $\partial/\partial t = 0$, applied to a sphere of radius R with the center at point $\mathbf{r}'$:

[87] As the Bernoulli equation (8.27) shows, at the later points, the fluid's pressure tends to $-\infty$.

[88] See, e.g., EM Secs. 2.9, 3.4, and 4.3.

[89] Indeed, an even very small viscosity leads to eventual decay of any possible initial vorticity – unless it is continually created by either an external source or spontaneously arising turbulence – see Sec. 8.5 of the lecture notes.

$$Q = \rho \oint_R v_n d^2 r \equiv -\rho \oint_R (\nabla \phi)_n d^2 r \equiv -\rho \oint_R \frac{\partial \phi}{\partial R} d^2 r \equiv -\rho 4\pi R^2 \frac{\partial \phi}{\partial R}, \quad \text{i.e. } \frac{\partial \phi}{\partial R} = -\frac{Q}{4\pi \rho R^2}.$$

An elementary integration of the last equation yields

$$\phi = -\frac{Q}{4\pi\rho}\int \frac{dR}{R^2} = \frac{Q}{4\pi\rho R} + \text{const}, \qquad \text{for } R \neq 0.$$

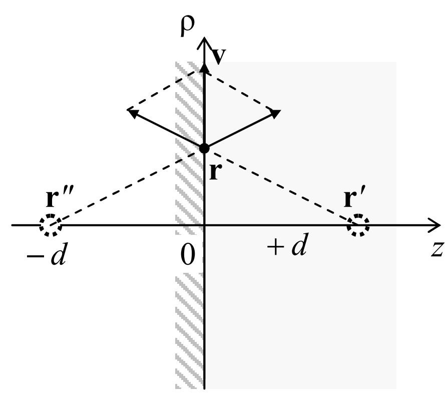
(In what follows, I will follow the convenient custom of taking this inconsequential constant for zero, so $\phi(R) \to 0$ at $R \to \infty$.)

For our system, this solution cannot be strictly valid everywhere, because the normal component of the velocity, and hence the normal derivative of the velocity potential, have to vanish on the wall's surface:[90]

$$\left.\frac{\partial \phi}{\partial n}\right|_{\text{wall}} = 0. \qquad (**)$$

From here comes the basic idea of a simple solution of our problem: The Laplace equation (*) is evidently satisfied by the potential distribution describing two rather than one point sources:

$$\phi(\mathbf{r}) = \frac{Q}{4\pi\rho|\mathbf{r} - \mathbf{r}'|} + \frac{Q''}{4\pi\rho|\mathbf{r} - \mathbf{r}''|}, \quad \text{for } \mathbf{r} \neq \mathbf{r}', \mathbf{r}'',$$



with any Q'' and $\mathbf{r}''$, provided that the additional source point is not located inside the actual fluid. If we use this discretion to take $Q'' = Q$, and the additional source's position $\mathbf{r}''$ at the point of mirror reflection of the real source point $\mathbf{r}'$ in the wall (see the figure on the right), then the boundary condition (**) becomes satisfied as well, because the real source and its image give equal and opposite contributions to the derivative $\partial\phi/\partial n$ on the wall surface located exactly between them. In another language, the radial flows created by two sources add up creating a flow parallel to the wall – see, e.g., the velocity vectors shown in the figure on the right.

In the cylindrical coordinates $\{\text{ρ}, \varphi, z\}$,[91] the resulting solution is

$$\phi = \frac{Q}{4\pi\rho}\left\{\frac{1}{\left[\text{ρ}^2 + (z-d)^2\right]^{1/2}} + \frac{1}{\left[\text{ρ}^2 + (z+d)^2\right]^{1/2}}\right\}, \qquad \text{for } z \geq 0;$$

in particular, at the wall's surface ($z = 0$),

$$\phi = \frac{Q}{2\pi\rho}\frac{1}{\left(\text{ρ}^2 + d^2\right)^{1/2}}.$$

[90] A very similar situation may be met in electrostatics (where the similar boundary condition for the electrostatic potential is valid on the surface of a dielectric with a very high dielectric constant $\kappa >> 1$), and at the Ohmic conduction (on the interface between a conductor and an insulator).

[91] I hope that the difference between the two fonts used for the same Greek letter "ro" is sufficient to distinguish their different meanings.

Since this distribution is axially symmetric, the fluid's velocity at the wall is purely radial, and its magnitude is

$$v\big|_{z=0} = -\frac{\partial \phi}{\partial \mathrm{\rho}}\Big|_{z=0} = \frac{Q}{2\pi\rho}\frac{\mathrm{\rho}}{\left(\mathrm{\rho}^2 + d^2\right)^{3/2}}.$$

Now the change of the pressure on the wall, due to the fluid's flow, may be calculated from the Bernoulli equation (8.27):

$$\Delta\mathcal{P} = -\frac{\rho}{2}v^2\big|_{z=0} = -\frac{Q^2}{8\pi^2\rho}\frac{\mathrm{\rho}^2}{\left(\mathrm{\rho}^2 + d^2\right)^3}.$$

According to this result, the fluid's velocity is the largest, and hence its pressure change is the lowest at the distance $\mathrm{\rho}_0 = d/\sqrt{2}$ from the z-axis.

Problem 8.9. Calculate the average kinetic, potential, and full energies (per unit area) of a traveling sinusoidal wave, of a small amplitude q_A, on the surface of an ideal, incompressible, deep liquid of density ρ, in a uniform gravity field **g**.

Solution: The full energy of such a wave, on an infinite surface, is infinite as well, so let us calculate its kinetic and potential components per unit area of the surface (in Fig. 8.9 of the lecture notes, at the [x, z]-plane), by time-averaging them over one wave period $\mathcal{T} = 2\pi/\omega$:[92]

$$\frac{\overline{T}}{A} = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} dt \int_{y_{\min}}^{y_{\max}(z,t)} dy \frac{\rho v^2}{2} = \frac{\rho}{2}\frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} dt \int_{y_{\min}}^{y_{\max}(z,t)} dy \left(v_y^2 + v_z^2\right), \qquad \frac{\overline{U}}{A} = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} dt \int_{y_{\min}}^{y_{\max}(z,t)} dy\, u = \rho g \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} dt \int_{y_{\min}}^{y_{\max}(z,t)} dy\, y,$$

where $y_{\max}(z, t)$ is the local height of the liquid's surface, and $y_{\min}$ is an artificial depth level necessary to keep U finite. (At $k\,|\,y_{\min}| >> 1$, the wave-related part of U does not depend on the choice of $y_{\min}$.) Per the discussion in Sec. 8.4 of the lecture notes, in the small-amplitude limit $A << \lambda$, we may replace $y_{\max}$ (referred to the unperturbed liquid's surface) with just zero in the first integral, and with $q_y(y = 0, z, t)$ in the second integral. Indeed, the function under the first integral is already proportional to A^2, so the further refining of $y_{\max}$ would give us just small corrections to the non-zero result. However, if we took $y_{\max} = 0$ in the second integral, we would get only the background energy U_0 of the liquid in the gravity field, independent of the wave, so we do need to use a better approximation there. Now using Eqs. (8.43)-(8.44), and then the dispersion relation (8.47): $\omega^2/k = g$, we get

$$\frac{\overline{T}}{A} = \frac{\rho}{2}\left(k\Phi_A\right)^2 \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} dt \int_{-\infty}^{0} dy\, e^{2ky} = \frac{\rho}{2}\left(k\Phi_A\right)^2 \frac{1}{2k} \equiv \frac{\rho}{2}\left(q_A\omega\right)^2 \frac{1}{2k} = \frac{\rho g q_A^2}{4},$$

$$\frac{\overline{U}}{A} = \rho g \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} dt \left(\int_{y_{\min}}^{0} dy\, y + \int_0^{q_y(y=0,z,t)} dy\;\; y\right) = \frac{U_0}{A} + \rho g \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} dt \frac{q_y^2(y=0,z,t)}{2}$$

$$= \frac{U_0}{A} + \frac{\rho g q_A^2}{2}\frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} \sin^2(kz - \omega t)dt = \frac{U_0}{A} + \frac{\rho g q_A^2}{4}.$$

[92] It is almost evident (and will be confirmed by this calculation) that this average does not depend on z.

So, as we could expect from the analogy with the waves discussed in Chapter 7, the average kinetic and potential energies of the wave are equal, and its total energy on a surface of area A is

$$\overline{E} = \frac{\rho g q_A^2}{2} A .$$

The result shows that the energy is of the order of the gravitational energy of the liquid in the volume Aq_A lifted by the height q_A. Figure 8.9 of the lecture notes illustrates how exactly the much larger but sign-alternating contributions to the potential energy, of the order of $dU/dA \sim \rho g q_A \lambda$, cancel each other at their averaging over the spatial (or temporal) period of the wave.

Problem 8.10. Calculate the average power carried by the surface wave discussed in the previous problem (per unit width of its front), and relate the result to the wave's energy.

Solution: The energy flow (i.e. the wave's power) per unit area of the surface (in our case, of the [x, z]-plane) is given by the z-component of the Umov vector (7.123), where z is the direction of the wave's propagation, which is normal to the wave's front x = const:

$$\wp_z = \sigma_{zx} \frac{\partial q_x}{\partial t} + \sigma_{zy} \frac{\partial q_y}{\partial t} + \sigma_{zz} \frac{\partial q_z}{\partial t} .$$

In an ideal liquid, the tensor σ is diagonal, with $\sigma_{jj} = -\mathcal{P}$ (see Eq. (8.2) of the lecture notes), so this expression is reduced to

$$\wp_z = \sigma_{zz} \frac{\partial q_z}{\partial t} = -\mathcal{P} \frac{\partial q_z}{\partial t} . \qquad (*)$$

According to Eq. (8.45) of the lecture notes, the sum $(-\rho\partial\phi/\partial t + \mathcal{P} + \rho g y)$ should remain constant through all ideal liquid's volume. Equating its values at an inner point $\{x, y, z\}$ and at the surface point $\{x, q_y|_{y=0}, z\}$ just above it,

$$-\rho \frac{\partial \phi}{\partial t}(y,z,t) + \mathcal{P}(y,z,t) + \rho g y = -\rho \frac{\partial \phi}{\partial t}\left(q_y\big|_{y=0}, z, t\right) + \mathcal{P}_0 + \rho g q_y\big|_{y=0} ,$$

we get

$$\mathcal{P}(y,z,t) = \rho \frac{\partial \phi}{\partial t}(y,z,t) - \rho \frac{\partial \phi}{\partial t}\left(q_y\big|_{y=0}, z, t\right) + \mathcal{P}_0 + \rho g q_y\big|_{y=0} - \rho g y .$$

Just as at the derivation of Eq. (8.46) in the lecture notes, due to the condition $k\,|\,q_y\,| << 1$, the second term on the right-hand side of this equality may be replaced with $\rho\partial\phi/\partial t(0, z, t)$, so using Eqs. (8.42) and (8.43), we get

$$\begin{aligned}\mathcal{P}(y,z,t) &= \rho \frac{\partial \phi}{\partial t}(y,z,t) - \rho \frac{\partial \phi}{\partial t}(0,z,t) + \mathcal{P}_0 + \rho g q_y\big|_{y=0} - \rho g y \\ &= \rho \Phi_A \left(-\omega e^{ky} + \omega - g\frac{k}{\omega}\right) \sin(kz - \omega t) + \overline{\mathcal{P}}, \qquad \text{where } \overline{\mathcal{P}} \equiv \mathcal{P}_0 - \rho g y ,\end{aligned}$$

so $\overline{\mathcal{P}}$ has the sense of the pressure's static part. According to the dispersion relation (8.47), the y-independent terms in the first parentheses of the last expression cancel, so

$$\mathcal{P}(y,z,t) = -\rho\omega\Phi_A e^{ky} \sin(kz - \omega t) + \overline{\mathcal{P}} .$$

With this result, and with the last of Eqs. (8.43) for $\partial q_z/\partial t \equiv v_z$, Eq. (*) yields

$$\not{p}_z = -\mathcal{P}\frac{\partial q_z}{\partial t} = -\left[-\rho\omega\Phi_A e^{ky}\sin(kz-\omega t)+\overline{\mathcal{P}}\right]\left[k\Phi_A e^{ky}\sin(kz-\omega t)\right].$$

At the averaging over time, the term proportional to $\overline{\mathcal{P}}$ vanishes, and we get

$$\overline{\not{p}}_z = \frac{\rho}{2}\omega k\Phi_A^2 e^{2ky}.$$

To get the full average energy flow (per unit width of the wave front), this expression should be integrated over y, giving

$$\frac{\overline{\mathscr{P}}}{w} \equiv \int_{-\infty}^{0}\overline{\not{p}}_z dy = \frac{\rho\omega\Phi_A^2}{4} \equiv \frac{\rho g\omega q_A^2}{4k}.$$

The physical meaning of this result may be revealed by its comparison with that of the wave's energy calculation in the previous problem:

$$\frac{\overline{\mathscr{P}}}{w}\Big/\frac{\overline{E}}{A} \equiv \overline{\mathscr{P}}\Big/\frac{d\overline{E}}{dz} = \frac{\omega}{2k}.$$

But according to the dispersion relation (8.47), this ratio is just the surface wave's group velocity:

$$u_{\text{gr}} \equiv \frac{d\omega}{dk} = \frac{d}{dk}(gk)^{1/2} = \frac{1}{2}\left(\frac{g}{k}\right)^{1/2} \equiv \frac{\omega}{2k}.$$

Hence our results show, in particular, that the energy of a narrow wave packet would move with the same velocity as its envelope – just as it should.

Problem 8.11. Derive Eq. (8.48) of the lecture notes for the surface waves on a finite-thickness layer of an incompressible ideal liquid.

Solution: Let us review the derivation of Eq. (8.47) in Sec. 8.4 of the lecture notes, modifying it for the case of waves on a uniform liquid layer of a finite thickness h. In this case, the vertical component $v_y = -\partial\phi/\partial y$ of the liquid's velocity has to vanish not at $y \to -\infty$, but at the layer's bottom, $y = -h$, so the solution of Eq. (8.41) for wave potential's amplitude Φ has to be modified as

$$\Phi = \Phi_A \cosh[k(y+h)],$$

(thus ensuring that $\partial\Phi/\partial y = 0$ at $y = -h$), and instead of Eq. (8.42), we get

$$\phi = \Phi_A \cosh[k(y+h)]\, e^{i(kz-\omega t)},$$

leading to the corresponding modification of Eqs. (8.43) and (8.44). In particular,

$$v_y = -\frac{\partial\phi}{\partial y} = -k\Phi_A \sinh[k(y+h)]e^{i(kz-\omega t)}, \quad \text{and} \quad q_y = \int v_y dt = -i\frac{k}{\omega}\Phi_A \sinh[k(y+h)]e^{i(kz-\omega t)}.$$

This result shows, in particular, that the liquid particle trajectories are not circles as at $h \to \infty$ (see Fig. 8.9 of the lecture notes), but vertically-squeezed ellipses with the major axes ratio

$$\frac{A_y}{A_x} = \tanh k(y+h),$$

tending to zero near the bottom of the layer.

Now let us repeat the reasoning given in the lecture notes: based on wave amplitude smallness, from Eq. (8.45) applied to liquid's surface (with $y \approx 0$ in the first term and $y = q_y(y = 0, z, t)$ in the last term), we get the following generalization of Eq. (8.46):

$$-\rho(-i\omega\Phi_A \cosh kh)\, e^{i(kz-\omega t)} + \rho g\left(-i\frac{k}{\omega}\Phi_A \sinh kh\right) e^{i(kz-\omega t)} = 0\,.$$

This relation is satisfied for all z and t (and any Φ_A) only if

$$-\rho(-i\omega \cosh kh) + \rho g\left(-i\frac{k}{\omega}\sinh kh\right) = 0\,,$$

immediately giving the dispersion relation (8.48):

$$\omega^2 = gk \tanh kh\,.$$

The reader may like to revisit the brief discussion of this relation in the lecture notes.

Problem 8.12. The utmost simplicity of Eq. (8.49) of the lecture notes for the velocity of waves on a relatively shallow ($h << \lambda$) layer of an ideal incompressible liquid implies that they may be described using a simple physical picture. Develop such a picture, and verify that it yields the same expression for the velocity.

Solution: According to the solution of the previous problem, at $h << \lambda$, i.e. $kh << 1$, the liquid particles' motion is nearly *horizontal*, so in the notation used in Fig. 8.9 of the lecture notes, $\mathbf{v} \approx v\mathbf{n}_z$. The small *vertical* displacements $q(z, t)\mathbf{n}_y$ of the surface are important only because according to the Pascal equation (8.6), they create the pressure changes,

$$\tilde{P}(z,t) = \rho g q(z,t)\,,$$

which sustain the waves. Per the z-component of the Euler equation (8.23), the longitudinal gradient of this pressure creates the liquid's acceleration

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial \tilde{\mathcal{P}}}{\partial z}, \qquad \text{i.e. } \frac{\partial v}{\partial t} = -g\frac{\partial q}{\partial z}. \qquad (*)$$

One more equation relating the variables v and q may be provided by the continuity relation (8.20a) applied to the mass $dM = \rho(h + q)dz$ of an elementary fragment of the liquid layer with a unit width and length dz:

$$\frac{dM}{dt} \equiv \rho\frac{\partial q}{\partial t}dz = -\rho h dv \equiv -\rho h\frac{\partial v}{\partial z}dz, \qquad \text{i.e. } \frac{\partial q}{\partial t} = -h\frac{\partial v}{\partial z}. \qquad (**)$$

Combining Eqs. (*) and (**),[93] we get the usual 1D wave equations (6.40a) for each of these variables, with the velocity given by Eq. (8.49):

$$u = (gh)^{1/2}. \qquad (***)$$

Note, once again, a different notation for the *wave*'s velocity, used to avoid its confusion with the liquid *particles'* velocity v. Note also a remarkable analogy between Eqs. (*)-(**) and the so-called telegrapher's equations describing TEM waves in electrodynamics.[94]

Problem 8.13. Extend the solution of the previous problem to calculate the energy and power of the shallow-layer waves, and use the result to explain the high tides on some ocean shores, for two models:

(i) the water depth h decreases gradually toward the shore, and
(ii) h decreases sharply, at some distance l from the shore – as it does on the ocean shelf border.

Solution: The simple picture developed in the previous solution enables us to calculate the wave's energy much more simply than it was done (for a more general case) in the solution of Problem 9. Indeed, from this simple picture, we readily get the following expressions for the kinetic and potential energies (again per unit width of the wave front) of a wave's fragment of length dz:

$$dT = \frac{\rho v^2}{2} h dz, \qquad dU = \rho g\left(\int_{-h}^{q} y dy\right)dz = \rho g\left(hq + \frac{q^2}{2} + \text{const}\right)dz. \qquad (*)$$

For a sinusoidal traveling wave, with $v, q \propto \exp\{i(kz - \omega t)\}$, Eqs. (*) and (**) of the previous problem's solution yield two linear homogeneous equations relating the complex amplitudes (and, due to the lack of dispersion, also the instant values) of these variables:

$$\omega v = gkq, \qquad \omega q = hkv. \qquad (**)$$

The condition of their consistency gives $\omega^2 = ghk^2$, i.e. confirms the expression $u \equiv \omega/k = (gh)^{1/2}$, while plugging this equality back into any of Eqs. (**), we get

$$\frac{q}{v} = \left(\frac{h}{g}\right)^{1/2}.$$

With these relations, Eq. (*) gives the following expressions for the full energy of the wave:[95]

$$\frac{dE}{dz} \equiv \frac{dT + dU}{dz} = \rho g q^2 = \rho h v^2.$$

In a "monochromatic" (sinusoidal) wave, $q^2 = q_A^2 \sin^2(\pm kz - \omega t + \text{const})$, so its time (or space) average is $q_A^2/2$, and we get the following average energy per unit length and width of the system:

[93] For example, by differentiating Eq. (*) over z, Eq. (**) over t, and then requiring the two results for the mixed derivative $\partial^2 v/\partial t\partial z$ to coincide.
[94] See, e.g., EM Eqs. (7.110)-(7.111).
[95] The dU term linear in q vanishes at the averaging over the wavelength – see the solution of Problem 9.

$$\frac{d\overline{E}}{dz}=\frac{\rho g q_A^2}{2}.$$

The average power carried by the wave is just the average energy crossing some plane z = const per unit time, i.e.

$$\overline{\mathscr{P}}\equiv\frac{d\overline{E}}{dt}=\pm\frac{d\overline{E}}{dz}\frac{dz}{dt}=\pm\frac{d\overline{E}}{dz}u=\pm\frac{\rho g q_A^2}{2}(gh)^{1/2}\propto q_A^2 h^{1/2}. \qquad (***)$$

Now we can address the two last tasks of the assignment.

(i) As we know from Sec. 6.4, relatively slow changes of the wave system's parameters (in our case, of the liquid layer's thickness h) do not lead to wave reflection, so the product (***) should not depend on z, and the wave's height q_A grows as it approaches the shore:

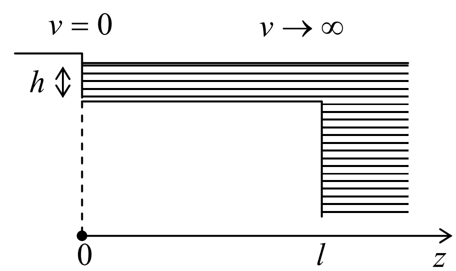
$$q_A\propto h^{-1/4}.$$

Unfortunately, this scaling cannot predict the limit of this increase at $h\to 0$ quantitatively. Its crude estimate may be obtained by using it up to a distance $\Delta z\sim\lambda$ from the shore (where $|dh/dz|\lambda$ becomes comparable to h, so that the approximation of slow parameter variation stops being valid), where $h\sim h_0 \equiv |dh/dz|_{\text{shore}}\lambda\sim|dh/dz|_{\text{shore}}(gh_0)^{1/2}\mathcal{T}$, where $\mathcal{T}$ is the wave's time period. This equality yields

$$h_0\sim g\mathcal{T}^2\left(\frac{dh}{dz}\right)^2_{\text{shore}},\qquad \frac{(q_A)_{\text{shore}}}{(q_A)_{\text{ocean}}}\approx\left(\frac{h_{\text{ocean}}}{h_0}\right)^{1/4}\sim\left(\frac{h_{\text{ocean}}^{1/2}}{g^{1/2}\mathcal{T}|dh/dz|_{\text{shore}}}\right)^{1/2}.$$

For the tidal waves ($\mathcal{T}\approx 4\times10^4$s) on a typical Earth ocean shore, with $h_{\text{ocean}}\sim 4$ km, and $|dh/dz|_{\text{shore}}\sim 0.05$, this estimate gives an amplitude ratio close to 10, in a very reasonable agreement with observations. For example, on the South Shore of our Long Island, the tides' full swings are close to two meters, the number to be compared with ~0.5 m on deep oceans – see the solution of Problem 4.33.

(ii) From Sec. 6.4, we know that a sharp and strong change of the wave system's leads to a nearly complete reflection of the wave. causes a partial reflection of the wave. The wave reflection analysis may be quantified by the introduction of an appropriate wave impedance but for our simple case (see the figure on the right), we do not need this notion. Indeed, it is evident that at the shallow layer's boundary with the shore, the particle velocity has to vanish, while at its step border with a deep ocean, the velocity has to be the highest because just beyond the step, the water's flow is redistributed to a much thicker layer, while its total discharge $Q\propto vh$ has to remain constant. As a result, as the discussion in Sec. 6.5 shows, the standing waves[96] with the velocity profile



$$v(z)=v_A\sin kz,\qquad q(z)=q_A\cos kz \qquad (****)$$

may be almost self-sustained if their wave number takes one of the resonance values given by Eq. (6.72) – see also Fig. 6.10b:

$$k_n=\left(n-\frac{1}{2}\right)\frac{\pi}{l},\qquad \text{with } n=1,2,3,...,$$

[96] In geophysics, such standing-wave resonances are called *seiches*.

corresponding to the standing wave frequencies $\omega_n = k_n u = k_n(gh)^{1/2}$.

The standing waves (****) may be induced, for example, by solar/lunar tides, if their period is close to one ω_n. At the Earth's oceans, such a situation is created by the continental shelf, whose bottom is nearly horizontal with the typical depth $h \approx 150$ m, corresponding to $u \approx 40$ m/s. For this value and the tide time period $\mathcal{T} = \mathcal{T}_{\text{day}}/2 \approx (24\times60\times60)/2$ s $\approx 0.43\times10^5$ s, the fundamental resonant value of l is $l_1 \equiv \pi/2k_1 \equiv \lambda_1/4 = u\mathcal{T}/4 \approx 430$ km. The continental shelf is on average more narrow; however, the resonance condition $l \approx l_1$ is approached in a few long bays, resulting in very high tides. For example, in the Bay of Fundy on the eastern coast of Canada, the tide swings reach 17 meters.

Problem 8.14.* Derive the differential equation describing 2D propagation of relatively long ($\lambda >> h$) surface waves in a plane layer of thickness h, of an ideal incompressible liquid. Use this equation to calculate the longest standing wave modes in a layer covering a spherical planet of radius $R >> h$, and their frequencies.

Hint: The last task requires some familiarity with the basic properties of spherical harmonics.[97]

Solution: Starting with a plane layer, we may generalize the approach discussed in Sec. 8.4 of the lecture notes and in the model solution of Problem 11, by looking for a solution of the Euler equation (8.26) in the form

$$\phi(\mathbf{r},t) = f(\boldsymbol{\rho},t)\cosh[k(y+h)]$$

(where $\boldsymbol{\rho}$ is the 2D radius vector of the points on the liquid's horizontal surface, $y = 0$), which satisfies the boundary condition at the bottom of the layer: $v_y = 0$ at $y = -h$. The substitution of this expression into Eq. (8.26) yields the following 2D equation,[98]

$$\nabla^2_{\boldsymbol{\rho}} f + k^2 f = 0, \qquad (*)$$

where $\nabla_{\boldsymbol{\rho}}^2$ is the 2D Laplace operator acting in the layer's plane.

For relatively long waves, with $\lambda >> h$, there is a broad range of lateral distances ($h << \Delta\rho << \lambda$) at which any sinusoidal wave ($f \propto \text{Re}[F_\omega(\boldsymbol{\rho})\exp\{-i\omega t\}]$) may be considered as "1D plane", i.e. propagating in one dimension. This means that at these distances any sinusoidal wave may be represented as

$$f = \text{Re}\left[F_\omega e^{i(\mathbf{k}\cdot\boldsymbol{\rho}-\omega t)}\right],$$

where $\mathbf{k}$ is some locally constant 2D wave vector, with the magnitude $|\,\mathbf{k}\,| = k$ – meaning the same k as in Eq. (*). Selecting the z-axis in the direction of this vector, we get, for the scalar potential ϕ, Eq. (8.40) of the lecture notes, and may repeat all the calculations made in the model solution of Problem 11 to get the dispersion relation (8.48):

$$\omega^2 = gk\tanh kh\,;$$

[97] See, e.g., QM Sec. 3.6.

[98] It is formally similar to the *Helmholtz equation* (see, e.g., EM Secs. 7.6-7.9 and QM Secs. 1.5, 3.1, and 9.1); however, typically this term is reserved for the case when the function f is time-independent, i.e. when it is the time-independent wave's amplitude, rather than the 2D wave as such – as in the case of Eq. (*). For example, Eq. (***) below is the "genuine" Helmholtz equation.

due to the condition $\lambda >> h$, i.e. $kh << 1$, this relation becomes free of dispersion (see also Eq. (8.49) of the lecture notes and the solutions of two previous problems):

$$\omega = uk, \qquad \text{with } u = (gh)^{1/2}. \tag{**}$$

Hence, a 2D Helmholtz equation similar to Eq. (*) is also valid for the complex amplitude of the wave:

$$\nabla^2_{\boldsymbol{\rho}} F_\omega + k^2 F_\omega = 0, \tag{***}$$

with the condition that ω and k in this equation are related by Eq. (**). The values of the wave number k should be calculated from the solution of Eq. (***) with appropriate boundary conditions on the lateral borders of the layer.

For a uniform liquid layer covering a planet of radius $R >> h$, Eqs. (**) and (***) are still valid because the gravity acceleration g is virtually constant through the layer, and the planet's curvature is negligible on the spatial scale $\sim h$ used at the derivation of Eq. (**). If the layer covers a spherical planet as a whole, the boundary conditions for Eq. (***) are those of the angular periodicity:

$$F_\omega(\theta, \varphi + 2\pi) = F_\omega(\theta, \varphi), \tag{****}$$

where θ and φ are the usual angular spherical coordinates (with $0 \leq \theta \leq \pi$). In these coordinates, Eq. (***) takes the form[99]

$$\frac{1}{R^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]F_\omega + k^2 F_\omega = 0.$$

The eigenfunctions of this equation, with the periodicity (****), are either the so-called *spherical harmonics* $Y_l^m(\theta, \varphi)$, or their real linear combinations $Y_{lm}(\theta, \varphi)$, with integer numbers l and m (restricted by relations $l \geq 0$, and $-l \leq m \leq +l$);[100] the corresponding eigenvalues of k^2 are

$$k_l^2 = \frac{l(l+1)}{R^2}, \qquad \text{so that } \omega_l = uk_l = \frac{u}{R}\left[l(l+1)\right]^{1/2}.$$

(As this formula shows, these eigenvalues are independent on m, i.e. are $(2l + 1)$-degenerate.)

However, not all these modes of this spectrum may be implemented for the surface waves because the liquid's incompressibility (i.e. the liquid volume conservation) imposes the following additional restriction on the surface displacement $(q_y)_\omega \propto \text{Re}[F_\omega]$:

$$\oint_{4\pi} (q_y)_\omega d\Omega \equiv \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi (q_y)_\omega = 0.$$

This requirement bans the lowest mode with $l = m = 0$ with $Y_0^0(\theta, \varphi) = Y_{00}(\theta, \varphi) = \text{const}$, which would correspond to a uniform swelling of the liquid layer and have the frequency $\omega_0 = 0$; all other modes satisfy the requirement. Hence the lowest-frequency standing waves are the three modes with $l = 1$ and $m = -1$, 0, and +1, which may be selected, for example, in the form

[99] See, e.g., MA Eq. (10.9), with $\partial/\partial r = 0$ and $r = R$.

[100] See, e.g., QM Sec. 3.6. In particular, the explicit expressions for the harmonics with the lowest l, and hence the lowest k, are listed in QM Eq. (3.174)-(3.176).

$$Y_{10} = a\cos\theta, \qquad Y_{1(+1)} = a\sin\theta\cos\varphi, \qquad Y_{1(-1)} = a\sin\theta\sin\varphi\,.$$

(Since these standing waves differ only by their spatial orientation,[101] and have the same frequency $\omega_1 = \sqrt{2}u/R$, any their linear combination also describes a possible standing wave.)

If this simple model is applied to the Earth's oceans, taking $R = R_E \approx 6.37\times10^6$ m, $g \approx 9.8$ m/s^2, and h equal to the oceans' average depth of 3.7×10^3 m (thus completely ignoring the scattering by landmasses and other depth irregularities), it gives $u \approx 190$ m/s and $\omega_1 \approx 4.2\times10^{-5}$ s^{-1}, corresponding to a time period $\mathcal{T}_1 \equiv 2\pi/\omega_1$ of 1.5×10^5 s $\approx$ 41 hours. Note, however, that the calculation of the wave's attenuation due to the water's viscosity (see Problem 24 below), shows that the Q-factor of such a standing-wave resonance would be rather low, even in the absence of scattering.

Problem 8.15. Calculate the velocity distribution and the dispersion relation of the waves propagating along the horizontal interface of two thick layers of ideal, incompressible, non-mixing liquids of different densities.

Solution: Since the equation describing the fluid dynamics inside each liquid (located, say, below and above the horizontal plane $y = 0$), the wave structure is the same as on the surface of a single liquid (see Sec. 8.4, in particular, Fig. 8.7 and Eq. (8.42) of the lecture notes) with the amplitude decaying into the bulk of each liquid:

$$\phi_\pm = \mathrm{Re}\left[\Phi_\pm e^{i(kz-\omega t)\mp ky}\right], \qquad (*)$$

where the upper sign refers to the top liquid (located at $y > 0$), while the lower sign, to the bottom liquid ($y < 0$). Note that the wave vector k (also playing the role of the amplitude decrease constant) and the frequency ω have to be the same for both liquids because their vertical velocities (and hence the vertical displacements) at the interface (at $y = 0$) should match for all z and t:[102]

$$v_y\Big|_{y=0} \equiv -\frac{\partial\phi_\pm}{\partial y}\Big|_{y=0} = \mathrm{Re}\left[k\Phi_+ e^{i(kz-\omega t)}\right] = \mathrm{Re}\left[-k\Phi_- e^{i(kz-\omega t)}\right].$$

The last equality can hold for all z and t only if

$$\Phi_- = -\Phi_+\,. \qquad (**)$$

The second matching condition at the interface is the equality of the corresponding values $\mathcal{P}_\pm$ of pressure in each liquid (valid at negligible surface tension). These values may be calculated from Eq. (8.45) written for each liquid. Integrated over coordinates (within each liquid), it yields

$$\mathcal{P}_\pm = \rho_\pm\left(\frac{\partial\phi_\pm}{\partial t} - gy\right)$$

– besides inconsequential constants. With the substitution of Eq. (*) for $\phi_\pm$, and the same small-amplitude approximation as was discussed for a single liquid in Sec. 8.4:

[101] See, e.g., the second row in QM Fig. 3.20.

[102] Please note that we cannot impose a similar requirement on the "horizontal" velocities v_z, because at the interface, the two ideal fluids can slip relative to each other without consequences – just as at the fluid-solid interfaces – see the discussion in Sec. 8.4 of the lecture notes.

$$y \to q_y = \int v_y\big|_{y=0} dt = \text{Re}\left[i\frac{k}{\omega}\Phi_+ e^{i(kz-\omega t)} \right] = \text{Re}\left[-i\frac{k}{\omega}\Phi_- e^{i(kz-\omega t)} \right],$$

the requirement $\mathcal{P}_+ = \mathcal{P}_-$ yields the following condition:

$$\rho_+(-i\omega\Phi_+) - \rho_+ g\left(i\frac{k}{\omega}\Phi_+ \right) = \rho_-(-i\omega\Phi_-) - \rho_- g\left(-i\frac{k}{\omega}\Phi_- \right).$$

Plugging it Eq. (**), we get the dispersion relation,

$$\omega^2 = \frac{\rho_- - \rho_+}{\rho_- + \rho_+} gk, \qquad (***)$$

which is evidently a generalization of Eq. (8.47) – to which Eq. (***) reduces at $\rho_+ \to 0$.

According to this result, as the top liquid (with the density ρ_+) becomes heavier, the same k requires a lower ω, i.e. the interface wave becomes slower, and its velocity formally vanishes at $\rho_+ = \rho_-$. This is natural because if the liquids have the same density, a static deformation of their interface does not carry any energy penalty (provided that we ignore the surface tension – as was done in the above analysis), and may persist forever.

However, the most interesting corollary of Eq. (***) is that if the top liquid is made heavier than the lower one ($\rho_+ > \rho_-$), then ω^2 becomes negative, i.e. ω becomes purely imaginary. This means that the interface is unstable; indeed, in this case, the exponents $\exp\{i(kz - \omega t)\}$ in our solution become exponential functions of time. This result is also natural because any attempt to put a layer of a heavier liquid on top of a lighter one would result in a spontaneous transient process, which would eventually lead to the flipping of these two layers. (Our linear equations may adequately describe only the initial stage of this process.)

Problem 8.16. Derive Eq. (8.50) of the lecture notes for the capillary waves ("ripples").

Solution: To account for the surface tension effects, we may repeat the surface wave analysis carried out in Sec. 8.4 of the lecture notes, all the way up to Eq. (8.45). However, the surface tension $\gamma \neq 0$ makes pressure $\mathcal{P}$ right under the surface uneven: according to the Young-Laplace formula (8.13),

$$\mathcal{P} - \mathcal{P}_0 = -\gamma\left(\frac{1}{R_1} + \frac{1}{R_2} \right), \qquad (*)$$

where $R_{1,2}$ are the surface curvature radii. For the "1D-plane" waves (with $\partial/\partial x = 0$) analyzed in Sec. 8.4, one of these radii is infinite, while for the remaining one we may use the wave smallness ($k|q_y| << 1$, meaning in particular that $|\partial q_y/\partial z| << 1$), and write[103]

$$\frac{1}{R_1} = \frac{\partial^2 q_y}{\partial z^2}.$$

For the sinusoidal waves (8.40), the double differentiation over z is equivalent to the multiplication by $(-k^2)$. As a result, the addition of the term (*) to Eq. (8.46) of the lecture notes yields

[103] See, e.g., MA Eq. (4.3) with $df/dx \to 0$.

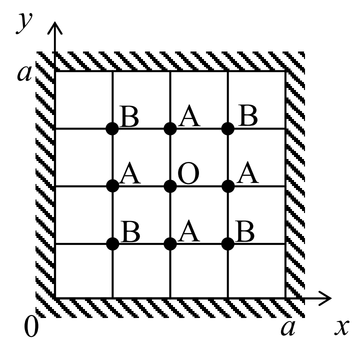
$$-\rho(-i\omega\Phi_A)e^{i(kz-\omega t)}+\gamma k^2\left(-i\frac{k}{\omega}\Phi_A\right)e^{i(kz-\omega t)}+\rho g\left(-i\frac{k}{\omega}\Phi_A\right)e^{i(kz-\omega t)}=0,$$

so requiring this relation to be satisfied for all z and t (and any amplitude Φ_A), we get

$$-\rho(-i\omega)+\gamma k^2\left(-i\frac{k}{\omega}\right)+\rho g\left(-i\frac{k}{\omega}\right)=0,$$

immediately giving the dispersion relation (8.50):

$$\omega^2=gk+\frac{\gamma k^3}{\rho}.$$



Problem 8.17. Use the finite-difference approximation for the Laplace operator, with the mesh step $h = a/4$, to find the maximum velocity and the total discharge Q of an incompressible viscous fluid's flow through a long tube with a square-shaped cross-section of side a. Compare the results with those described in Sec. 8.5 of the lecture notes for the same problem with the mesh step $h = a/2$ and for a tube with a circular cross-section of the same area.

Solution: Due to the evident symmetry of the problem, besides the points on the wall (where $v = 0$), there are only three species of points (O, A, and B) with different values of fluid velocity, on the $a/4$-mesh grid – see the figure on the right. Using Eq. (8.66) of the lecture notes to write, for each of these points, the finite-difference version,

$$\frac{v_{\rightarrow}+v_{\leftarrow}+v_{\uparrow}+v_{\downarrow}-4v}{h^2}=-\frac{\chi}{\eta},\quad \text{with}\;\; \chi\equiv-\frac{\partial P}{\partial z}=\text{const},$$

of the nonvanishing part of the reduced Navier-Stokes equation (8.57), we get the following system of three linear equations for the values of v in those points:

$$\begin{aligned}\text{Point O}:&\;\; 4v_\text{A}-4v_\text{O}=C,\\ \text{Point A}:&\;\; v_\text{O}+2v_\text{B}-4v_\text{A}=C,\\ \text{Point B}:&\;\; 2v_\text{A}-4v_\text{B}=C,\end{aligned}$$

where C is a constant:

$$C\equiv-\frac{\chi}{\eta}h^2=-\frac{a^2\chi}{16\eta}.$$

Solving this (easy) system of three linear equations, we get:

$$v_\text{O}=-\frac{9}{8}C,\quad v_\text{A}=-\frac{7}{8}C,\quad v_\text{B}=-\frac{11}{16}C.$$

The fluid discharge may be calculated as

$$Q=\rho\int_0^a dx\int_0^a dy\, v(x,y)\equiv 4\rho\int_0^{a/2}I(x)dx,\qquad \text{with}\;\; I(x)\equiv\int_0^{a/2}v(x,y)dy,$$

where the chosen Cartesian coordinates are shown in the figure above. To calculate the integrals $I(x)$ and then Q with a comparable precision, we can use the Simpson formula,[104] getting

$$I(0) \approx \frac{a}{12}(0+0+0) = 0,$$

$$I\left(\frac{a}{4}\right) \approx \frac{a}{12}(0 + 4v_B + v_A) = -\frac{a}{12}\frac{29}{8}C,$$

$$I\left(\frac{a}{2}\right) \approx \frac{a}{12}(0 + 4v_A + v_O) = -\frac{a}{12}\frac{37}{8}C,$$

$$Q \approx 4\rho\frac{a}{12}\left[I(0) + 4I\left(\frac{a}{4}\right) + I\left(\frac{a}{2}\right)\right] = -4\rho\left(\frac{a^2}{12}\right)^2\frac{153}{8}C \equiv \frac{17}{512}A^2\rho\frac{\chi}{\eta} \approx 0.0332\, A^2\rho\frac{\chi}{\eta},$$

where $A = a^2$ is the area of the tube's cross-section. We may see that the maximum velocity,

$$v_{\max} = v_O \approx -\frac{9}{8}C = \frac{9}{128}A\frac{\chi}{\eta} \approx 0.0703\, A\frac{\chi}{\eta},$$

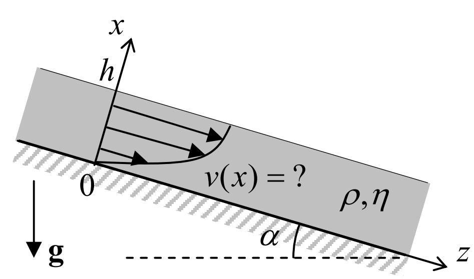
in this approximation, is ~12% higher than that ($v_{\max} = 0.0625\, A\chi/\eta$) calculated in Sec. 8.5 of the lecture notes for the $a/2$ mesh step, while the fluid discharge Q is ~19% larger.

On the other hand, the analytical results given by Eqs. (8.60) and (8.62) of the lecture notes, for the Poiseuille problem of a pipe with a circular cross-section expressed in terms of its cross-section area $A = \pi R^2$, look as follows:

$$v_{\max} = \frac{1}{4\pi}A\frac{\chi}{\eta} \approx 0.0796\, A\frac{\chi}{\eta}, \qquad Q = \frac{1}{8\pi}A^2\frac{\chi}{\eta} \approx 0.0398\, A^2\rho\frac{\chi}{\eta}.$$

Thus the discharge through the tube with a square-shaped cross-section is ~20% lower than that through a round pipe of the same cross-section area A. This difference may be attributed to the tube's parts near its cross-section's corners where the fluid's velocity is relatively low – please compare the calculated values of v_B and v_A.

Problem 8.18. A layer, of thickness h, of a heavy, viscous, incompressible liquid flows down a long and wide inclined plane, under its own weight – see the figure on the right. Calculate the liquid's stationary velocity distribution profile and its discharge per unit width.



Solution: Assuming the laminar flow and with the coordinate choice shown in the figure on the right, we may write $\mathbf{v} = \mathbf{n}_z v(x)$, so the z-component of the Navier-Stokes equation (8.53), in the stationary case ($\partial v/\partial t = 0$), becomes

$$\rho g \sin\alpha + \eta\frac{d^2 v}{dx^2} = 0.$$

[104] See, e.g., MA Eq. (5.3) with $N = 2$.

(Due to the open surface of the liquid, it cannot sustain any pressure gradient in the z-direction.) This equation should be solved with boundary conditions

$$v\Big|_{x=0} = 0, \qquad \frac{dv}{dx}\Big|_{x=h} = 0,$$

the latter one resulting from the absence of forces in the z-direction (and hence the stress tensor element σ_{zx}) on the surface. The (easy) solution of this boundary problem yields

$$v(x) = \frac{\rho g \sin\alpha}{\eta}\frac{x(2h-x)}{2}, \qquad \frac{Q}{w} \equiv \rho\int_0^h v(x)dx = \frac{\rho^2 g h^3 \sin\alpha}{3\eta}.$$

Note that the velocity scales as the reciprocal kinematic viscosity η/ρ rather than the reciprocal dynamic viscosity η – the fact illustrating the popularity of the former notion in all problems where the fluid's flow is caused by gravity.

Problem 8.19. An external force moves two coaxial round disks of radius R, with an incompressible viscous fluid in the gap between them, toward each other with a constant velocity u. Calculate the applied force in the limit when the gap's thickness t is already much smaller than R.

Solution: When the disks are being pressed together, the incompressible fluid is being squeezed out of the gap. Due to the axial symmetry of the problem, the fluid moves radially outward, driven by the local pressure gradient $\chi(\rho) \equiv -\partial\mathcal{P}/\partial\rho > 0$, where ρ is the distance from the system's symmetry axis.

Let us start with finding the distribution of the fluid's velocity across the gap. Due to the strong inequality $t << R$, the flow inside any quasi-rectangular surface element of the area $dA = d\rho\times\rho d\varphi$ with $t << d\rho$, $\rho d\varphi << \rho \leq R$, does not differ from that in an infinite-area gap. Hence repeating the arguments of Sec. 8.5 that have led to Eq. (8.57), we see that we may use the 1D version of this equation:

$$\eta\frac{d^2v}{dz^2} = -\chi(\rho), \qquad \text{for } 0 \leq z \leq t,$$

where z is the distance from one of the disk surfaces, with boundary conditions similar to Eq. (8.55): $v(0) = v(t) = 0$. The solution of this boundary problem is elementary:

$$v = \frac{\chi(\rho)}{\eta}\frac{z(t-z)}{2},$$

giving the following total volumic discharge from a fluid disk of radius ρ:[105]

$$q(\rho) \equiv 2\pi\rho\int_0^t vdz = 2\pi\rho\frac{\chi(\rho)}{\eta}\frac{t^3}{12} \equiv \frac{\pi t^3}{6\eta}\rho\chi(\rho). \qquad (*)$$

Since the fluid is incompressible, the radius-dependent discharge (*) should be equal to the decrease of the volume $\pi\rho^2 t$ of the fluid disk per unit time, due to the disks' approach with velocity u, i.e. to $\pi\rho^2 u$. As a result, Eq. (*) yields the following differential equation for the pressure inside the gap:

105 In order to avoid using the same letter ρ for two different quantities, I am using the *volumic* discharge q. The *mass* discharge Q discussed in the lecture notes equals this q multiplied by the fluid's density.

$$\frac{d\mathcal{P}}{d\rho} \equiv -\chi(\rho) = -\frac{6\eta u}{t^3}\rho,$$

with the boundary condition $\mathcal{P}(R) = \mathcal{P}_0$, where $\mathcal{P}_0$ is the pressure just outside the gap. An easy integration yields the result,

$$\mathcal{P} = \mathcal{P}_0 + \frac{3\eta u}{t^3}\left(R^2 - \rho^2\right),$$

which enables us to calculate the external force necessary to compensate for the pressure difference:

$$F = \int_{\text{disk area}} (\mathcal{P} - \mathcal{P}_0) d^2\rho = \frac{3\eta}{t^3} 2\pi \int_0^R \left(R^2 - \rho^2\right)\rho d\rho = \frac{3\pi}{2}\frac{\eta R^4}{t^3} u.$$

Very naturally, the necessary force increases fast as the gap is narrowed, due to the increasing fluid's drag.

Problem 8.20. Calculate the drag torque exerted on a unit length of a solid round cylinder, of radius R, that rotates about its axis with an angular velocity ω, inside an incompressible fluid with viscosity η, kept static far from the cylinder.

Solution: It is natural to work in cylindrical coordinates, with the z-axis coinciding with the cylinder's axis. Due to the axial and longitudinal symmetry of the problem, the fluid's velocity may be looked for in the form $\mathbf{v}(\mathbf{r}) = \mathbf{n}_\varphi v(\rho)$, so the full expression for the Laplace operator of a vector variable[106] is reduced to

$$\nabla^2 \mathbf{v} = \mathbf{n}_\varphi \left(\nabla^2 v - \frac{v}{\rho^2}\right) = \mathbf{n}_\varphi \left(\frac{d^2 v}{d\rho^2} + \frac{1}{\rho}\frac{dv}{d\rho} - \frac{v}{\rho^2}\right).$$

Also, in this stationary situation ($\partial/\partial t = 0$), the circular integrals of the pressure gradient and the distributed potential force $\mathbf{f}$ (if any) have to equal zero, so they cannot have any effect on the fluid's rotation. Because of that, the Navier-Stokes equation (8.53) is reduced to just $\nabla^2\mathbf{v} = 0$, i.e. to an ordinary differential equation

$$\frac{d^2 v}{d\rho^2} + \frac{1}{\rho}\frac{dv}{d\rho} - \frac{v}{\rho^2} = 0, \qquad (*)$$

with the following boundary conditions

$$v(R) = \omega R, \qquad v(\infty) = 0.$$

Looking for the solution of the homogeneous linear equation (*) in the natural form $v = C\rho^\alpha$ with constant C and α, we see that it is satisfied if the latter constant obeys the following characteristic equation:

$$\alpha(\alpha - 1) + \alpha - 1 = 0, \qquad \text{giving } \alpha = \pm 1.$$

Thus Eq. (*) is satisfied by any linear combination of the functions ρ and $1/\rho$.[107]

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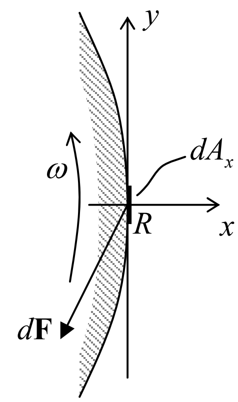
[106] See, e.g., MA Eqs. (10.6) and (10.3). Note that $\nabla^2(\mathbf{n}_\varphi v)$ is not equal to $\mathbf{n}_\varphi \nabla^2 v$ even in this simple case!

[107] Such a linear-combination solution allows a ready generalization of this problem to the case of two concentric cylinders rotating with different angular velocities (the so-called *Couette flow*) – an additional exercise highly recommended to the reader.

Our boundary conditions may be only satisfied by the latter partial solution, with $C/R = \omega R$, so

$$v = \frac{\omega R^2}{\rho}. \qquad (**)$$

It may look like the problem is virtually solved, but the recalculation of this velocity distribution into the stress tensor $\sigma_{jj'}$ by using Eq. (8.52a) of the lecture notes still requires some care because Eq. (8.52b) for the strain derivative tensor $e_{jj'}$ is valid only in the Cartesian but not in curvilinear (such as the cylindrical) coordinates.[108] Perhaps the easiest way to do this calculation is to rewrite Eq. (**) in the Cartesian coordinates



$$x = \rho\cos\varphi, \qquad y = \rho\sin\varphi, \qquad (***)$$

with the x-axis passing through the point at which we want to calculate the tangential component dF_y of the force acting on a small surface element dA_x – see the figure on the right. For an arbitrary point $\{x, y\}$ (with $\rho = (x^2 + y^2)^{1/2} \geq R$), the differentiation of Eqs. (***) over time, for our velocity distribution (**), and with $\dot{\rho} = 0$ and $\rho\dot{\varphi} = v$, gives

$$v_x \equiv \dot{x} = -v\sin\varphi \equiv -\frac{\omega R^2 y}{x^2 + y^2}, \qquad v_y \equiv \dot{y} = v\cos\varphi = \frac{\omega R^2 x}{x^2 + y^2},$$

so the needed element of the tensor (8.52b), taken at point $\{x = R, y = 0\}$, is

$$e_{yx}\Big|_{\substack{x=R\\y=0}} \equiv \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)\Big|_{\substack{x=R\\y=0}} = -\omega R^2 \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2}\Big|_{\substack{x=R\\y=0}} = -\omega,$$

and Eqs. (7.18), (8.51), and (8.52a) yield

$$\frac{dF_y}{dA_x} \equiv \sigma_{yx} = \tilde{\sigma}_{yx} = 2\eta e_{yx} = -2\eta\omega. \qquad (****)$$

Evidently, due to the arbitrary direction of the x-axis (within the cylinder cross-section plane), this expression is valid for the tangential force dF_τ exerted on an arbitrary elementary area dA of the cylinder's surface. From here, the drag torque per unit length of the cylinder is

$$\frac{\tau}{l} = \frac{1}{l}\int_A R\, dF_\tau = \frac{1}{l}\int_A R\frac{dF_\tau}{dA}dA = 2\pi R^2\frac{dF_\tau}{dA} = -4\pi\eta\omega R^2.$$

The negative sign in this expression shows that the drag force is directed, naturally, against the rotation.

<u>Problem 8.21</u>. Solve a similar problem for a sphere of radius R, rotating about one of its principal axes.

Hint: You may like to use the following expression for the relevant element of the strain derivative tensor $e_{jj'}$ in spherical coordinates:

[108] For the reader's reference, the tensor element we would need to solve our problem is $e_{\rho\varphi} = ½ (dv/d\rho - v/\rho)$. As you may readily verify, its use gives the same result (****).

$$e_{r\varphi} = \frac{1}{2}\left(\frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r}\right).$$

Solution: In this case, it is natural to work in spherical coordinates, with the z-axis coinciding with the sphere's rotation axis. Hence we need to look for the solution of the vector Laplace equation $\nabla^2\mathbf{v} = 0$ (see the previous problem's solution) in a similar but more general form $\mathbf{v} = \mathbf{n}_\varphi v(r, \theta)$, so the equation remains partial rather than ordinary:[109]

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) - \frac{v}{r^2\sin^2\theta} = 0, \qquad \text{for } R \leq r < \infty; \qquad (*)$$

It has to be solved with boundary conditions

$$v \to 0, \quad \text{at } r \to \infty; \qquad v = \omega R \sin\theta, \quad \text{at } r = R.$$

Besides that, from the problem's symmetry, v has to vanish at the rotation axis (just as the boundary condition at $r = R$ does) for any $r \geq R$. This is why it is natural to look for the solution of this boundary problem in the form typical for the variable separation method (already discussed in Secs. 6.5, 6.6, and 8.4 of the lecture notes), but with a specific form of the polar-angle functions:

$$v(r,\theta) = \sum_n \mathcal{R}_n(r)\Theta_n(\theta), \qquad \text{with } \Theta_n(\theta) \equiv \sin n\theta, \quad n = 1, 2, ... \qquad (**)$$

(For any given r, this is just the general Fourier series for a function turning to zero at $\theta = 0$ and $\theta = \pi$.) Plugging the n^{th} term of this sum into Eq. (*), and then multiplying all its terms by $r^2/\mathcal{R}_n\Theta_n$ and carrying the two last terms over to the right-hand side, we get

$$\frac{1}{\mathcal{R}_n}\frac{d}{dr}\left(r^2\frac{d\mathcal{R}_n}{dr}\right) = -\frac{1}{\Theta_n \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta_n}{d\theta}\right) + \frac{1}{\sin^2\theta}.$$

The left-hand side of this equation depends only on r, while its right-hand side depends only on θ, so they both have to be equal to some constant C_n.

First, let us consider the resulting equation for the function $\Theta_n(\theta)$:

$$-\frac{1}{\Theta_n \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta_n}{d\theta}\right) + \frac{1}{\sin^2\theta} = C_n.$$

The direct substitution shows that only the function $\Theta_1(\theta) \equiv \sin\theta$ satisfies it (giving the variable separation constant $C_1 = 2$), so all other functions $\mathcal{R}_n(r)$ have to equal zero. Now let us look for the resulting equation for $\mathcal{R}_1$,

$$\frac{1}{\mathcal{R}_1}\frac{d}{dr}\left(r^2\frac{d\mathcal{R}_1}{dr}\right) = 2,$$

in the form similar to that used in the previous problem: $\mathcal{R}_1 \propto r^\alpha$. The direct substitution shows that the equation is satisfied, provided that the constant α is a root of the following quadratic equation:

$$\alpha(\alpha+1) = 2, \qquad \text{i.e. } \alpha^2 + \alpha - 2 = 0.$$

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[109] See, e.g., MA Eqs. (10.9) and (10.12) with $f_r = f_\theta = 0$ and $\partial/\partial\varphi = 0$.

The standard solution of this characteristic equation shows that it has two roots, $\alpha_+ = 1$ and $\alpha_- = -2$, so the function $\mathcal{R}_1$ is generally a linear combination of two terms proportional, respectively, to r and r^{-2}. The first of these terms does not satisfy our boundary condition at $r \to \infty$,[110] so we have to take $v = \text{const}\times r^{-2}\sin\theta$. Selecting the constant so that this solution satisfies the boundary condition at $r = R$, we finally get

$$v = \frac{\omega R^3}{r^2}\sin\theta.$$

Now using Eq. (8.52) of the lecture notes, with the tensor element provided in the *Hint*,[111] we may calculate the tangential force exerted on an elementary area of the sphere's surface, located at a certain polar angle θ:

$$\frac{dF_\varphi}{dA_r} \equiv \sigma_{\theta r} = \tilde{\sigma}_{\theta r} = 2\eta e_{\theta r} = \eta\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)_{r=R} = -3\eta\omega\sin\theta .$$

What remains is to use this result to calculate the net torque exerted on the sphere:

$$\tau = \int_{\text{sphere}} R\sin\theta\, dF_\varphi = \int_S R\sin\theta \frac{dF_\varphi}{dA_r} dA_r = -\int_0^\pi R\sin\theta \times 3\eta\omega\sin\theta \times 2\pi R^2 \sin\theta d\theta = -8\pi\eta\omega R^3 .$$

As in the previous problem, the negative sign in this expression shows that the drag torque tries to stop the rotation.

Note that this result coincides with that of the previous problem if, in the latter, we formally take the cylinder's length l equal to the extension $2R$ of the sphere along the rotation axis. Of course, such an operation is quantitatively illegitimate because the previous problem was solved by assuming that $l >> R$, but the coincidence indicates that the order of magnitude of both results makes sense, thus serving as a good sanity check.

Problem 8.22. Calculate the tangential force (per unit area) exerted by an incompressible fluid, with density ρ and viscosity η, on a broad solid plane placed over its surface and forced to oscillate along it with amplitude a and frequency ω.

Solution: Let us direct the y-axis vertically, i.e. normally to the oscillating plane, and the z-axis in the direction of plane oscillations – as in Fig. 8.10 of the lecture notes. Then, due to the system's symmetry, in the absence of turbulence, the fluid's velocity is horizontal at any point: $\mathbf{v}(\mathbf{r}, t) = \mathbf{n}_z v(y, t)$, while the distributed force $\mathbf{f} = \rho\mathbf{g}$ and the gradient of pressure $\mathcal{P} = \mathcal{P}(y)$ do not have horizontal components. As a result, the Navier-Stokes equation (8.53) is reduced to just

$$\rho\frac{\partial v}{\partial t} = \eta\frac{\partial^2 v}{\partial y^2}. \qquad (*)$$

This equation has to be solved with the following boundary conditions:

[110] This term may be essential for the solution of other similar problems, for example, the flow of a fluid confined between two concentric spheres rotating, with different angular velocities, about the same axis.
[111] Otherwise, the force may be re-calculated using the Cartesian coordinates like in the previous problem, but in our current case of spherical coordinates, such calculation is a tad more cumbersome.

$$v(0,t)=\frac{\partial}{\partial t}q(0,t)=\frac{\partial}{\partial t}\mathrm{Re}\left[ae^{-i\omega t}\right]=\mathrm{Re}\left[-i\omega a e^{-i\omega t}\right],\qquad v(-\infty,t)=0,$$

where the position of the fluid's interface with the vibrating plane is taken for $y = 0$. Since Eq. (*) is linear, we may look for its solution in the form $v(y, t) = \mathrm{Re}\,[\,v_0(y)e^{-i\omega t}\,]$. Plugging it into Eq. (*), we get a linear ordinary differential equation for the complex amplitude v_0 of the fluid's velocity oscillations:

$$-i\omega\rho v_0=\eta\frac{d^2v_0}{dy^2},$$

which has to be solved with the boundary conditions $v_0(0) = -i\omega a$, and $v_0(-\infty) = 0$. Due to the latter condition, in the general solution of the equation, we need to keep just one exponent,

$$v_0(y)=v_0(0)e^{\kappa y},\qquad \text{where } -i\omega\rho=\eta\kappa^2,$$

with positive Reκ:[112]

$$\kappa=\left(\frac{-i\omega\rho}{\eta}\right)^{1/2}\equiv\left(\frac{\omega\rho}{2\eta}\right)^{1/2}(1-i).$$

For the only oscillating element of the stress tensor (8.52) at the interface,

$$\sigma_{zy}=\tilde{\sigma}_{zy}=\eta\frac{\partial v}{\partial y}\Big|_{y=0},$$

this solution yields

$$\sigma_{zy}=\eta\,\mathrm{Re}\left[\kappa v_0 e^{-i\omega t}\right]=\left(\frac{\omega\rho\eta}{2}\right)^{1/2}\mathrm{Re}\left[v_0(0)(1-i)e^{-i\omega t}\right].$$

Per the definition (7.18) of the stress tensor, this is just the tangential force per unit area, in the top layer of the fluid. By the 3rd Newton law, the tangential force F exerted by the fluid on a unit area of the plane is equal and opposite:

$$\frac{F}{A}=-\left(\frac{\omega\rho\eta}{2}\right)^{1/2}\mathrm{Re}\left[v_0(0)(1-i)e^{-i\omega t}\right]\equiv\left(\frac{\omega\rho\eta}{2}\right)^{1/2}\left[-\mathrm{Re}\left(v_0(0)e^{-i\omega t}\right)+\mathrm{Re}\left(\omega a e^{-i\omega t}\right)\right].$$

Perhaps the most interesting feature of this result is the separation of the force into two *quadrature components*, phase-shifted by $\pi/2$. Namely, besides the purely viscous force, acting in antiphase with the plane's velocity (and described by the first term in the last square brackets), there is another force of the same magnitude, acting in phase with the plane's displacement $\mathrm{Re}\,[\,ae^{-i\omega t}\,]$, i.e. in antiphase with its acceleration. The latter force may be interpreted as a result of the oscillating plane's dragging with it the following effective mass of the fluid (per unit area):

$$\frac{M}{A}=\frac{F/\left(-\omega^2 a\right)}{A}=\left(\frac{\rho\eta}{2\omega}\right)^{1/2}\equiv\frac{\rho\delta}{2},\qquad \text{with } \delta\equiv\left(\frac{2\eta}{\rho\omega}\right)^{1/2}=\frac{1}{\mathrm{Re}\,\kappa},$$

independent of the oscillation amplitude a – if the amplitude is not high enough to induce turbulence.

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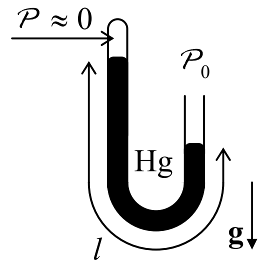
112 Note that this solution, with $\mathrm{Im}\kappa = -\mathrm{Re}\kappa$, i.e. with the oscillation phase growing linearly with distance as $-y/\delta$ (where $\delta \equiv 1/\mathrm{Re}\kappa$ is the depth of the oscillations' penetration into the fluid), is similar to that at the *skin-effect*: the penetration of oscillating electromagnetic field and electric current into a conductor – see, e.g. EM Sec. 6.3.

Using the data in Table 1 in Sec. 8.5 of the lecture notes, we may calculate that, for example, at 1-Hz oscillations on a water surface, the penetration depth δ is close to 0.56 mm, and the dragged mass is about 0.23 kg/m^2.



Problem 8.23. Calculate the frequency and the damping factor of longitudinal oscillations of a mercury column, of the total length l, in a U-shaped mercury manometer (see the figure on the right), assuming that its tube has a round cross-section with a relatively small radius R. Formulate the quantitative conditions of validity of your result and check whether they are fulfilled for the following parameters: l = 1 m and R = 0.25 mm.

Solution: As the solution of the previous problem implies, if

$$\frac{\rho\omega_{\text{ef}}}{2\eta}R^2 << 1, \qquad (*)$$

where ω_{ef} is the effective frequency of oscillations, then the distribution of the mercury's velocity over the cross-section is virtually the same as at its stationary flow, i.e. is given by Eq. (8.60) of the lecture notes, with the discharge Q expressed by Eq. (8.62). Assuming that this condition is satisfied, we may use Eq. (8.62) to express the pressure drop along the mercury column due to the fluid's viscosity:

$$\Delta\mathcal{P}_\eta = \chi l = \frac{8}{\pi}\frac{\eta}{\rho}\frac{l}{R^4}Q. \qquad (**)$$

Let the column be displaced from the equilibrium position (in which the difference of external pressures on its surfaces is counter-balanced by the height difference h_0 in accordance with the Pascal equation (8.6): $\mathcal{P}_0 = \rho g h_0$) by some small distance q. Then the height difference h between the mercury column ends becomes equal to $h_0 + 2q$, leading to the following imbalance between the pressure and gravity forces:

$$F_p = -2q\rho g A = -2\pi R^2 \rho g q,$$

directed toward the equilibrium. Since we may use the virtual incompressibility of mercury in human-scale experiments to express the discharge Q via the time derivative of the same displacement q,

$$Q = -\rho A\dot{q} = -\pi R^2 \rho\dot{q},$$

the total viscous drag force due to the pressure difference (**), directed against the column's velocity $\dot{q}$, is

$$F_\eta = A\Delta\mathcal{P}_\eta = \pi R^2 \Delta\mathcal{P}_\eta = -\pi R^2 \frac{8}{\pi}\frac{\eta}{\rho}\frac{l}{R^4}\pi R^2 \rho\dot{q} \equiv -8\pi\eta l\dot{q}.$$

The total mass of the mercury column is $M = \rho A l = \pi R^2 \rho l$, so we may spell out the 2$^{\text{nd}}$ Newton law for the column's motion, $M\ddot{q} = F_\eta + F_p$,[113] as

$$\pi R^2 \rho l\,\ddot{q} = -8\pi\eta l\dot{q} - 2\pi R^2 \rho g q,$$

[113] Note that for this problem, the effects of surface tension cancel, because it adds equal additional forces to both sides of the column.

giving us the standard differential equation (5.6b) of a damped oscillator,

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = 0, \qquad \text{with } \omega_0 = \left(\frac{2g}{l}\right)^{1/2}, \quad \text{and } \delta = \frac{4\eta}{\rho R^2} .$$

The effective frequency of the height variation process is[114]

$$\omega_{\text{ef}} \approx \min\left[\omega_0, \frac{\omega_0^2}{2\delta}\right], \qquad (***)$$

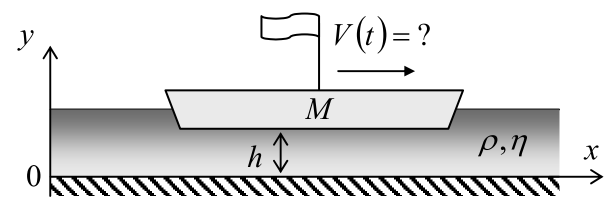
and this is the frequency that should be used in the condition (*).

For the parameters given in the assignment, using the data given in Table 8.1 of the lecture notes, we get $\omega_0 \approx 4.43\ \text{s}^{-1}$ (i.e. $f \equiv \omega/2\pi \approx 0.7$ Hz), $\delta \approx 7.25\ \text{s}^{-1}$, so the system is overdamped and for estimates, we need to use for ω_{ef} the second term in the square brackets of Eq. (***). The substitution shows that the left-hand side of Eq. (*) is close to 0.4, so the condition is reasonably well satisfied.

Another validity condition is the absence of turbulence:

$$Re \equiv \frac{2R\rho v_{\max}}{\eta} \approx \frac{2R\rho\omega q_{\max}}{\eta} < 2{,}100,$$

where $q_{\max}$ is the oscillation amplitude. For our parameters, the result is $q_{\max} \approx 0.3$ m ~ l.



Problem 8.24. A barge, with a flat bottom of area A, floats in shallow water, with clearance $h << A^{1/2}$ – see the figure on the right. Analyze the time dependence of the barge's velocity $V(t)$, and the water's velocity profile, after the barge's engine has been turned off. Discuss the limits of large and small values of the dimensionless parameter $M/\rho Ah$.

Solution: Because of the smallness of h in comparison with the lateral dimensions of the barge, at virtually all the area under the barge's bottom, the water velocity may have only one substantial component: $\mathbf{v} = \mathbf{n}_x v(y,t)$, so the Navier-Stokes equation (8.53) is reduced to

$$\rho\frac{\partial v}{\partial t} = \eta\frac{\partial^2 v}{\partial y^2}.$$

This equation may be satisfied by a sum of factorized (variable-separated) solutions of the type $v(y,t) = T(t)Y(y)$. In fact, by plugging such a solution into the equation, and dividing all terms by TY, we get

$$\rho\frac{1}{T}\frac{dT}{dt} = \eta\frac{1}{Y}\frac{d^2Y}{dy^2}. \qquad (*)$$

The left-hand side may be only a function of t, while the right-hand side, of y; this is only possible if each part is a constant. It is convenient to denote this constant as $-\rho/\tau$ because then the solution of the equation for T has a natural form $T = \text{const}\times\exp\{-t/\tau\}$, i.e. τ has the physical sense of the motion's decay constant. Now, solving the equation for $Y(y)$, with appropriate boundary conditions ($v = 0$ at $y = 0$,

[114] See, e.g., Footnote 5 in Chapter 5 of the lecture notes.

and $v = V(t)$ at $y = h$), we get $Y = \text{const}\times\sin(y/\delta)$, where the physical sense of δ is the characteristic depth of the barge's speed spread under its bottom, so finally the solution is

$$v = V(t)\frac{\sin(y/\delta)}{\sin(h/\delta)}, \qquad V(t) = V(0)e^{-t/\tau}.$$

We still need to find the constants τ and δ. One relation between them follows from Eq. (*):

$$\frac{\rho}{\tau} = \frac{\eta}{\delta^2},$$

while the second relation is similar to Eq. (8.56), in our case for the water-to-barge bottom interface:

$$\eta\frac{\partial v}{\partial y}\bigg|_{y=h} = \frac{F_{wb}}{A},$$

where F_{wb} is the drag force exerted on the water by the barge. This force satisfies the (horizontal component of the) 2nd Newton law for the barge:

$$M\frac{dV}{dt} = F_{bw} = -F_{wb}.$$

Combining these relations, we get the following algebraic characteristic equation for the constant δ:

$$\tan\frac{h}{\delta} = \frac{\rho A\delta}{M}, \qquad (**)$$

which, a bit counter-intuitively, does not depend explicitly on water's viscosity η. (The viscosity, however, does affect the time constant τ of the barge's slowdown; see below.) This transcendental equation allows for simple solutions in two limits:

(i) $M/\rho Ah >> 1$, i.e. the barge is much more massive than the layer of water below it. Then we may guess that $M/\rho A\delta >> 1$ as well, so the tangent function in Eq. (**) is much smaller than 1, and hence is virtually equal to its argument. In this case, the equation yields

$$\delta = h\left(\frac{M}{\rho Ah}\right)^{1/2} >> h, \qquad \text{so that } \frac{M}{\rho A\delta} = \left(\frac{M}{\rho Ah}\right)^{1/2} >> 1,$$

thus confirming the above guess. In this case, the water speed profile is approximately linear, $v \approx (y/h)v_b(t)$, as at the static flow, and the barge's slow-down time constant $\tau = \rho\delta^2/\eta = Mh/\eta$ is proportional to its mass.

(ii) In the opposite limit of relatively deep water (or a relatively light barge), $M/\rho A\delta << 1$, the tangent function in Eq. (**) is much larger than 1, so its argument is very close to $\pi/2$, giving

$$\delta = \frac{2}{\pi}h, \qquad \tau = \frac{4}{\pi^2}\frac{\rho h^2}{\eta}.$$

This means that the water velocity profile corresponds to one-quarter of the sine function's period. The corresponding value of the slow-down time does not depend on the barge's mass, i.e. the solution describes the process of the water layer flow slowing down by itself.

Please note that in the latter limit, our simple factored solution $v = T(t)Y(y)$ is not always strictly valid. In fact, let the barge move, at $t < 0$, with a constant velocity (say, driven by its engine). At this steady motion, Eq. (*) gives $d^2Y/dy^2 = 0$, i.e. the water velocity profile is linear, regardless of the ratio $M/\rho Ah$. Thus, at least immediately after the engine has been stopped, the velocity profile cannot be sinusoidal. In order to describe the transient from the linear to the sinusoidal profile, the solution of Eq. (*) should be sought in a more general form (typical for the variable separation method):

$$v(z,t) = \sum_{n=1}^{\infty} T_n(t)Y_n(y) = \sum_{n=1}^{\infty} C_n \exp(-t/\tau_n)\sin(y/\delta_n), \qquad (***)$$

where δ_n are the roots of Eq. (**), while the coefficients C_n may be found from an expansion of the initial (linear) velocity profile in the series (***) at $t = 0$. However, according to Eq. (**), all δ_n with $n > 1$ are relatively small ($< h/\pi$), so the corresponding time scales $\tau_n = \rho(\delta_n)^2/\eta$ are rather short. This result shows that the higher terms (with $n > 1$) in the series (***) decay in time rapidly, so the simple solution discussed above, which corresponds to $n = 1$, is a very reasonable approximation.

Problem 8.25.* Derive a general expression for mechanical energy loss rate in an incompressible fluid that obeys the Navier-Stokes equation, and use this expression to calculate the attenuation coefficient of the surface waves, assuming that the viscosity is small. (Quantify this condition).

Solution: Perhaps the simplest (though not the most rigorous[115]) way to calculate the energy loss rate, is to use the general Eq. (7.30) for the elementary work of stress forces (per unit volume)

$$\delta w \equiv -\sum_{j,j'=1}^{3} \sigma_{jj'} \delta s_{j'j},$$

where, according to Eq. (7.9b), the variation of the symmetric strain tensor $s_{jj'}$ may be represented as

$$\delta s_{jj'} \equiv \frac{1}{2}\left(\frac{\partial(\delta q_j)}{\partial r_{j'}} + \frac{\partial(\delta q_{j'})}{\partial r_j}\right). \qquad (*)$$

Let us apply these expressions to the variation $\delta s_{jj'}$ taking place in a unit volume of a fluid during a small time interval δt. Plugging into Eq. (*) the stress tensor given by Eq. (8.51), and dividing both sides by δt, we get

$$\frac{\delta w}{\delta t} = \mathcal{P}\frac{1}{\delta t}\sum_j \delta\sigma_{jj} - \sum_{j,j'=1}^{3} \tilde{\sigma}_{jj'}(\mathbf{v})\frac{1}{2}\left(\frac{\partial(\delta q_j/\delta t)}{\partial r_{j'}} + \frac{\partial(\delta q_{j'}/\delta t)}{\partial r_j}\right).$$

But according to Eq. (7.13), the first sum is just $\delta V/V$, while the last operand in the second term describes elements of the tensor e defined by Eq. (8.52b):

$$e_{jj'} = \frac{1}{2}\left(\frac{\partial v_j}{\partial r_{j'}} + \frac{\partial v_{j'}}{\partial r_j}\right),$$

so we may write

[115] A (slightly :-) more strict derivation may be found, for example, in Sec. 16 of L. Landau and L. Lifshitz, *Fluid Mechanics*, 2nd ed., Butterworth-Heinemann, 1987.

$$\frac{\delta w}{\delta t} = \frac{\mathcal{P}}{V}\frac{\delta V}{\delta t} + \frac{\delta \varepsilon}{\delta t}, \quad \text{where} \quad \frac{\delta \varepsilon}{\delta t} \equiv -\sum_{j,j'=1}^{3} \tilde{\sigma}_{jj'} e_{j'j} \ .$$

The first term evidently describes a reversible change of the bulk compression energy of the fluid, while the second term, which depends only on the fluid's velocity distribution (and is constant in time if $\mathbf{v} = \mathbf{v}(\mathbf{r})$ only), represents the rate of irreversible energy losses (per unit volume).

For an incompressible fluid, Tr (e) = 0, so Eq. (8.52) describing the viscosity effects in the Navier-Stokes approximation is reduced to

$$\tilde{\sigma}_{jj'}(\mathbf{v}) = 2\eta e_{jj'} ,$$

and the above expression for energy loss rate may be rewritten as

$$\frac{\delta \varepsilon}{\delta t} = -2\eta \sum_{j,j'=1}^{3} e_{jj'}^2 \ . \qquad (*)$$

Now moving to the second task: as it follows from the discussion in Sec. 6.6 of the lecture notes, the surface wave's attenuation along the direction (say, z) of its propagation may be characterized by the coefficient

$$\alpha \equiv -\frac{d\overline{E} / dz}{\overline{E}} ,$$

where E is the local energy of a wave with a time-independent spatial distribution.[116] If the attenuation is so weak that α is much smaller than the wave number k (i.e. if the wavelength $\lambda = 2\pi/k$ is much smaller than the decay length $l_d \equiv 1/\alpha$), we may calculate the energy dissipation rate ignoring the viscosity effect on the local ($\Delta z \sim \lambda << l_d$) distribution of the fluid velocity components. For the surface waves analyzed in Sec. 8.4 of the lecture notes, these distributions are given by Eq. (8.43), and since in that "1D plane" wave, $\partial/\partial x = 0$, the only nonvanishing elements of the tensor e are:

$$e_{22} = \frac{\partial v_y}{\partial y} = k^2 \Phi_A e^{ky} \cos(kz - \omega t), \quad e_{33} = \frac{\partial v_z}{\partial z} = -k^2 \Phi_A e^{ky} \cos(kz - \omega t),$$

$$e_{23} = e_{32} = \frac{1}{2}\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) = k^2 \Phi_A e^{ky} \sin(kz - \omega t),$$

so in this case the energy loss rate (*) does not depend on z and t:

$$\frac{\delta \varepsilon}{\delta t} = -2\eta \sum_{j,j'=1}^{3} e_{jj'}^2 = -4\eta k^4 \Phi_A^2 e^{2ky} \ .$$

According to this formula, the rate of energy dissipation per unit area,

$$\frac{\delta}{\delta t}\frac{dE}{dxdz} = \int_{-\infty}^{0} \frac{\delta \varepsilon}{\delta t} dy = -4\eta k^4 \Phi_A^2 \int_{-\infty}^{0} e^{2ky} dy = -2\eta k^3 \Phi_A^2 ,$$

[116] This definition implies that the wave is being continuously induced on one end of a long (formally, semi-infinite) system, and propagates inside it, gradually losing energy to dissipation. As was discussed in Sec. 6.6, $l_d \equiv 1/\alpha$ has the physical sense of the length scale of such wave's decay.

is proportional to the wave's energy (also per unit area), whose calculation was the task of Problem 7:

$$\frac{d\overline{E}}{dxdz} = \frac{\rho g q_A^2}{2} \equiv \frac{\rho k}{2}\Phi_A^2,$$

so their ratio does not depend on the wave's amplitude (Φ_A), and we may calculate the attenuation constant as[117]

$$\alpha \equiv -\frac{d\overline{E}/dz}{\overline{E}} = -\frac{1}{v_{\text{gr}}}\frac{\delta\overline{E}/\delta t}{\overline{E}} = -\frac{1}{v_{\text{gr}}}\frac{\delta}{\delta t}\left(\frac{d\overline{E}}{dxdz}\right)\Bigg/\frac{d\overline{E}}{dxdz} = \left(\frac{2k}{\omega}\right)\left(2\eta k^3\Phi_A^2\right)\Bigg/\left(\frac{\rho k}{2}\Phi_A^2\right) = 8\frac{\eta}{\rho}\frac{k^3}{\omega},$$

where, according to Eq. (8.47), the wave's group velocity $u_{\text{gr}} \equiv d\omega/dk = \omega/2k$. Using the same dispersion relation (8.47) again, we may recast this result as a function of the wave number k alone:

$$\alpha = 8\frac{\eta}{\rho}\frac{k^{5/2}}{g^{1/2}}.$$

This formula (with the constants η and ρ taken from Table 8.1) shows that for very long waves on water, the viscosity's contribution to the attenuation may be very low. For example, for the longest waves, with $k \sim 0.1/h_{\max} \sim 10^{-3}$ m^{-1}, i.e. $\lambda \sim 6$ km, for which, according to Eq. (8.48), the finite-depth effects may still be neglected in the deepest parts of the ocean, we get $\alpha \sim 10^{-13}$ m^{-1} – i.e., almost no attenuation on the ocean's width scale $\sim 10^6$ m. This estimate explains, in particular, the virtually attenuation-free propagation of tsunami waves between the continents. (Their amplitude's drop with distance from the source is mostly due to the circular-like propagation.)

Finally, as was discussed above, our approach based on the assumption of viscosity smallness, is valid only if $\alpha << k$. Using the above result for α, we get the following applicability condition:

$$\eta << \rho\frac{g^{1/2}}{8k^{3/2}}, \qquad \text{i.e. } k << \frac{g^{1/3}}{(8\eta/\rho)^{2/3}}.$$

For the water waves on the Earth's surface, this condition is fulfilled for k below $\sim 10^3$ m^{-1}, i.e. for any wavelength larger than ~ 1 mm. (Actually, the surface tension effects, also neglected in our calculation, limit its applicability even earlier, at $\lambda \sim a_c \sim 4$ mm – see Eq. (8.14) and its discussion.)

Problem 8.26. Use the Navier-Stokes equation to calculate the attenuation of a sinusoidal plane acoustic wave.

Solution: As was already discussed in Sec. 7.7 of the lecture notes, in fluids (i.e. materials with vanishing shear modulus μ), only longitudinal acoustic waves may propagate. In a plane wave of this type, the fluid particle displacements $\mathbf{q}$ and velocities $\mathbf{v}$, and also all gradients, have only one Cartesian component: along the propagation direction – say, the z-axis: $\mathbf{q} = \mathbf{n}_z q$, $\mathbf{v} = \mathbf{n}_z v$, $\nabla = \mathbf{n}_z(\partial/\partial z)$. In this case, the Navier-Stokes equation (8.53), in the linear approximation in v, is reduced to

[117] The first of these steps is based on the equivalence of two expressions for the change dE of the local energy of a k-narrow wave packet at an observation point's shift by dz: either from the definition of the attenuation constant: $dE = -\alpha E dz$ or as a result of energy dissipation during the wave packet travel time $dt = dz/v_{\text{gr}}$ into the new position: $dE = (\delta E/\delta t)dt = (\delta E/\delta t)dz/v_{\text{gr}}$.

$$\rho\frac{\partial v}{\partial t} = -\frac{\partial \mathcal{P}}{\partial z} + \left(\zeta + \frac{4\eta}{3}\right)\frac{\partial^2 v}{\partial z^2}.$$

For the pressure gradient calculation, we may use Eq. (7.37) with the relative volume variations (7.13) having just one spatial component:

$$\frac{\partial \mathcal{P}}{\partial z} = \frac{\partial}{\partial z}\left(-K\frac{dV}{V}\right) = -K\frac{\partial}{\partial z}\mathrm{Tr}(\mathrm{s}) = -K\frac{\partial}{\partial z}s_{zz} = -K\frac{\partial^2 q}{\partial z^2},$$

so the Navier-Stokes equation is reduced to

$$\rho\frac{\partial v}{\partial t} = K\frac{\partial^2 q}{\partial z^2} + \left(\zeta + \frac{4\eta}{3}\right)\frac{\partial^2 v}{\partial z^2}.$$

Looking for its solution in the form of a usual sinusoidal wave,

$$q = \mathrm{Re}\left[ae^{i(kz-\omega t)}\right], \qquad \text{so that } v = \frac{\partial q}{\partial t} = \mathrm{Re}\left[-i\omega a e^{i(kz-\omega t)}\right],$$

we get the following dispersion relation:

$$-\omega^2\rho = -Kk^2 + i\omega k^2\left(\zeta + \frac{4\eta}{3}\right), \qquad \text{i.e. } k^2 = \frac{\omega^2}{u_1^2}\left[1 - i\frac{\omega}{K}\left(\zeta + \frac{4\eta}{3}\right)\right]^{-1}, \qquad (*)$$

where $u_1 \equiv (K/\rho)^{1/2}$ is the sound velocity – see Eq. (7.114) with the notation change $v \to u_1$.

At negligible viscosity (or rather both viscosities ζ and η), Eq. (*) yields the standard acoustic dispersion relation $k = \omega/u_1$. However, in the general case, it gives a complex k^2 and hence a complex wave number $k = k' + ik''$. As we know from the discussion in Sec. 6.6, the attenuation coefficient α is just twice the imaginary part of k, so Eq. (*) yields

$$\alpha = 2k'' = 2\frac{\omega}{u_1}\mathrm{Im}\left[1 - i\frac{\omega}{K}\left(\zeta + \frac{4\eta}{3}\right)\right]^{-1/2}.$$

In the most important case of small attenuation ($\alpha\lambda << 1$, i.e. $k'' << k'$),[118] this expression yields:

$$\alpha \approx 2\frac{\omega}{u_1}\mathrm{Im}\left[1 + i\frac{\omega}{2K}\left(\zeta + \frac{4\eta}{3}\right)\right] = \frac{2\omega^2}{u_1 K}\left(\zeta + \frac{4\eta}{3}\right) = \frac{2\omega^2}{u_1^3\rho}\left(\zeta + \frac{4\eta}{3}\right).$$

Its form without the ζ-term is called the *Stokes law* – not to be confused with the Stokes formula (8.71).

For example, for water $\zeta \approx 3.1\times10^{-3}$ Pa·s, $\eta \approx 0.9\times10^{-3}$ Pa·s, $K \approx 2.1\times10^{9}$ Pa, and ρ = 1,000 kg/m^3, so at a typical sound frequency of 1 kHz ($\omega \approx 6\times10^3$ s^{-1}), at which $u_1 = (K/\rho)^{1/2} \approx 1.45$ km/s, the attenuation constant is very small: $\alpha \approx 0.83\times10^{-7}$ m^{-1}, i.e. ~0.36×10^{-3} dm/km in engineering units,[119] in good agreement with experimental data for distilled water. (In typical seawater, the attenuation is much higher, ~60×10^{-3} dm/km, due to the vibration damping by dissolved molecules.)

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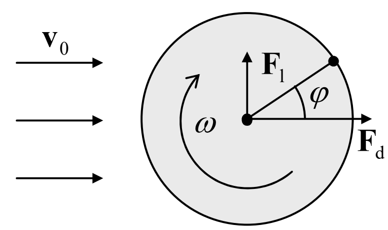
[118] In typical media, this condition is violated only at extremely high frequencies above ~10^{12} Hz.

[119] Their definition is $\alpha\big|_{\text{db/km}} \equiv 10\log_{10}\dfrac{\mathcal{P}(z=0)}{\mathcal{P}(z=1\,\text{km})} = 10\log_{10} e^{10^3\alpha} = \dfrac{10}{\ln 10}10^3\alpha \approx 4.34\times10^3\alpha$.

Note that Eq. (*) does not give accurate results for most gases: due to a strong temperature dependence of their bulk modulus on temperature, their sound velocity depends on whether the compression/extension cycles of the wave are isothermal (as it happens only at extremely low frequencies) or virtually adiabatic – as it takes place for the usual (audible) and higher sound frequencies. In the latter case, the attenuation constant has an additional contribution due to heat transfer from the compressed to extended parts of the wave. In addition, in gases of diatomic and more complex molecules, the attenuation may be dominated, just as in seawater, by molecular damping. For example, due to the latter effect, the sound attenuation in the air (at ambient conditions) is as high as ~5 db/km (at the same frequency of 1 kHz).

Problem 8.27.* Use two different approaches for a semi-quantitative calculation of the Magnus lift force $\mathbf{F}_l$ exerted by an incompressible fluid of density ρ on a round cylinder of radius R, with its axis normal to the fluid's unperturbed velocity $\mathbf{v}_0$, which rotates about the axis with an angular velocity ω – see Fig. 8. 17 whose augmented version is reproduced on the right. Discuss the relation of the results.



Solution: One approach may be based on the fact (mentioned in Sec. 8.6 of the lecture notes) that at high Reynolds numbers, there is a thin turbulent boundary layer that matches the fluid velocity at the very surface and that of the external part of the fluid, which moves as a nearly ideal one. In our current case, the layer adds the surface velocity $\mathbf{v}_r = -\omega R\mathbf{n}_\varphi$ to the near-surface velocity of the potential flow that was calculated in Sec. 8.4 – see Eq. (8.37):

$$\mathbf{v}_p = -\mathbf{n}_\varphi \frac{\partial}{\partial(R\varphi)}\phi\Big|_{\rho=R} = \mathbf{n}_\varphi \frac{d}{d\varphi} 2v_0 R\cos\varphi = -\mathbf{n}_\varphi 2v_0 \sin\varphi . \qquad (*)$$

If the cylinder's rotation speed is not too high, $|\omega| < 2|v_0|/R$, the net velocity $\mathbf{v}_r + \mathbf{v}_p$ turns to zero at two stagnation points (in the figure above, both on the lower side of the cylinder) with

$$\sin\varphi_0 = -\frac{\omega R}{2v_0} . \qquad (**)$$

Now we may calculate the fluid's pressure at the surface by using the Bernoulli equation (8.27):

$$\mathcal{P} = \mathcal{P}_0 + \frac{\rho}{2}v_0^2 - \frac{\rho}{2}v^2 = \mathcal{P}_0 + \frac{\rho}{2}\left[v_0^2 - (\omega R + 2v_0\sin\varphi)^2\right].$$

If we assume that this formula is valid at all points of the cylinder's surface, we may use it to calculate the drag and lift components of the pressure force (see the figure above), per unit length of the cylinder:

$$\frac{F_d}{l} = \oint_{r=R} \mathcal{P}\cos\varphi\, dr = \frac{\rho}{2}R\int_0^{2\pi}\left[v_0^2 - (\omega R + 2v_0\sin\varphi)^2\right]\cos\varphi\, d\varphi = 0,$$

$$\frac{F_l}{l} = -\oint_{r=R} \mathcal{P}\sin\varphi\, dr = -\frac{\rho}{2}R\int_0^{2\pi}\left[v_0^2 - (\omega R + 2v_0\sin\varphi)^2\right]\sin\varphi\, d\varphi = 2\pi\rho R^2\omega v_0 . \qquad (***)$$

The first of these results, repeating the D'Alembert paradox mentioned in the lecture notes, contradicts the experimental data for the turbulent flow, which show a very substantial drag force see Fig. 8.15):

$$\frac{F_\mathrm{d}}{l} \approx \rho R v_0^2 .$$

This disagreement implies that Eq. (***) for the Magnum lift force is not strictly valid either. Indeed, both wind-tunnel experiments and numerical simulations based on the Navier-Stokes equation show that Eq. (***), while giving its correct sign and a reasonable functional dependence of the lift, overestimates its magnitude. The reason is that, as Fig. 8.16 illustrates, for a typical object moving in a fluid, the thin boundary layer covers only its *leading edge*, while the *trailing edge* is directly exposed to the vortices of the turbulent wake, where the Bernoulli equation is not valid at all.

This is why we may use a different approach, guessing[120] that the rotation with not too high speed, $|\omega| < 2|v_0|/R$, just deflects the turbulent wake by the angle φ_0 given by Eq. (**), without changing its structure too much. For the (formally, negative) rotation direction shown in the figure above, the wake is deflected down. By the 3rd Newton's law, its reaction creates an upward lift force

$$\frac{F_\mathrm{l}}{l} \approx -\frac{F_\mathrm{d}}{l}\sin\varphi_0 \approx \frac{1}{2}\rho R^2 \omega v_0 . \qquad (****)$$

This formula is functionally similar to Eq. (***) but has a much smaller numerical coefficient. This is also natural because the derivation of Eq. (****) does not take into account the part of the Bernoulli force exerted on the leading edge of the cylinder, where the boundary-layer picture is valid. Experimental and numerical-simulation results reveal a rather complicated behavior of F_l as a function of parameters,[121] with its magnitude generally in between the estimates (***) and (****).

However, the first approach outlined above is very valuable (and gives almost exact results) for thin objects with a sharp trailing edge – most importantly, typical airplane wings and other *airfoils*. In this case, the turbulent wake does not contact the moving body, and the boundary-layer picture is valid for the whole surface. Since such objects are not rotated, it makes sense to generalize our analysis for this case. For doing that informally, we may notice that the velocity $\mathbf{v}_\mathrm{r} = -\omega R \mathbf{n}_\varphi$ we have added to the potential flow (*), is not itself a potential one because according to the Stokes theorem,[122] it has a nonvanishing average curl:

$$\Gamma \equiv -\int_S (\nabla \times \mathbf{v}_\mathrm{r})\cdot d^2\mathbf{r} = -\oint_C \mathbf{v}_\mathrm{r} \cdot d\mathbf{r} = \int_0^{2\pi} \omega R\, R d\varphi = 2\pi\omega R^2,$$

where S is the cross-section of the cylinder, and C is its outer contour drawn immediately outside of the boundary layer. So, our result (***) may be rewritten as the *Kutta-Joukowsky theorem*[123]

$$\frac{F_\mathrm{l}}{l} = \rho v_0 \Gamma, \qquad \text{with } \Gamma \equiv -\oint_C \mathbf{v}_\mathrm{r} \cdot d\mathbf{r} .$$

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[120] This guess finds an at least semi-quantitative confirmation in the results of numerical simulations of the Magnus effect – see, for example, a nice video available at https://en.wikipedia.org/wiki/Magnus_effect and/or very informative still pictures in http://repository.ias.ac.in/24676/1/316.pdf.

[121] Indeed, for some values of v_0 and ω, the Magnus force F_l may even change its sign – see, e.g., Z. Zheng *et al.*, *Flow, Turbulence, and Combustion* **98**, 109 (2017) and references therein.

[122] See, e.g. MA Eq. (12.1).

[123] It was derived in 1906 by N. Zhukovsky (legacy-spelled "Joukowsky") and (apparently, independently) by M. Kutta in 1910.

Since F_l is the "global" characteristic of the flow, it should be no surprise that this theorem is approximately applicable to any cylindrical object with a sharp trailing edge, regardless of the source of the *circulation* Γ. In a traditional airplane wing, the circulation is created by making the wing's upper surface convex, so the air speed near it is larger than that near the lower surface, so

$$\Gamma = -\left(-\int_{\text{upper}} \mathbf{v}_{\text{r}} d\mathbf{r} + \int_{\text{lower}} \mathbf{v}_{\text{r}} d\mathbf{r} \right) > 0 ,$$

giving an upward lift force.

Chapter 9. Deterministic Chaos

Problem 9.1. Generalize the reasoning of Sec. 9.1 of the lecture notes to an arbitrary 1D map $q_{n+1} = f(q_n)$, with the function $f(q)$ differentiable at all points of interest. In particular, derive the condition of stability of an N-point limit cycle $q^{(1)} \to q^{(2)} \to \ldots \to q^{(N)} \to q^{(1)} \ldots$.

Solution: Let a 1D map have an N-point limit cycle $\{q^{(1)}, q^{(2)}, \ldots, q^{(N)}\}$:

$$q^{(2)} = f\left(q^{(1)}\right), \quad q^{(3)} = f\left(q^{(2)}\right), \quad \ldots, \quad q^{(N)} = f\left(q^{(N-1)}\right), \quad q^{(1)} = f\left(q^{(N)}\right).$$

Now let one of the points be slightly displaced from its initial position, for example, $q_1 = q^{(1)} + \varepsilon$, with $\varepsilon \to 0$. Then in the linear approximation in ε, we may write:

$$q_2 = f(q_1) = f\left(q^{(1)} + \varepsilon\right) \approx f\left(q^{(1)}\right) + \left.\frac{df}{dq}\right|_{q=q^{(1)}} \varepsilon \equiv q^{(2)} + \left.\frac{df}{dq}\right|_{q=q^{(1)}} \varepsilon,$$

$$q_3 = f(q_2) = f\left(q^{(2)} + \left.\frac{df}{dq}\right|_{q=q^{(1)}} \varepsilon\right) \approx f(q^{(2)}) + \left.\frac{df}{dq}\right|_{q=q^{(2)}} \left.\frac{df}{dq}\right|_{q=q^{(1)}} \varepsilon \equiv q^{(2)} + \left.\frac{df}{dq}\right|_{q=q^{(2)}} \left.\frac{df}{dq}\right|_{q=q^{(1)}} \varepsilon, \ldots$$

Repeating this process N times, we return to the initial point q_1 with the small displacement

$$\varepsilon' = \varepsilon \prod_{k=1}^{N} \left.\frac{df}{dq}\right|_{q=q^{(k)}} .$$

Evidently, the limit cycle is stable if $|\,\varepsilon'\,| \leq |\,\varepsilon\,|$, i.e. if[124]

$$\left| \prod_{k=1}^{N} \left.\frac{df}{dq}\right|_{q=q^{(k)}} \right| \leq 1 . \qquad (*)$$

For the simplest cases of a "one-point limit cycle", i.e. a fixed point $q^{(k)}$, and a two-point cycle $\{q_+, q_-\}$, Eq. (*) is reduced, respectively, to

$$\left| \left.\frac{df}{dq}\right|_{q=q^{(k)}} \right| \leq 1, \qquad (**)$$

and

$$\left| \left.\frac{df}{dq}\right|_{q=q_+} \left.\frac{df}{dq}\right|_{q=q_-} \right| \leq 1 . \qquad (***)$$

As the simplest sanity check, for the logistic map defined by Eq. (9.2) of the lecture notes, with $f(q) = rq(1 - q)$, i.e. $df/dq = r(1 - 2q)$, the trivial fixed point $q^{(0)} = 0$ exists for any r. Applying to it the condition (**), we see that it becomes unstable at $r > 1$, i.e. exactly when another fixed point $q^{(1)} = 1 - 1/r$ appears. For the latter point, Eq. (**) reads

[124] Notice that according to Eqs. (9.8)-(9.9) of the lecture notes, the logarithm of this product, for $N >> 1$, is just a particular case of the Lyapunov exponent.

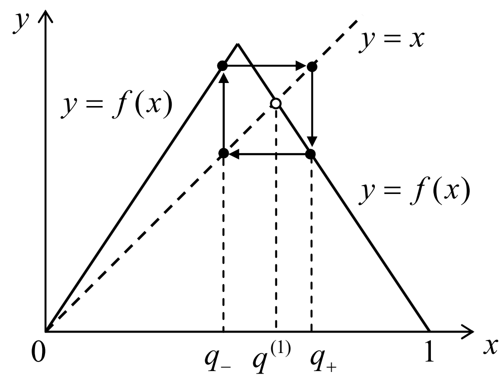
$$\left|\frac{df}{dq}\bigg|_{q=q^{(1)}}\right| \equiv \left|r\left(1-2q^{(1)}\right)\right| \equiv \left|r\left[1-2\left(1-\frac{1}{r}\right)\right]\right| \equiv |1-r| \leq 1,$$

so the nontrivial fixed point becomes unstable at $r > 2$ – as was already mentioned in Sec. 9.1.

For this map, the spelling out Eq. (***) would require using the bulky *Tartaglia-Cardano formulas*[125] roots, but in the next problem, we will use it for another map.

Problem 9.2. Use the stability condition derived in the previous problem, to analyze the possibility of deterministic chaos in the so-called *tent map*, with

$$f(q) = \begin{cases} rq, & \text{for } 0 \leq q \leq ½, \\ r(1-q), & \text{for } ½ \leq q \leq 1, \end{cases} \qquad \text{with } 0 \leq r \leq 2.$$



Solution: The tent map (shown in the figure on the right for the particular value $r = 1.5$) coincides with the logistic map at $q \to 0$, so the stability of the trivial point $q^{(0)} = 0$ is absolutely similar. Also, as soon as it becomes unstable (at $r > 1$), there is a nontrivial fixed point $q^{(1)}$, located at the "falling" branch of the function $f(q)$: $q > ½$, which has a constant negative slope: $df/dq = -r$. Hence, though $q^{(1)}$ may be readily calculated from the stationary condition $q^{(1)} = f(q^{(1)}) \equiv r(1 - q^{(1)})$, giving $q^{(1)} = r/(r + 1)$, this calculation is actually unnecessary to conclude, using the stability condition (**) derived in Problem 1,

$$\left|\frac{df}{dq}\bigg|_{q=q^{(k)}}\right| \leq 1,$$

that this point is unstable as soon as it exists (at $r > 1$). This means that a small deviation from $q^{(1)}$ grows, by magnitude, from iteration to iteration, at least until q reaches the positive-slope branch $f = rq$.

Superficially, it may look that since every other iteration point sits on that branch, a stable two-point limit cycle $\{q_+, q_-\}$ may be established – see the arrows in the figure above. Indeed, the set of Eqs. (9.5) of the lecture notes for this case,

$$q_+ = f(q_- < ½) \equiv rq_-, \quad q_- = f(q_+ > ½) \equiv r(1-q_+),$$

has a solution: $q_+ = r^2/(1 + r^2)$, $q_- = r/(1 + r^2)$, which satisfies our assumption $q_- < ½ < q_+$. However, its stability condition, given by Eq. (***) of the previous problem's solution,

$$\left|\frac{df}{dq}\bigg|_{q=q_+} \frac{df}{dq}\bigg|_{q=q_-}\right| \leq 1,$$

shows that this limit cycle is also unstable because, for the tent map, its left-hand side equals r^2. Moreover, the general Eq. (*) of that solution shows that for this map, *any* N-point cycle is unstable,

125 See, e.g., https://mathworld.wolfram.com/CubicFormula.html.

because $|df/dq| = r > 1$ at any regular point ($q \neq ½$)! As a result, in contrast with the logistic map, the tent map with $r > 1$ is always chaotic.[126]

Problem 9.3. Find the conditions of existence and stability of fixed points of the so-called *standard circle map*:

$$q_{n+1} = q_n + \Omega - \frac{K}{2\pi}\sin 2\pi q_n,$$

where q_n are real numbers defined modulo 1 (i.e. with $q_n + 1$ identified with q_n), while Ω and K are constant parameters. Discuss the relevance of the result for phase locking of self-oscillators – see, e.g., Sec. 5.4 of the lecture notes.

Solution: Per the discussion in Sec. 9.1 of the lecture notes, fixed points $q^{(0)}$ of this map are the roots of the following equation:

$$q^{(0)} = q^{(0)} + \Omega - \frac{K}{2\pi}\sin 2\pi q^{(0)}, \qquad \text{i.e. } \sin 2\pi q^{(0)} = \frac{2\pi\Omega}{K}.$$

Obviously, on any segment of unit length, e.g., $-½ \leq q \leq +½$, there are two solutions to this equation:

$$q_+^{(0)} = \frac{1}{2\pi}\sin^{-1}\frac{2\pi\Omega}{K}, \quad \text{and} \quad q_-^{(0)} = \frac{1}{2} - \frac{1}{2\pi}\sin^{-1}\frac{2\pi\Omega}{K}, \qquad \text{with } \cos 2\pi q_\pm^{(0)} = \pm\,\text{sgn}\left(\frac{\Omega}{K}\right)\left[1 - \left(\frac{2\pi\Omega}{K}\right)^2\right]^{1/2},$$

both existing only within the parameter range

$$-1 \leq \frac{2\pi\Omega}{K} \leq +1. \qquad (*)$$

With this map represented in the form used in Sec. 9.1 of the lecture notes (and in Problem 1):

$$q_{n+1} = f(q_n), \quad \text{with } f(q) = q + \Omega - \frac{K}{2\pi}\sin 2\pi q,$$

the fixed point stability condition derived in that solution becomes

$$\left|\frac{df}{dq}\right|_{q=q_\pm^{(0)}} \equiv \left|1 + K\cos 2\pi q_\pm^{(0)}\right| \equiv \left|1 \pm \text{sgn}\left(\frac{\Omega}{K}\right)\left[K^2 - (2\pi\Omega)^2\right]^{1/2}\right| \leq 1.$$

Within the range given by Eq. (*), this condition is always satisfied for one of the points $q_\pm^{(0)}$, which is therefore stable as soon as it exists, and is never satisfied for its counterparts.

This result has implications for the theory of phase locking. Indeed, rewriting the differential equation (5.68) describing this effect[127] as

$$d\varphi = (\xi + \Delta\cos\varphi)dt, \qquad (**)$$

[126] Note that in the special case when the parameter r equals exactly 2, the tent map may be explored using binary arithmetic (see, e.g., Sec. 2.1 in the book J. L. *McCauley*, *Chaos, Dynamics, and Fractals*, Cambridge U. Press, 1994) and hence is sometimes called *binary tent map*.

[127] As the reader could see from the solution of Problems 6.20 and 6.21, this equation describes not only the phase locking of a single self-oscillator by an external force but also the mutual phase locking of two or more oscillators.

we see that the standard circle map may be interpreted as its finite-difference version within the first-order Euler approximation (5.96) in the small time step $\Delta t \equiv h$:

$$\varphi_{n+1} = \varphi_n + (\xi + \Delta \cos \varphi_n) h, \qquad \textbf{(***)}$$

with the following notation correspondence:

$$\varphi_n \leftrightarrow 2\pi q_n - \frac{\pi}{2}, \quad \xi \leftrightarrow \frac{2\pi\Omega}{h}, \quad \Delta \leftrightarrow \frac{K}{h}.$$

Thus the above stability condition means that the transfer from the differential to the finite-difference version of the equation does not affect stable phase locking within its whole range $-\Delta \leq \xi \leq +\Delta$. This fact is important because some digital phase-locking systems, in which the external signal is applied to the oscillator not continuously but only at discrete times, are better described by Eq. (***) than by Eq. (**).

Problem 9.4. Find the conditions of existence and stability of fixed points of the so-called *Hénon map*:[128]

$$q_{n+1} = 1 - a q_n^2 + p_n,$$
$$p_{n+1} = b q_n, \quad \text{with } 0 < b < 1.$$

Solution: The Hénon map is *not* one-dimensional, because it transforms the n^{th} value of the two-component vector $\{q, p\}$ into the $(n + 1)^{\text{th}}$ value of this vector. However, a direct substitution of the second equation, with the replacement $n \to n - 1$, into the first one transforms it into a 1D map:

$$q_{n+1} = 1 - a q_n^2 + b q_{n-1}, \qquad (*)$$

albeit with the right-hand part depending on two previous values of q. Obviously, a fixed point $q^{(0)}$ of this map should satisfy the quadratic equation

$$q^{(0)} = 1 - a\left(q^{(0)}\right)^2 + b q^{(0)},$$

which may be readily solved, giving two roots:

$$q_{\pm}^{(0)} = \frac{1}{2a}\left\{\pm\left[(1-b)^2 + 4a\right]^{1/2} - (1-b)\right\}. \qquad (**)$$

The expression under the square root is positive, and hence both fixed points exist if

$$a > -\frac{1}{4}(1-b)^2.$$

Considering their stability, we need to mind that the map (**) has a structure different from that assumed in Sec. 9.1 of the lecture notes, so we need to repeat the analysis while using the same general approach. Linearizing Eq. (*) with respect to small deviations $\tilde{q}_n \equiv q_n - q^{(0)} \to 0$, we get

$$\tilde{q}_{n+1} = -2aq^{(0)}\tilde{q}_n + b\tilde{q}_{n-1}. \qquad (***)$$

[128] This map, first explored by M. Hénon in 1976 (for a particular set of the constants a and b), has played an important historic role in the study of strange attractors.

The general solution of this linear algebraic equation is a linear superposition of two exponential solutions proportional to $e^{\lambda n}$. Plugging such a solution into Eq. (***), we get the following characteristic equation for these two constants λ:

$$e^{\lambda} = -2aq^{(0)} + be^{-\lambda}, \qquad \text{i.e. } \xi^2 + 2aq^{(0)}\xi - b = 0, \qquad \text{where } \xi \equiv e^{\lambda}.$$

Solving this quadratic equation, we get two roots:

$$\xi_{\pm} = \pm\left[\left(aq^{(0)}\right)^2 + b\right]^{1/2} - aq^{(0)}.$$

The deviation from a fixed point does not have an infinitely growing exponent (i.e. the point is stable) only if both corresponding roots satisfy the condition Re $\lambda_{\pm} \leq 0$, i.e. $|\xi_{\pm}| \leq 1$. Since $b > 0$, both roots $\xi_{\pm}$ are real, the stability condition is just $-1 \leq \xi_{\pm} \leq +1$, and is satisfied if

$$-(1-b) \leq 2aq_{\pm}^{(0)} \leq +(1-b). \qquad (****)$$

Now comparing these two conditions with Eq. (**) rewritten as

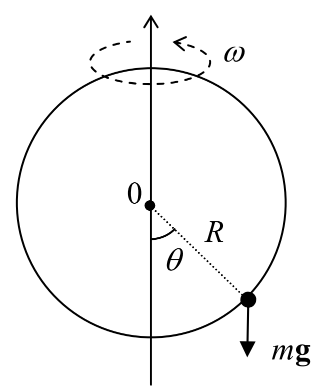
$$2aq_{\pm}^{(0)} = \pm\left[(1-b)^2 + 4a\right]^{1/2} - (1-b),$$

we may readily see that for the fixed point $q_-^{(0)}$, the left of the inequalities (****) is never satisfied (and hence that point is always unstable), while for the counterpart point $q_+^{(0)}$, both of them are satisfied if

$$a \leq \frac{3}{4}(1-b)^2.$$

Just for the reader's reference: as a is increased just beyond this limit, the Hénon map has a stable two-point cycle, and for even higher values of a, exhibits multi-point cycles and chaotic dynamics.[129]

Problem 9.5. Is the deterministic chaos possible in our "testbed" problem shown in Fig. 2.1 of the lecture notes (reproduced on the right)? What if an additional periodic external force is applied to the bead? Explain your answers.



Solution: As was discussed in Sec. 2. 2, this system is described by one differential equation (2.25) of the second order, which is equivalent to two equations of the first order, for example[130]

$$\dot{\theta} = p_{\theta} / mR^2,$$

$$\dot{p}_{\theta} = mR^2\omega^2 \sin\theta\cos\theta - mgR\sin\theta.$$

Hence, the situation is the same as in the system discussed in Problem 3: deterministic chaos is impossible without an external force but is possible in its presence. Actually, the simple, externally-driven pendulum, whose chaotic dynamics was discussed in Sec. 2, is just a particular case (for $\omega = 0$, but with an additional sinusoidal external force) of the testbed problem, i.e. ring's rotation does not create a qualitative difference in this aspect.

[129] For details, including illustrations, see, e.g., D. Gulick, *Encounters with Chaos*, McGraw-Hill (1992).

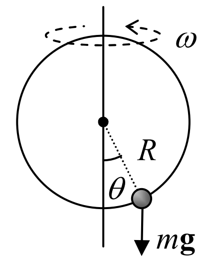
[130] Note that the p_{θ} so introduced is just the generalized momentum corresponding to the generalized coordinate θ – see Eq. (2.39) of the lecture notes, as well as Problem 10.1.

Chapter 10. A Bit More of Analytical Mechanics

In each of Problems 1-3, for the given system:

(i) derive the Hamilton equations of motion, and

(ii) check whether these equations are equivalent to those derived from the Lagrangian formalism.



Problem 10.1. Our testbed system: a bead on a ring rotated, with a fixed angular velocity ω, about its vertical diameter – see Fig. 2.1 of the lecture notes, partly reproduced on the right.

Solution: The generalized momentum p_θ (corresponding to the generalized coordinate θ) and the Hamiltonian function of this system were already derived in Sec. 2.3 of the lecture notes – see Eqs. (2.39)-(2.40):

$$p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = mR^2 \dot{\theta}, \qquad (*)$$

$$H = \frac{m}{2} R^2 \left(\dot{\theta}^2 - \omega^2 \sin^2 \theta \right) - mgR \cos\theta + \text{const} . \qquad (**)$$

Thus, to compose the Hamilton equations (10.7), we only need to express H via θ and p_θ. (In this particular problem, time is not explicitly involved.) This is easy to do by using Eq. (*) to write

$$\dot{\theta} = \frac{p_\theta}{mR^2}, \qquad (***)$$

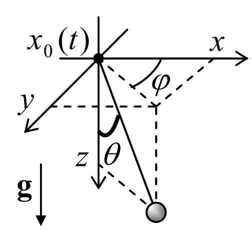
and plugging this expression into Eq. (**), getting

$$H = \frac{m}{2} R^2 \left(\frac{p_\theta^2}{m^2 R^4} - \omega^2 \sin^2 \theta \right) - mgR \cos\theta \equiv \frac{p_\theta^2}{2mR^2} - \frac{m}{2} R^2 \omega^2 \sin^2 \theta - mgR \cos\theta + \text{const} .$$

Now, Eqs. (10.7), with $q_j = \theta$ and $p_j = p_\theta$, yield the following equations of motion:

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2}, \qquad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = mR^2 \omega^2 \sin\theta \cos\theta - mgR \sin\theta .$$

The former of these equations merely reproduces Eq. (*), while the latter one, taking into account Eq. (*), coincides with Eq. (2.25) derived using the Lagrangian formalism.



Problem 10.2. The system considered in Problem 2.3: a pendulum hanging from a point whose motion $x_0(t)$ in the horizontal direction is fixed – see the figure on the right. (No vertical-plane constraint.)

Solution: From the model solution of Problem 2.3, we already know the Lagrangian and Hamiltonian functions of the system (besides arbitrary constants):

$$L = \frac{m}{2}\left[l^2\dot\theta^2 + l^2\dot\varphi^2\sin^2\theta + \dot x_0^2(t) + 2l\dot x_0(t)\left(\dot\theta\cos\theta\cos\varphi - \dot\varphi\sin\theta\sin\varphi\right)\right] + mgl\cos\theta, \qquad (*)$$

$$H = \frac{m}{2}\left[l^2\dot\theta^2 + l^2\sin^2\theta\,\dot\varphi^2 - \dot x_0^2(t)\right] - mgl\cos\theta, \qquad (**)$$

where θ and φ are, respectively, the polar and azimuthal coordinates of the pendulum, with the polar axis directed vertically down – see the figure above. From Eq. (*), we can readily calculate the generalized momenta corresponding to these two generalized coordinates:

$$p_\theta \equiv \frac{\partial L}{\partial\dot\theta} = ml^2\dot\theta + ml\dot x_0(t)\cos\theta\cos\varphi, \qquad p_\varphi \equiv \frac{\partial L}{\partial\dot\varphi} = ml^2\dot\varphi\sin^2\theta - ml\dot x_0(t)\sin\theta\sin\varphi. \quad (***)$$

By solving these linear equations for the generalized velocities:

$$\dot\theta = \frac{p_\theta}{ml^2} - \frac{\dot x_0(t)}{l}\cos\theta\cos\varphi, \qquad \dot\varphi = \frac{p_\varphi}{ml^2\sin^2\theta} + \frac{\dot x_0(t)}{l}\frac{\sin\varphi}{\sin\theta}, \qquad (****)$$

and plugging the result into Eq. (**), we get the Hamiltonian function expressed in its canonical form, i.e. via the generalized coordinates and momenta:

$$H = \frac{m}{2}\left[\left(\frac{p_\theta}{ml} - \dot x_0(t)\cos\theta\cos\varphi\right)^2 + \left(\frac{p_\varphi}{ml\sin\theta} + \dot x_0(t)\sin\varphi\right)^2 - \dot x_0^2(t)\right] - mgl\cos\theta.$$

From this expression and Eqs. (10.7) of the lecture notes applied to the generalized coordinates θ and φ, we get the following four equations of motion:

$$\dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} - \frac{\dot x_0(t)}{l}\cos\theta\cos\varphi,$$

$$\dot\varphi = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{ml^2\sin^2\theta} + \frac{\dot x_0(t)}{l}\frac{\sin\varphi}{\sin\theta},$$

$$\dot p_\theta = -\frac{\partial H}{\partial\theta} = -\left(\frac{p_\theta}{l} - m\dot x_0(t)\cos\theta\cos\varphi\right)\dot x_0(t)\sin\theta\cos\varphi + \left(\frac{p_\varphi}{l\sin\theta} + m\dot x_0(t)\sin\varphi\right)\frac{p_\varphi\cos\theta}{ml\sin^2\theta} - mgl\sin\theta,$$

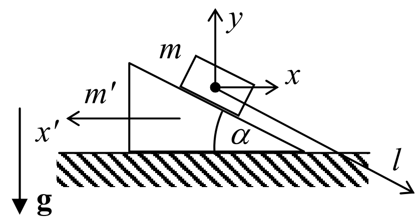
$$\dot p_\varphi = -\frac{\partial H}{\partial\varphi} = -\left(\frac{p_\theta}{l} - m\dot x_0(t)\cos\theta\cos\varphi\right)\dot x_0(t)\cos\theta\sin\varphi - \left(\frac{p_\varphi}{l\sin\theta} + m\dot x_0(t)\sin\varphi\right)\dot x_0(t)\cos\varphi.$$

The first two equations just reproduce Eqs. (****), while the two last ones, after plugging in Eqs. (***), a straightforward differentiation and several cancellations, give the same equations of motion,

$$\ddot\theta - \dot\varphi^2\sin\theta\cos\theta + \frac{g}{l}\sin\theta + \frac{\ddot x_0(t)}{l}\cos\theta\cos\varphi = 0,$$

$$\ddot\varphi\sin^2\theta + 2\dot\theta\dot\varphi\sin\theta\cos\theta - \frac{\ddot x_0(t)}{l}\sin\theta\sin\varphi = 0,$$

as those obtained in the solution of Problem 2.3 from the Lagrangian formalism. Note, however, how much longer was the way toward these equations in this solution. This is a good illustration of why the Lagrangian approach is discussed first in most classical mechanics courses – including this one.



Problem 10.3. The system considered in Problem 2.8 – a block of mass m that can slide, without friction, along the inclined surface of a heavy wedge of mass m'. The wedge is free to move, also without friction, along a horizontal surface – see the figure on the right. (Both motions are within the vertical plane containing the steepest slope line.)

Solution: The Lagrangian function $L = T - U$ of this system was already calculated in the model solution of Problem 2.8:

$$L = \frac{m}{2}\dot{l}^2 + \frac{m+m'}{2}\dot{x}'^2 - m\dot{l}\dot{x}'\cos\alpha + mgl\sin\alpha,$$

where the coordinates l and x' are indicated in the figure above. Differentiating L over the corresponding generalized velocities $\dot{l}$ and $\dot{x}'$, we get the generalized momenta:

$$p_l \equiv \frac{\partial L}{\partial \dot{l}} = m\dot{l} - m\dot{x}'\cos\alpha, \qquad p_{x'} \equiv \frac{\partial L}{\partial \dot{x}'} = (m+m')\dot{x}' - m\dot{l}\cos\alpha. \qquad (*)$$

Solving this system of two equations for $\dot{l}$ and $\dot{x}'$,

$$\dot{l} = \frac{p_l(1+m'/m) + p_{x'}\cos\alpha}{m' + m\sin^2\alpha}, \qquad \dot{x}' = \frac{p_l\cos\alpha + p_{x'}}{m' + m\sin^2\alpha}, \qquad (**)$$

and plugging this result into the expression for the Hamiltonian function,[131] also derived in Problem 2.8,

$$H = \frac{m}{2}\dot{l}^2 + \frac{m+m'}{2}\dot{x}'^2 - m\dot{l}\dot{x}'\cos\alpha - mgl\sin\alpha$$

we get H in its canonical form, i.e. expressed as a function of the generalized coordinates and momenta:

$$H = \frac{1}{2(m' + m\sin^2\alpha)}\left[\left(1+\frac{m'}{m}\right)p_l^2 + 2p_l p_{x'}\cos\alpha + p_{x'}^2\right] - mgl\sin\alpha.$$

Now we may use Eqs. (10.7) of the lecture notes to compose two Hamiltonian equations of motion for each of the two degrees of freedom of this system:

$$\dot{l} = \frac{\partial H}{\partial p_l} = \frac{p_l(1+m'/m) + p_{x'}\cos\alpha}{m' + m\sin^2\alpha}, \qquad \dot{p}_l = -\frac{\partial H}{\partial l} = mg\sin\alpha;$$
$$\dot{x}' = \frac{\partial H}{\partial p_{x'}} = \frac{p_l\cos\alpha + p_{x'}}{m' + m\sin^2\alpha}, \qquad \dot{p}_{x'} = -\frac{\partial H}{\partial x'} = 0.$$

The left two relations just reproduce Eqs. (**), and it is straightforward to verify that the two right equations, with an account of Eqs. (*), are equivalent to the two Lagrange equations derived in the solution of Problem 2.8:

$$m\ddot{l} - m\ddot{x}'\cos\alpha - mg\sin\alpha = 0,$$
$$(m+m')\ddot{x}' - m\ddot{l}\cos\alpha = 0.$$

[131] In this case, H coincides with the system's energy $E = T + U$.

Problem 10.4. Derive and solve the equations of motion of a particle with the following Hamiltonian function:

$$H = \frac{1}{2m}(\mathbf{p} + a\mathbf{r})^2,$$

where a is a constant scalar.

Solution: Rewriting H in the Cartesian form

$$H = \frac{1}{2m}\left[(p_x + ax)^2 + (p_y + ay)^2 + (p_z + az)^2\right],$$

and using the coordinates $\{x, y, z\}$ of the particle and the corresponding components $\{p_x, p_y, p_z\}$ of its momentum $\mathbf{p}$ as, respectively, the generalized coordinates and momenta, we may use Eqs. (10.7) of the lecture notes to derive the following equations of motion:

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x + ax}{m}, \qquad \dot{p}_x = -\frac{\partial H}{\partial x} = -a\frac{p_x + ax}{m}, \qquad (*)$$

with absolutely similar equations for two other coordinates.

Since the equations for different coordinate-momentum pairs are decoupled, it is sufficient to analyze just one pair of them, for example, Eqs. (*). The easiest way to do this is, first, to use these equations to calculate the time derivative of the combination ($p_x + ax$):

$$\frac{d}{dt}(p_x + ax) = \dot{p}_x + a\dot{x} = -a\frac{p_x + ax}{m} + a\frac{p_x + ax}{m} = 0.$$

So, all three such combinations, and hence the whole vector ($\mathbf{p} + a\mathbf{r}$), are the integrals of motion:

$$\mathbf{p} + a\mathbf{r} = \mathbf{C}, \qquad (**)$$

where $\mathbf{C} = \{C_x, C_y, C_z\}$ is a vector constant.[132]

Next, we may eliminate one of the variables from the system (*) – say by the differentiation of the first equation over time, and then plugging Eqs. (*) into the right-hand side of the result:

$$\ddot{x} = \frac{1}{m}(\dot{p}_x + a\dot{x}) = \frac{1}{m}\left(-a\frac{p_x + ax}{m} + a\frac{p_x + ax}{m}\right) = 0.$$

Hence, x is a linear function of time. Since the same is true for all other Cartesian coordinates, the general solution of the equations of motion may be represented exactly in the same form as for a free particle:

$$\mathbf{r}(t) = \mathbf{v}(0)t + \mathbf{r}(0), \qquad \mathbf{v}(t) \equiv \dot{\mathbf{r}}(t) = \mathbf{v}(0) = \text{const}.$$

Note, however, that per Eq. (**), at $a \neq 0$, the particle's momentum $\mathbf{p}$ is *not* necessarily constant:

$$\mathbf{p}(t) = \mathbf{C} - a\mathbf{r}(t) = \mathbf{C}' - a\mathbf{v}(0)t, \qquad \text{where } \mathbf{C}' \equiv \mathbf{C} - a\mathbf{r}(0),$$

[132] This fact is evidently compatible with the time independence of our Hamiltonian function, which follows from Eq. (10.8) of the lecture notes because H does not depend on time explicitly: $dH/dt = \partial H/\partial t = 0$.

and hence is generally different from the product $m\mathbf{v}(t)$ = const. As we already know from Sec. 4.6, such a situation, when the "canonical momentum" **p** participating in the Hamiltonian function is different from the "kinetic momentum" $m\mathbf{v}$, is typical for motion description in non-inertial reference frames.[133]

Problem 10.5. Let L be the Lagrangian function, and H the Hamiltonian function, of the same system. What three of the following four statements,

$$\text{(i)}\ \frac{dL}{dt}=0, \qquad \text{(ii)}\ \frac{\partial L}{\partial t}=0, \qquad \text{(iii)}\ \frac{dH}{dt}=0, \qquad \text{(iv)}\ \frac{\partial H}{\partial t}=0,$$

are equivalent? Give an example of when three of these equalities hold, but the fourth one does not.

Solution: According to Eq. (2.35) of the lecture notes, Eqs. (ii) and (iii) are equivalent, while according to Eq. (10.8), Eqs. (iii) and (iv) are equivalent – so all three of them are. On the other hand, dL/dt does not necessarily vanish even if all three other derivatives do. For example, according to Eq. (2.46), in a system with the kinetic energy T being a quadratic-homogeneous function of coordinates, H equals energy $E = T + U$. In a potential system of this kind (say, with the potential energy U depending only on the generalized coordinates), the full energy E is conserved but may be constantly transferred between its components T and U – say as happens at the usual linear ("harmonic") oscillator. However, according to Eq. (2.19b), for such a system $L = T - U = E - 2U$, i.e. if the potential energy changes in time (as it does in all nontrivial cases), so does L, i.e. $dL/dt \neq 0$.

Problem 10.6. Calculate the Poisson brackets of a Cartesian component of the angular momentum **L** of a particle moving in a central force field and its Hamiltonian function H, and discuss the most important implication of the result.

Solution: The Hamiltonian function of such a particle is perhaps obvious, but just for one more exercise in the Hamilton formalism, let us derive it formally from the particle's Lagrangian

$$L = T - U(r) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U(r).$$

According to Eq. (2.31) of the lecture notes, the generalized momenta corresponding to the generalized coordinates of the particle (in this particular case, to its Cartesian coordinates x, y, and z), are

$$p_x \equiv \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad p_y \equiv \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \qquad p_x \equiv \frac{\partial L}{\partial \dot{z}} = m\dot{z},$$

i.e. are just the Cartesian components of the usual linear momentum $\mathbf{p} \equiv m\dot{\mathbf{r}}$. The reciprocal relations,

$$\dot{x} = \frac{p_x}{m}, \qquad \dot{y} = \frac{p_y}{m}, \qquad \dot{z} = \frac{p_z}{m},$$

enable us to recast the Hamiltonian function defined by Eq. (2.32) in its canonical form, i.e. as a function of the generalized coordinates and momenta (and generally, but not in this case, of time):

[133] It is also typical for charged particles moving in the magnetic field – see, e.g., EM Sec. 9.7 and QM Sec. 3.1.

$$H \equiv p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - L = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + U(r).$$

On the other hand, the angular momentum vector is defined by Eq. (1.31): $\mathbf{L} \equiv \mathbf{r}\times\mathbf{p}$, so its Cartesian components are

$$L_x = yp_z - zp_y, \qquad L_y = zp_x - xp_z, \qquad L_z = xp_y - yp_x.$$

Let us calculate the Poissonian bracket of L_x and H. Since, according to its definition (10.18), the bracket is a linear function of each of its operands, it falls apart into a sum of the partial brackets of each operand pair:

$$\begin{aligned}\{L_x, H\} &= \left\{ yp_z - zp_y, \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + U(r) \right\} = \{yp_z, U(r)\} - \{zp_y, U(r)\} \\ &+ \left\{ yp_z, \frac{p_x^2}{2m} \right\} + \left\{ yp_z, \frac{p_y^2}{2m} \right\} + \left\{ yp_z, \frac{p_z^2}{2m} \right\} - \left\{ zp_y, \frac{p_x^2}{2m} \right\} - \left\{ zp_y, \frac{p_y^2}{2m} \right\} - \left\{ zp_y, \frac{p_z^2}{2m} \right\}.\end{aligned} \qquad (*)$$

This sum of eight terms may look intimidating but actually, these partial brackets are of only three different types. Let us start by using Eq. (10.18) to spell out the first bracket:

$$\{yp_z, U(r)\} = \left[\frac{\partial(yp_z)}{\partial p_x}\frac{\partial U}{\partial x} - \frac{\partial(yp_z)}{\partial x}\frac{\partial U}{\partial p_x} \right] + \left[\frac{\partial(yp_z)}{\partial p_y}\frac{\partial U}{\partial y} - \frac{\partial(yp_z)}{\partial y}\frac{\partial U}{\partial p_y} \right] + \left[\frac{\partial(yp_z)}{\partial p_z}\frac{\partial (U)}{\partial z} - \frac{\partial(yp_z)}{\partial z}\frac{\partial (U)}{\partial p_z} \right].$$

Carrying out the partial differentiation, we should remember that in the Hamiltonian formalism, all generalized coordinates (x, y, z) and momenta (p_x, p_y, p_z) have to be considered as independent arguments, so their mutual partial derivatives do not vanish only if the operands are identical. On the other hand, the radius vector's modulus $r \equiv (x^2 + y^2 + z^2)^{1/2}$ and hence the potential energy $U(r)$ are functions of all three coordinates. As a result, we get

$$\{yp_z, U(r)\} = \left(0\cdot\frac{\partial U}{\partial x} - 0\cdot 0\right) + \left(0\cdot\frac{\partial U}{\partial y} - p_z\cdot 0\right) + \left(y\frac{\partial U}{\partial z} - 0\cdot 0\right) = y\frac{\partial U}{\partial z}.$$

An absolutely similar calculation of the second term of Eq. (*) yields

$$-\{zp_y, U(r)\} = -z\frac{\partial U}{\partial y}.$$

Now using the central character of the field, $U = U(r)$, the sum of these two terms may be calculated as

$$\begin{aligned}y\frac{\partial U}{\partial z} - z\frac{\partial U}{\partial y} &= y\frac{dU}{dr}\frac{\partial r}{\partial z} - z\frac{dU}{dr}\frac{\partial r}{\partial y} = \frac{dU}{dr}\left[y\frac{\partial(x^2+y^2+z^2)^{1/2}}{\partial z} - z\frac{\partial(x^2+y^2+z^2)^{1/2}}{\partial y} \right] \\ &= \frac{dU}{dr}\frac{1}{(x^2+y^2+z^2)^{1/2}}(yz - zy) = 0.\end{aligned}$$

The remaining six terms of the sum (*) are of two types. In the brackets of the first type, all Cartesian directions are different; for example, according to Eq. (10.18), the very first one equals

$$\left\{yp_z, \frac{p_x^2}{2m}\right\} = \frac{1}{2m}\left[\left(\frac{\partial(yp_z)}{\partial p_x}\frac{\partial(p_x^2)}{\partial x} - \frac{\partial(yp_z)}{\partial x}\frac{\partial(p_x^2)}{\partial p_x}\right) + \left(\frac{\partial(yp_z)}{\partial p_y}\frac{\partial(p_x^2)}{\partial y} - \frac{\partial(yp_z)}{\partial y}\frac{\partial(p_x^2)}{\partial p_y}\right)\right.$$
$$\left. + \left(\frac{\partial(yp_z)}{\partial p_z}\frac{\partial(p_x^2)}{\partial z} - \frac{\partial(yp_z)}{\partial z}\frac{\partial(p_x^2)}{\partial p_z}\right)\right]$$
$$= \frac{1}{2m}\left[(0\cdot 0 - 0\cdot 2p_x) + (0\cdot 0 - p_z\cdot 0) + (y\cdot 0 - 0\cdot 0)\right] = 0.$$

In the brackets of the second type, the same Cartesian direction participates in both operands; for example,

$$\left\{yp_z, \frac{p_y^2}{2m}\right\} = \frac{1}{2m}\left[\left(\frac{\partial(yp_z)}{\partial p_x}\frac{\partial(p_y^2)}{\partial x} - \frac{\partial(yp_z)}{\partial x}\frac{\partial(p_y^2)}{\partial p_x}\right) + \left(\frac{\partial(yp_z)}{\partial p_y}\frac{\partial(p_y^2)}{\partial y} - \frac{\partial(yp_z)}{\partial y}\frac{\partial(p_y^2)}{\partial p_y}\right)\right.$$
$$\left. + \left(\frac{\partial(yp_z)}{\partial p_z}\frac{\partial(p_y^2)}{\partial z} - \frac{\partial(yp_z)}{\partial z}\frac{\partial(p_y^2)}{\partial p_z}\right)\right] \qquad (**)$$
$$= \frac{1}{2m}\left[(0\cdot 0 - 0\cdot 0) + (0\cdot 0 - p_z\cdot 2p_y) + (y\cdot 0 - 0\cdot 0)\right] = -\frac{p_z p_y}{m}.$$

The last two calculations clearly show that each of the last six Poisson brackets of the sum (*) vanishes unless one of its operands contains a certain Cartesian coordinate and its counterpart – the corresponding momentum. In the whole sum (*), there is only one more bracket of this type:

$$-\left\{zp_y, \frac{p_z^2}{2m}\right\} = -\frac{1}{2m}\left[\left(\frac{\partial(zp_y)}{\partial p_x}\frac{\partial(p_z^2)}{\partial x} - \frac{\partial(zp_y)}{\partial x}\frac{\partial(p_z^2)}{\partial p_x}\right) + \left(\frac{\partial(zp_y)}{\partial p_y}\frac{\partial(p_z^2)}{\partial y} - \frac{\partial(zp_y)}{\partial y}\frac{\partial(p_z^2)}{\partial p_y}\right)\right.$$
$$\left. + \left(\frac{\partial(zp_y)}{\partial p_z}\frac{\partial(p_z^2)}{\partial z} - \frac{\partial(zp_y)}{\partial z}\frac{\partial(p_z^2)}{\partial p_z}\right)\right]$$
$$= -\frac{1}{2m}\left[(0\cdot 0 - 0\cdot 0) + (z\cdot 0 - 0\cdot 0) + (0\cdot 0 - p_y\cdot 2p_z)\right] = \frac{p_y p_z}{m}.$$

The sum of this bracket and that in Eq. (**) equals zero, so the whole bracket (*) vanishes:

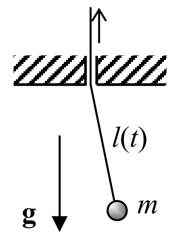
$$\{L_x, H\} = 0. \qquad (***)$$

Due to the invariance of H to the simultaneous rotational replacement of directions, $x \to y \to z \to x$, which results in the replacement $L_x \to L_y \to L_z \to L_x$, even without repeating the similar calculations for L_y and L_z, we may conclude that

$$\{L_y, H\} = \{L_z, H\} = 0 \qquad (****)$$

as well.

Per Eq. (10.20) of the lecture notes, since the Cartesian components of **L** do not depend on time explicitly, Eqs. (***) and (****) mean that the angular momentum is an integral of motion. Admittedly, for this simple case, this fact may be proved by much simpler means – see the derivation of Eq. (1.35).



Problem 10.7. After small oscillations had been initiated in the point pendulum shown in the figure on the right, the supporting string is being pulled up slowly, so the pendulum's length l is being reduced. Neglecting dissipation,

(i) prove by a direct calculation that the oscillation energy is indeed changing proportionately to the oscillation frequency, as it follows from the constancy of the corresponding adiabatic invariant (10.40); and

(ii) find the l-dependence of the amplitudes of the angular and linear deviations from the equilibrium.

Solutions:

(i) The work done by the external force F pulling up the string during a small change dl is $d\mathscr{W} = -Fdl$, where F may be found from the 2nd Newton law's component along the string's direction:

$$m\frac{v^2}{l} = F - mg\cos\theta \approx F - mg\left(1 - \frac{\theta^2}{2}\right), \quad \text{with } v = l\dot{\theta}, \qquad (*)$$

where the radial velocity dl/dt was neglected due to its smallness. Due to the same slowness of the function $l(t)$, for small oscillations, we may take $\theta = A\cos\Psi$ (where $\Psi \equiv \omega t + \text{const}$, $\omega^2 \equiv g/l$), so $v = -l\omega A\sin\Psi$, and Eq. (*) yields[134]

$$F \approx mg + mgA^2\left(\sin^2\Psi - \frac{1}{2}\cos^2\Psi\right).$$

Averaging the work $d\mathscr{W} = -Fdl$ over the oscillation period $\Delta\Psi = 2\pi$, we get

$$\overline{d\mathscr{W}} = -mgdl - \frac{1}{4}mgA^2 dl.$$

The first term describes the reversible change of the pendulum's potential energy at the point of its equilibrium, but the second term is the work due to the oscillations. Due to the absence of dissipation in the system, this work may only go into the change of the oscillation energy $E = mglA^2/2$, so

$$dE = -\frac{1}{4}mgA^2 dl \equiv -\frac{1}{2}E\frac{dl}{l}.$$

This result has to be compared with the change of the oscillation frequency $\omega = (g/l)^{1/2}$,

$$d\omega = \frac{d\omega}{dl}dl = -\frac{1}{2}\frac{g^{1/2}}{l^{3/2}}dl \equiv -\frac{1}{2}\omega\frac{dl}{l}.$$

The comparison gives $d\omega/\omega = dE/E$, i.e. exactly the result expected from the adiabatic invariance of the action variable $J = E/\omega$.

(ii) Since the total energy $E = mglA^2/2$ scales as $\omega \propto l^{-1/2}$, the angular deviation amplitude $A \propto (E/l)^{1/2}$ is proportional to $(l^{-1/2}/l)^{1/2} \equiv l^{-3/4}$, while the linear deviation amplitude, Al, to $l^{-3/4}\times l \equiv l^{1/4}$. Hence,

134 Actually, the extreme values of F, given by this formula: $F_{\max} = F|_{\theta=0} = F|_{\Psi=\pi(n+½)} = mg - mgA^2/2 = mg - E/l$, and $F_{\min} = F|_{\theta=\pm A} = F|_{\Psi=\pi n} = mg + mgA^2 = mg + 2E/l$, were already used at the beginning of Sec. 5.5 of the lecture notes. (In that calculation, the sinusoidal shape of the function $\theta(t)$ was not due to the slowness of the function $l(t)$, but due to the constancy of l between its abrupt changes.)

if the string is pulled up, the oscillations of the angular deviations from the equilibrium (vertical) position are growing, while the linear deviations are decreasing, albeit rather slowly.

Problem 10.8. The mass m of a small body that performs 1D oscillations in the potential well $U(x) = ax^{2n}$, with $n > 0$, is being changed slowly, without exerting any additional direct force. Calculate the oscillation energy E as a function of m.

Solution: As was discussed in Sec. 10.2 of the lecture notes, the adiabatic invariant

$$\oint_C pdq, \qquad (*)$$

where C is the closed contour on the plane of the generalized coordinate (q) and momentum (p), remains constant at a slow change of the system's parameters. In our current case, we can take the coordinate x for q, and calculate p from the system's Lagrangian,

$$L \equiv T - U = \frac{m\dot{x}^2}{2} - U(x),$$

in the usual way – see Eq. (2.31):

$$p \equiv \frac{\partial L}{\partial \dot{x}} = m\dot{x}.$$

At constant m, the oscillations conserve the system's energy

$$E = T + U = \frac{m\dot{x}^2}{2} + U(x) = \frac{p^2}{2m} + ax^{2n},$$

so p may be calculated as a function of x and E as

$$p = \pm\left[2m\left(E - ax^{2n}\right)\right]^{1/2}.$$

The two signs describe two similar parts of the oscillation cycle, which give equal contributions to the contour integral (*), so we may write

$$\oint_C pdq = 2\int_{x_{\min}}^{x_{\max}} pdx = 2\int_{-(E/a)^{1/2n}}^{+(E/a)^{1/2n}} \left[2m\left(E - ax^{2n}\right)\right]^{1/2} dx = 4\int_0^{(E/a)^{1/2n}} \left[2m\left(E - ax^{2n}\right)\right]^{1/2} dx$$

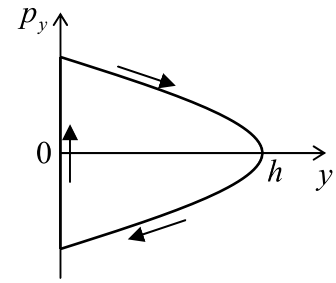
$$= 4(2mE)^{1/2}(E/a)^{1/2n}\int_0^1 \left(1 - \xi^{2n}\right)^{1/2} d\xi, \qquad \text{where } \xi \equiv (a/E)^{1/2n} x.$$

The definite integral in the last expression is just a dimensionless constant, and we do not need to work it out to solve this particular problem, because the adiabatic invariance already yields the requested relation between E and m (and moreover, the parameter a as well):

$$(mE)^{1/2}(E/a)^{1/2n} \equiv m^{1/2}E^{(1/2+1/2n)}/a^{1/2n} = \text{const}, \qquad \text{so that } E \propto a^{1/(n+1)}/m^{n/(n+1)}.$$

For the particular case $n = 1$, i.e. for the motion in the quadratic potential $U(x) = ax^2$, this result is reduced to $E \propto (a/m)^{1/2} = \omega_0$ – the result that was already derived (in different ways) in Sec. 10.2 of the lecture notes and in the previous problem.

<u>Problem 10.9</u>. A stiff ball is bouncing vertically from the floor of an elevator whose upward acceleration changes very slowly. Neglecting the energy dissipation, calculate how much the bounce height h changes during the acceleration's increase from 0 to g. Is your result valid for an equal but abrupt increase of the elevator's acceleration?



Solution: Per Eq. (4.92) of the lecture notes, from the point of view of the non-inertial reference frame moving with the elevator, its upward acceleration a may be described by adding it to the constant gravity acceleration g. The phase plane trajectory of a single bounce of the ball, under the effect of a constant acceleration ($g + a$), is shown in the figure on the right, where y is the vertical coordinate referred to the elevator's floor, and p_y is the corresponding momentum – i.e. that in the moving reference frame. The curved part of the trajectory is described by the parabola corresponding to the conservation of the ball's effective energy (i.e. its energy as measured in the moving reference frame):[135]

$$\frac{p_y^2}{2m} + m(g+a)y = \text{const} = m(g+a)h, \qquad \text{i.e. } p_y = \pm m[2(g+a)(h-y)]^{1/2}.$$

Let us calculate the action variable (10.39) for this trajectory:

$$J \equiv \frac{1}{2\pi}\oint p_y dy = \frac{1}{2\pi}2\int_0^h |p_y| dy = \frac{1}{2\pi}2\int_0^h m[2(g+a)(h-y)]^{1/2} dy$$
$$\equiv \frac{\sqrt{2}m(g+a)^{1/2}}{\pi}\int_0^h (h-y)^{1/2} dy = \frac{\sqrt{2}m(g+a)^{1/2}}{\pi}\frac{2}{3}h^{3/2}.$$

Now according to the discussion in Sec. 10.2 of the lecture notes, at a slow change of a, this variable stays constant, i.e.

$$(g+a)^{1/2}h^{3/2} = \text{const}, \qquad \text{so } h \propto \frac{1}{(g+a)^{1/3}}.$$

Hence, under the effect of elevator acceleration's increase of a from 0 to g, the bounce height becomes

$$h = h_0\left(\frac{g}{g+a}\right)^{1/3} = h_0\left(\frac{1}{2}\right)^{1/3} \approx 0.794\, h_0, \qquad (*)$$

i.e. decreases by only ~20%.

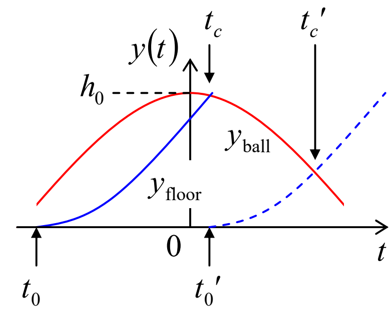
This result is *not* valid in the case of an abrupt increase of acceleration. Indeed, let us write the evident expressions for the coordinates y of the ball and the floor, now in an inertial reference frame that was moving with the floor *before* the moment t_0 of the increase:

$$y_{\text{ball}}(t) = h_0 - \frac{gt^2}{2}, \qquad y_{\text{floor}}(t) = \frac{a(t-t_0)^2}{2}, \qquad \text{for } t_0 \le t \le t_c,$$

---

[135] The fact that such effective energy is conserved at constant a may be readily proved by repeating the calculations at the beginning of Sec. 1.3 of the lecture notes after adding, to the potential energy mgy of the real gravity force $\mathbf{F}_g = -mg\mathbf{n}_y$, the contribution may from the inertial "force" $-m\mathbf{a}_0 = -ma\mathbf{n}_y$. (See also the solution of Problem 4.37.)

where the time is referred to the moment of the last time when the ball reached its maximal height (h_0), and t_c is the time of their first collision after t_0, defined by the equation $y_{\text{ball}}(t_c) = y_{\text{floor}}(t_c)$. Even the schematic plot of these two functions, the latter one for a couple of t_0 values (see the figure on the right), shows clearly that the relative velocity of the ball and the floor at their collision at time t_c, and hence the energy transferred to the ball at this collision (and hence the resulting height h of the ball bounces at $t > t_c$) depends on the exact moment t_0 of the acceleration's increase – the dependence absent from the result (*) for the adiabatic process.



Problem 10.10.* A 1D particle of a constant mass m moves in a time-dependent potential $U(q, t) = m\omega^2(t)q^2/2$, where $\omega(t)$ is a slow function of time, with $|\dot{\omega}| << \omega^2$. Develop the approximate method for the solution of the corresponding equation of motion, similar to the WKB approximation used in quantum mechanics.[136] Use the approximation to confirm the conservation of the action variable (10.40) for this system.

Hint: You may like to look for the solution to the equation of motion in the form

$$q(t) = \exp\{\Lambda(t) + i\Psi(t)\},$$

where Λ and Ψ are some real functions of time, and then make proper approximations in the resulting equations for these functions.

Solution: The differential equation describing the system's dynamics is

$$\ddot{q} + \omega^2(t)q = 0.$$

Plugging the solution suggested in the *Hint* into this equation, performing the double differentiation of q (so far, without any approximations), and canceling the exponent in all terms, we get

$$\left(\ddot{\Lambda} + i\ddot{\Psi}\right) + \left(\dot{\Lambda} + i\dot{\Psi}\right)^2 + \omega^2 = 0.$$

Opening the parentheses and requiring the real and imaginary parts of this equation to be separately satisfied, we get a system of two equations for the two real functions $\Lambda(t)$ and $\Psi(t)$:

$$\ddot{\Lambda} + \dot{\Lambda}^2 - \dot{\Psi}^2 + \omega^2 = 0, \qquad (*)$$

$$\ddot{\Psi} + 2\dot{\Psi}\dot{\Lambda} = 0.$$

The last equation, after the division of both terms by $\dot{\Psi}$, may be readily integrated over time:

$$\int^t \frac{\ddot{\Psi}(t')}{\dot{\Psi}(t')}dt' \equiv \int^t \frac{d}{dt'}\ln\dot{\Psi}(t')\,dt' \equiv \ln\dot{\Psi}(t) + \text{const} = -2\int^t \dot{\Lambda}(t')dt' \equiv -2\Lambda(t) + \text{const},$$

so

$$\Lambda = -\frac{1}{2}\ln\dot{\Psi} + \text{const} \equiv \ln\frac{1}{\dot{\Psi}^{1/2}} + \text{const}, \qquad \text{i.e. } \exp\{\Lambda\} = \frac{C}{\dot{\Psi}^{1/2}}, \qquad (**)$$

[136] See, e.g., QM Sec. 2.4 and/or a similar approach in the solution of Problem 6.18 of this (CM) course.

where C is a constant.

In contrast with this exact solution, the nonlinear differential equation (*) with an arbitrary function $\omega^2(t)$ may be readily solved only approximately, using the condition $|\dot{\omega}| << \omega^2$ specified in the assignment. This condition allows us, in the first approximation in small d/dt, to neglect the first two terms. The resulting simple equation

$$-\dot{\Psi}^2 + \omega^2 = 0$$

has two solutions:

$$\dot{\Psi}_\pm = \pm\omega(t), \qquad \text{i.e. } \Psi_\pm(t) = \pm\int^t \omega(t')dt' + \text{const}.$$

Since our differential equation is linear, its general solution is a sum of two terms – each with one of these functions. Using Eq. (**), we get

$$q(t) = \frac{C_+}{\omega^{1/2}(t)}\exp\left\{+i\int^t \omega(t')dt'\right\} + \frac{C_-}{\omega^{1/2}(t)}\exp\left\{-i\int^t \omega(t')dt'\right\},$$

where the constants $C_\pm$ are determined by initial conditions.[137] Since here we are only interested in real functions $q(t)$, it is instrumental to rewrite this solution in a manifestly real form, for example

$$q(t) = \frac{a}{\omega^{1/2}(t)}\cos\left(\int^t \omega(t')dt' + \varphi\right), \qquad (***)$$

with real constants a and φ. (Note that because of the denominator of the pre-exponential term, the constant a is *not* the oscillation amplitude and, moreover, has a dimensionality different from q.)

Now let us use this solution to explore the adiabatic invariance in the given system. Since it describes nearly-sinusoidal equations of the particle near the equilibrium, the system's full energy,

$$E \equiv T + U = \frac{m}{2}\dot{q}^2 + U(x,t) = \frac{m}{2}\dot{q}^2 + \frac{m\omega^2(t)}{2}q^2,$$

periodically moves from its kinetic form to the potential form and back. Due to the slowness of the function $\omega(t)$, we may evaluate the energy, for example, at a moment when $T = 0$, and hence

$$E = U_{\max} = \frac{m\omega^2}{2}q_{\max}^2.$$

According to Eq. (***), this particle displacement's maximum equals $a/\omega^{1/2}$, and hence

$$E = \frac{m\omega^2}{2}\left(\frac{a}{\omega^{1/2}}\right)^2 \equiv \frac{ma^2}{2}\omega.$$

Since the factors m and a do not change at slow changes of ω, we see that at such changes, $E/\omega = \text{const}$. This is exactly the result (10.40), which was obtained in Sec. 10.2 of the lecture notes from the general theory of adiabatic invariance.

[137] This solution is the exact temporal analog of the 1D WKB approximation in quantum wave mechanics – see, e.g., QM Eq. (2.94).