

Chapter 3. Dipoles and Dielectrics

In contrast to conductors, the motion of charges in dielectrics is restricted to the atom/molecule interiors, so the electric polarization of these materials by an external field takes a different form. This issue is the main subject of this chapter, but in preparation for its analysis, we have to start with a general discussion of the electric field induced by spatially restricted systems of charges.

3.1. Electric dipole

Let us consider a localized system of charges, of a linear scale a , and derive a simple approximate expression for the electrostatic field induced by the system at a distant point \mathbf{r} . For that, let us select a reference frame with the origin either somewhere inside the system, or at a distance of the order of a from it (Fig. 1).

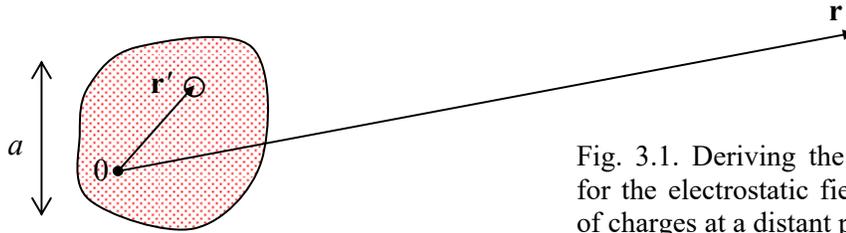


Fig. 3.1. Deriving the approximate expression for the electrostatic field of a localized system of charges at a distant point ($r \gg r' \sim a$).

Then the positions of all charges of the system satisfy the following condition:

$$r' \ll r. \quad (3.1)$$

Using this condition, we can expand the general expression (1.38) for the electrostatic potential $\phi(\mathbf{r})$ of the system into the Taylor series in small parameter r' . For any function of type $f(\mathbf{r} - \mathbf{r}')$, the expansion may be represented as¹

$$f(\mathbf{r} - \mathbf{r}') = f(\mathbf{r}) - \sum_{j=1}^3 r'_j \frac{\partial f}{\partial r_j}(\mathbf{r})|_{r'=0} + \frac{1}{2!} \sum_{j,j'=1}^3 r'_j r'_{j'} \frac{\partial^2 f}{\partial r_j \partial r_{j'}}(\mathbf{r})|_{r'=0} - \dots \quad (3.2)$$

Applying this formula to the fraction $1/|\mathbf{r} - \mathbf{r}'|$ in Eq. (1.38) (i.e. essentially to the free-space Green's function), we get the so-called *multipole expansion* of the electrostatic potential:

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{1}{r} Q + \frac{1}{r^3} \sum_{j=1}^3 r_j p_j + \frac{1}{2r^5} \sum_{jj'=1}^3 r_j r_{j'} Q_{jj'} + \dots \right), \quad (3.3)$$

whose \mathbf{r} -independent parameters are defined as follows:

$$Q \equiv \int \rho(\mathbf{r}') d^3 r', \quad p_j \equiv \int \rho(\mathbf{r}') r'_j d^3 r', \quad Q_{jj'} \equiv \int \rho(\mathbf{r}') (3r'_j r'_{j'} - r'^2 \delta_{jj'}) d^3 r'. \quad (3.4)$$

¹ See, e.g., MA Eq. (2.11b).

(Indeed, the two leading terms of the expansion (2) may be rewritten in the vector form $f(\mathbf{r}) - \mathbf{r}' \cdot \nabla f(\mathbf{r})$, and the gradient of such a spherically-symmetric function $f(r) = 1/r$ is just $\mathbf{n} df/dr$, so

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{1}{r} - \mathbf{r}' \cdot \mathbf{n}_r \frac{d}{dr} \left(\frac{1}{r} \right) = \frac{1}{r} + \mathbf{r}' \cdot \frac{\mathbf{r}}{r^3}, \tag{3.5}$$

immediately giving the two first terms of Eq. (3). The proof of the third, *quadrupole* term in Eq. (3) is similar but a bit longer, and is left for the reader's exercise.)

Evidently, the scalar parameter Q in Eqs. (3)-(4) is just the total charge of the system. The constants p_j may be considered as Cartesian components of the following vector:

$$\mathbf{p} \equiv \int \rho(\mathbf{r}') \mathbf{r}' d^3 r', \tag{3.6}$$

Electric dipole moment

called the system's *electric dipole moment*, and \mathcal{Q}_{ij} are Cartesian elements of a tensor – system's *electric quadrupole moment*. If $Q \neq 0$, all higher terms on the right-hand side of Eq. (3), at large distances (1), are just small corrections to the first one, and in many cases may be ignored. However, the net charge of many systems is exactly zero, the most important examples being neutral atoms and molecules. For such neutral systems, the second (*dipole*) term in Eq. (3) is, most frequently, the leading one. Such systems are called *electric dipoles*. Due to their importance, let us rewrite the expression for the dipole term in three other, mathematically equivalent forms:

$$\phi_d \equiv \frac{1}{4\pi\epsilon_0} \frac{\mathbf{r} \cdot \mathbf{p}}{r^3} \equiv \frac{1}{4\pi\epsilon_0} \frac{p \cos \theta}{r^2} \equiv \frac{1}{4\pi\epsilon_0} \frac{p z}{(x^2 + y^2 + z^2)^{3/2}}, \tag{3.7}$$

Electric dipole's potential

that are more convenient for some applications. Here θ is the angle between the vectors \mathbf{p} and \mathbf{r} , and in the last (Cartesian) representation, the z -axis is directed along the vector \mathbf{p} . Fig. 2a shows equipotential surfaces of the dipole field – or rather their cross-sections by any plane in which the vector \mathbf{p} resides.

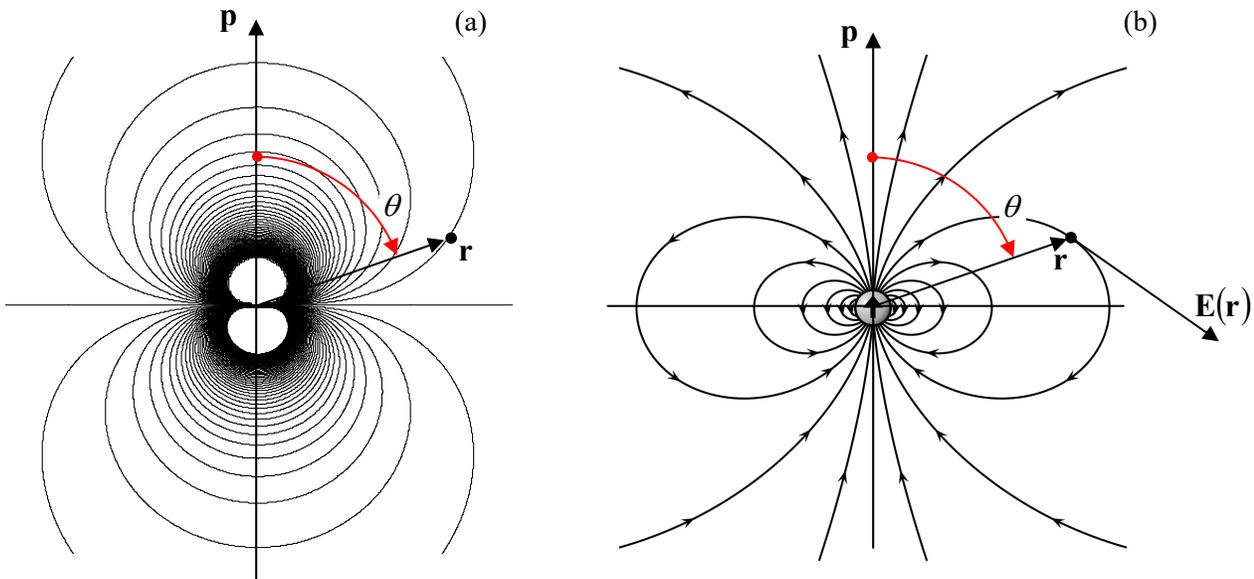


Fig. 3.2. (a) The equipotential surfaces and (b) the electric field lines of a dipole. (Panel (b) adapted from <http://en.wikipedia.org/wiki/Dipole> under the GNU Free Documentation License.)

The simplest example of a system whose field, at large distances, approaches the dipole field (7), is two equal but opposite point charges (“poles”), $+q$ and $-q$, with the radius vectors, respectively, \mathbf{r}_+ and \mathbf{r}_- :

$$\rho(\mathbf{r}) = (+q)\delta(\mathbf{r} - \mathbf{r}_+) + (-q)\delta(\mathbf{r} - \mathbf{r}_-). \quad (3.8)$$

For this system (sometimes called the *physical dipole*), Eq. (4) yields

$$\mathbf{p} = (+q)\mathbf{r}_+ + (-q)\mathbf{r}_- = q(\mathbf{r}_+ - \mathbf{r}_-) = q\mathbf{a}, \quad (3.9)$$

where \mathbf{a} is the vector connecting the points \mathbf{r}_- and \mathbf{r}_+ . Note that in this case (and indeed for all systems with $Q = 0$), the dipole moment does not depend on the choice of the reference frame’s origin.

A less trivial example of a dipole is a conducting sphere of radius R in a uniform external electric field \mathbf{E}_0 . As a reminder, its field was calculated in Sec. 2.8, and its result is expressed by Eq. (2.176). The first term in the parentheses of that relation describes just the external field (2.173), so the field of the sphere itself (i.e. that of the surface charge induced by \mathbf{E}_0) is given by the second term:

$$\phi_s = \frac{E_0 R^3}{r^2} \cos \theta. \quad (3.10)$$

Comparing this expression with the second form of Eq. (7), we see that the sphere has an *induced* dipole moment

$$\mathbf{p} = 4\pi\epsilon_0 \mathbf{E}_0 R^3. \quad (3.11)$$

This is an interesting example of a virtually pure dipole field: at all points outside the sphere ($r > R$), the field has neither a quadrupole moment nor any higher moments.

Other examples of dipole fields are given by two more systems discussed in Chapter 2 – see Eqs. (2.215) and (2.219). Those systems, however, do have higher-order multipole moments, so for them, Eq. (7) gives only the long-distance approximation.

Now returning to the general properties of the dipole field (7), let us calculate its major characteristics. First of all, we may use Eq. (7) to calculate the electric field of a dipole:

$$\mathbf{E}_d = -\nabla\phi_d = -\frac{1}{4\pi\epsilon_0} \nabla \left(\frac{\mathbf{r} \cdot \mathbf{p}}{r^3} \right) = -\frac{1}{4\pi\epsilon_0} \nabla \left(\frac{p \cos \theta}{r^2} \right). \quad (3.12)$$

This differentiation is easiest in the spherical coordinates, using the well-known expression for the gradient of a scalar function in these coordinates² and taking the z -axis parallel to the dipole moment \mathbf{p} . From the last form of Eq. (12), we immediately get

$$\mathbf{E}_d = \frac{p}{4\pi\epsilon_0 r^3} (2\mathbf{n}_r \cos \theta + \mathbf{n}_\theta \sin \theta) \equiv \frac{1}{4\pi\epsilon_0} \frac{3\mathbf{r}(\mathbf{r} \cdot \mathbf{p}) - \mathbf{p}r^2}{r^5}. \quad (3.13)$$

Electric
dipole’s
field

Fig. 2b above shows the electric field lines given by Eqs. (13). The most important features of this result are a faster drop of the field’s magnitude ($E_d \propto 1/r^3$, rather than $E \propto 1/r^2$ for a point charge), and the change of the signs of its radial component as a function of the polar angle $\theta \in [0, \pi]$.

² See, e.g., MA Eq. (10.8) with $\partial/\partial\varphi = 0$.

Next, let us use Eq. (1.55) to calculate the potential energy of interaction between a dipole and an external electric field. Assuming that this field does not change much at distances of the order of a (Fig. 1), we may expand its potential $\phi_{\text{ext}}(\mathbf{r})$ into the Taylor series, and keep only two leading terms:

$$U = \int \rho(\mathbf{r})\phi_{\text{ext}}(\mathbf{r})d^3r \approx \int \rho(\mathbf{r})[\phi_{\text{ext}}(0) + \mathbf{r} \cdot \nabla\phi_{\text{ext}}(0)]d^3r \equiv Q\phi_{\text{ext}}(0) - \mathbf{p} \cdot \mathbf{E}_{\text{ext}}. \quad (3.14)$$

The first term is the potential energy the system would have if it were just a point charge. If the net charge Q is zero, that term disappears, and the leading contribution is due to the dipole moment:

$$U = -\mathbf{p} \cdot \mathbf{E}_{\text{ext}}, \quad \text{for } \mathbf{p} = \text{const.} \quad (3.15a)$$

Note that this result is only valid for a *fixed* dipole, with \mathbf{p} independent of \mathbf{E}_{ext} . In the opposite limit, when the dipole is *induced* by the field, i.e. $\mathbf{p} \propto \mathbf{E}_{\text{ext}}$ (you may have one more look at Eq. (11) to see an example of such a proportionality), we need to start with Eq. (1.60) rather than Eq. (1.55), getting

$$U = -\frac{1}{2}\mathbf{p} \cdot \mathbf{E}_{\text{ext}}, \quad \text{for } \mathbf{p} \propto \mathbf{E}_{\text{ext}}. \quad (3.15b)$$

Dipole's
energy in
external
field

In particular, combining Eqs. (13) and Eq. (15a), we may get the following important formula for the interaction of two independent dipoles:

$$U_{\text{int}} = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}_1 \cdot \mathbf{p}_2 r^2 - 3(\mathbf{r} \cdot \mathbf{p}_1)(\mathbf{r} \cdot \mathbf{p}_2)}{r^5} = \frac{1}{4\pi\epsilon_0} \frac{p_{1x}p_{2x} + p_{1y}p_{2y} - 2p_{1z}p_{2z}}{r^3}, \quad (3.16)$$

where \mathbf{r} is the vector connecting the dipoles, and the z -axis is directed along this vector. It is easy to prove (this exercise is left for the reader) that if the magnitude p of each dipole moment is fixed (the approximation valid, in particular, for weak interaction of so-called *polar molecules*), this potential energy reaches its minimum at the parallel orientation of the dipoles along the line connecting them. Note also that in this case, U_{int} is proportional to $1/r^3$. On the other hand, if each moment \mathbf{p} has a random value plus a component due to its polarization by the electric field of its counterpart: $\Delta\mathbf{p}_{1,2} \propto \mathbf{E}_{2,1} \propto 1/r^3$, their average interaction energy (which may be calculated from Eq. (16) with the additional factor $1/2$) is always negative and is proportional to $1/r^6$. Such negative potential describes, in particular, the long-range, attractive part (the so-called *London dispersion force*) of the interaction between electrically neutral atoms and molecules.³

According to Eqs. (15), the electric field should “try” to reach the minimum of U by aligning the dipole vector's direction with its own. The direct quantitative description of this effect is the torque $\boldsymbol{\tau}$ exerted by the field. The simplest way to calculate it is to sum up all the elementary torques $d\boldsymbol{\tau} = \mathbf{r} \times d\mathbf{F}_{\text{ext}} = \mathbf{r} \times \mathbf{E}_{\text{ext}}(\mathbf{r})\rho(\mathbf{r})d^3r$ exerted on all elementary charges of the system:

$$\boldsymbol{\tau} = \int \mathbf{r} \times \mathbf{E}_{\text{ext}}(\mathbf{r})\rho(\mathbf{r})d^3r \approx \mathbf{p} \times \mathbf{E}_{\text{ext}}(0), \quad (3.17)$$

where at the last step, the spatial dependence of the external field $\mathbf{E}_{\text{ext}}(\mathbf{r})$ was again neglected. This dependence cannot, however, be ignored at the calculation of the *total force* exerted by the field on the dipole (with $Q = 0$). Indeed, Eqs. (15) shows that if the field is constant, the dipole's energy is

³ Several calculations of this force, using various models, are described in the QM and SM parts of this series.

independent of its spatial location and hence the net force is zero. However, if the field has a non-zero gradient, a total force does appear; for a field-independent dipole,

$$\mathbf{F} = -\nabla U = \nabla(\mathbf{p} \cdot \mathbf{E}_{\text{ext}}), \quad (3.18)$$

where the derivative has to be taken at the dipole's position (in our notation, at $\mathbf{r} = 0$). If the dipole that is being moved in a field retains its magnitude and orientation, then the last formula is equivalent to⁴

$$\mathbf{F} = (\mathbf{p} \cdot \nabla) \mathbf{E}_{\text{ext}}. \quad (3.19)$$

Alternatively, the last expression may be obtained similarly to Eq. (14):

$$\mathbf{F} = \int \rho(\mathbf{r}) \mathbf{E}_{\text{ext}}(\mathbf{r}) d^3r \approx \int \rho(\mathbf{r}) [\mathbf{E}_{\text{ext}}(0) + (\mathbf{r} \cdot \nabla) \mathbf{E}_{\text{ext}}] d^3r = Q \mathbf{E}_{\text{ext}}(0) + (\mathbf{p} \cdot \nabla) \mathbf{E}_{\text{ext}}. \quad (3.20)$$

Finally, let me add a note on the so-called *coarse-grain model* of the dipole. The dipole approximation explored above is asymptotically correct only *at large distances*, $r \gg a$. However, for some applications (including the forthcoming discussion of the molecular field effects in Sec. 3) it is beneficial to have an expression that might be formally used *everywhere* in space, though maybe without exact details at $r \sim a$, giving the correct result for the space average of the electric field,

$$\bar{\mathbf{E}} \equiv \frac{1}{V} \int_V \mathbf{E} d^3r, \quad (3.21)$$

where V is a regularly-shaped volume much larger than a^3 , for example, a sphere of radius $R \gg a$, with the dipole at its center. For the field \mathbf{E}_d given by Eq. (13), such an average is zero. Indeed, let us consider the Cartesian components of that vector in a reference frame with the z -axis directed along the vector \mathbf{p} . Due to the axial symmetry of the field, the averages of the components E_x and E_y vanish. Let us use Eq. (13) to spell out the “vertical” component of the field (parallel to the dipole moment vector):

$$E_z \equiv \mathbf{E}_d \cdot \frac{\mathbf{p}}{p} = \frac{1}{4\pi\epsilon_0 r^3} (2\mathbf{n}_r \cdot \mathbf{p} \cos\theta - \mathbf{n}_\theta \cdot \mathbf{p} \sin\theta) = \frac{p}{4\pi\epsilon_0 r^3} (2\cos^2\theta - \sin^2\theta). \quad (3.22)$$

Integrating this expression over the whole solid angle $\Omega = 4\pi$, at fixed r , using a convenient variable substitution $\cos\theta \equiv \xi$, we get

$$\oint_{4\pi} E_z d\Omega = 2\pi \int_0^\pi E_z \sin\theta d\theta = \frac{p}{2\epsilon_0 r^3} \int_0^\pi (2\cos^2\theta - \sin^2\theta) \sin\theta d\theta = \frac{p}{2\epsilon_0 r^3} \int_{-1}^{+1} (3\xi^2 - 1) d\xi = 0. \quad (3.23)$$

On the other hand, the *exact* electric field of an *arbitrary* charge distribution, with the total dipole moment \mathbf{p} , obeys the following equality:

$$\int_V \mathbf{E}(\mathbf{r}) d^3r = -\frac{\mathbf{p}}{3\epsilon_0} \equiv -\frac{1}{4\pi\epsilon_0} \frac{4\pi}{3} \mathbf{p}, \quad (3.24)$$

where the integration is over *any* sphere containing all the charges. (A proof of this formula by using Eqs. (1.9) and (1.22) is left for the reader's exercise.) The origin of the difference is illustrated in Fig. 3 on the example of a physical dipole, i.e. a system of two equal but opposite charges – see Eqs. (8)-(9).

⁴ The equivalence may be proved, for example, by using MA Eq. (11.6) with $\mathbf{f} = \mathbf{p} = \text{const}$ and $\mathbf{g} = \mathbf{E}_{\text{ext}}$, taking into account that according to the general Eq. (1.28), $\nabla \times \mathbf{E}_{\text{ext}} = 0$.

The zero average (23) of the dipole field (13) does not take into account the contribution from the region between the charges where Eq. (13) is not valid, and the field is directed mostly against the dipole vector (9).

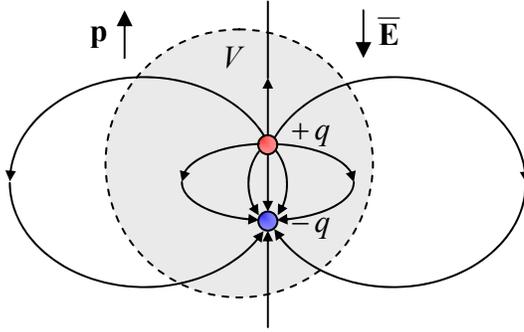


Fig. 3.3. A sketch illustrating the origin of Eq. (24) for a physical dipole.

So, in order to be used as a reasonable coarse-grain model, Eq. (13) may be modified as follows:

$$\mathbf{E}_{\text{cg}} = \frac{1}{4\pi\epsilon_0} \left[\frac{3\mathbf{r}(\mathbf{r} \cdot \mathbf{p}) - \mathbf{p}r^2}{r^5} - \frac{4\pi}{3} \mathbf{p} \delta(\mathbf{r}) \right], \quad (3.25)$$

with the average (21) satisfying Eq. (24). Evidently, such a modification does not change the field at large distances $r \gg a$, i.e. in the region where the expansion (3), and hence Eq. (13), are valid.

3.2. Dipole media

Now let us generalize Eq. (7) to the case of several (possibly, many) dipoles \mathbf{p}_j located at arbitrary points \mathbf{r}_j . Using the linear superposition principle, we get

$$\phi_d(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_j \mathbf{p}_j \cdot \frac{\mathbf{r} - \mathbf{r}_j}{|\mathbf{r} - \mathbf{r}_j|^3}. \quad (3.26)$$

If our system (medium) contains many similar dipoles, distributed in space with density $n(\mathbf{r})$, we may approximate the last sum with a *macroscopic potential*, which is the average of the genuine (“microscopic”) potential (26) over a local volume much larger than the distance between the dipoles, and as a result, is given by the integral

$$\phi_d(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \mathbf{P}(\mathbf{r}') \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3r', \quad \text{with } \mathbf{P}(\mathbf{r}) \equiv n(\mathbf{r})\mathbf{p}, \quad (3.27)$$

where the vector $\mathbf{P}(\mathbf{r})$, called the *electric polarization*, has the physical meaning of the net dipole moment per unit volume. (Note that by its definition, $\mathbf{P}(\mathbf{r})$ is also a “macroscopic” field.)

Now comes a very impressive trick, which is the basis of all the theory of “macroscopic” electrostatics (and eventually, “macroscopic” electrodynamics). Just as was done at the derivation of Eq. (5), Eq. (27) may be rewritten in the equivalent form

$$\phi_d(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \mathbf{P}(\mathbf{r}') \cdot \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} d^3r', \quad (3.28)$$

where ∇' means the del operator (in this particular case, the gradient) acting in the “source space” of vectors \mathbf{r}' . The right-hand side of Eq. (28), applied to any volume V limited by a closed surface S , may be readily integrated by parts to give⁵

$$\phi_d(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_S \frac{P_n(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2r' - \frac{1}{4\pi\epsilon_0} \int_V \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'. \quad (3.29)$$

If the surface does not carry an infinitely dense (δ -functional) sheet of additional dipoles,⁶ or it is just very distant, the first term on the right-hand side is negligible. Now comparing the second term with the basic equation (1.38) for the electric potential, we see that this term may be interpreted as the field of certain *effective* electric charges with density

$$\rho_{\text{ef}} = -\nabla \cdot \mathbf{P}. \quad (3.30)$$

Effective
charge
density

Figure 4 illustrates the physics of this key relation for a cartoon model of a simple multi-dipole system: a layer of uniformly distributed two-point-charge units oriented normally to the layer’s surface. (In this case, $\nabla \cdot \mathbf{P} = dP/dx$.) One can see that the ρ_{ef} defined by Eq. (30) may be interpreted as the density of the uncompensated surface charges of polarized elementary dipoles.

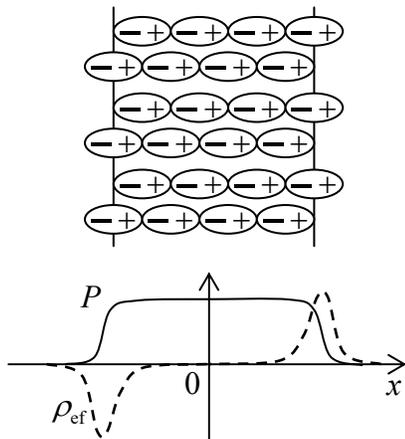


Fig. 3.4. The spatial distributions of the polarization and effective charges in a layer of similar elementary dipoles (schematically).

Next, from Sec. 1.2, we already know that Eq. (1.38) is equivalent to the inhomogeneous Maxwell equation (1.27) for the electric field, so the *macroscopic* electric field of the dipoles (defined as $\mathbf{E}_d = -\nabla\phi_d$, where ϕ_d is given by Eq. (27)) obeys a similar equation, with the effective charge density (30).

Now let us consider a more general case when a system, besides the compensated charges of the dipoles, also has certain stand-alone charges – not parts of the dipoles already taken into account in the polarization \mathbf{P} . As was discussed in Sec. 1.1, if we average this charge over the inter-point-charge distances, i.e. approximate it with a continuous “macroscopic” density $\rho(\mathbf{r})$, then its macroscopic

⁵ To prove this (almost evident) formula strictly, it is sufficient to apply the divergence theorem given by MA Eq. (12.2), to the vector function $\mathbf{f} = \mathbf{P}(\mathbf{r}')/|\mathbf{r} - \mathbf{r}'|$, in the “source space” of radius-vectors \mathbf{r}' .

⁶ Just like in the case of Eq. (1.9), we may always describe such a dipole sheet using the second term in Eq. (29), by including a delta-functional part into the polarization distribution $\mathbf{P}(\mathbf{r}')$.

electric field also obeys Eq. (1.27), but with the stand-alone charge density. Due to the linear superposition principle, for the *total macroscopic field* \mathbf{E} of these charges and dipoles, we may write

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} (\rho + \rho_{\text{ef}}) = \frac{1}{\varepsilon_0} (\rho - \nabla \cdot \mathbf{P}). \quad (3.31)$$

This is already the main result of the “macroscopic” electrostatics. However, it is evidently tempting (and very convenient for applications) to rewrite Eq. (31) in a different form by carrying the dipole-related term of this equality over to its left-hand side. The resulting formula is called the *macroscopic Maxwell equation for \mathbf{D}* :

$$\nabla \cdot \mathbf{D} = \rho, \quad (3.32)$$

Maxwell
equation
for \mathbf{D}

where $\mathbf{D}(\mathbf{r})$ is a new “macroscopic” field, called the *electric displacement* (in some older texts, “electric induction”), defined as⁷

$$\mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P}. \quad (3.33)$$

Electric
displacement

The comparison of Eqs. (32) and (1.27) shows that \mathbf{D} (or more strictly, the fraction \mathbf{D}/ε_0) may be interpreted as the “would-be electric field” that *would be* created by stand-alone charges in the absence of the dipole medium polarization. It should be distinguished from the \mathbf{E} participating in Eqs. (31) and (33), i.e. from the genuine electric field, if averaged over a spatial scale of the order of the distance between elementary charges and dipoles.

In order to get an even better gut feeling of the fields \mathbf{E} and \mathbf{D} , let us first rewrite the macroscopic Maxwell equation (32) in the integral form. Applying the divergence theorem to an arbitrary volume V limited by surface S , we get the following *macroscopic Gauss law*:

$$\oint_S D_n d^2 r = \int_V \rho d^3 r \equiv Q, \quad (3.34)$$

Macroscopic
Gauss law

where Q is the *stand-alone* charge inside volume V .

This general result may be used to find the boundary conditions for \mathbf{D} at a sharp interface between two different dielectrics. (The analysis is applicable to a dielectric/free-space boundary as well.) For that, let us apply Eq. (34) to a flat pillbox formed at the interface (see the solid rectangle in Fig. 5), which is sufficiently small on the spatial scales of the dielectric’s nonuniformity and the interface’s curvature, but still contains many elementary dipoles. Assuming that the interface does not have stand-alone surface charges, we immediately get

$$(D_n)_1 = (D_n)_2, \quad (3.35)$$

Boundary
condition
for \mathbf{D}

i.e. the normal component of the electric displacement has to be continuous. Note that a similar statement for the macroscopic electric field \mathbf{E} is generally not valid, because the polarization vector \mathbf{P} may have, and typically does have a leap at a sharp interface (say, due to the different polarizability of

⁷ Note that according to its definition (33), the dimensionality of \mathbf{D} in the SI units is different from that of \mathbf{E} . In contrast, in the Gaussian units, the electric displacement is defined as $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$, so $\nabla \cdot \mathbf{D} = 4\pi\rho$ (the relation $\rho_{\text{ef}} = -\nabla \cdot \mathbf{P}$ remains the same as in the SI units), and the dimensionalities of \mathbf{D} and \mathbf{E} coincide. This coincidence is a certain perceptual handicap because it is frequently convenient to consider the scalar components of \mathbf{E} as generalized forces, and those of \mathbf{D} as generalized coordinates (see Sec. 5 below), and it is somewhat comforting to have their dimensionalities different, as they are in the SI units.

the two different dielectrics), providing a surface layer of the effective charges (30) – see again the example shown in Fig. 4.

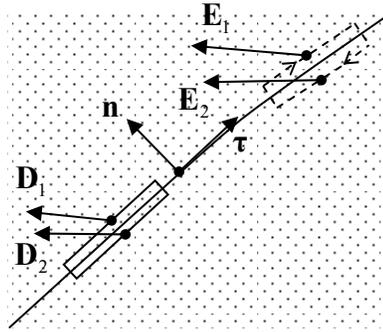


Fig. 3.5. Deriving the boundary conditions at an interface between two dielectrics, using a Gauss pillbox (shown as a solid-line rectangle) and a circulation contour (dashed-line rectangle). Here \mathbf{n} and $\boldsymbol{\tau}$ are the unit vectors that are, respectively, normal and tangential to the interface. Note that due to the leap of polarization, the field lines are generally “refracted” at the interface – see Fig. 11b for an example.

However, we still can make an important statement about the behavior of \mathbf{E} at the interface. Indeed, the macroscopic electric fields defined by Eqs. (29) and (31), are evidently still potential ones, and hence obey the macroscopic Maxwell equation similar to Eq. (1.28):

Macroscopic Maxwell equation for \mathbf{E}

$$\nabla \times \mathbf{E} = 0. \tag{3.36}$$

Integrating this equality along a narrow contour stretched along the interface (see the dashed rectangle in Fig. 5), we get

Boundary condition for \mathbf{E}

$$(E_\tau)_1 = (E_\tau)_2. \tag{3.37}$$

Note that this condition is compatible with (and may be derived from) the continuity of the macroscopic electrostatic potential ϕ related to the macroscopic field \mathbf{E} by the relation similar to Eq. (1.33), $\mathbf{E} = -\nabla\phi$, at each point of the interface: $\phi_1 = \phi_2$.

In order to see how these boundary conditions work, let us consider the simple problem shown in Fig. 6. A very broad plane capacitor, with zero voltage between its conducting plates (as may be enforced, for example, by their connection with an external wire), is partly filled with a material with a uniform polarization \mathbf{P}_0 ,⁸ oriented normal to the plates. Let us calculate the spatial distribution of the fields \mathbf{E} and \mathbf{D} , and also the surface charge density of each conducting plate.

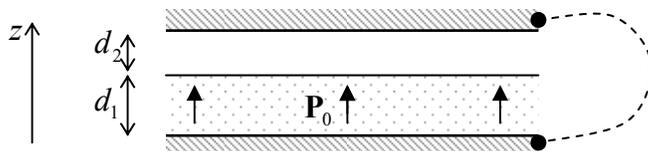


Fig. 3.6. A simple system whose analysis requires Eq. (35).

Due to the symmetry of the system, the vectors \mathbf{E} and \mathbf{D} are both normal to the plates and do not depend on the position in the capacitor’s plane, so we can limit the fields’ analysis to the calculation of their z -components $E(z)$ and $D(z)$. In this case, the Maxwell equation (32) is reduced to $dD/dz = 0$ inside each layer (but not at their border!), so within each of them, D is constant – say, some D_1 in the layer with $\mathbf{P} = \mathbf{P}_0$, and certain D_2 in the free-space layer, where $\mathbf{P} = 0$. As a result, according to Eq. (33), the (macroscopic) electric field inside each layer is also constant:

⁸ As will be discussed in the next section, this is a good approximation for the so-called *electrets*, and also for *hard ferroelectrics* in not very high electric fields.

$$D_1 = \varepsilon_0 E_1 + P_0, \quad D_2 = \varepsilon_0 E_2. \quad (3.38)$$

Since the voltage between the plates is zero, we may also require the integral of E , taken along a path connecting the plates, to vanish. This gives us one more relation:

$$E_1 d_1 + E_2 d_2 = 0. \quad (3.39)$$

Still, the three equations (38)-(39) are insufficient to calculate the four fields in the system ($E_{1,2}$ and $D_{1,2}$). The decisive help comes from the boundary condition (35):

$$D_1 = D_2. \quad (3.40)$$

(Note that it is valid because the layer interface does not carry *stand-alone* electric charges, even though it has a *polarization surface charge*, whose areal density may be calculated by integrating Eq. (30) across the interface: $\sigma_{\text{ef}} = P_0$. Note also that in our simple system, Eq. (37) is identically satisfied due to the system's symmetry, and hence does not give any additional information.)

Now solving the resulting system of four equations (38)-(40), we readily get

$$E_1 = -\frac{P_0}{\varepsilon_0} \frac{d_2}{d_1 + d_2}, \quad E_2 = \frac{P_0}{\varepsilon_0} \frac{d_1}{d_1 + d_2}, \quad D_1 = D_2 = D = P_0 \frac{d_1}{d_1 + d_2}. \quad (3.41)$$

The areal densities of the electrode surface charges may now be readily calculated by the integration of Eq. (32) across each surface:

$$\sigma_1 = -\sigma_2 = D = P_0 \frac{d_1}{d_1 + d_2}. \quad (3.42)$$

Note that due to the spontaneous polarization of the lower layer's material, the capacitor plates are charged even in the absence of voltage between them and that this charge is a function of the second electrode's position (d_2).⁹ Also notice a substantial similarity between this system (Fig. 6), and the one whose analysis was the subject of Problem 2.6.

3.3. Polarization of dielectrics

The general relations derived in the previous section may be used to describe the electrostatics of any dielectrics – materials with bound electric charges (and hence with negligible dc electric conduction). However, to form a full system of equations necessary to solve electrostatics problems, they have to be complemented by certain constitutive relations between the vectors \mathbf{P} and \mathbf{E} .¹⁰

In most materials, in the absence of an external electric field, the elementary dipoles \mathbf{p} either equal zero or have a random orientation in space, so the net dipole moment of each macroscopic volume

⁹ This effect is used in most modern microphones. In such a device, the sensed sound wave's pressure bends a thin conducting membrane playing the role of one of the capacitor's plates, and thus modulates the thickness (in Fig. 6, d_2) of the air gap adjacent to the electret layer. This modulation produces proportional variations of the charges (42), and hence the corresponding electric current flowing between the plates, which is picked up by readout electronics. According to J. West (who, together with G. Sessler, invented the electret microphone in 1962), currently more than 2 billion of these devices are fabricated each year.

¹⁰ In the problem solved at the end of the previous section, the role of such relation was played by the equality $\mathbf{P}_0 = \text{const}$.

(still containing many such dipoles) equals zero: $\mathbf{P} = 0$ at $\mathbf{E} = 0$. Moreover, if the field changes are sufficiently slow, most materials may be characterized by a unique dependence of \mathbf{P} on \mathbf{E} . Then using the Taylor expansion of function $\mathbf{P}(\mathbf{E})$, we may argue that in relatively low electric fields the function should be well approximated by a linear dependence between these two vectors. Such dielectrics are called *linear* (or “simple”). In an isotropic media, the coefficient of proportionality should be just a scalar. In the SI units, this scalar is defined by the following relation:

Electric susceptibility

$$\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}, \quad (3.43)$$

with the dimensionless constant χ_e called the *electric susceptibility*. However, it is much more common to use, instead of χ_e , another dimensionless parameter,¹¹

Dielectric constant

$$\kappa \equiv 1 + \chi_e, \quad (3.44)$$

which is sometimes called the “relative electric permittivity”, but much more often, the *dielectric constant*. This parameter is very convenient, because combining Eqs. (43) and (44),

$$\mathbf{P} = (\kappa - 1)\varepsilon_0 \mathbf{E}. \quad (3.45)$$

and then plugging the resulting relation into the general Eq. (33), we get simply

$$\mathbf{D} = \kappa \varepsilon_0 \mathbf{E}, \quad \text{or} \quad \mathbf{D} = \varepsilon \mathbf{E}, \quad (3.46)$$

where another popular parameter,¹²

Electric permittivity

$$\varepsilon \equiv \kappa \varepsilon_0 \equiv (1 + \chi_e) \varepsilon_0. \quad (3.47)$$

ε is called the *electric permittivity* of the material.¹³ Table 1 gives the approximate values of the dielectric constant for several representative materials.

In order to understand the range of these values, let me discuss (briefly and rather superficially¹⁴) the two simplest mechanisms of electric polarization. The first of them is typical for liquids and gases of *polar* atoms/molecules, which have their own, spontaneous dipole moments \mathbf{p} . (A typical example is the water molecule H_2O , with the negative oxygen ion offset from the line connecting two positive hydrogen ions, thus producing a spontaneous dipole moment $p = ea$, with $a \approx 0.38 \times 10^{-10} \text{m} \sim r_B$.) In the absence of an external electric field, the orientation of such dipoles may be random, with the average polarization $\mathbf{P} = n\langle \mathbf{p} \rangle$ equal to zero – see the top panel of Fig. 7a.

¹¹ In older physics literature, the dielectric constant is often denoted by the letter ε_r (with the index “r” meaning “relative”), while in electrical engineering publications, its notation is frequently K .

¹² The reader may be perplexed by the use of three different but uniquely related parameters (χ_e , $\kappa \equiv 1 + \chi_e$, and $\varepsilon \equiv \kappa \varepsilon_0$) for the description of just one scalar property. Unfortunately, such redundancy is typical for physics, whose different sub-field communities have different, well-entrenched traditions.

¹³ In the Gaussian units, χ_e is defined by the following relation: $\mathbf{P} = \chi_e \mathbf{E}$, while ε is defined just as in the SI units, $\mathbf{D} = \varepsilon \mathbf{E}$. Because of that, in the Gaussian units, the constant ε is dimensionless and equals $(1 + 4\pi\chi_e)$. As a result, $\varepsilon_{\text{Gaussian}} = (\varepsilon/\varepsilon_0)_{\text{SI}} \equiv \kappa$, so $(\chi_e)_{\text{Gaussian}} = (\chi_e)_{\text{SI}}/4\pi$, sometimes creating confusion between the numerical values of the latter parameter – dimensionless in both systems.

¹⁴ While I believe this discussion is very useful, it is quantitatively valid only for relatively sparse media, with low concentration ($n \ll 1/a^3$) of elementary atomic/molecular dipoles of size scale a . Indeed, in some condensed materials, with $na^3 \sim 1$, even the notion of the dipole moment \mathbf{p} with a single atomic cell is ambiguous.

Table 3.1. Dielectric constants of a few representative (and/or practically important) dielectrics

Material	κ
Air (at ambient conditions)	1.00054
Teflon (polytetrafluoroethylene, $[\text{C}_2\text{F}_4]_n$)	2.1
Silicon dioxide (amorphous)	3.9
Glasses (of various compositions)	3.7–10
Castor oil	4.5
Silicon ^(a)	11.7
Water (at 100°C)	55.3
Water (at 20°C)	80.1
Barium titanate (BaTiO_3 , at 20°C)	~1,600

^(a) Anisotropic materials, such as silicon crystals, require a *susceptibility tensor* to give an exact description of the linear relation of the vectors \mathbf{P} and \mathbf{E} . However, most important crystals (including Si) are only weakly anisotropic, so they may be reasonably well characterized with a scalar (angle-average) susceptibility.

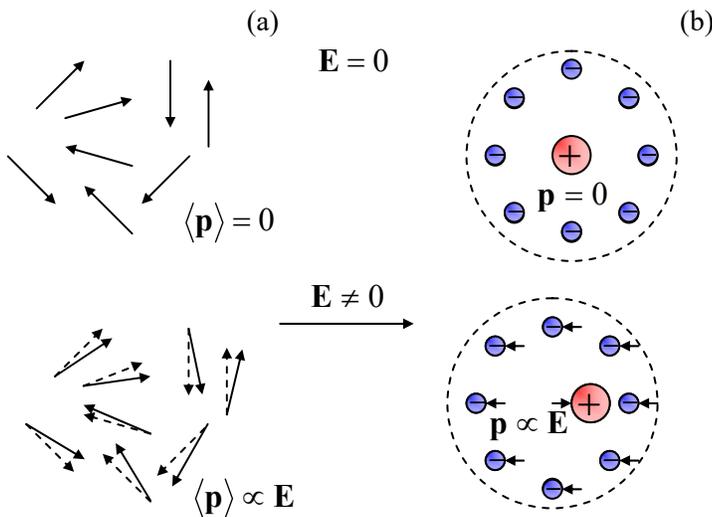


Fig. 3.7. Crude cartoons of two mechanisms of the induced electrical polarization: (a) a partial ordering of spontaneous elementary dipoles, and (b) an elementary dipole induction. The upper two panels correspond to $\mathbf{E} = 0$, and the lower two panels, to $\mathbf{E} \neq 0$.

A relatively weak external field does not change the magnitude of the dipole moments significantly, but according to Eqs. (15a) and (17), tries to orient them along the field, creating a non-zero vector average $\langle \mathbf{p} \rangle$ directed along the vector \mathbf{E}_m , where \mathbf{E}_m is the microscopic field at the point of the dipole's location – cf. two panels of Fig. 7a. If the field is not too high ($p\langle E_m \rangle \ll k_B T$), the induced average polarization $\langle \mathbf{p} \rangle$ is proportional to \mathbf{E}_m . If we write this proportionality relation in the following traditional form,

$$\langle \mathbf{p} \rangle = \alpha \mathbf{E}_m,$$

(3.48) Atomic polarizability

where α is called the *atomic* (or, sometimes, “molecular”) *polarizability*, this means that α is positive. If the concentration n of such elementary dipoles is low, the contribution of their own fields into the

microscopic field acting on each dipole is negligible, and we may identify \mathbf{E}_m with the macroscopic field \mathbf{E} . As a result, the second of Eqs. (27) yields

$$\mathbf{P} \equiv n \langle \mathbf{p} \rangle = \alpha n \mathbf{E}. \quad (3.49)$$

Comparing this relation with Eq. (45), we get

$$\kappa = 1 + \frac{\alpha n}{\epsilon_0}, \quad (3.50)$$

so $\kappa > 1$ (i.e. $\chi_e = \alpha n / \epsilon_0 > 0$). Note that at this particular polarization mechanism (illustrated on the lower panel of Fig. 7a), the thermal motion “tries” to randomize the dipole orientation, i.e. reduce its ordering by the field, so we may expect α , and hence $\chi_e \equiv \kappa - 1$ to increase as temperature T is decreased – the so-called *paraelectricity*. Indeed, the basic statistical mechanics¹⁵ shows that in this case, the electric susceptibility follows the so-called *Curie law* $\chi_e \propto 1/T$.

The materials of the second, much more common class consist of *non-polar* atoms without intrinsic spontaneous polarization. A crude classical image of such an atom is an isotropic cloud of negatively charged electrons surrounding a positively charged nucleus – see the top panel of Fig. 7b. The external electric field shifts the positive charge in the direction of the vector \mathbf{E} , and the negative charges in the opposite direction, thus creating a similarly directed average dipole moment $\langle \mathbf{p} \rangle$.¹⁶ At relatively low fields, this average moment is proportional to \mathbf{E} , so we again arrive at Eq. (48), with $\alpha > 0$, and if the dipole concentration n is sufficiently low, also at Eq. (50), with $\kappa - 1 > 0$. So, the dielectric constant is larger than 1 for both polarization mechanisms – please have one more look at Table 1.

In order to make a crude but physically transparent estimate of the difference $\kappa - 1$, let us consider the following toy model of a non-polar dielectric: a set of similar conducting spheres of radius R , distributed in space with a low density $n \ll 1/R^3$. At such density, the electrostatic interaction of the spheres is negligible, and we can use Eq. (11) for the induced dipole moment of a single sphere. Then the polarizability definition (48) yields $\alpha = 4\pi\epsilon_0 R^3$, so Eq. (50) gives

$$\kappa = 1 + 4\pi R^3 n. \quad (3.51)$$

Let us use this result for a crude estimate of the dielectric constant of air at the so-called *ambient conditions*, meaning the normal atmospheric pressure $\mathcal{P} = 1.013 \times 10^5$ Pa and temperature $T = 300$ K. At these conditions the molecular density n may be, with a few-percent accuracy, found from the well-known equation of state of an ideal gas:¹⁷ $n \approx \mathcal{P}/k_B T \approx (1.013 \times 10^5)/(1.38 \times 10^{-23} \times 300) \approx 2.45 \times 10^{25} \text{ m}^{-3}$. The molecule of the air’s main component, N_2 , has a van-der-Waals radius¹⁸ of 1.55×10^{-10} m. Taking this radius for the R of our crude model, we get $\chi_e \equiv \kappa - 1 \approx 1.15 \times 10^{-3}$. Comparing this number with the

¹⁵ See, e.g., SM Chapter 2.

¹⁶ Realistically, these effects are governed by quantum mechanics, so the average here should be understood not only in the statistical-mechanical but also (and mostly) in the quantum-mechanical sense. Because of that, for non-polar atoms, α is typically a very weak function of temperature, at least on the usual scale $T \sim 300\text{K}$.

¹⁷ If needed, see, e.g., SM Secs. 1.4 and 3.1.

¹⁸ Such radius is defined by the requirement that the volume of the corresponding sphere, if used in the van-der-Waals equation (see, e. g., SM Sec. 4.1), gives the best fit to the experimental equation of state $n = n(P, T)$.

first line of Table 1, we see that the model gives a surprisingly reasonable result: to get the experimental value, it is sufficient to decrease the effective R of the sphere by just $\sim 30\%$, to $\sim 1.2 \times 10^{-10}$ m.¹⁹

This result may encourage us to try using Eq. (51) for a larger density n . For example, as a crude model for a non-polar crystal, let us assume that the conducting spheres form a simple cubic lattice with the period $a = 2R$ (i.e., the neighboring spheres virtually touch). With this, $n = 1/a^3 = 1/8R^3$ and Eq. (44) yields $\kappa = 1 + 4\pi/8 \approx 2.5$. This estimate provides a reasonable semi-qualitative explanation for the values of κ listed in a few middle rows of Table 1. However, at such small distances, the electrostatic dipole-dipole interaction should be already essential, so this simple model cannot even approximately describe the values of κ much larger than 1, listed in the last rows of the table.

Such high values may be explained by the so-called *molecular field effect*: each elementary dipole is polarized not only by the external field, as Eq. (49) assumes, but by the field of neighboring dipoles as well. Ottavino-Fabrizio Mossotti in 1850 and (almost 30 years later) Rudolf Clausius suggested what is now known, rather unfairly, as the *Clausius-Mossotti formula*,²⁰ which describes this effect reasonably well in many non-polar materials. In our notation, it reads²¹

$$\frac{\kappa - 1}{\kappa + 2} = \frac{\alpha n}{3\epsilon_0}, \quad \text{so } \kappa = 1 + \frac{\alpha n / \epsilon_0}{1 - \alpha n / 3\epsilon_0}. \quad (3.52)$$

Clausius-Mossotti formula

If the dipole density is low in the sense $n \ll \epsilon_0/\alpha$, this relation is reduced to Eq. (50) corresponding to independent dipoles. However, at higher dipole density, κ and hence $\chi_e \equiv \kappa - 1$ increase faster and tend to infinity as the density-polarizability product approaches some critical value n_c , equal to $3\epsilon_0/\alpha$ in the Clausius-Mossotti approximation.²² This means that the zero-polarization state becomes unstable even in the absence of an external electric field.

This instability is a linear-theory (i.e. low-field) manifestation of a substantially nonlinear effect – the formation, in some materials, of spontaneous polarization even in the absence of an external electric field. Such materials are called *ferroelectrics*, and may be experimentally recognized by the hysteretic behavior of their polarization as a function of the applied (external) electric field – see Fig. 8. As the plots show, the polarization of a ferroelectric depends on the applied field's history. For example, the direction of its spontaneous *remnant polarization* P_R may be switched by first applying, and then removing a sufficiently high field (larger than the so-called *coercive field* E_C – see Fig. 8) of the opposite orientation. The physics of this switching is rather involved; the polarization vector \mathbf{P} of a ferroelectric material is typically constant only within each of the spontaneously formed spatial regions (called *domains*), with a typical size of a few tenths of a micron, and different (frequently, opposite) directions of the vector \mathbf{P} in adjacent domains. The change of the applied electric field results not in the

¹⁹ As will be discussed in QM Chapter 6, for a hydrogen atom in its ground state, the low-field polarizability may be calculated analytically: $\alpha = (9/2) \times 4\pi\epsilon_0 r_B^3$, corresponding to our metallic-ball model with a close value of the effective radius: $R = (9/2)^{1/3} r_B \approx 1.65 r_B \approx 0.87 \times 10^{-10}$ m.

²⁰ Applied to the high-frequency electric field, with κ replaced by the square of the refraction coefficient at the field's frequency (see Chapter 7), this formula is known as the *Lorenz-Lorentz relation*.

²¹ The proof of Eq. (52), by using Eq. (24) for the molecular field's evaluation, is left for the reader's exercise.

²² The Clausius-Mossotti formula does not give quantitatively correct results for many condensed materials, notably including ferroelectrics. For a review of modern approaches to the theory of their polarization, see, e.g., the paper by R. Resta and D. Vanderbilt in the review collection by K. Rabe, C. Ahn, and J.-M. Triscone (eds.), *Physics of Ferroelectrics: A Modern Perspective*, Springer, 2010.

switching of the direction of \mathbf{P} inside each domain, but rather in a shift of the domain walls, resulting in the change of the average polarization of the sample.

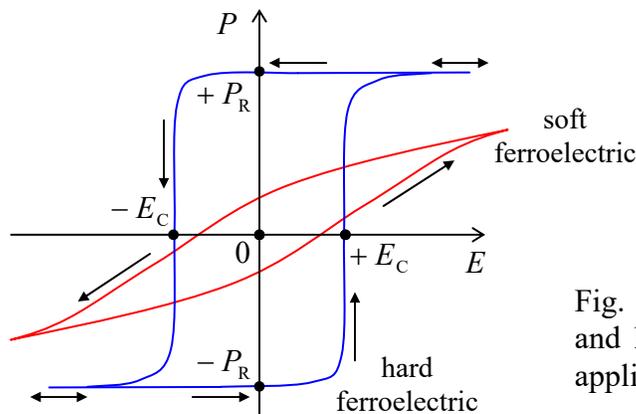


Fig. 3.8. The average polarization of soft and hard ferroelectrics as functions of the applied electric field (schematically).

Depending on the ferroelectric's material, temperature, and the sample's geometry (a solid crystal, a ceramic material, or a thin film), the hysteretic loops may be rather different, ranging from a rather smooth form in the so-called *soft ferroelectrics* (which include most ferroelectric thin films) to an almost rectangular form in *hard ferroelectrics* – see Fig. 8. In low fields, soft ferroelectrics behave essentially as linear paraelectrics, but with a very high average dielectric constant – see the bottom line of Table 1 for such a classical material as BaTiO_3 (which is a soft ferroelectric at temperatures below $T_c \approx 120^\circ\text{C}$, and a paraelectric above this critical temperature). On the other hand, the polarization of a hard ferroelectric in the fields below its coercive field remains virtually constant, and the analysis of their electrostatics may be based on the condition $\mathbf{P} = \mathbf{P}_R = \text{const}$ – already used in the problem discussed in the end of the previous section.²³ This condition is even more applicable to the so-called *electrets* – synthetic polymers with a spontaneous polarization that remains constant even in very high electric fields.

Some materials exhibit even more complex polarization effects, for example, *antiferroelectricity*, *helielectricity*, and (practically very valuable) *piezoelectricity*. Unfortunately, I do not have time for a discussion of these exotic phenomena in this course;²⁴ the main reason I am mentioning them is to emphasize again that the constitutive relation $\mathbf{P} = \mathbf{P}(\mathbf{E})$ is material-specific rather than fundamental. However, most insulators, in practicable fields, behave as linear dielectrics, so the next section will be committed to the discussion of their electrostatics.

²³ Due to this property, hard ferroelectrics, such as the lead zirconate titanate (PZT) and strontium bismuth tantalite (SBT), with high remnant polarization P_R (up to $\sim 1 \text{ C/m}^2$), may be used in nonvolatile random-access memories (dubbed either FRAM or FeRAM) – see, e.g., J. Scott, *Ferroelectric Memories*, Springer, 2000. In a cell of such a memory, binary information is stored in the form of one of two possible directions of spontaneous polarization at $\mathbf{E} = 0$ (see Fig. 8). Unfortunately, the time of spontaneous depolarization of ferroelectric thin films is typically well below 10 years – the industrial standard for data retention in nonvolatile memories, and this time may be decreased even more by “fatigue” from the repeated polarization recycling at information recording. Due to these reasons, the industrial production of FRAM is currently just a tiny fraction of the nonvolatile memory market, which is dominated by floating-gate memories – see, e.g., Sec. 4.2 below.

²⁴ For detailed coverage of ferroelectrics, I can recommend the encyclopedic monograph by M. Lines and A. Glass, *Principles and Applications of Ferroelectrics and Related Materials*, Oxford U. Press, 2001, and the recent review collection edited by K. Rabe *et al.*, that was cited above.

3.4. Electrostatics of linear dielectrics

First, let us consider the simplest but very important problem: how is the electrostatic field of a set of stand-alone charges of density $\rho(\mathbf{r})$ modified if it is placed into a uniform linear dielectric medium that obeys Eq. (46) with a dielectric constant κ constant in the whole region we are interested in. In this case, we may combine Eqs. (32) and (46) to write

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}. \quad (3.53)$$

As a reminder, in the free space, we had a similar equation (1.27), but with a different constant, $\varepsilon_0 = \varepsilon/\kappa$. Hence all the results discussed in Chapter 1 are valid inside a uniform linear dielectric, for the macroscopic field the \mathbf{E} (and the corresponding macroscopic electrostatic potential ϕ), if they are reduced by the factor of $\kappa > 1$. Thus, the most straightforward result of the induced polarization of a dielectric medium is the *electric field reduction*. This is a very important effect, especially taking into account the very high values of κ in such common dielectrics as water – see Table 1. Indeed, it is the reduction of the attraction between positive and negative ions (called, respectively, *cations* and *anions*) in water that enables their substantial dissociation and hence almost all biochemical reactions, which are the basis of the biological cell functions – and hence of the life itself.

Let us apply this general result to the important particular case of the plane capacitor (Fig. 2.3) filled with a linear, uniform dielectric. Applying the macroscopic Gauss law (34) to a pillbox-shaped volume on the conductor surface, we get the following relation,

$$\sigma = D_n = \varepsilon E_n = -\varepsilon \frac{\partial \phi}{\partial n}, \quad (3.54)$$

which differs from Eq. (2.3) only by the replacement $\varepsilon_0 \rightarrow \varepsilon \equiv \kappa\varepsilon_0$. Hence, for a fixed field E_n , the charge density calculated for the free-space case should be increased by the factor of κ – that's it. In particular, this means that the capacitance (2.28) has to be increased by this factor:

$$C = \frac{\kappa\varepsilon_0 A}{d} \equiv \frac{\varepsilon A}{d}. \quad (3.55)$$

C of a
planar
capacitor

(As a reminder, this increase of C by κ has been already incorporated, without proof, into some estimates made in Secs. 2.1 and 2.2, to make them realistic.)

If a linear dielectric is nonuniform, the situation is more complex. For example, let us consider the case of a sharp interface between two otherwise uniform dielectrics, free of stand-alone charges. In this case, we still may use Eq. (37) for the tangential component of the macroscopic electric field, and also Eq. (36), with $D_n = \varepsilon E_n$, for its normal component, getting

$$(\varepsilon E_n)_1 = (\varepsilon E_n)_2, \quad \text{i.e. } \varepsilon_1 \frac{\partial \phi_1}{\partial n} = \varepsilon_2 \frac{\partial \phi_2}{\partial n}. \quad (3.56)$$

Boundary
condition
for E_n

Let us apply these boundary conditions, first of all, to consider how carving a slit of some width d and a much smaller thickness $t \ll d$ from inside a dielectric, changes an initially uniform electric field \mathbf{E}_0 , depending on its orientation – see Fig. 9.

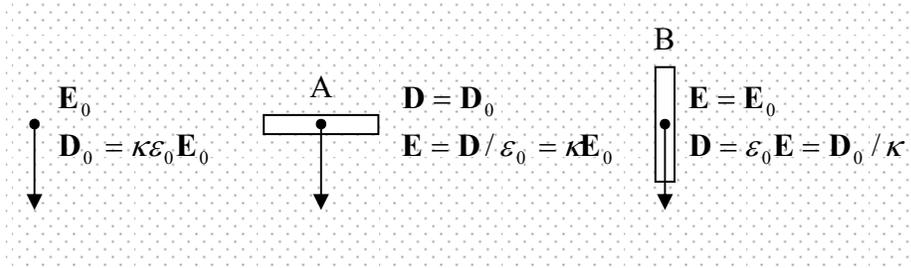


Fig. 3.9. Fields inside two narrow slits cut in a linear dielectric.

First of all, intuition tells us that regardless of its orientation, a slit cannot change the field far from it; moreover, at $t \rightarrow 0$, it cannot modify substantially even the field right outside its “major” (broader) surfaces. This conclusion may be supported either by direct calculations (see, e.g., the problem illustrated by Fig. 11 below), or by energy arguments: at $t \ll d$, any potential energy decrease due to the field change inside the slit’s volume (proportional to td) cannot compensate its increase in the outer volume proportional to d^2 . However, it may induce some local field changes – inside the slit, and even outside it, close to its “minor” surfaces.

To calculate the inner field for case A, with the slit’s plane normal to the applied field, we may apply Eq. (56) to its major surfaces (shown horizontal), to prove that the vector \mathbf{D} should be continuous. But according to Eq. (46), this means that in the free space inside the slit, the electric field should equal \mathbf{D}/ϵ_0 , and hence be κ times higher than the field $\mathbf{E}_0 = \mathbf{D}/\kappa\epsilon_0$ far from the slit. This field, and hence \mathbf{D} , may be measured by a sensor placed inside the gap, so the electric displacement is not an entirely mathematical construct.²⁵ On the contrary, for case B, with the slit’s plane parallel to the initial field, we may apply Eq. (37) to the major (now, vertical) interfaces of the slit, to see that now the electric field \mathbf{E} is continuous, while the electric displacement $\mathbf{D} = \epsilon_0\mathbf{E}$ inside the gap is a factor of κ lower than its value in the dielectric. (Similarly to case A, any perturbations of the field uniformity, caused by the compliance with Eq. (56) at the minor surfaces, settle down at distances $\sim t$ from them.)

For other problems with piecewise-constant ϵ , with more complex geometries, we may need to apply the methods studied in Chapter 2. In particular, in the simplest cases, we can select such a set of orthogonal coordinates that the electrostatic potential depends on just one of them. Consider, for example, two types of filling a plane capacitor with two different dielectrics – see Fig. 10.

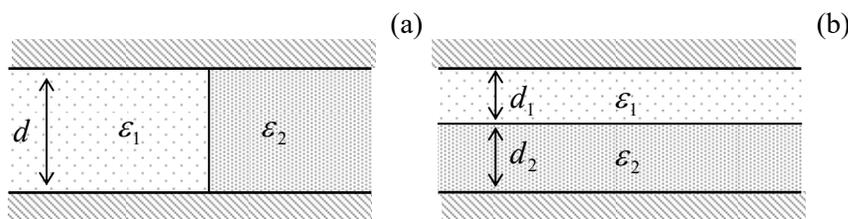


Fig. 3.10. Plane capacitors filled with two different dielectrics.

In case (a), the voltage V between the electrodes is the same for each part of the capacitor, telling us that at least far from the dielectric interface, the electric field is vertical, uniform, and constant ($E = V/d$). Hence the boundary condition (37) is satisfied even if such a distribution is valid near the surface

²⁵ Superficially, this result violates the boundary condition (37) at the vertical (“minor”) surfaces of the gap. This apparent contradiction is resolved by the fact the thin slit can deform the field both inside and outside it, at distances of the order of t around these interfaces, but not far beyond them, so the above relations for \mathbf{E} and \mathbf{D} are valid at most of the slit area.

as well, i.e. at any point of the system. The only effect of different values of ε in the two parts is that the electric displacement $D = \varepsilon E$ and hence electrodes' surface charge density $\sigma = D$ are different in them. Thus we can calculate the electrode charges $Q_{1,2}$ of the two parts independently, and then add up the results to get the total mutual capacitance

$$C = \frac{Q_1 + Q_2}{V} = \frac{1}{d}(\varepsilon_1 A_1 + \varepsilon_2 A_2). \quad (3.57)$$

Note that this formula may be interpreted as the total capacitance of two separate lumped capacitors connected (by wires) *in parallel*. This is natural, because we may cut the system along the dielectric interface, without any effect on the fields in either part, and then connect the corresponding electrodes by external wires, again without any effect on the system – besides very close vicinities of the capacitor's edges, where the fringe

Case (b) may be analyzed just as in the problem illustrated by Fig. 6, by applying Eq. (34) to a Gaussian pillbox with one lid inside the (for example) bottom electrode, and the other lid inside any of the layers. As a result, we see that D anywhere inside the system should be equal to the surface charge density σ of the electrode, i.e. constant. Hence, according to Eq. (46), the electric field E inside each dielectric layer is also constant: in the top layer, it is $E_1 = D_1/\varepsilon_1 = \sigma/\varepsilon_1$, while in bottom layer, $E_2 = D_2/\varepsilon_2 = \sigma/\varepsilon_2$. Integrating the field E across the whole capacitor, we get

$$V = \int_0^{d_1+d_2} E(z) dz = E_1 d_1 + E_2 d_2 = \left(\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2} \right) \sigma, \quad (3.58)$$

so the mutual capacitance per unit area

$$\frac{C}{A} \equiv \frac{\sigma}{V} = \left[\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2} \right]^{-1}. \quad (3.59)$$

Note that this result is similar to the total capacitance of an *in-series* connection of two plane capacitors based on each of the layers. This is also natural because we could insert an uncharged, thin conducting sheet (rather than a cut as in the previous case) at the layer interface, which is an equipotential surface, without changing the field distribution in any part of the system. Then we could thicken the conducting sheet as much as we liked (and possibly shape its internal part into a thin wire), also without changing the fields in the dielectric parts of the system, and hence the capacitance.

Proceeding to problems with more complex geometry, let us consider the system shown in Fig. 11a: a dielectric sphere placed into an initially uniform external electric field \mathbf{E}_0 . According to Eq. (53) for the macroscopic electric field, and the definition of the macroscopic electrostatic potential, $\mathbf{E} = -\nabla\phi$, the potential satisfies the Laplace equation both inside and outside the sphere, though not at its border. Due to the spherical symmetry of the dielectric sample, this problem invites the variable separation method in spherical coordinates, which was discussed in Sec. 2.8. From that discussion, we already know, in particular, the general solution (2.172) of the Laplace equation outside of the sphere. To satisfy the uniform-field condition at $r \rightarrow \infty$, we have to reduce this solution to

$$\phi_{r \geq R} = -E_0 r \cos \theta + \sum_{l=1}^{\infty} \frac{b_l}{r^{l+1}} \mathcal{P}_l(\cos \theta). \quad (3.60)$$

Inside the sphere, we can also use Eq. (2.172), but keeping only the radial functions finite at $r \rightarrow 0$:

$$\phi_{r \leq R} = \sum_{l=1}^{\infty} a_l r^l \mathcal{P}_l(\cos \theta). \quad (3.61)$$

Now, spelling out the boundary conditions (37) and (56) at $r = R$, we see that for all coefficients a_l and b_l with $l \geq 2$, we get homogeneous linear equations (just like for the conducting sphere discussed in Sec. 2.8) that have only trivial solutions. Hence, all these terms may be dropped, while for the only surviving terms with $l = 1$, proportional to the Legendre polynomial $\mathcal{P}_1(\cos \theta) \equiv \cos \theta$, we get two equations:

$$-E_0 - \frac{2b_1}{R^3} = \kappa a_1, \quad -E_0 R + \frac{b_1}{R^2} = a_1 R. \quad (3.62)$$

Solving this simple system of linear equations for a_1 and b_1 , and plugging the result into Eqs. (60) and (61), we get the final solution of the problem:

$$\phi_{r \geq R} = E_0 \left(-r + \frac{\kappa - 1}{\kappa + 2} \frac{R^3}{r^2} \right) \cos \theta, \quad \phi_{r \leq R} = -E_0 \frac{3}{\kappa + 2} r \cos \theta. \quad (3.63)$$

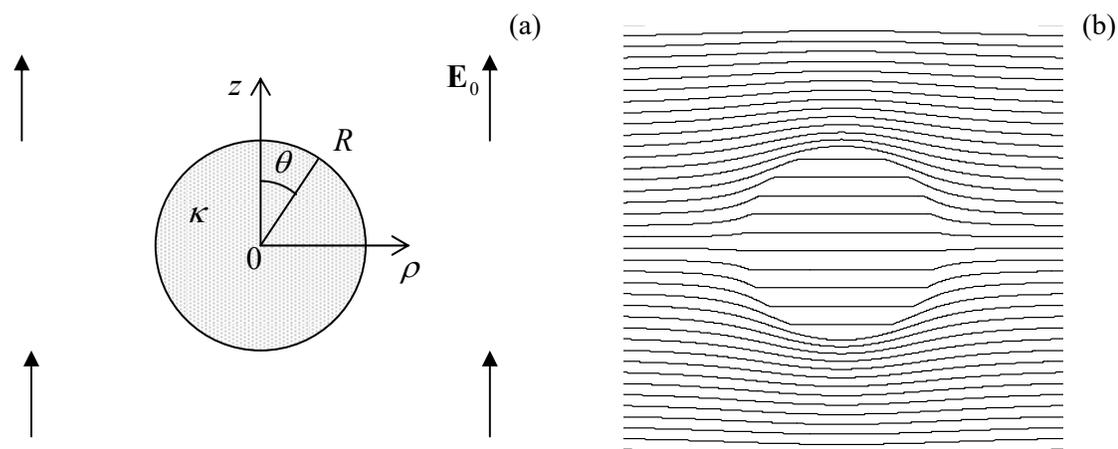


Fig. 3.11. A dielectric sphere in an initially uniform electric field: (a) the problem, and (b) the equipotential surfaces, as given by Eq. (63), for $\kappa = 3$.

Figure 11b shows the equipotential surfaces given by this solution, for a particular value of the dielectric constant κ . Note that according to Eq. (62), at $r \geq R$ the dielectric sphere, just as the conducting sphere in a similar problem, produces (on top of the uniform external field) a pure dipole field, with the dipole moment

$$\mathbf{p} = 4\pi R^3 \frac{\kappa - 1}{\kappa + 2} \varepsilon_0 \mathbf{E}_0 \equiv 3V \frac{\kappa - 1}{\kappa + 2} \varepsilon_0 \mathbf{E}_0, \quad \text{where } V = \frac{4\pi}{3} R^3. \quad (3.64)$$

This is an evident generalization of Eq. (11), to which Eq. (64) tends at $\kappa \rightarrow \infty$. By the way, this property is common: for their electrostatic properties, conductors may be adequately described as dielectrics with $\kappa \rightarrow \infty$.

Another remarkable feature of Eqs. (63) is that the electric field and polarization inside the sphere are uniform, with R -independent values

$$\mathbf{E} = \frac{3}{\kappa + 2} \mathbf{E}_0, \quad \mathbf{D} \equiv \kappa \varepsilon_0 \mathbf{E} = \varepsilon_0 \frac{3\kappa}{\kappa + 2} \mathbf{E}_0, \quad \mathbf{P} \equiv \mathbf{D} - \varepsilon_0 \mathbf{E} = 3\varepsilon_0 \frac{\kappa - 1}{\kappa + 2} \mathbf{E}_0. \quad (3.65)$$

In the limit $\kappa \rightarrow 1$ (for example, the “sphere made of free space”, i.e. no sphere at all), the electric field inside it naturally tends to the external one, and its polarization vanishes. In the opposite limit $\kappa \rightarrow \infty$, the electric field inside the sphere vanishes. Curiously enough, in this limit the electric displacement inside the sphere remains finite: $\mathbf{D} \rightarrow 3\varepsilon_0 \mathbf{E}_0$.

More complex problems with piecewise-uniform dielectrics also may be addressed by the methods discussed in Chapter 2, and hopefully, the reader will be able to use them to solve a few such problems offered in Sec. 6, on their own. Let me discuss just one of such problems because it exhibits a new feature of the charge image method that was discussed in Secs 2.9 (and is the basis of Green’s function approach – see Sec. 2.10). Consider the system shown in Fig. 12: a point charge near a dielectric half-space; it obviously parallels the system discussed in Sec. 2.9 – see Fig. 2.26.

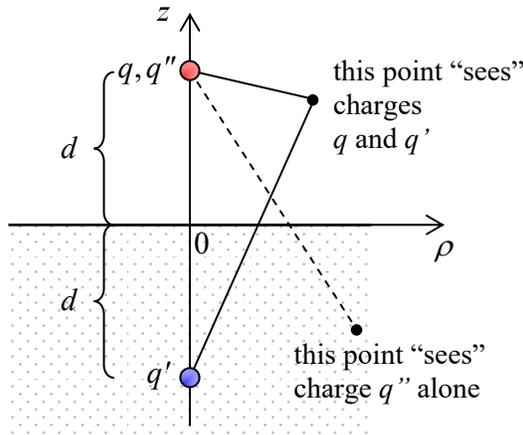


Fig. 3.12. Charge images for a dielectric half-space.

As for the case of a conducting half-space, the Laplace equation for the electrostatic potential in the upper half-space $z > 0$ (besides the charge point $\rho = 0, z = d$) may be satisfied using a single image charge q' at the point with $\rho = 0$ and $z = -d$, but now q' may differ from $(-q)$. In addition, in contrast to the case analyzed in Sec. 2.9, we should also calculate the field inside the dielectric (at $z \leq 0$). This field cannot be contributed by the image charge q' , because that would give a potential divergence at its location. Thus, in the dielectric-filled half-space we should try to use the real point source only, but with a re-normalized charge q'' rather than the genuine charge q – see Fig. 12. As a result, we may look for the potential distribution in the form

$$\phi(\rho, z) = \frac{1}{4\pi\varepsilon_0} \times \begin{cases} \left[\frac{q}{(\rho^2 + (z-d)^2)^{1/2}} + \frac{q'}{(\rho^2 + (z+d)^2)^{1/2}} \right], & \text{for } z \geq 0, \\ \frac{q''}{(\rho^2 + (z-d)^2)^{1/2}}, & \text{for } z \leq 0, \end{cases} \quad (3.66)$$

at this stage of solution, with unknown q' and q'' . Plugging this equality into the boundary conditions (37) and (56) at $z = 0$ (with $\partial/\partial n = \partial/\partial z$), we see that they are indeed satisfied (so Eq. (66) does express the solution of the boundary problem), provided that the effective charges q' and q'' obey the following relations:

$$q - q' = \kappa q'', \quad q + q' = q'' . \quad (3.67)$$

Solving this simple system of linear equations, we get

$$q' = -\frac{\kappa - 1}{\kappa + 1} q, \quad q'' = \frac{2}{\kappa + 1} q . \quad (3.68)$$

If $\kappa \rightarrow 1$, then $q' \rightarrow 0$, and $q'' \rightarrow q$ – both facts very natural because in this limit (no polarization at all) we have to recover the unperturbed field of the initial point charge in both semi-spaces. In the opposite limit $\kappa \rightarrow \infty$ (which, as was discussed above, may describe a conducting half-space), $q' \rightarrow -q$ (repeating the result we have discussed in detail in Sec. 2.9), and $q'' \rightarrow 0$. The last result means that in this limit, the electric field \mathbf{E} in the dielectric tends to zero – as it should.

In conclusion of this section, please note that if the permittivity ε of a linear dielectric is a continuous rather than piecewise function of coordinates, the distribution of the electrostatic potential ϕ may be found from Eq. (32) with the electric displacement given by Eq. (46): $\mathbf{D} = \varepsilon(\mathbf{r})\mathbf{E} = -\varepsilon(\mathbf{r})\nabla\phi$. However, analytical solutions of the resulting partial differential equation of the second order may be found only for rare particular cases; one of them is offered in Sec. 6 for the reader's exercise.

3.5. Electric field energy in a dielectric

In Chapter 1, we have obtained two key results for the electrostatic energy: Eq. (1.55) for a charge interaction with an independent (“external”) field, and a similarly structured formula (1.60), but with an additional factor $\frac{1}{2}$, for the field induced by the charges under consideration. These relations are universal, i.e. valid for dielectrics as well, provided that the charge density includes *all* charges – including those bound into the elementary dipoles. However, for most applications, it is convenient to recast them into a form where these bound charges participate not explicitly, but only via the macroscopic polarization effects they create.

If a field is created only by the stand-alone charges under consideration and is proportional to $\rho(\mathbf{r})$ (requiring that we deal with linear dielectrics), we can repeat all the argumentation of the beginning of Sec. 1.3, and again arrive at Eq. (1.60), provided that ϕ is now the macroscopic field's potential. Now we can recast this result in the terms of fields – essentially as this was done in Eqs. (1.62)-(1.64), but now making a clear difference between the macroscopic electric field $\mathbf{E} = -\nabla\phi$ and the electric displacement field \mathbf{D} , which obeys the macroscopic Maxwell equation (32). Plugging $\rho(\mathbf{r})$ expressed from that equation, into Eq. (1.60), we get

$$U = \frac{1}{2} \int (\nabla \cdot \mathbf{D}) \phi d^3 r . \quad (3.69)$$

Using the fact²⁶ that for differentiable functions ϕ and \mathbf{D} ,

$$(\nabla \cdot \mathbf{D}) \phi = \nabla \cdot (\phi \mathbf{D}) - (\nabla \phi) \cdot \mathbf{D} , \quad (3.70)$$

we may rewrite Eq. (69) as

$$U = \frac{1}{2} \int \nabla \cdot (\phi \mathbf{D}) d^3 r - \frac{1}{2} \int (\nabla \phi) \cdot \mathbf{D} d^3 r . \quad (3.71)$$

²⁶ See, e.g., MA Eq. (11.4a).

The divergence theorem, applied to the first term on the right-hand side, reduces it to a surface integral of ϕD_n . (As a reminder, in Eq. (1.63) the integral was of $\phi(\nabla\phi)_n \propto \phi E_n$.) If the surface of the volume we are considering is sufficiently far, this surface integral vanishes. On the other hand, the gradient in the second term of Eq. (71) is just (minus) field \mathbf{E} , so it gives

$$U = \frac{1}{2} \int \mathbf{E} \cdot \mathbf{D} d^3r = \frac{1}{2} \int E(\mathbf{r}) \varepsilon(\mathbf{r}) E(\mathbf{r}) d^3r \equiv \frac{\varepsilon_0}{2} \int \kappa(\mathbf{r}) E^2(\mathbf{r}) d^3r. \quad (3.72)$$

This expression is a natural generalization of Eq. (1.65), and shows that we can, as we did in free space, represent the electrostatic energy in a local form:²⁷

$$U = \int u(\mathbf{r}) d^3r, \quad \text{with } u = \frac{1}{2} \mathbf{E} \cdot \mathbf{D} = \frac{\varepsilon}{2} E^2 = \frac{D^2}{2\varepsilon}. \quad (3.73)$$

Field energy in a linear dielectric

As a sanity check, in the trivial case $\varepsilon = \varepsilon_0$ (i.e. $\kappa = 1$), this result is reduced to Eq. (1.65).

Of course, Eq. (73) is valid only for linear dielectrics, because our starting point, Eq. (1.60), is only valid if ϕ is proportional to ρ . To make our calculation more general, we should intercept the calculations of Sec. 1.3 at an earlier stage, at which this proportionality had not yet been used. For example, the first of Eqs. (1.56) may be rewritten, in the continuous form, as

$$\delta U = \int \phi(\mathbf{r}) \delta \rho(\mathbf{r}) d^3r, \quad (3.74)$$

where the symbol δ means a small variation of the function – e.g., its change in time, sufficiently slow to ignore the relativistic and magnetic-field effects. Applying such variation to Eq. (32), and plugging the resulting relation $\delta \rho = \nabla \cdot \delta \mathbf{D}$ into Eq. (74), we get

$$\delta U = \int (\nabla \cdot \delta \mathbf{D}) \phi d^3r. \quad (3.75)$$

(Note that in contrast to Eq. (69), this expression does not have the front factor $\frac{1}{2}$.) Now repeating the same calculations as in the linear case, for the energy density's *variation* we get a remarkably simple (and general!) formula,

$$\delta u = \mathbf{E} \cdot \delta \mathbf{D} \equiv \sum_{j=1}^3 E_j \delta D_j, \quad (3.76)$$

Energy density's variation

where the last expression uses the Cartesian components of the vectors \mathbf{E} and \mathbf{D} . This is as far as we can go for the general dependence $\mathbf{D}(\mathbf{E})$. If the dependence is linear and isotropic, as in Eq. (46), then $\delta \mathbf{D} = \varepsilon \delta \mathbf{E}$ and

$$\delta u = \varepsilon \mathbf{E} \cdot \delta \mathbf{E} \equiv \varepsilon \delta \left(\frac{E^2}{2} \right). \quad (3.77)$$

The integration of this expression over the whole variation, from the field equal to zero to a certain final distribution $\mathbf{E}(\mathbf{r})$, brings us back to Eq. (73).

An important role of Eq. (76), in its last form, is to indicate that from the point of view of analytical mechanics, the Cartesian coordinates of \mathbf{E} may be interpreted as generalized forces, and those

²⁷ In the Gaussian units, each of the last three expressions should be divided by 4π .

of \mathbf{D} as generalized coordinates of the field's effect on a unit volume of the dielectric. This allows one, in particular, to form the proper *Gibbs potential energy*²⁸ of a system with an electric field $\mathbf{E}(\mathbf{r})$ fixed, at every point, by some external source:

Gibbs
potential
energy

$$U_G = \int_V u_G(\mathbf{r}) d^3r, \quad u_G(\mathbf{r}) = u(\mathbf{r}) - \mathbf{E}(\mathbf{r}) \cdot \mathbf{D}(\mathbf{r}). \quad (3.78)$$

The essence of this notion is that if the generalized external force (in our case, \mathbf{E}) is fixed, the stable equilibrium of the system corresponds to the minimum of U_G , rather than of the potential energy U as such – in our case, that of the field in our system.

As the simplest illustration of this important concept, let us consider a very long cylinder (with an arbitrary cross-section shape), made of a uniform linear dielectric, placed into a uniform external electric field parallel to the cylinder's axis – see Fig. 13.

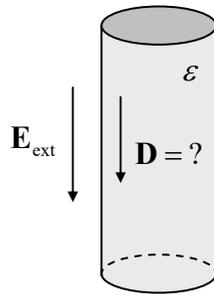


Fig. 3.13. A cylindrical dielectric sample in a longitudinal external electric field.

For this simple problem, the equilibrium value of \mathbf{D} inside the cylinder may be, of course, readily found without any appeal to energies. Indeed, the solution of the Laplace equation inside the cylinder, with the boundary condition (37) is evident: $\mathbf{E}(\mathbf{r}) = \mathbf{E}_{\text{ext}}$, and so Eq. (46) immediately yields $\mathbf{D}(\mathbf{r}) = \epsilon \mathbf{E}_{\text{ext}}$. One may wonder why the minimum of the potential energy U , given by Eq. (73) in its last form,

$$\frac{U}{V} = \frac{D^2}{2\epsilon}, \quad (3.79)$$

corresponds to a different (zero) value of \mathbf{D} , but let us recall that Eq. (73) was derived for the case when the electric field is created by the stand-alone charges in the system under consideration. If it is created by external sources, we have to use the Gibbs potential energy (78) instead. For our current uniform case, this energy per unit volume of the cylinder is

$$\frac{U_G}{V} = \frac{U}{V} - \mathbf{E} \cdot \mathbf{D} = \frac{D^2}{2\epsilon} - \mathbf{E} \cdot \mathbf{D} \equiv \sum_{j=1}^3 \left(\frac{D_j^2}{2\epsilon} - E_j D_j \right), \quad (3.80)$$

and its minimum as a function of every Cartesian component of \mathbf{D} corresponds to the correct value of the displacement: $D_j = \epsilon E_j$, i.e. to $\mathbf{D} = \epsilon \mathbf{E} = \epsilon \mathbf{E}_{\text{ext}}$. So, the systems' equilibrium indeed corresponds to the minimum of the Gibbs potential energy (78) rather than of the energy (73).

²⁸ See, e.g., CM Sec. 1.4, in particular Eq. (1.41). Note that as Eq. (78) clearly illustrates, once again, that the difference between the potential energies U_G and U , usually discussed in courses of thermodynamics and statistical physics as the difference between the Gibbs and Helmholtz free energies (see, e.g., SM 1.4), is much more general than the effects of random thermal motion addressed by these disciplines.

Now note that Eq. (80), at this equilibrium point (only!), may be rewritten as

$$\frac{U_G}{V} = \frac{U}{V} - \mathbf{E} \cdot \mathbf{D} = \frac{D^2}{2\epsilon} - \frac{\mathbf{D}}{\epsilon} \cdot \mathbf{D} \equiv -\frac{D^2}{2\epsilon}, \quad (3.81)$$

i.e. formally coincides with Eq. (79), besides the (perhaps, somewhat counter-intuitive) *opposite sign*. A similar but more general relation (not limited to linear dielectrics and uniform fields) may be obtained by taking the variation of the u_G expressed by Eq. (78), and then using Eq. (76):

$$\delta u_G = \delta u - \delta(\mathbf{E} \cdot \mathbf{D}) = \mathbf{E} \cdot \delta \mathbf{D} - (\delta \mathbf{E} \cdot \mathbf{D} + \mathbf{E} \cdot \delta \mathbf{D}) \equiv -\mathbf{D} \cdot \delta \mathbf{E}. \quad (3.82)$$

In order to see how this expression works, let us plug \mathbf{D} from Eq. (33):

$$\delta u_G = -(\epsilon_0 \mathbf{E} + \mathbf{P}) \cdot \delta \mathbf{E} \equiv -\delta \left(\frac{\epsilon_0 E^2}{2} \right) - \mathbf{P} \cdot \delta \mathbf{E}. \quad (3.83)$$

So far, this relation is general. In the particular case when the polarization \mathbf{P} is field-independent, we may integrate Eq. (83) over the full electric field's variation, say from 0 to some finite value \mathbf{E} , getting

$$u_G = -\frac{\epsilon_0 E^2}{2} - \mathbf{P} \cdot \mathbf{E}. \quad (3.84)$$

Again, the Gibbs energy is relevant only if \mathbf{E} is dominated by an external field \mathbf{E}_{ext} independent of the orientation of \mathbf{P} . If, in addition, $\mathbf{P}(\mathbf{r}) \neq 0$ only in some finite volume V , we may integrate Eq. (84) over that volume, getting

$$U_G = -\mathbf{p} \cdot \mathbf{E}_{\text{ext}} + \text{const}, \quad \text{with } \mathbf{p} \equiv \int_V \mathbf{P}(\mathbf{r}) d^3 r, \quad (3.85)$$

where the “const” means the terms independent of \mathbf{p} . In this expression, we may readily recognize Eq. (15a) for an electric dipole \mathbf{p} of a fixed magnitude, which was obtained in Sec. 1 in a different way. This comparison illustrates again that U_G is nothing mysterious; it is just the relevant part of the potential energy of the system in a fixed external field, including the energy of its interaction with the field.

Finally, in the other important case of a linear dielectric, when according to Eqs. (45) and (47), $\mathbf{P} = (\epsilon - \epsilon_0)\mathbf{E}$, the similar integration of the general Eq. (83) over the field yields the additional factor $\frac{1}{2}$:

$$U_G = -\frac{1}{2} \int_V \mathbf{P} \cdot \mathbf{E}_{\text{ext}} d^3 r + \text{const}. \quad (3.86)$$

This expression may be very convenient for analyses of the forces exerted by electric fields on linear dielectric media – see, for, example, a few exercises on this topic, offered at the end of this chapter.

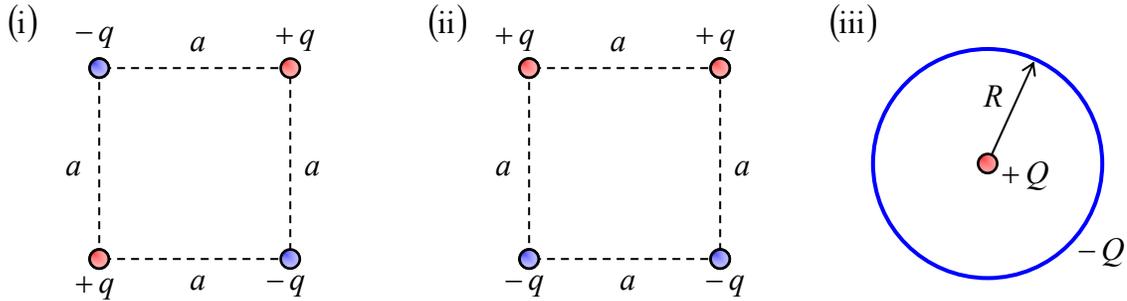
3.6. Exercise problems

3.1. Prove Eqs. (3)-(4), starting from Eqs. (1.38) and (3.2).

3.2. A thin ring of radius R is charged with a constant linear density λ . Calculate the exact electrostatic potential distribution along the symmetry axis of the ring, and prove that at large distances, $r \gg R$, the three leading terms of its multipole expansion are indeed correctly described by Eqs. (3)-(4).

3.3. In suitable reference frames, calculate the dipole and quadrupole moments of the following systems (see the figures below):

- (i) four point charges of the same magnitude but alternating signs, placed in the corners of a square;
- (ii) a similar system but with a pair charge sign alternation; and
- (iii) a point charge in the center of a thin ring carrying a similar but opposite charge uniformly distributed along its circumference.



3.4. Calculate the dipole and quadrupole moments of a thin spherical shell of radius R , carrying an electric charge with the areal density $\sigma = \sigma_0 \cos \theta$. Discuss the relation between the results and the solution of Problem 2.28.

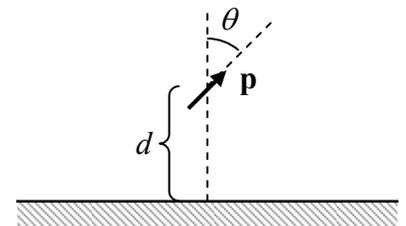
3.5. For a regular cubic lattice of similarly oriented identical dipoles, calculate the electric field it creates at the location of each dipole.

3.6. Without carrying out an exact calculation, can you predict the spatial dependence of the interaction between various electric multipoles, including point charges (in this context, frequently called electric *monopoles*), dipoles, and quadrupoles? Based on these predictions, what is the functional dependence of the interaction between *homonuclear* diatomic molecules such as H_2 , N_2 , O_2 , etc., on the distance between them when the distance is much larger than the molecular size?

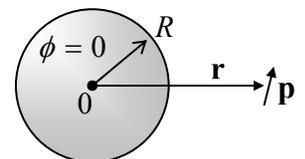
3.7. Two similar electric dipoles, of a fixed magnitude p , located at a fixed distance r from each other, are free to change their directions. What stable equilibrium position(s) they may take as a result of their electrostatic interaction?

3.8. An electric dipole is located above a grounded infinite conducting plane (see the figure on the right). Calculate:

- (i) the distribution of the induced charge in the conductor,
- (ii) the dipole-to-plane interaction energy, and
- (ii) the force and the torque exerted on the dipole.



3.9. Calculate the net charge Q induced in a grounded conducting sphere of radius R by a dipole \mathbf{p} located at point \mathbf{r} outside the sphere – see the figure on the right.



3.10. Use two different approaches to calculate the energy of interaction between a grounded conductor and an electric dipole \mathbf{p} placed in the center of a spherical cavity of radius R , carved in the conductor.

3.11. A plane separating two halves of otherwise free space is densely and uniformly (with a constant areal density n) filled with electric dipoles, with similar moments \mathbf{p} oriented normally to the plane.

(i) Use two different approaches to calculate the electrostatic potential at distances $d \gg 1/n^{1/2}$ on both sides of the plane.

(ii) Give a physical interpretation of your result.

(iii) Use the result to calculate the potential distribution created in space by a spherical surface of radius R , densely and uniformly filled with radially oriented dipoles.

3.12. Prove Eq. (24).

Hint: You may like to use the basic Eq. (1.9) to spell out the left-hand side of Eq. (1.24), change the order of integration over \mathbf{r} and \mathbf{r}' , and then contemplate the physical sense of the inner integral.

3.13. A sphere of radius R is made of a material with a uniform spontaneous polarization \mathbf{P}_0 . Calculate the electric field everywhere in space – both inside and outside the sphere, and compare the result for the internal field with Eq. (24).

3.14. Calculate the electric field at the center of a cube made of a material with the uniform spontaneous polarization \mathbf{P}_0 of arbitrary orientation.

3.15. Derive the Clausius-Mossotti formula (52) by combining Eq. (24) with the result of the solution of Problem 5.

3.16. Stand-alone charge Q is distributed, in some way, within the volume of a body made of a uniform linear dielectric with a dielectric constant κ . Calculate the total polarization charge Q_{ef} residing on the surface of the body, provided that it is surrounded by free space.

3.17. In two separate experiments, a thin plane sheet of a linear dielectric with $\kappa = \text{const}$ is placed into a uniform external electric field \mathbf{E}_0 , in two different ways:

(i) with the sheet's surfaces parallel to the electric field, and

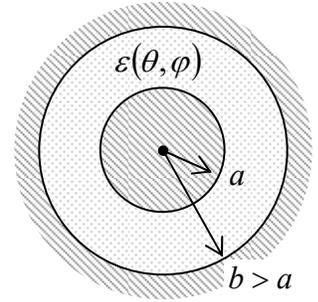
(ii) with its surfaces normal to the field.

For each case, find the electric field \mathbf{E} , the electric displacement \mathbf{D} , and the polarization \mathbf{P} inside the dielectric, sufficiently far from the sheet's edges.

3.18. A fixed dipole \mathbf{p} is placed in the center of a spherical cavity of radius R , carved inside a uniform linear dielectric. Calculate the electric field distribution everywhere in the system.

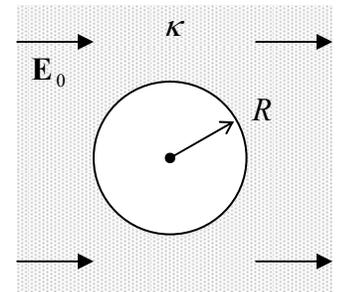
Hint: You may start with the assumption that the field at $r > R$ has a distribution typical for a dipole. However, be ready for surprises.

3.19. A spherical capacitor (see the figure on the right) is filled with a linear dielectric whose permittivity ε depends on the spherical angles θ and φ , but not on the distance r from the system's center. Derive an explicit expression for its capacitance C .



3.20. A spherical capacitor similar to that considered in the previous problem is now filled with a linear dielectric whose permittivity depends only on the distance from the center. Obtain an explicit expression for its capacitance, and spell it out for the particular case $\varepsilon(r) = \varepsilon(a)(r/a)^n$.

3.21. A uniform electric field \mathbf{E}_0 has been created (by distant external sources) inside a uniform linear dielectric. Find the electric field's change created by carving out a cavity in the shape of a round cylinder of radius R , with its axis normal to the external field – see the figure on the right.



3.22. Similar small spherical particles, made of a linear dielectric, are dispersed in free space with a low concentration $n \ll 1/R^3$, where R is the particle's radius. Calculate the average dielectric constant of such a medium. Compare the result with the apparent but wrong answer

$$\bar{\kappa} - 1 = (\kappa - 1)nV, \quad \text{(WRONG!)}$$

(where κ is the dielectric constant of the particle's material and $V = (4\pi/3)R^3$ is its volume), and explain the origin of the difference.

3.23. A straight thin filament, uniformly charged with linear density λ , is positioned parallel to the plane separating two uniform linear dielectrics, at a distance d from it. Calculate the electric potential's distribution everywhere in the system.

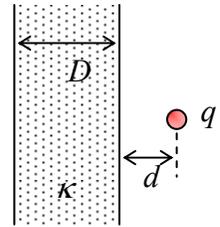
3.24. A point charge q is located at a distance $d > R$ from the center of a sphere of radius R , made of a uniform linear dielectric with permittivity ε .

(i) Calculate the electrostatic potential's distribution in all the space, for an arbitrary ratio d/R .

(ii) For large d/R , use two different approaches to calculate the interaction force and the energy of interaction between the sphere and the charge, in the first nonzero approximation in $R/d \ll 1$.

Hint: Task (i) cannot be carried out using the method of charge images, so you may like to use the expansion of the function $1/|\mathbf{r} - \mathbf{r}'|$ in the series over the Legendre polynomials, whose proof was the subject of Problem 2.40.

3.25. Calculate the spatial distribution of the electrostatic potential induced by a point charge q located at distance d from a very wide parallel plate, of thickness D , made of a uniform linear dielectric – see the figure on the right.

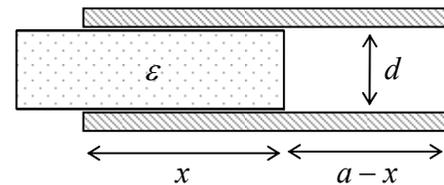


3.26. Discuss the physical nature of Eq. (76). Apply your conclusions to a material with a fixed (field-independent) polarization $\mathbf{P}_0(\mathbf{r})$, and calculate the electric field's energy of a uniformly polarized sphere (see Problem 13 above).

3.27. Use Eqs. (73) and (82) to calculate the force of attraction of a plane capacitor's plates (per unit area), for two cases:

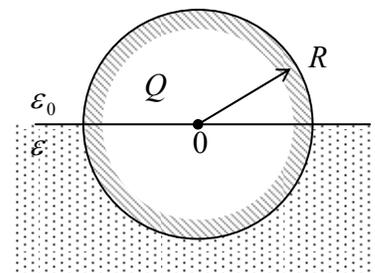
- (i) the capacitor is charged to voltage V , and then disconnected from the battery,²⁹ and
- (ii) the capacitor remains connected to the battery.

3.28. A slab made of a linear dielectric is partly inserted into a plane capacitor – see the figure on the right. Assuming the simplest (cylindrical) geometry of the system, calculate the force exerted by the field on the slab, for the same two cases as in the previous problem



3.29. For each of the two capacitors shown in Fig. 10, calculate the electric force exerted on the interface between two different dielectrics, in terms of the fields in the system.

3.30. One half of a conducting sphere of radius R , carrying electric charge Q , is submerged into a half-space filled with a linear dielectric with permittivity ϵ – see the figure on the right. Calculate the electric force exerted on the sphere by the dielectric.



²⁹ “Battery” is a common if misleading term for what is usually a single *galvanic element*. (The last term stems from the name of Luigi Galvani, a pioneer of electric current studies. Another term derived from his name is the *galvanic connection*, meaning a direct connection of two conductors, enabling a dc current flow – see the next chapter.) The term “battery” had to be, in all fairness, reserved for the connection of several galvanic elements in series – as was pioneered in 1800 by L. Galvani’s friend Alexander Volta.