Chapter 4. DC Currents

The goal of this chapter is to discuss the distribution of stationary ("*dc*") *currents in conducting samples and their "global" characteristics such as resistance. In the most important case of linear* ("*Ohmic*") *conductivity, the current distribution is governed by the same Laplace and Poisson equations whose solution methods were discussed in detail in the previous chapters. Because of that, we can piggyback on most approaches discussed earlier, enabling me to keep this chapter rather brief.*

4.1. Continuity equation and the Kirchhoff laws

Until this point, our discussion of conductors has been limited to the cases when they are separated with *insulators* (meaning either the free space or some dielectric media), preventing any continuous motion of charges from one conductor to another, even if there is a non-zero voltage (and hence electric field) between them – see Fig. 1a.

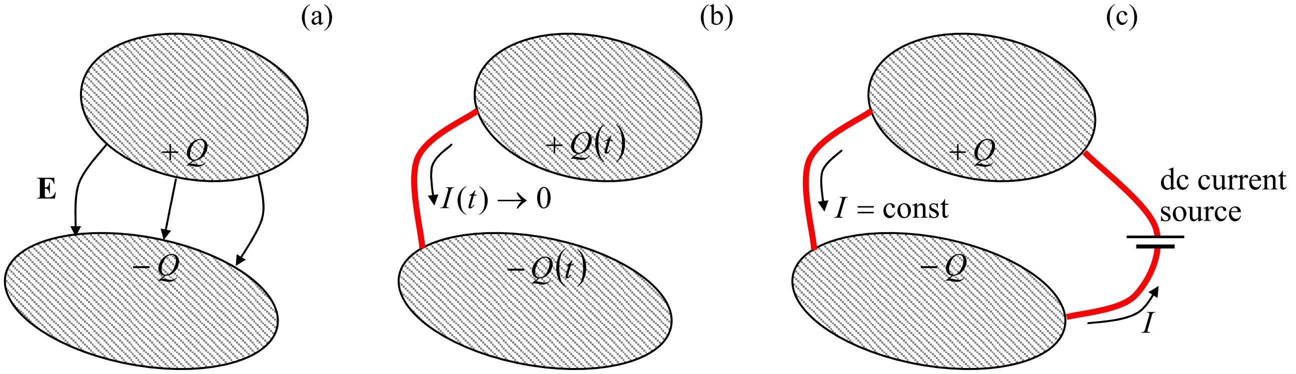

Fig. 4.1. Two oppositely charged conductors: (a) in the electrostatic situation, (b) at the charge relaxation through an additional narrow conductor ("wire"), and (c) in a system sustaining a dc current *I*.

Now let us connect the two conductors with a *wire* – a thin, elongated conductor (Fig. 1b). Then the electric field causes the motion of charge carriers in the wire, from the conductor with a higher electrostatic potential toward that with lower potential, until the potentials equilibrate. Such a process is called *charge relaxation*. The main equation governing this process may be obtained from the fundamental experimental fact (already mentioned in Sec. 1.1) that electric charges cannot appear or disappear – though opposite charges may recombine with the conservation of the net charge. As a result, the charge Q in a conductor may change only due to the *electric current* I through the wire:

$$\frac{dQ}{dt} = -I(t); \tag{4.1}$$

this relation may be understood as the definition of the current.[1]

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[1] Just as a (hopefully, unnecessary :-) reminder, in the SI units the current is measured in amperes (A). In legal metrology, the ampere (rather than the coulomb, which is defined as 1C = 1A × 1s) is a primary unit. (Its formal definition will be discussed in the next chapter.) In the Gaussian units, Eq. (1) remains the same, so the current's unit is the statcoulomb per second – the so-called *statampere*.

Let us express Eq. (1) in a differential form, introducing the notion of the *current density* $\mathbf{j}(\mathbf{r})$. This vector may be defined via the following relation for the elementary current dI crossing an elementary area dA (Fig. 2):

$$dI = j dA \cos\theta \equiv (j\cos\theta) dA \equiv j_n dA, \tag{4.2}$$

where θ is the angle between the direction normal to the surface and the charge carrier motion direction, which is taken for the direction of the vector $\mathbf{j}$.

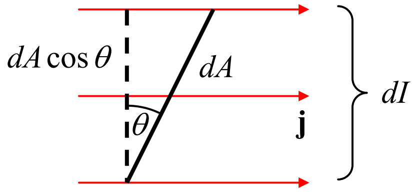

Fig. 4.2. The current density vector **j**.

With that definition, Eq. (1) may be rewritten as

$$\frac{d}{dt}\int_V \rho\, d^3r = -\oint_S j_n d^2r, \tag{4.3}$$

where V is an arbitrary but stationary volume limited by the closed surface S. Applying to this volume the same divergence theorem as was repeatedly used in previous chapters, we get

$$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j}\right] d^3r = 0. \tag{4.4}$$

Since the volume V is arbitrary, this equation may be true only if

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0.} \tag{4.5}$$

Continuity equation

This is the fundamental *continuity equation* – which is true even for time-dependent phenomena.[2]

The charge relaxation, illustrated by Fig. 1b, is of course a dynamic, time-dependent process. However, electric currents may also exist in stationary situations, when a certain *current source*, for example a battery, drives the current against the electric field, and thus replenishes the conductor charges and sustains currents at a certain time-independent level – see Fig. 1c. (This process requires a persistent replenishment of the electrostatic energy of the system from either a source or a large storage of energy of a different kind – say, the chemical energy of the battery.) Let us discuss the laws governing the distribution of such *dc currents*. In this case ($\partial/\partial t = 0$), Eq. (5) reduces to a very simple equation

$$\nabla\cdot\mathbf{j} = 0. \tag{4.6}$$

This relation acquires an even simpler form in the particular but important case of *dc electric circuits* (Fig. 3) – the systems that may be fairly represented as direct ("galvanic") connections of components of two types:

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[2] Similar differential relations are valid for the density of any conserved quantity, for example for mass in classical dynamics (see, e.g., CM Sec. 8.3), and for the probability, as it is defined in statistical physics (SM Sec. 5.6) and in quantum mechanics (QM Sec. 1.4).

(i) relatively-small-size (*lumped*) *circuit elements*, meaning either a passive resistor, or a current source, etc. – generally, any "black box" with two or more *terminals*, and

(ii) *perfectly conducting wires*, with a negligible drop of the electrostatic potential along them, that are galvanically connected at certain points called *nodes* (or "junctions").

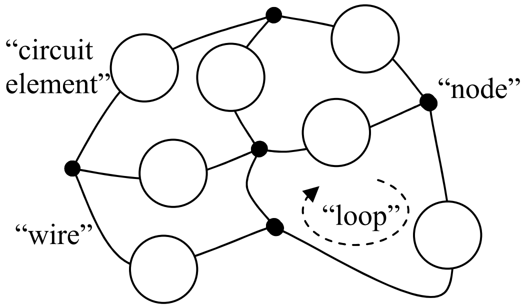

Fig. 4.3. A typical system obeying Kirchhoff laws.

In the standard circuit theory, the electric charges of the nodes are considered negligible,[3] and we may integrate Eq. (6) over the closed surface drawn around any node to get a simple equality

$$\sum_j I_j = 0, \tag{4.7a}$$

where the summation is over all the wires (numbered with index j) connected in the node. On the other hand, according to its definition (2.25), the voltage V_k across each circuit element may be represented as the difference of the electrostatic potentials of the adjacent nodes, $V_k = \phi_k - \phi_{k-1}$. Summing such differences around any closed loop of the circuit (Fig. 3), we get all terms canceled, so

$$\sum_k V_k = 0. \tag{4.7b}$$

These relations are called, respectively, the *1st* and *2nd Kirchhoff laws*[4] – or sometimes the *node rule* (7a) and the *loop rule* (7b). They may seem elementary, and their genuine power is in the mathematical fact that any set of Eqs. (7) covering every node and every circuit element of the system at least once, gives a system of equations sufficient for the calculation of all currents and voltages in it – provided that the relation between the current and voltage is known for each circuit element.

It is almost evident that in the absence of current sources, the system of equations (7) has only the trivial solution: $I_j = 0$, $V_k = 0$ – with the exotic exception of superconductivity, to be discussed in Sec. 6.3. The current sources that allow non-zero current flows may be described by their *electromotive forces* (*e.m.f.*) $\mathscr{V}_k$, having the dimensionality of voltage, which have to be taken into account in the corresponding terms V_k of the sum (7b). Let me hope that the reader has some experience of using Eqs. (7) for analyses of simple circuits – say, consisting of several resistors and batteries, so I can save our time by skipping their discussion. Still, due to their practical importance, I would recommend the reader to carry out a self-test by solving a couple of problems offered at the beginning of Sec. 6.

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[3] In many cases, the charge accumulation/relaxation may be described without an explicit violation of Eq. (7a), just by adding other circuit elements, *lumped capacitors* (see Fig. 2.5 and its discussion), to the circuit under analysis. The resulting circuit may be used to describe not only the transient processes but also periodic ac currents. However, it is convenient for me to postpone the discussion of such *ac circuits* until Chapter 6, where one more circuit element type, *lumped inductances*, will be introduced.

[4] Named after Gustav Kirchhoff (1824-1887) – who also suggested the differential form (8) of the Ohm law.

4.2. The Ohm law

As was mentioned above, the relations spelled out in Sec. 1 are sufficient for forming a closed system of equations for finding electric current and field in a system only if they are complemented with some constitutive relations between the scalars I and V in each lumped circuit element, or alternatively between the macroscopic (atomic–scale–averaged) vectors **j** and **E** at each point of the material of such an element. The simplest of such relations is the famous *Ohm law* whose differential (or "local") form is

$$\boxed{\mathbf{j} = \sigma \mathbf{E},} \qquad (4.8)$$

Ohm law

where σ is a constant called the *Ohmic conductivity* (or just the "conductivity" for short).[5] Though the Ohm law (discovered, in its simpler form, by Georg Simon Ohm in 1827) is one of constitutive rather than fundamental relations, and is approximate for *any* conducting medium, we can argue that if:

(i) the medium carries no current at $\mathbf{E} = 0$ (mind superconductors!),
(ii) the medium is isotropic or virtually isotropic (a notable exception: some organic conductors),
(iii) the *mean free path* l of the current carriers (the notion to be discussed in detail in SM Ch. 6) in this medium is much smaller than the characteristic scale a of the spatial variations of **j** and **E**,

then the law may be viewed as the leading, linear term of the Taylor expansion of the local relation **j**(**E**), and thus is general for relatively low fields.

Table 1 gives approximate experimental values of σ for some representative (and/or practically important) materials. Note that the range of these values is very broad, even without going to such extremes as very pure metallic crystals at very low temperatures, where σ may reach $\sim 10^{12}$ S/m.

Table 4.1. Ohmic dc conductivities for some materials at 20°C.

Material	σ (S/m)
Teflon (PTFE, $[C_2F_4]_n$)	10^{-22}-10^{-24}
Silicon dioxide	10^{-16}-10^{-19}
Various glasses	10^{-10}-10^{-14}
Deionized water	$\sim 10^{-6}$
Seawater	5
Silicon *n*-doped to 10^{16}cm^{-3}	2.5×10^{2}
Silicon *n*-doped to 10^{19}cm^{-3}	1.6×10^{4}
Silicon *p*-doped to 10^{19}cm^{-3}	1.1×10^{4}
Nichrome (alloy 80% Ni + 20% Cr)	0.9×10^{6}
Aluminum	3.8×10^{7}
Copper	6.0×10^{7}
Zinc crystal along *a*-axis	1.65×10^{7}
Zinc crystal along *c*-axis	1.72×10^{7}

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[5] In SI units, the conductivity is measured in S/m, where one siemens (S) is the reciprocal of the ohm: $1\text{S} \equiv (1\Omega)^{-1} \equiv 1\text{A}/1\text{V}$. The constant reciprocal to conductivity, $1/\sigma$, is called *resistivity* and is commonly denoted by the letter ρ. I will, however, try to avoid using this notion, because in these notes this letter is already overused.

In order to get a better feeling of what these values mean, let us consider a very simple system (Fig. 4): a plane capacitor of area $A >> d^2$, filled with a material that has not only a dielectric constant κ, but also some Ohmic conductivity σ, with much more conductive electrodes.

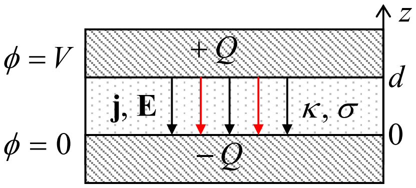

Fig. 4.4. A "leaky" plane capacitor.

Assuming that these properties are compatible with each other,[6] we may assume that the distribution of the electric potential (not too close to the capacitor's edges) still obeys Eq. (2.39), so the electric field is normal to the electrode surfaces and uniform, with $E = V/d$. Then, according to Eq. (6), the current density is also uniform, $j = \sigma E = \sigma V/d$. From here, the total current between the plates is

$$I = jA = \sigma EA = \sigma \frac{V}{d} A. \tag{4.9}$$

On the other hand, from Eqs. (2.26) and (3.45), the instantaneous value of the total charge of the top electrode is $Q = CV = (\kappa\varepsilon_0 A/d)V$. Plugging these relations into Eq. (1), we see that the speed of charge (and voltage) relaxation is independent of the geometric parameters A and d of the capacitor:

$$\frac{dV}{dt} = -\frac{V}{\tau_\mathrm{r}}, \qquad \text{with } \tau_\mathrm{r} \equiv \frac{\varepsilon_0 \kappa}{\sigma} \equiv \frac{\varepsilon}{\sigma}, \tag{4.10}$$

so the *relaxation time constant* τ_r may be used to characterize the gap-filling material as such.

As we already know (see Table 3.1), for most practical materials the dielectric constant κ is within one order of magnitude from 10, so the numerator in the second of Eqs. (10) is of the order of 10^{-10} (SI units). As a result, according to Table 1, the charge relaxation time ranges from $\sim 10^{14}$s (more than a million years!) for the best insulators like Teflon (polytetrafluoroethylene, PTFE),[7] to $\sim 10^{-18}$s for the least resistive metals. What is the physics behind such a huge range of σ, and why, for some materials, Table 1 gives them with such a large uncertainty? As in Chapters 2 and 3, in this course, I have time only for a brief, admittedly superficial discussion of these issues.[8]

If the charge carriers move almost as classical particles (e.g., in plasmas or non-degenerate semiconductors), a very reasonable description of the conductivity is given by the famous *Drude formula*.[9] In his picture, due to a weak electric field, the charge carriers are accelerated in its direction (on top of their random motion in all directions, with the average velocity vector equal to zero):

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}\mathbf{E}, \tag{4.11}$$

and as a result, their velocity acquires the average value

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[6] As will be discussed in Chapter 6, this is true only if σ is not too high.

[7] This polymer is broadly used in engineering and physical experiment, due to its many remarkable properties.

[8] A more detailed discussion of this issue may be found in SM Chapter 6.

[9] It was suggested by Paul Drude in 1900.

$$\mathbf{v} = \frac{d\mathbf{v}}{dt}\tau = \frac{q}{m}\mathbf{E}\tau, \qquad (4.12)$$

where the phenomenological parameter $\tau = l/v$ (not to be confused with τ_r!) may be understood as the average time since the last scattering event. From here, the current density:[10]

$$\boxed{\mathbf{j} = qn\mathbf{v} = \frac{q^2 n\tau}{m}\mathbf{E}, \qquad \text{i.e. } \sigma = \frac{q^2 n\tau}{m}.} \qquad (4.13a)$$

Drude formula: two versions

(Notice the independence of σ of the charge sign.) Another form of the same result, more popular in the physics of semiconductors, is

$$\boxed{\sigma = q^2 n\mu, \qquad \text{with } \mu = \frac{\tau}{m},} \qquad (4.13b)$$

where the parameter μ, defined by the relation $\mathbf{v} \equiv \mu\mathbf{E}$, is called the *charge carrier mobility.*

Most good conductors (e.g., metals) are essentially degenerate Fermi gases (or liquids), in which the average thermal energy of a particle, k_BT is much lower than the Fermi energy ε_F. In this case, a quantum theory is needed for the calculation of σ. Such a theory was developed by A. Sommerfeld in 1927 (and is sometimes called the *Drude-Sommerfeld model*). I have no time to discuss it in this course,[11] and here will only notice that for a nearly ideal, isotropic Fermi gas the result is reduced to Eq. (13), with a certain effective value of τ, so it may be used for estimates of σ, with due respect to the quantum theory of scattering. In a typical metal, n is very high ($\sim 10^{23}$ cm^{-3}) and is fixed by the atomic structure, so the sample quality may only affect σ via the scattering time τ.

At room temperature, the scattering of electrons by thermally-excited lattice vibrations (*phonons*) dominates, so τ and σ are high but finite, and do not change much from one sample to another. (Hence the relatively accurate values given for metals in Table 1.) On the other hand, at $T \to 0$, quantum mechanics says a perfect crystal should not exhibit scattering at all, and its conductivity should be infinite. In practice, this is never true (for one, due to electron scattering from imperfect boundaries of finite-size samples), and the effective conductivity σ is infinite (or practically infinite, at least above the largest measurable values $\sim 10^{20}$ S/m) only in superconductors.[12]

On the other hand, the conductivity of quasi-insulators (including deionized water) and semiconductors depends mostly on the carrier density n, which is much lower than in metals. From the point of view of quantum mechanics, this happens because the ground-state wavefunctions of charge carriers are localized within an atom (or molecule), and their energies are separated from those of excited states, with space-extended wavefunctions, by a large energy gap – often called the *bandgap*. For example, in SiO_2 the bandgap approaches 9 eV, equivalent to ~4,000 K. This is why even at room temperatures the density of thermally-excited free charge carriers in good insulators is negligible. In these materials, n is determined by impurities and vacancies, and may depend on a particular chemical synthesis or other fabrication technology, rather than on the fundamental properties of the material. (On the contrary, the carrier mobility μ in these materials is almost technology-independent.)

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[10] Note that $\mathbf{j}$ in Eq. (8) is defined as an already macroscopic variable, averaged over inter-particle distances, so no additional average sign is necessary in the first of Eqs. (13a).

[11] For such a discussion see, e.g., SM Sec. 6.3.

[12] The electrodynamic properties of superconductors are so interesting (and fundamentally important) that I will discuss them in more detail in Chapter 6.

The practical importance of the fabrication technology may be illustrated by the following example. In the cells of the so-called *floating-gate memories*, in particular, the *flash memories*, which currently dominate the nonvolatile digital memory technology, data bits are stored as small electric charges ($Q \sim 10^{-16}$ C $\sim 10^{3}\ e$) of highly doped silicon islands (so-called *floating gates*) separated from the rest of the integrated circuit with ~10-nm-thick layers of silicon dioxide, SiO_2. Such layers are fabricated by high-temperature oxidation of virtually perfect silicon crystals. The conductivity of the resulting high-quality (though amorphous) material is so low, $\sigma \sim 10^{-19}$ S/m, that the relaxation time τ_r, defined by Eq. (10), is well above 10 years – the industrial standard for data retention in nonvolatile memories. To appreciate how good this technology is, the cited value should be compared with the typical conductivity $\sigma \sim 10^{-16}$ S/m of the usual, bulk SiO_2 ceramics.[13]

To conclude this chapter, let me note that the Ohm law, for all its importance, is not a universal law of nature. As a reminder of this fact, in Sec. 5 below I describe two very simple systems (leaving their analysis for the reader's exercise) whose *I-V* relation is nonlinear even for very small currents.

4.3. Boundary problems

For an Ohmic conducting medium, we may combine Eqs. (6) and (8) to get the following differential equation

$$\nabla \cdot (\sigma \nabla \phi) = 0 . \qquad (4.14)$$

For a uniform conductor (σ = const), Eq. (14) is reduced to the Laplace equation for the (macroscopic) electrostatic potential ϕ. As we already know from Chapters 2 and 3, its solution depends on the boundary conditions. These conditions, in turn, depend on the interface type.

(i) Conductor-conductor interface. Applying the continuity equation (6) to a Gauss-type pillbox at the interface of two different conductors (Fig. 5), we get

$$(j_n)_1 = (j_n)_2, \qquad (4.15)$$

so if the Ohm law (8) is valid inside each medium, then

$$\sigma_1 \frac{\partial \phi_1}{\partial n} = \sigma_2 \frac{\partial \phi_2}{\partial n} . \qquad (4.16)$$

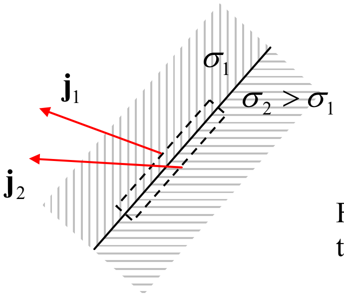

Fig. 4.5. DC current's "refraction" at the interface between two different conductors.

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[13] This course is not an appropriate platform to discuss details of the floating-gate memory technology. However, I think that every educated physicist should know its basics, because such memories are presently the driver of all semiconductor integrated circuit technology development, and hence of the whole information technology progress. Perhaps the best available general book on this topic is still the relatively old review collection by J. Brewer and M. Gill (eds.), *Nonvolatile Memory Technologies with Emphasis on Flash*, IEEE Press, 2008.

Also, since the electric field should be finite, its potential ϕ has to be continuous across the interface – the condition that may also be written as

$$\frac{\partial \phi_1}{\partial \tau} = \frac{\partial \phi_2}{\partial \tau}. \tag{4.17}$$

Both these conditions (and hence the solutions of the boundary problems using them) are similar to those for the interface between two dielectrics – cf. Eqs. (3.46)-(3.47). Note that using the Ohm law, Eq. (17) may be rewritten as

$$\frac{1}{\sigma_1}(j_\tau)_1 = \frac{1}{\sigma_2}(j_\tau)_2. \tag{4.18}$$

Comparing it with Eq. (15) we see that, generally, the current density's magnitude changes at the interface: $j_1 \neq j_2$. It is also curious that if $\sigma_1 \neq \sigma_2$, the current line slope changes at the interface (Fig. 5), qualitatively similar to the refraction of light rays in optics – see Chapter 7.

(ii) Conductor-electrode interface. An *electrode* is defined as a body made of a "perfect conductor", i.e. of a medium with $\sigma \to \infty$. Then, at a fixed current density at the interface, the electric field in the electrode tends to zero, and hence it may be described by the equality

$$\phi = \phi_j = \text{const}, \tag{4.19}$$

where constants ϕ_j may be different for different electrodes (numbered with index j). Note that with such boundary conditions, the Laplace boundary problem becomes exactly the same as in electrostatics – see Eq. (2.35) – and hence we can use the methods (and some solutions :-) discussed in Chapter 2 for finding the dc current distribution.

(iii) Conductor-insulator interface. For the description of a good insulator, we can use the equality $\sigma = 0$, so Eq. (16) yields the following boundary condition,

$$\frac{\partial \phi}{\partial n} = 0, \tag{4.20}$$

for the potential derivative *inside the conductor*. From the Ohm law (8) in the form $\mathbf{j} = -\sigma\nabla\phi$, we see that this is just the very natural requirement for the dc current not to flow into an insulator. Now note that this condition makes the Laplace problem inside the conductor completely well-defined, and independent of the potential distribution in the adjacent insulator. On the contrary, due to the continuity of the electrostatic potential at the border, its distribution inside the surrounding insulator has to follow that inside the conductor.

Let us discuss this conceptual issue on the following (apparently, trivial) example: dc current in a uniform wire of length l and a cross-section of area A. The reader certainly knows the answer:

$$\boxed{I = \frac{V}{R}, \qquad \text{where } R \equiv \frac{V}{I} = \frac{l}{\sigma A},} \tag{4.21}$$

Uniform wire's resistance

where the constant R is called the wire's *resistance*.[14]

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[14] The first of Eqs. (21) is essentially the (historically, initial) integral form of the Ohm law, and is valid not only for a uniform wire but also for Ohmic conductors of any geometry in that I and V may be clearly defined.

However, let us derive this result formally from our theoretical framework. For the simple geometry shown in Fig. 6a, this is easy to do. Here the potential evidently has a linear 1D distribution

$$\phi = \text{const} - \frac{x}{l}V, \tag{4.22}$$

both in the conductor and the surrounding free space, with both boundary conditions (16) and (17) satisfied at the conductor-insulator interfaces, and the condition (20) satisfied at the conductor-electrode interfaces. As a result, the electric field is constant and has only one Cartesian component: $E_x = V/l$, so inside the conductor

$$j_x = \sigma E_x, \quad I = j_x A, \tag{4.23}$$

giving us the well-known Eq. (21).

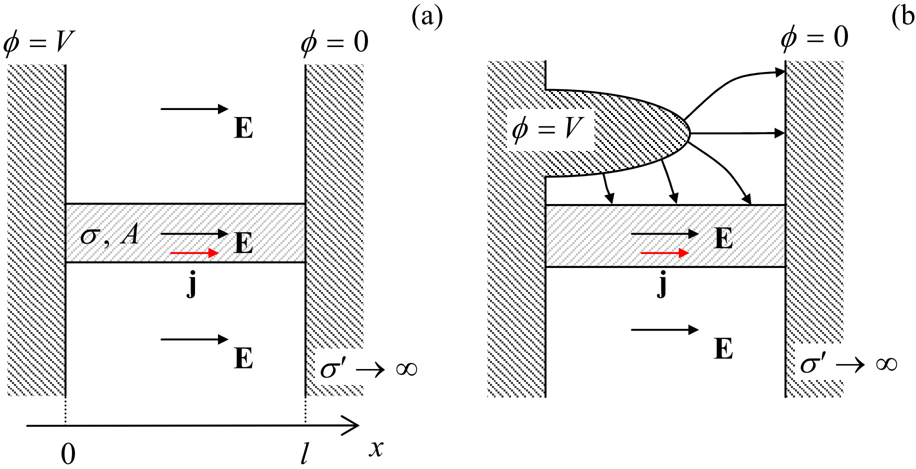

Fig. 4.6. (a) An elementary problem and (b) a (slightly) less obvious problem of the field distribution at dc current flow (schematically).

However, what about the geometry shown in Fig. 6b? In this case, the field distribution in the free space around the conductor is dramatically different, but according to the boundary problem defined by Eqs. (14) and (20), inside the conductor, the solution is exactly the same as it was in the former case. Now, the Laplace equation in the surrounding insulator has to be solved with the boundary values of the electrostatic potential, "dictated" by the distribution of the current (and hence potential) in the conductor. Note that as a result, the electric field lines are generally *not* normal to the conductor's surface, because the surface is not equipotential – see Eq. (22) again.

Let us solve a problem in that this *conduction hierarchy* may be followed analytically to the very end. Consider an empty spherical cavity carved in a conductor with an initially uniform current flow with a constant density $\mathbf{j}_0 = \mathbf{n}_z j_0$ (Fig. 7a). Following the hierarchy, we have to solve the boundary problem in the conducting part of the system, i.e. outside the sphere (at $r \geq R$), first. Since the problem is evidently axially symmetric, we already know the general solution of the Laplace equation – see Eq. (2.172). Moreover, we know that in order to match the uniform field distribution at $r \to \infty$, all coefficients a_l but one ($a_1 = -E_0 = -j_0/\sigma$) have to be zero, and that the boundary conditions at $r = R$ will give zero solutions for all coefficients b_l but one (b_1), so

$$\phi = -\frac{j_0}{\sigma} r\cos\theta + \frac{b_1}{r^2}\cos\theta, \qquad \text{for } r \geq R. \tag{4.24}$$

In order to find the remaining coefficient b_1, we have to use the boundary condition (20) at $r = R$:

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=R} = \left(-\frac{j_0}{\sigma} - \frac{2b_1}{R^3}\right)\cos\theta = 0\,. \qquad (4.25)$$

This gives $b_1 = -j_0R^3/2\sigma$, so, finally,

$$\phi(r,\theta) = -\frac{j_0}{\sigma}\left(r + \frac{R^3}{2r^2}\right)\cos\theta, \qquad \text{for } r \geq R\,. \qquad (4.26)$$

(Note that this potential distribution corresponds to the dipole moment $\mathbf{p} = -\mathbf{E}_0R^3/2$. It is straightforward to check that if the spherical cavity was cut in a dielectric, the potential distribution outside it would be similar, with $\mathbf{p} = -\mathbf{E}_0R^3(\kappa - 1)/(\kappa + 2)$. In the limit $\kappa \to \infty$, these two results coincide, despite the rather different type of the problem: in the dielectric case, there is no current at all.)

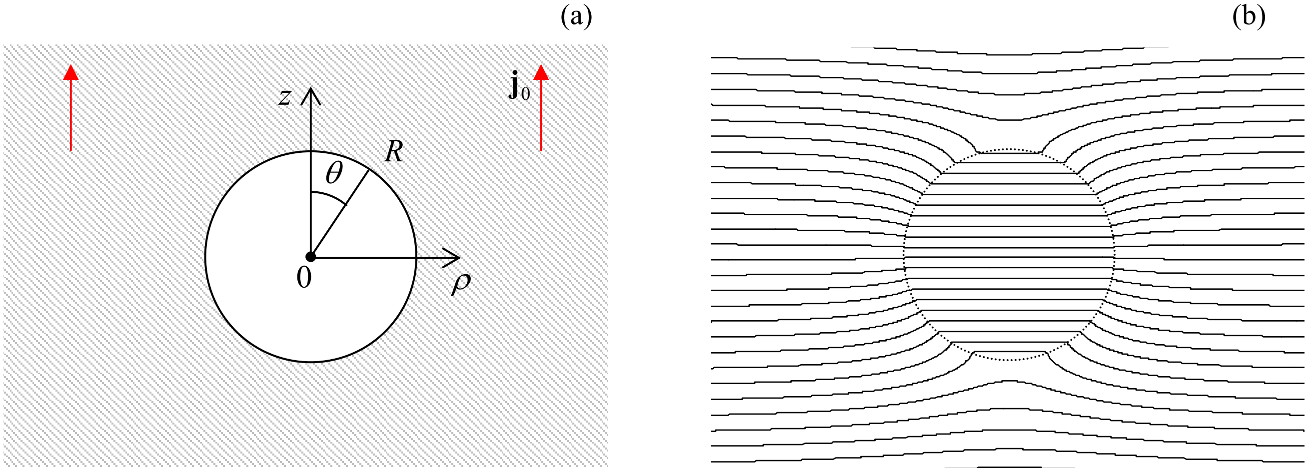

Fig. 4.7. A spherical cavity carved in a uniform conductor: (a) the problem's geometry, and (b) the equipotential surfaces as given by Eqs. (26) and (28).

Now, as the second step in the conductivity hierarchy, we may find the electrostatic potential distribution $\phi(r,\theta)$ in the insulator, in this particular case inside the empty cavity (at $r \leq R$). It should also satisfy the Laplace equation with the boundary values at $r = R$, "dictated" by the distribution (26):

$$\phi(R,\theta) = -\frac{3}{2}\frac{j_0}{\sigma}R\cos\theta\,. \qquad (4.27)$$

We could again solve this problem by the formal variable separation (keeping in the general solution (2.172) only the term proportional to a_1, which does not diverge at $r \to 0$), but if we notice that the boundary condition (27) depends on just one Cartesian coordinate, $z = R\cos\theta$, the solution may be just guessed:

$$\phi(r,\theta) = -\frac{3}{2}\frac{j_0}{\sigma}z = -\frac{3}{2}\frac{j_0}{\sigma}r\cos\theta, \qquad \text{at } r \leq R\,. \qquad (4.28)$$

Indeed, it evidently satisfies the Laplace equation and the boundary condition (27), and corresponds to a constant electric field parallel to the vector $\mathbf{j}_0$ and equal to $3j_0/2\sigma$ – see Fig. 7b. Again, the cavity surface is *not* equipotential, and the electric field lines at $r \leq R$ are *not* normal to it at almost all points.

More generally, the conductivity hierarchy says that static electrical fields and charges outside conductors (e.g., electric wires) do not affect currents flowing in the wires, and it is physically very clear why. For example, if a charge in the free space is slowly moved close to a wire, it (in accordance with the linear superposition principle) only induces an additional surface charge (see Sec. 2.1) that screens the external charge's field, without participating in the current flow inside the conductor.

Besides this conceptual issue, the two examples given above may be considered as applications of the first two methods discussed in Chapter 2 – the orthogonal coordinates (Fig. 6) and the variable separation (Fig. 7) – to dc current distribution problems. As the reader may recall, in that chapter we also discussed the method of charge images. It turns out that its analog may be also used for the solution of some dc conductivity problems. Indeed, let us consider a spherically-symmetric potential distribution of the electrostatic potential, similar to that given by the basic Eq. (1.35):

$$\phi = \frac{c}{r}. \tag{4.29}$$

As we know from Chapter 1, this is a particular solution of the 3D Laplace equation at all points but $r = 0$. In free space, this distribution would correspond to a point charge $q = 4\pi\varepsilon_0 c$; but what about a uniform Ohmic conductor? Calculating the corresponding electric field and current density,

$$\mathbf{E} = -\nabla\phi = \frac{c}{r^3}\mathbf{r}, \qquad \mathbf{j} = \sigma\mathbf{E} = \sigma\frac{c}{r^3}\mathbf{r}, \tag{4.30}$$

we see that the total current flowing from the origin through a sphere of an arbitrary radius r does not depend on the radius:

$$I = Aj = 4\pi r^2 j = 4\pi\sigma c. \tag{4.31}$$

Plugging the resulting coefficient c into Eq. (29), we get

$$\phi = \frac{I}{4\pi\sigma r}. \tag{4.32}$$

Hence the Coulomb-type distribution of the electric potential in a conductor is possible (at least at some distance from the singular point $r = 0$), and describes the dc current I flowing out of a small-size electrode – or *into* such an electrode if the coefficient c is negative. Such *current injection* may be readily implemented experimentally; think for example about an insulated wire with a small bare end, inserted into a poorly conducting soil – an important method in geophysical research. Such point injection is even simpler in 2D situations – think about a wire attached, within a small spot, to a thin resistive layer, such as the thin films used for wiring in microelectronics.[15]

Now let the current injection point $\mathbf{r}'$ be close to a plane interface between the conductor and an insulator (Fig. 8). In this case, besides the Laplace equation, we should satisfy the boundary condition,

$$j_n = \sigma E_n = -\sigma\frac{\partial\phi}{\partial n} = 0, \tag{4.33}$$

at the interface. It is clear that this can be done by replacing the insulator with an imaginary similar conductor with an additional current injection point, at the mirror image point $\mathbf{r}''$. Note, however, that in contrast to charge images, the sign of the imaginary current has to be *similar*, not opposite, to the initial one, so the total electrostatic potential inside the conducting semi-space is

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[15] Note that in such layers, the current distribution near the injection point is different, $j \propto 1/r$ rather than $1/r^2$.

$$\phi(\mathbf{r}) = \frac{I}{4\pi\sigma}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{|\mathbf{r}-\mathbf{r}''|}\right). \tag{4.34}$$

(The image current's sign would be opposite at the interface between a conductor with moderate conductivity and a perfect conductor ("electrode"), whose potential should be virtually constant.)

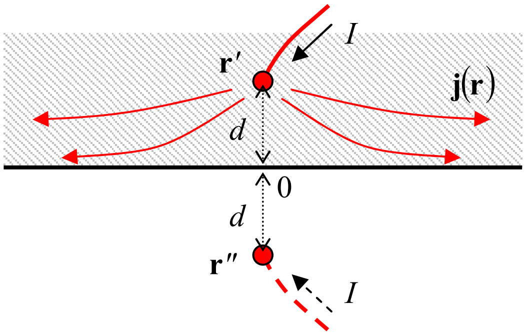

Fig. 4.8. Applying the method of images for the current injection analysis.

This result may be readily used, for example, to calculate the current density at a plane surface of a uniform conductor, as a function of distance ρ from point 0 – the surface's point closest to the current injection site – see Fig. 8. At such surface, Eq. (34) yields

$$\phi = \frac{I}{2\pi\sigma}\frac{1}{\left(\rho^2 + d^2\right)^{1/2}}, \tag{4.35}$$

so the current density is:

$$j_\rho = \sigma E_\rho = -\sigma\frac{\partial\phi}{\partial\rho} = \frac{I}{2\pi}\frac{\rho}{\left(\rho^2 + d^2\right)^{3/2}}. \tag{4.36}$$

Deviations from Eqs. (35) and (36) may be used to find and characterize conductance inhomogeneities, say, those due to mineral deposits in the Earth's crust.[16]

So, the methods used in electrostatics to calculate the potential distribution in linear dielectrics may be also used to find such distributions in Ohmic conductors. Moreover, some of these methods are more valuable in this field. For example, in electrostatics, the effective methods of solution of the 2D Laplace equation, discussed in Secs. 2.3-2.6, could be only applied to cylindrical geometries. At Ohmic conduction, this equation is also valid in some 3D cases. A practically important example is the current flow in thin resistive layers where, due to the conductivity hierarchy principle, the 3D-distributed field outside a layer, induced by the 2D-distributed current in it, does not affect the flow and in many cases is not important. A few problems of this kind, formulated in Sec. 5, are left for the reader's exercise.

4.4. Energy dissipation

Let me conclude this brief chapter with an ultra-short discussion of energy dissipation in conductors. In contrast to the electrostatic situations in insulators (vacuum or dielectrics), at dc

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[16] The current injection may be also produced, due to electrochemical reactions, by an ore mass itself, so one need only measure (and correctly interpret :-) the resulting potential distribution – the so-called *self-potential method* – see, e.g., Sec. 6.1 in W. Telford *et al.*, *Applied Geophysics*, 2nd ed., Cambridge U. Press, 1990.

conduction, the electrostatic energy U is "dissipated" (i.e. transferred to heat) at a certain rate $\mathscr{P} \equiv -dU/dt$, with the dimensionality of power.[17] The rate of this *energy dissipation* may be evaluated by calculating the power of the electric field's work on a single moving charge:

$$\mathscr{P}_1 = \mathbf{F}\cdot\mathbf{v} = q\mathbf{E}\cdot\mathbf{v}\,. \tag{4.37}$$

After the summation over all charges, Eq. (37) gives us the average power of energy dissipation. If the charge density n is uniform, multiplying by it both parts of this relation, and taking into account that $qn\mathbf{v} = \mathbf{j}$, for the energy dissipation rate in a unit volume we get the following *Joule law*[18]

General Joule law

$$\boxed{\mathscr{p} \equiv \frac{\mathscr{P}}{V} = \frac{\mathscr{P}_1 N}{V} = \mathscr{P}_1 n = q\mathbf{E}\cdot\mathbf{v}n = \mathbf{E}\cdot\mathbf{j}\,.} \tag{4.38}$$

In the case of the Ohmic conductivity (8), this expression may be also rewritten in two other forms:

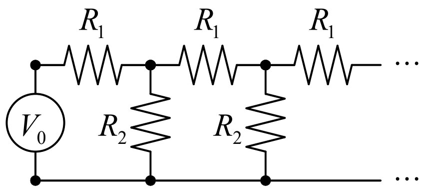
Joule law for Ohmic conductivity

$$\boxed{\mathscr{p} = \sigma E^2 = \frac{j^2}{\sigma}\,.} \tag{4.39}$$

With our electrostatics background, it is also straightforward (and hence left for the reader's exercise) to prove that the dc current distribution in a uniform Ohmic conductor, at a fixed voltage distribution along its surface, corresponds to the minimum of the total dissipation in the sample,

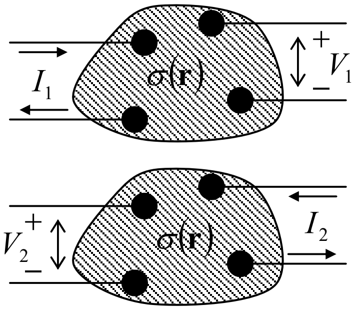
$$\mathscr{P} \equiv \int_V \mathscr{p}\, d^3r = \sigma \int_V E^2 d^3r\,. \tag{4.40}$$

4.5. Exercise problems



4.1. DC voltage V_0 is applied to the end of a semi-infinite chain of lumped Ohmic resistors, shown in the figure on the right. Calculate the voltage across the j^{th} link of the chain.

4.2. It is well known that properties of many dc current sources (e.g., batteries) may be reasonably well represented as a connection in series of a *perfect voltage source* and an Ohmic *internal resistance*. Discuss the option, and possible advantages, of using a different equivalent circuit that would include a *perfect current source*.



4.3. Prove the following *Rayleigh-Lorentz-Carson reciprocity relation*: the results of the two separate experiments shown schematically in the figure on the right, with an arbitrary Ohmic conductor with four electrodes/terminals, are related as $I_1V_2 = I_2V_1$.

Hint: Try to apply the same approach as was used to prove Green's reciprocity relation of electrostatics in Problem 1.18, but with proper modifications.

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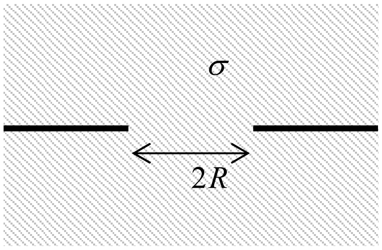
[17] If this electric field and hence the electrostatic energy are time-independent, the energy is replenished at the same rate from the current source(s).

[18] Named after James Prescott Joule, who quantified this effect in 1841.

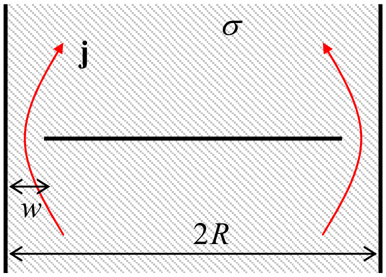
4.4. Calculate the resistance between two large uniform Ohmic conductors separated by a very thin, plane, insulating partition with a circular hole of radius R in it – see the figure on the right.

Hint: You may like to use the oblate spheroidal coordinates that were discussed in Sec. 2.4.



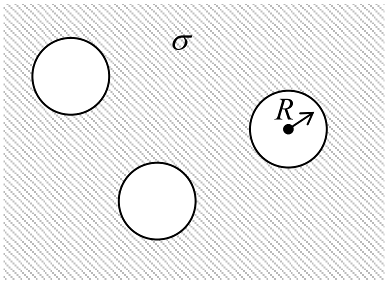
4.5. A very narrow plane crack inside a round conducting wire of radius R does not reach its surface by a small distance w – see the figure on the right. Assuming that the Ohmic conductivity σ of the wire's material is otherwise constant, calculate the electric resistance of the obstacle in the first approximation in small $w/R << 1$.

Hint: You may like to use the same elliptic coordinates as were employed at the solution of Problem 2.12.



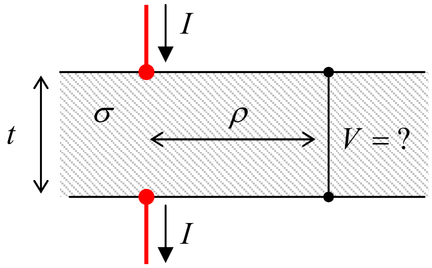
4.6. Calculate the effective (average) conductivity σ_{ef} of a medium with many empty spherical cavities of radius R, carved at random positions in a uniform Ohmic conductor (see the figure on the right), in the limit of a low density $n << R^{-3}$ of the cavities.

Hint: You may like to use the analogy with an electric-dipole medium – see, e.g., Sec. 3.2.

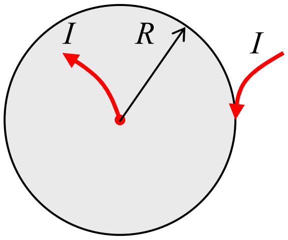


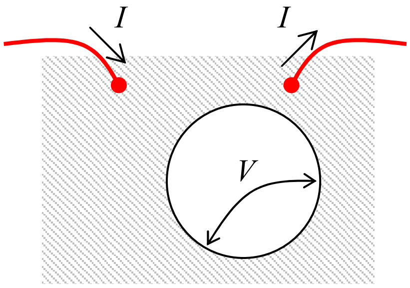
4.7. In two separate experiments, a narrow gap, possibly of irregular width, between two close, perfectly conducting electrodes is filled with some material: in the first case, a uniform linear dielectric with an electric permittivity ε, and in the second case, a uniform conducting material with an Ohmic conductivity σ. Neglecting the fringe effects, calculate the relation between the mutual capacitance C between the electrodes (in the first case) and the dc resistance R between them (in the second case).

4.8. Calculate the voltage V across a uniform, wide resistive slab of thickness t, at distance ρ from the points of injection/pickup of the dc current I passed across the slab – see the figure on the right.

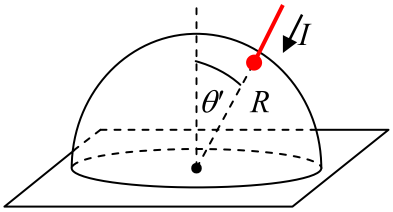


4.9. Calculate the distribution of the dc current's density in a thin, round, uniform resistive disk, if the current is inserted into some point at the disk's rim, and picked up in its center – see the figure on the right.



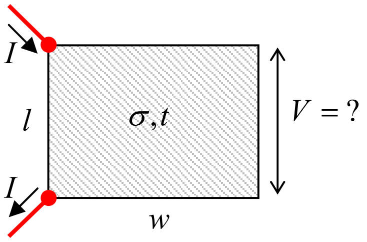


4.10. DC current is passed between two point electrodes connected to a wide, thin, uniform resistive sheet – see the figure on the right. Use the model solution of the previous problem to prove, without much new calculation, that cutting a round hole in the sheet (outside of the current injection/extraction points) doubles the voltage between any two points on its border.



4.11. The rim of a hemispherical thin shell, of radius R and thickness $t << R$, made of a uniform Ohmic conductor, is connected to a plane ground electrode. Calculate the distribution of the electrostatic potential created in the shell by a dc current I injected into it through a small-size electrode located at a polar angle $\theta' < \pi/2$ from the symmetry axis – see the figure on the right.

Hint: You may like to use the variable substitution $\rho \equiv \tan(\theta/2)$ to map the hemisphere onto a unit circle.



4.12. A rectangle of area $l \times w$ is cut from a uniform resistive sheet of thickness $t << l, w$. Use two different approaches to calculate the voltage V between its two adjacent corners, induced by the dc current I passed between the two other corners – see the figure on the right.

Hint: Besides the charge/current image method, you may like to consider using the variable separation method, with due respect to the current injection/extraction points.

4.13.* The simplest reasonable model of a vacuum diode consists of two parallel planar metallic electrodes of area A, separated by a gap of thickness $d << A^{1/2}$: a "cathode" that emits electrons into the gap, and an "anode" that absorbs the electrons arriving from the gap at its surface. Calculate the dc I-V curve of the diode, i.e. the relation between the average current I flowing between the electrodes and the dc voltage V applied between them, using the following simplifying assumptions:

(i) due to the effect of the negative space charge of the emitted electrons, the current I is much lower than the emission ability of the cathode,

(ii) the initial velocity of the emitted electrons is negligible, and

(iii) the direct Coulomb interaction of electrons (besides the space charge effect) is negligible.

4.14.* Calculate the space-charge-limited current in a system with the same geometry as in the previous problem, and using the same assumptions besides that now the emitted charge carriers do not fly ballistically, but rather drift in accordance with the Ohm law, with the conductivity given by Eq. (13): $\sigma = q^2 \mu n$, with a constant mobility μ.[19]

Hint: In order to get a realistic result, assume that the medium in which the charge carriers move has a certain dielectric constant κ unrelated to the carriers.

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[19] As was mentioned in Sec. 2, the approximation of a constant (in particular, field- and charge-density-independent) mobility is most suitable for semiconductors.

4.15. Prove that the distribution of dc currents in a uniform Ohmic conductor with a given voltage distribution along its surface corresponds to the minimum of the total energy dissipation rate ("Joule heat").