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Essential Graduate Physics

Lecture Notes and Problems

Exercise Problems with Model Solutions

Part EM: Classical Electrodynamics

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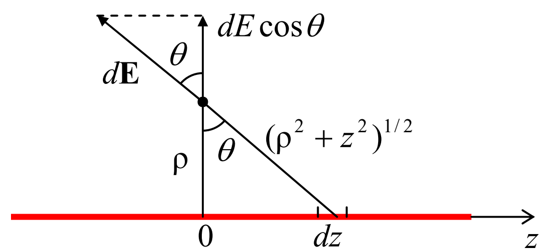
Chapter 1. Electric Charge Interaction

Problem 1.1. Calculate the electric field of a thin, long, straight filament, electrically charged with a constant linear density λ, by using two approaches:

(i) directly from the Coulomb law, and
(ii) from the Gauss law.

Solutions:

(i) From the translational and axial symmetries of the problem, we can conclude that $\mathbf{E}(\mathbf{r}) = \mathbf{n}_\rho E(\rho)$, where ρ is the shortest distance between the observation point and the filament, i.e. its 2D radius.[1] Let us select the plane of drawing so it contains both the filament and the observation point, and take the line of filament for the z-axis – see the figure on the right. Then, according to the linear superposition principle, the field's magnitude may be calculated as



$$E(d) = \int_{z=-\infty}^{z=+\infty} dE_\rho = \int_{z=-\infty}^{z=+\infty} dE\cos\theta = \int_{z=-\infty}^{z=+\infty} dE \frac{\rho}{\left(\rho^2 + z^2\right)^{1/2}},$$

where dE is the magnitude of the elementary contribution to the field, created by a small segment dz of the filament, with the electric charge λdz. According to Eq. (1.7) of the lecture notes,

$$dE = \lambda dz \frac{1}{4\pi\varepsilon_0} \frac{1}{\rho^2 + z^2},$$

so the total field[2]

$$E(\rho) = \frac{\lambda\rho}{4\pi\varepsilon_0} \int_{-\infty}^{+\infty} \frac{dz}{\left(\rho^2 + z^2\right)^{3/2}} \equiv \frac{\lambda}{2\pi\varepsilon_0\rho} \int_0^{+\infty} \frac{d\xi}{\left(1 + \xi^2\right)^{3/2}} = \frac{\lambda}{2\pi\varepsilon_0\rho}. \qquad (*)$$

(ii) Taking a round cylinder of radius ρ and length l, with its axis on the filament, for the Gaussian volume, we ensure that on its round walls, the electric field E(ρ) is constant and normal to the volume's boundary, and the field flux through the cylinder's "lids" is zero. As a result, Eq. (1.16) yields

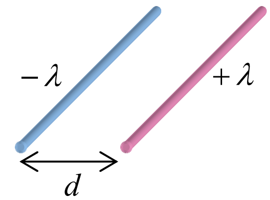
$$2\pi\rho\, l E(\rho) = \frac{\lambda l}{\varepsilon_0},$$

immediately giving the same result (*).

[1] Let me hope that the difference between the fonts used for the 2D radius ρ and the volumic density ρ of the electric charge is sufficient to avoid confusion. (Similar notation is used in all solutions where both these notions are encountered.)

[2] See, e.g., MA Eq. (6.5c). Actually, this integral may be easily worked out by the substitution $\xi \equiv \tan\varphi$, giving $d\xi = d\varphi/\cos^2\varphi \equiv d\varphi\,(1 + \tan^2\varphi) \equiv d\varphi\,(1 + \xi^2)$, so $d\xi/(1 + \xi^2)^{3/2} = d\varphi/(1 + \xi^2)^{1/2} \equiv \cos\varphi\, d\varphi \equiv d(\sin\varphi)$.

We see that for this highly symmetric problem, both approaches are straightforward, but the Gauss law makes calculations simpler.



Problem 1.2. Two thin, straight, parallel filaments separated by distance d carry equal and opposite uniformly distributed charges with a linear density λ – see the figure on the right. Calculate the force (per unit length) of the Coulomb interaction of the filaments. Compare its functional dependence on d with the Coulomb law for two point charges, and interpret their difference.

Solution: Using the result of Problem 1 with the notation replacement $\rho \to d$, and Eq. (1.6) of the lecture notes, we get

$$\frac{F}{l} = \frac{qE(d)}{l} = \lambda E(d) = \frac{\lambda^2}{2\pi\varepsilon_0 d}.$$

So, the force drops with distance as $1/d$, rather than as $1/r^2$ for point charges. This difference may be interpreted just as Eq. (1.23) was interpreted in Sec. 1.2 of the lecture notes: the farther is one of the filaments from a fixed elementary charge of the counterpart filament, the more elementary charges $dQ = \lambda dz$ are "visible" to it under a modest angle to the normal direction, thus mitigating the drop of the field induced by each of them. Such different scaling of the total magnitude of the same interaction in systems of different dimensionality is very typical for physics at large.

Problem 1.3. Calculate the electric field of the following spherically symmetric charge distribution: $\rho(r) = \rho_0 \exp\{-\lambda r\}$.

Solution: The solution of this problem is essentially given by Eq. (1.20) of the lecture notes:

$$E(r) = \frac{Q_r}{4\pi\varepsilon_0 r^2},$$

so what remains is just to spell out the charge Q_r inside the sphere of the radius r:

$$Q_r \equiv \int_{r'<r} \rho(r') d^3r' = 4\pi \int_0^r \rho(r') r'^2 dr' = 4\pi\rho_0 \int_0^r \exp\{-\lambda r'\} r'^2 dr'.$$

Perhaps the easiest way to calculate the last integral is to notice that it may be represented as the second partial derivative of another (elementary) integral over the parameter λ:

$$\int_0^r \exp\{-\lambda r'\} r'^2 dr' = \frac{\partial^2}{\partial\lambda^2} \int_0^r \exp\{-\lambda r'\} dr' = \frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{\lambda}\left(1 - \exp\{-\lambda r\}\right)\right] = \frac{2}{\lambda^3}\left[1 - \left(1 + \lambda r + \frac{\lambda^2 r^2}{2}\right)\exp\{-\lambda r\}\right],$$

so, finally:

$$E(r) = \frac{\rho_0}{\varepsilon_0 \lambda^3 r^2}\left[1 - \left(1 + \lambda r + \frac{\lambda^2 r^2}{2}\right)\exp\{-\lambda r\}\right].$$

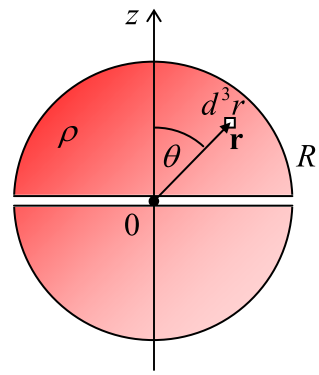
Note that at large distances ($r >> 1/\lambda$), the field decreases as $1/r^2$, i.e. as that of a point charge, because of a fast (exponential) drop of the charge density $\rho(r)$.

Problem 1.4. A sphere of radius R, whose volume had been charged with a constant density ρ, is split with a very narrow planar gap passing through the sphere's center. Calculate the force of mutual electrostatic repulsion of the resulting two hemispheres.

Solution: Since the gap is very narrow, we may neglect its effect on the distribution of the total electric field **E** inside the sphere, and use Eq. (1.22) of the lecture notes:

$$\mathbf{E}(\mathbf{r}) = \mathbf{n}_r E(r), \qquad \text{with } E(r) = \frac{\rho r}{3\varepsilon_0},$$

where r is the distance of the observation point from the sphere's center 0 – see the figure on the right. Acting on an elementary volume $dV = d^3r$ of the sphere's material, with the elementary charge $dQ = \rho d^3r$, this electric field produces a radially-directed force of magnitude



$$dF = E(r)dQ = \frac{\rho r}{3\varepsilon_0}\rho\, d^3r \equiv \frac{\rho^2 r}{3\varepsilon_0}d^3r.$$

Due to the axial symmetry of the problem, the net (repulsive) Coulomb force **F** acting on each hemisphere has to be normal to the gap's plane – in the figure on the right, has to be directed along the z-axis. Hence only the z-component of the elementary force dF,

$$dF_z = dF\cos\theta = \frac{\rho^2 r}{3\varepsilon_0}\cos\theta\, d^3r,$$

where θ is the polar angle of the radius vector **r** of the elementary volume d^3r (see the figure above), can contribute to the net force **F**. As a result, we may use the standard spherical coordinates, with the origin in the sphere's center, to calculate the net force magnitude as

$$F = F_z = \int_{z>0} dF_z = \frac{\rho^2}{3\varepsilon_0}\int_0^{2\pi} d\varphi \int_0^{\pi/2}\cos\theta\sin\theta d\theta\int_0^R r^3 dr = \frac{\rho^2}{3\varepsilon_0}2\pi\frac{1}{2}\frac{R^4}{4} \equiv \frac{\pi}{12}\frac{\rho^2 R^4}{\varepsilon_0}.$$

Another way to represent this result is to express it via the charge $Q = (2\pi/3)R^3\rho$ of each hemisphere:

$$F = \frac{1}{4\pi\varepsilon_0}\frac{3}{4}\frac{Q^2}{R^2} \equiv \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{R_{\text{ef}}^2}, \qquad \text{with } R_{\text{ef}} \equiv \frac{2}{\sqrt{3}}R,$$

showing that the effective distance between two point charges Q with the same interaction is $R_{\text{ef}} \approx 1.155\, R$ – a very reasonable value.

One may worry whether the above calculation properly excludes the effect of the electric field of a hemisphere on itself, since the field we have used is produced by the sphere as a whole. A proper response to this concern is that due to the 3rd Newton law, the internal forces between elementary charges of the same hemisphere compensate each other, so these forces are automatically canceled at the integration over the hemisphere's volume:

$$\int_{z>0}\rho\mathbf{E}_{\text{full}}d^3r = \int_{z>0}\rho(\mathbf{E}_{\text{other}} + \mathbf{E}_{\text{self}})d^3r = \int_{z>0}\rho\mathbf{E}_{\text{other}}d^3r + 0 = \int_{z>0}\rho\mathbf{E}_{\text{other}}d^3r = \mathbf{F}.$$

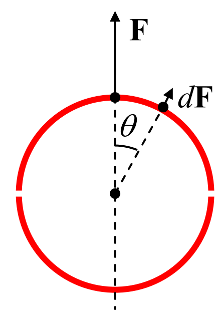
Problem 1.5. A thin spherical shell of radius R, which had been charged with a constant areal density σ, is split into two equal halves with a very thin planar cut passing through the sphere's center. Calculate the force of the mutual electrostatic repulsion of the resulting hemispheric shells, and compare the result with that of the previous problem.

Solution: A very thin cut does not alter substantially the electric field distribution, so it has the same radial direction and spherically-symmetric distribution $\mathbf{E}(\mathbf{r}) = \mathbf{n}_r E(r)$ as for the uncut shell. The function $E(r)$ may be readily found using the Gauss law applied to spheres with $r < R$ and $r > R$:

$$E(r) = \begin{cases} 0, & \text{for } r < R, \\ \sigma R^2 / \varepsilon_0 r^2, & \text{for } R < r. \end{cases}$$

Hence the force $d\mathbf{F}$ applied to an elementary area dA of the shell is normal to it, and its magnitude equals[3]

$$dF = \overline{E} dQ = \frac{E(R-0) + E(R+0)}{2} \sigma dA = \frac{\sigma^2}{2\varepsilon_0} dA .$$



Due to the axial symmetry of the problem, only the "vertical" components (meaning the direction normal to the cut plane – see the figure on the right) of these elementary forces may contribute to the total force $\mathbf{F}$ acting on each hemisphere – for example, the top one:

$$F = \int_{z>0} \left(\frac{d\mathbf{F}}{dA} \right)_z d^2 r = \int_{z>0} \frac{dF}{dA} \cos\theta \, d^2 r = R^2 \int_0^{2\pi} d\varphi \int_0^{\pi/2} \frac{\sigma^2}{2\varepsilon_0} \cos\theta \sin\theta d\theta = \frac{\pi}{2} \frac{\sigma^2 R^2}{\varepsilon_0}. \qquad (*)$$

For a fair comparison of this result with the solution of the previous problem, let us assume that the radius R and the total charge Q of each hemisphere are the same in both cases:

$$2\pi R^2 \sigma = \frac{2\pi}{3} R^3 \rho = Q .$$

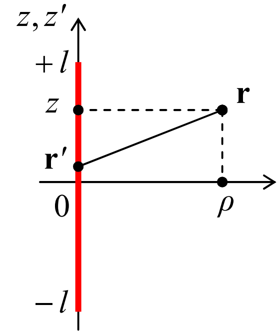
In this case, Eq. (*) becomes

$$F = \frac{1}{4\pi\varepsilon_0} \frac{1}{2} \frac{Q^2}{R^2} \equiv \frac{1}{4\pi\varepsilon_0} \frac{Q^2}{R_{\text{ef}}^2}, \qquad \text{with } R_{\text{ef}} \equiv \sqrt{2} R \approx 1.414 R ,$$

while for a volume-distributed charge, the similarly defined effective distance is close to $1.155R$. This difference is natural, because in the case of thin hemisphere shells, the elementary charges of the counterpart shells are, on average, farther from each other.

Problem 1.6. Calculate the spatial distribution of the electrostatic potential created by a straight thin filament of a finite length $2l$, charged with a constant linear density λ, and explore the result in the limits of very small and very large distances from the filament.

[3] A strict proof of the correctness of such field averaging will be given in the model solution of Problem 2.1.



Solution: Due to the limited (only axial) symmetry of the problem, applying the Gauss law to it is not very productive, so let us resort to the direct summation (actually, integration) of the component charge fields. Let us select the reference frame so that the filament coincides with segment [$-l$, $+l$] of the z-axis, and the observation point $\mathbf{r}$ has Cartesian coordinates $\{\rho, 0, z\}$ – see the figure on the right. Then the field source point $\mathbf{r}'$ has coordinates $\{0, 0, z'\}$, and Eq. (1.38) of the lecture notes, integrated across the filament's cross-section, reads

$$\phi(\mathbf{r}) = \frac{\lambda}{4\pi\varepsilon_0}\int_{-l}^{+l}\frac{dz'}{|\mathbf{r}-\mathbf{r}'|} \equiv \frac{\lambda}{4\pi\varepsilon_0}\int_{-l}^{+l}\frac{dz'}{\left[(z-z')^2+\rho^2\right]^{1/2}}.$$

By introducing the dimensionless variable $\xi \equiv (z'-z)/\rho$, we may reduce the integral to a table one:[4]

$$\phi(\mathbf{r}) = \frac{\lambda}{4\pi\varepsilon_0}\int_{(-l-z)/\rho}^{(+l-z)/\rho}\frac{d\xi}{\left(\xi^2+1\right)^{1/2}} = \frac{\lambda}{4\pi\varepsilon_0}\ln\left|\left(\xi^2+1\right)^{1/2}+\xi\right|_{(-l-z)/\rho}^{(+l-z)/\rho}$$

$$\equiv \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{\left[(l-z)^2+\rho^2\right]^{1/2}+(l-z)}{\left[(l+z)^2+\rho^2\right]^{1/2}-(l+z)}. \qquad (*)$$

For observation points very close to the filament and not very close to any of its ends, the denominator of the fraction is much smaller than its numerator. Expanding these expressions in small parameters $\rho/(l \pm z)$ and keeping only the leading terms, we get the result obtained in the solution of Problem 1:

$$\phi(\mathbf{r}) \approx \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{2(l-z)}{\rho^2/2(l+z)} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\rho + f(z), \quad \text{for } |z|<l, \text{ and } \rho \ll l,\ l-|z|.$$

On the other hand, at large distances from the filament, the numerator and denominator of the fraction approach each other. Expanding both expressions in the Taylor series in the small parameter l/r, where $r \equiv (z^2 + \rho^2)^{1/2}$ is the 3D distance from the filament's middle point, in the first nonvanishing approximation we get

$$\phi(\mathbf{r}) \approx \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{1+l/r}{1-l/r} \approx \frac{\lambda}{4\pi\varepsilon_0}\ln\left(1+\frac{2l}{r}\right) \approx \frac{\lambda}{4\pi\varepsilon_0}\frac{2l}{r}, \quad \text{for } r \gg l,$$

i.e. Eq. (1.35) for the point charge $q = 2l\lambda$.

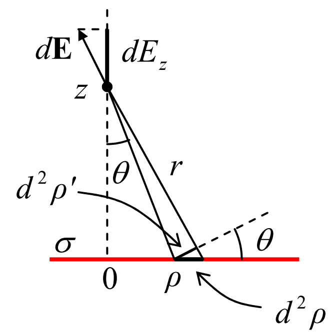
Problem 1.7. A thin planar sheet, perhaps of an irregular shape, carries an electric charge with a constant areal density σ.

(i) Express the electric field's component normal to the plane, at a certain distance from it, via the solid angle Ω at which the sheet is visible from the observation point.

(ii) Use the result to calculate the field in the center of a cube with one face charged with a constant density σ.

[4] See, e.g., MA Eq. (6.2a).

Solutions:



(i) Let us place the coordinate origin 0 at the charged plane's point that is closest to the field observation point (at a distance z from the plane) – see the figure on the right. Then, according to Eq. (1.7) of the lecture notes, the normal (z-) component of the electric field component induced by charge element $dQ = \sigma d^2\rho$ (where $\boldsymbol{\rho}$ is the 2D radius vector within the plane) equals

$$dE_z = dE\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{r^2}\cos\theta = \frac{\sigma}{4\pi\varepsilon_0}\frac{d^2\rho}{r^2}\cos\theta, \qquad (*)$$

where $r = (\rho^2 + z^2)^{1/2}$ is the distance between the elementary area $d^2\rho$ and the observation point, while $\cos\theta = z/r$. But as the figure above shows, the product $d^2\rho\cos\theta$ equals $d^2\rho'$ – the projection of the elementary area $d^2\rho$ on the plane normal to the vector $\mathbf{r}$. In turn, the ratio $d^2\rho'/r^2$ is just $d\Omega$ – the solid angle at which the elementary area $d^2\rho$ is visible from the field measurement point.[5] As a result, Eq. (*) may be rewritten simply as

$$dE_z = \frac{\sigma}{4\pi\varepsilon_0}d\Omega,$$

and its integration over the charged sheet yields the requested result:

$$E_z = \frac{\sigma}{4\pi\varepsilon_0}\Omega. \qquad (**)$$

(ii) Since all six faces of a cube are visible from its center at equal angles Ω, and their sum has to be equal to the full solid angle 4π, each Ω is equal to $4\pi/6$, and Eq. (**) yields

$$E_z = \frac{\sigma}{4\pi\varepsilon_0}\frac{4\pi}{6} \equiv \frac{\sigma}{6\varepsilon_0}.$$

Due to the obvious symmetry of the system, the field cannot have other Cartesian components, so we may also write

$$\mathbf{E} = \frac{\sigma}{6\varepsilon_0}\mathbf{n}_z.$$

Problem 1.8. Can one create, in an extended region of space, electrostatic fields with the Cartesian components proportional to the following products of the Cartesian coordinates $\{x, y, z\}$:

(i) $\{yz, xz, xy\}$,

(ii) $\{xy, xy, yz\}$?

Solution: Let us calculate the curl of both supposed fields, using the definition of that operator:[6]

$$\nabla\times\mathbf{E} = \left\{\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right\}.$$

[5] Note that this calculation is similar to the one made in the proof of the Gauss law in Sec. 1.2 of the lecture notes – see the transition from Eq. (1.13) to Eq. (1.14).

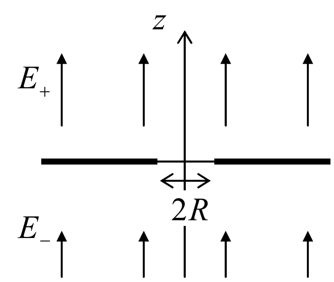
[6] See, e.g., MA Eq. (8.5).

For field (i), we get $\nabla\times\mathbf{E} \propto \{x - x,\ y - y, z - z\} \equiv 0$, while for field (ii), $\nabla\times\mathbf{E} \propto \{z - 0, 0 - 0, y - x\}$ vanishes only on one line: $x = y$, $z = 0$. However, according to the homogeneous Maxwell equation (Eq. (1.28) of the lecture notes), the curl of the electrostatic field has to equal zero at *any* point where it exists; hence field (i) can exist in a region of finite size, while field (ii) cannot.

The fact that field (i) has zero divergence as well, i.e. requires $\rho(\mathbf{r}) \equiv 0$ within the region of its existence, does not prevent it from being realistic, because the field may be created by electric charges outside of that particular region. Note also the electric field (ii) may be induced by a certain magnetic field changing (linearly) in time – see Sec. 6.1 of the lecture notes, but such $\mathbf{E}$ cannot be called an *electrostatic* field – and this was the term used in the assignment.

Problem 1.9. Distant sources have been used to create different uniform electrostatic fields in two half-spaces:

$$\mathbf{E}(\mathbf{r})\big|_{r>>R} = \mathbf{n}_z \times \begin{cases} E_+, & \text{at } z < 0, \\ E_-, & \text{at } z > 0, \end{cases}$$



except for a transitional region of scale R near the origin, where the field is perturbed but still axially symmetric. (As will be discussed in the next chapter, this may be done, for example, using a thin conducting membrane with a round hole of radius R in it – see the figure on the right.) Prove that such field may serve as an electrostatic lens for charged particles flying along the z-axis, at distances $\rho << R$ from it, and calculate the focal distance f of this lens. Spell out the conditions of validity of your result.

Solution: Let us select the orientation of the mutually perpendicular axes $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$ so that for the particular flying particle, $y = 0$. At the axis of this axially-symmetric system (i.e. at $\rho = 0$), the partial derivatives $\partial E_x/\partial x$ and $\partial E_y/\partial y$ have to be equal due to the symmetry, so the Maxwell equation (1.27), $\nabla\cdot\mathbf{E} = 0$, yields

$$2\frac{\partial E_x}{\partial x}\Big|_{\rho=0} = -\frac{\partial E_z}{\partial z}\Big|_{\rho=0}.$$

Since at the axis, E_x has to vanish due to the same symmetry, the Taylor expansion of the function $E_x(x, y)$ at $x = 0$ starts with the terms linear in x and y, so at small x and $y = 0$:

$$E_x\Big|_{\substack{x<<R\\ y=0}} \approx x\frac{\partial E_x}{\partial x}\Big|_{\rho=0} = -\frac{x}{2}\frac{\partial E_z}{\partial z}\Big|_{\rho=0}.$$

According to this expression, at sufficiently small x, the field E_x is small, so the particle's deviation from its initial trajectory during its flight through the perturbed field region (of the length $\Delta z \sim R$) is negligible. However, during this flight, the particle picks up the transverse momentum created by the field:

$$p_x = \int F_x dt = q\int E_x dt = q\int E_x \frac{dt}{dz}dz = -q\frac{x}{2v_z}\int_{-\infty}^{+\infty}\frac{\partial E_z}{\partial z}\Big|_{\rho=0}dz = -q\frac{x}{2v_z}\left(E_+ - E_-\right).$$

(Here the change of particle's longitudinal velocity v_z during its flight through the hole was neglected – which is legitimate if the length of the uniform field region is much larger than R, as implied by the problem's conditions.) If this momentum, and hence the acquired radial velocity $v_x = p_x/m$, are directed toward the axis, which is true at the proper sign of the combination $q(E_+ - E_-)/v_z$,[7] it would lead to the particle's crossing the system's axis at the z-distance

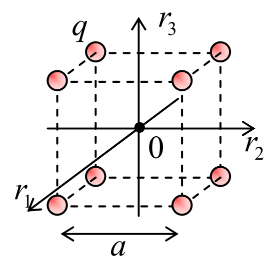
$$f = v_z \frac{x}{(-v_x)} = v_z \frac{x}{(-p_x / m)} = \frac{2mv_z^2}{q(E_+ - E_-)} = \frac{4T}{q(E_+ - E_-)},$$

where $T = mv_z^2/2$ is the particle's kinetic energy. The most important feature of this result is that the initial coordinate x of the particle has dropped out of it. This means that all particles with any small x (and, by the axial symmetry, all particles flying sufficiently close to axis z and parallel to it) will be directed to the same focal point f. This is the key property of any lens – in this case, the electrostatic one.

It is remarkable that this result does not depend on the exact distribution of the field in the transitional region. (For a thin-membrane implementation shown in the figure above, this distribution may be calculated analytically – for example, using the oblate spheroidal coordinates to be discussed in Sec. 2.4 of the lecture notes.) The result is valid if two strong conditions,

$$R << f << \frac{T}{qE_\pm},$$

are satisfied. The first of them has allowed us to treat x as a constant during the integration of the force F_x over time, while the second one, to neglect the change of v_z during the focusing process. Note, however, that a violation of the second condition would not ruin the focusing effect; it would only make the expression for f more involved and dependent on the specific distribution of the field at $z \sim f$.



Problem 1.10. Eight equal point charges q are located in the corners of a cube of side a. Calculate all Cartesian components E_j of the electric field, and their spatial derivatives $\partial E_j/\partial r_{j'}$, in the cube's center, where r_j are the Cartesian coordinates oriented along the cube's sides – see the figure on the right. Are all of your results valid for the center of a planar square, with four equal charges at its corners?

Solution: Per Eq. (1.33) of the lecture notes, the Cartesian components of the field and their derivatives may be expressed via spatial derivatives of the electrostatic potential:

$$E_j = -\frac{\partial \phi}{\partial r_j}, \qquad \frac{\partial E_j}{\partial r_{j'}} = -\frac{\partial^2 \phi}{\partial r_j \partial r_{j'}}. \tag{*}$$

Due to the cube's symmetry, at its center, the whole vector **E** should vanish. (Indeed, if it did not, the vector would be directed toward either some face or some corner of the cube, but that would violate the equivalence of all faces and corners.) Hence, according to the first of Eqs. (*), all field components, i.e. the partial derivatives of the scalar potential have to vanish at the center. So, if we take this point for the origin ($\mathbf{r} = 0$, i.e. $r_1 = r_2 = r_3 = 0$), the Taylor expansion[8] of the function $\phi(r_1, r_2, r_3)$, besides an arbitrary constant $\phi(0, 0, 0)$, should start with quadratic terms. Due to the cube's symmetry with respect to coordinate swaps, the coefficients at these terms have to be independent of the coordinates:

[7] If the sign is opposite, the system disperses the parallel particle beam, but in a highly ordered way – with the distance from the axis proportional to the initial distance x. Such a system still may work as a lens, though a diverging ("negative") one, with $f < 0$.

[8] See, e.g., MA Eq. (2.11b).

$$\phi(r_1, r_2, r_3) - \phi(0,0,0) = \frac{A}{2}\sum_{j=1}^{3} r_j^2 + \frac{B}{2}\sum_{\substack{j,j'=1 \\ j'\neq j}}^{3} r_j r_{j'} + O(r_j^3), \qquad (**)$$

where A and B are constants:

$$A = \frac{\partial^2\phi}{\partial r_j^2}\Big|_{\mathbf{r}=0} \equiv -\frac{\partial E_j}{\partial r_j}\Big|_{\mathbf{r}=0}, \quad \text{while } B = \frac{\partial^2\phi}{\partial r_j \partial r_{j'}}\Big|_{\mathbf{r}=0} \equiv -\frac{\partial E_j}{\partial r_{j'}}\Big|_{\mathbf{r}=0} \quad \text{for any } j \neq j'. \qquad (***)$$

But because of the system's symmetry, the potential has to be also invariant with respect to the change of sign of any single coordinate, $r_j \to -r_j$, which geometrically corresponds to its mirror reflection in the common plane of two other coordinate axes.[9] As Eq. (**) shows, this is only possible if $B = 0$. Moreover, since there is no charge in the vicinity of the cube's center, the potential has to satisfy the Laplace equation (1.42), in the Cartesian coordinates reading

$$\sum_{j=1}^{3} \frac{\partial^2\phi}{\partial r_j^2} = 0. \qquad (****)$$

Comparing this requirement with the first of Eqs. (***), we get $A = 0$ as well, so all the derivatives $\partial E_j/\partial r_{j'}$ at $\mathbf{r} = 0$ have to equal zero.

For the field in the center of a plane square, with four similar charges in its corners, the situation is somewhat different, due to the reduced symmetry. The electric field $\mathbf{E}$ at the center (and hence all its Cartesian components) still has to equal zero, because its in-plane component (say, $\mathbf{E}_{12}$) cannot be directed toward any side of the square, and its normal component $\mathbf{E}_3$ cannot be directed to either side of the plane, due to the $r_3 \leftrightarrow -r_3$ symmetry. However, the potential may not remain the same if the coordinate r_3 directed normally to the square's plane is swapped with one of the in-plane coordinates (r_1 or r_2). As a result, Eq. (**), still with $B = 0$ due to the mirror symmetry, has to be generalized as

$$\phi(r_1, r_2, r_3) - \phi(0,0,0) = \frac{A}{2}\left(r_1^2 + r_2^2\right) + \frac{A'}{2} r_3^2 + O\left(r_j^4\right),$$

and the Laplace equation (****) imposes only the following requirement: $2A + A' = 0$, i.e.

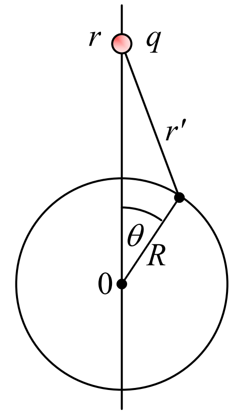
$$\frac{\partial E_1}{\partial r_1} = \frac{\partial E_2}{\partial r_2} = -\frac{1}{2}\frac{\partial E_3}{\partial r_3}.$$

This relation is similar to the one used for the solution of Problem 9 (where it was true due to the axial symmetry of the system).

Problem 1.11. By a direct calculation, find the average electric potential of a spherical surface of radius R, created by a point charge q located at a distance $r > R$ from the sphere's center. Use the result to prove the following general *mean value theorem*: the electric potential at any point is always equal to

[9] Actually, because of this symmetry, the higher terms of the expansion (**) should start with $O(r_j^4)$, rather than $O(r_j^3)$.

its average value on any spherical surface with the center at that point while containing no electric charges inside it.



Solution: Using the evident axial symmetry of the problem (see the figure on the right) and Eq. (1.35) of the lecture notes, we get:

$$\phi_{\text{ave}} \equiv \frac{1}{4\pi}\oint \phi(\theta)d\Omega = \frac{1}{4\pi}2\pi\int_0^{\pi}\phi(\theta)\sin\theta d\theta = \frac{1}{2}\int_0^{\pi}\frac{q}{4\pi\varepsilon_0 r'}\sin\theta d\theta,$$

where r' is the distance between the considered point charge and the observation point:

$$(r')^2 = R^2 + r^2 - 2Rr\cos\theta.$$

The integral may be readily worked out by the introduction of a new variable $\xi \equiv \cos\theta$ (so that $\sin\theta\, d\theta = -d\xi$):

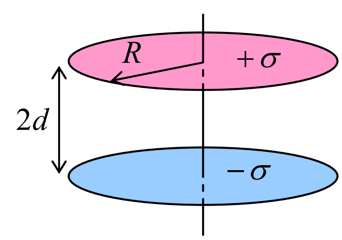
$$\begin{aligned}\phi_{\text{ave}} &= \frac{1}{2}\frac{q}{4\pi\varepsilon_0}\int_{-1}^{+1}\frac{d\xi}{\left(R^2+r^2-2Rr\xi\right)^{1/2}} = -\frac{1}{2}\frac{q}{4\pi\varepsilon_0 r'}\frac{1}{Rr}\left(R^2+r^2-2Rr\xi\right)^{1/2}\Big|_{\xi=-1}^{\xi=+1} \\ &\equiv -\frac{1}{2}\frac{q}{4\pi\varepsilon_0}\frac{1}{Rr}\left[\left(R^2+r^2-2Rr\right)^{1/2} - \left(R^2+r^2+2Rr\right)^{1/2}\right] = -\frac{q}{4\pi\varepsilon_0}\frac{|r-R|-(r+R)}{2Rr} \\ &= \frac{q}{4\pi\varepsilon_0}\times\begin{cases}1/r, & \text{for } R<r, \\ 1/R, & \text{for } r<R.\end{cases}\end{aligned}$$

We see that for our case $r > R$, the average potential of the sphere indeed coincides with the potential's value in its center. Now the proof of the mean value theorem becomes straightforward by using the linear superposition principle: since the relation in question,

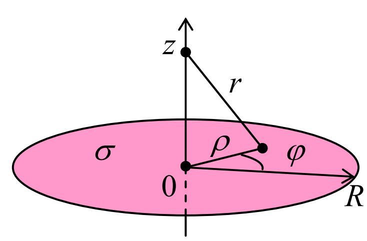
$$\frac{q}{4\pi\varepsilon_0 r} = \phi_{\text{ave}}, \quad \text{for } r > R,$$

holds for each point charge located outside the sphere, it is also true for any system of such charges.

<u>Problem 1.12</u>. Two similar thin, circular, coaxial disks of radius R, separated by distance $2d$, are uniformly charged with equal and opposite areal densities $\pm\sigma$ – see the figure on the right. Calculate and sketch the distribution of the electrostatic potential and the electric field of the disks along their common axis.



Solution: Let us start by calculating the electrostatic potential of one of the disks, with a constant areal charge density σ, at the axial point separated by distance z from the disk's plane – see the figure on the right. In the polar coordinates $\{\rho, \varphi\}$ within the plane of the disk (with angle φ referred to an arbitrary horizontal axis}, the elementary disk area is $\rho d\rho d\varphi$, and its electric charge is $\sigma\rho d\rho d\varphi$, so Eq. (1.38) of the lecture notes takes the following form:



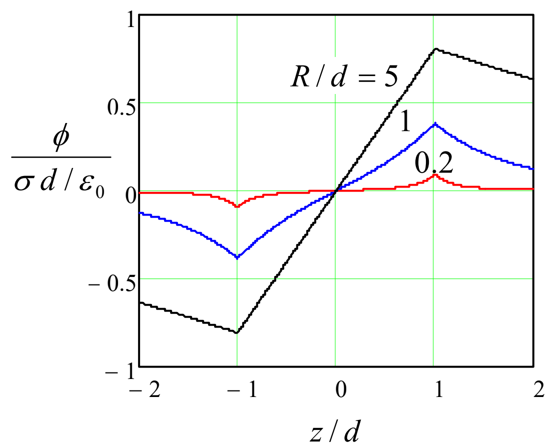
$$\phi(z) = \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi} d\varphi \int_0^R \frac{\rho d\rho}{r},$$

where r is the distance between the elementary charge's location and the observation point on the disks' common axis. As the figure above shows, $r = (\rho^2 + z^2)^{1/2}$, so the function under the integral is independent of φ, and it may be easily worked out:

$$\phi(z) = \frac{\sigma}{4\pi\varepsilon_0} 2\pi \int_0^R \frac{\rho\, d\rho}{\left(\rho^2 + z^2\right)^{1/2}} = \frac{\sigma}{4\varepsilon_0}\int_{\rho=0}^{\rho=R} \frac{d\left(\rho^2 + z^2\right)}{\left(\rho^2 + z^2\right)^{1/2}} = \frac{\sigma}{2\varepsilon_0}\left(\rho^2 + z^2\right)^{1/2}\Big|_{\rho=0}^{\rho=R} \equiv \frac{\sigma}{2\varepsilon_0}\left[\left(R^2 + z^2\right)^{1/2} - |z|\right].$$

Now by using this formula and the linear superposition principle, we may readily write down the expression describing the potential created by both disks, at a distance z from the center of the system (in this new reference frame, the disk center positions are $\pm d$):

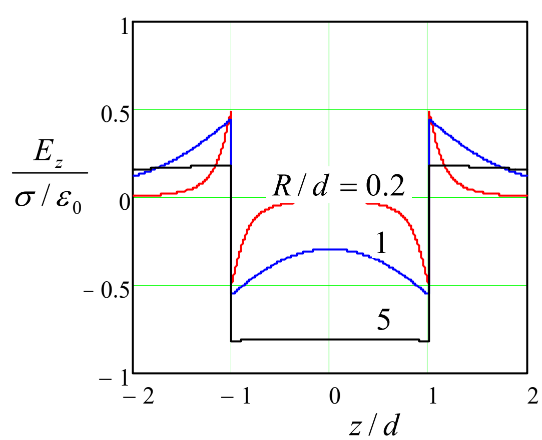
$$\phi = \frac{\sigma}{2\varepsilon_0}\times\begin{Bmatrix} \left[R^2 + (z-d)^2\right]^{1/2} - |z-d| \\ -\left[R^2 + (z+d)^2\right]^{1/2} + |z+d| \end{Bmatrix}.$$



This function is plotted in the figure on the right for three different values of the R/d ratio. For small values of this ratio, the potential is clearly separated into two peaks, of opposite polarity, created by each disk. On the other hand, at $R >> d$, the result tends to the one for two infinite planes, with ϕ between the disks being a linear function of z, with the slope corresponding to the electric field – see the model solution of Problem 15 below.

Now the electric field at the axis (which has only one, vertical component, due to the axial symmetry of the problem) may be calculated by the partial differentiation of the electrostatic potential – see Eq. (1.33) of the lecture notes:

$$E_z = -\frac{\partial\phi}{\partial z} = \frac{\sigma}{2\varepsilon_0}\left\{-\frac{z-d}{\left[R^2 + (z-d)^2\right]^{1/2}} + \operatorname{sgn}(z-d) + \frac{z+d}{\left[R^2 + (z+d)^2\right]^{1/2}} - \operatorname{sgn}(z+d)\right\}.$$



In the figure on the right, this function is plotted for the same three values of the R/d ratio as the potential in the figure above. The plots show that at $R/d = 5$, the field between the disks is already pretty uniform, though its magnitude is still noticeably smaller than that (σ/ε_0) between two infinite planes. Note also that the ratio R/d does not affect the electric field's jump by $\pm\sigma/\varepsilon_0$ as the observation point crosses a disk – just as it has to be per Eq. (1.24) of the lecture notes.

On the technical side, this solution illustrates again the advantage of calculating the electrostatic potential (a scalar function) first, and only then the electric field from it.

Problem 1.13. The electrostatic potential created by an electric charge distribution is

$$\phi(\mathbf{r}) = C\left(\frac{1}{r} + \frac{1}{2r_0}\right)\exp\left\{-\frac{r}{r_0}\right\},$$

where C and r_0 are constants, and $r \equiv |\,\mathbf{r}\,|$ is the distance from the origin. Calculate the charge distribution in space.

Solution: According to the Poisson equation (see Eq. (1.41) of the lecture notes), the charge density may be calculated as

$$\rho(\mathbf{r}) = -\varepsilon_0 \nabla^2 \phi(\mathbf{r}).$$

Since the given potential distribution is spherically symmetric, $\phi(\mathbf{r}) = \phi(r)$, the general expression for the Laplace operator of such scalar function in spherical coordinates[10] reduces to

$$\nabla^2 \phi(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right), \qquad (*)$$

so by performing a straightforward differentiation, we get a very simple, exponential charge density distribution:

$$\rho(\mathbf{r}) = -\frac{\varepsilon_0 C}{2r_0^3}\exp\left\{-\frac{r}{r_0}\right\}. \qquad \text{(INCOMPLETE!)}$$

Note, however, that since Eq. (*) is invalid at $r = 0$, this point has to be explored separately. At $r \to 0$, the given potential distribution tends to C/r, and as Eq. (1.35) of the lecture notes shows, such potential is created by a point charge $q = 4\pi\varepsilon_0 C$, located at the origin. As a result, the complete solution of our problem is

$$\rho(\mathbf{r}) = \varepsilon_0 C\left[4\pi\delta(\mathbf{r}) - \frac{1}{2r_0^3}\exp\left\{-\frac{r}{r_0}\right\}\right] \equiv q\left[\delta(\mathbf{r}) - \frac{1}{8\pi r_0^3}\exp\left\{-\frac{r}{r_0}\right\}\right],$$

with the total charge[11]

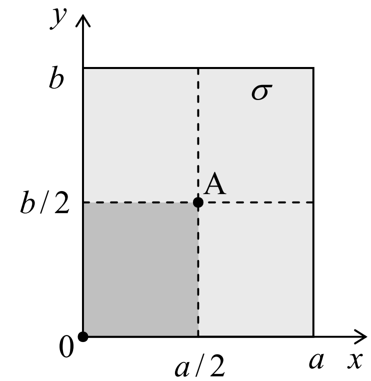
$$Q = \int \rho(\mathbf{r})d^3r = q\left(1 - \frac{1}{8\pi r_0^3}\int \exp\left\{-\frac{r}{r_0}\right\}d^3r\right) = q\left(1 - \frac{1}{8\pi r_0^3}4\pi\int_0^\infty \exp\left\{-\frac{r}{r_0}\right\}r^2 dr\right)$$
$$= q\left(1 - \frac{1}{2}\int_0^\infty \exp\{-\xi\}\xi^2 d\xi\right) = 0\,.$$

Just for the reader's reference, such $\rho(\mathbf{r})$ is an approximate model for the charge distribution in a neutral atom, describing screening of the positive nuclear charge $q = Ze$ by Z negatively charged electrons. Such an approximation, as well as a more accurate *Thomas-Fermi model* of the screening,[12] takes into account the quantum-mechanical properties of electrons only partly, and as a result, works reasonably well only for heavy atoms ($Z >> 1$).

[10] See, e.g., MA Eq. (10.9).
[11] The last step of this calculation uses the table integral MA Eq. (6.7d) with $n = 2$.
[12] See, e.g., either QM Problems 8.30-8.31 or SM Problems 3.12-3.13.

Problem 1.14. A thin, flat, rectangular sheet of size $a\times b$ is electrically charged with a constant areal density σ. Without an explicit calculation of the spatial distribution $\phi(\mathbf{r})$ of the electrostatic potential induced by this charge, find the ratio of its values in the center and in the corners of the rectangle.

Hint: Consider partitioning the rectangle into several similar parts and using the linear superposition principle.



Solution: Selecting the Cartesian coordinates as shown in the figure on the right, we may use Eq. (1.38) of the lecture notes to express the potential at the origin (i.e., in one of the rectangle's corners) as

$$\phi_0 = \frac{\sigma}{4\pi\varepsilon_0}\int_0^a dx \int_0^b dy \frac{1}{\left(x^2+y^2\right)^{1/2}} \equiv \frac{\sigma a}{4\pi\varepsilon_0} I,$$

where I is the following dimensionless integral:

$$I \equiv \int_0^1 d\xi \int_0^{b/a} d\zeta \frac{1}{\left(\xi^2+\zeta^2\right)^{1/2}},$$

with $\xi \equiv x/a$, $\zeta \equiv y/a$.

Now let us calculate what potential *would* be induced, at the same point, only by the adjacent quarter of the rectangle, of size $(a/2)\times(b/2)$, marked with darker shading in the figure above:

$$\phi_0' = \frac{\sigma}{4\pi\varepsilon_0}\int_0^{a/2} dx \int_0^{b/2} dy \frac{1}{\left(x^2+y^2\right)^{1/2}}.$$

Evidently, this result may be reduced to the same dimensionless integral I:

$$\phi_0' = \frac{\sigma(a/2)}{4\pi\varepsilon_0} I,$$

so even without the calculation of the integral[13] we see that $\phi_0' = \phi_0/2$.

Next, we may note that due to the linear superposition principle, the potential ϕ_A at the center of the rectangle may be represented as a sum of four potentials induced by such quarters in their corners. Due to the system's symmetry, each of these partial potentials is equal to ϕ_0', so, finally,

$$\phi_A = 4\phi_0' = 2\phi_0.$$

This solution illustrates the power of *scaling* arguments, broadly used for symmetric systems in all fields of physics.

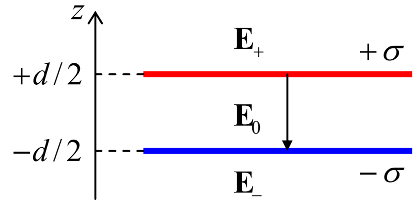
Problem 1.15**.** Calculate the electrostatic energy per unit area of a system of two thin, parallel planes with equal and opposite charges of a constant areal density σ, separated by distance d.

---

[13] Just for reference, this integral may be worked out analytically: $I = \frac{b}{a}\ln\frac{\left(a^2+b^2\right)^{1/2}+a}{b} + \sinh^{-1}\frac{b}{a}$.

Solution: From the similar problem solved in the lecture notes (see Fig. 1.4 and its discussion), it is clear that the electric field everywhere should be normal to the planes and constant within each of three ranges:

$$\mathbf{E} = \mathbf{n}_z \times \begin{cases} E_-, & \text{at } z < -d/2, \\ E_0, & \text{at } -d/2 < z < +d/2, \\ E_+, & \text{at } +d/2 < z, \end{cases}$$



where the z-axis and its origin are selected as shown in the figure on the right, and E_-, E_0, and E_+ are some scalar constants. These constants may be calculated by applying the Gauss law to three pillboxes with all three possible combinations of different lid positions, giving three equations:

$$E_+ - E_0 = +\sigma/\varepsilon_0, \qquad E_0 - E_- = -\sigma/\varepsilon_0, \qquad E_+ - E_- = 0.$$

An (easy) solution of this system of equations yields[14]

$$E_+ = E_- = 0, \qquad E_0 = -\frac{\sigma}{\varepsilon_0}. \tag{*}$$

Thus, the field exits only between the planes, producing the electrostatic potential

$$\phi = \phi_{|z=0} - \int_0^z E(z')dz' = \phi_{|z=0} + \frac{\sigma}{\varepsilon_0} \times \begin{cases} z, & \text{for } |z| < d/2, \\ (d/2)\,\mathrm{sgn}(z), & \text{for } |z| > d/2, \end{cases}$$

so the potential difference (*voltage*) between the planes is

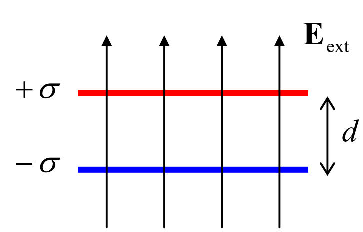
$$V \equiv \phi\left(+\frac{d}{2}\right) - \phi\left(-\frac{d}{2}\right) = \frac{\sigma}{\varepsilon_0} d \equiv \frac{\sigma}{C_0},$$

where $C_0 = \varepsilon_0/d$ is the *specific capacitance* of this simple system (which is equivalent to a *plane capacitor* – for more, see Chapter 2).

Now using Eq. (*) in Eq. (1.65) for the potential energy of the field, we get

$$\frac{U}{A} = \frac{\varepsilon_0}{2} \int_{-d/2}^{+d/2} E^2 dz = \frac{\varepsilon_0 E_0^2}{2} d = \frac{\sigma^2}{2\varepsilon_0} d \equiv \frac{C_0 V^2}{2}.$$

(We will repeatedly run into the last expression in Chapter 2.) Note that at fixed σ (of any sign) the energy grows with d. This is natural, because the oppositely charged planes attract each other, so following the force (reducing d) corresponds to the way toward the potential energy's minimum.



Problem 1.16. The system analyzed in the previous problem (two thin, parallel, oppositely charged planes) is now placed into an external, uniform, normal electric field $E_{\text{ext}} = \sigma/\varepsilon_0$ – see the figure on the right. Find the force (per unit area) acting on each plane, by two methods:

[14] An even simpler way to get Eq. (*) is to employ the linear superposition principle by summing up the fields of the same magnitude (1.23), $|E| = |\sigma|/2\varepsilon_0$, induced by each of the planes. Due to the difference in plane charge signs, the fields add up inside the gap between the planes but cancel each other outside of this gap.

(i) directly from the electric field distribution, and
(ii) from the potential energy of the system.

Solutions:

(i) In order to calculate the force F acting on the top plane, we have to neglect the field of this plane itself,[15] summing up the external field directed up and equal to σ/ε_0, and the field of the bottom plane, directed down and equal to $\sigma/2\varepsilon_0$ – see Fig. 1.4 and Eq. (1.24) of the lecture notes. The net field equals $\sigma/2\varepsilon_0$ and is directed up, so the force per unit area is

$$\frac{F}{A}=\frac{q}{A}\frac{\sigma}{2\varepsilon_0}=\frac{\sigma^2}{2\varepsilon_0}, \qquad (*)$$

and is directed up. A similar calculation for the bottom plane yields a force of the same magnitude but directed down.

(ii) Using the linear superposition principle, we may calculate the total field in each spatial region by adding the external field to that of the planes, which was calculated in the previous problem:

$$\mathbf{E}=\mathbf{n}_z\frac{\sigma}{\varepsilon_0}+\mathbf{n}_z\frac{\sigma}{\varepsilon_0}\times\begin{Bmatrix} 0, & \text{for } z<-d/2, \\ -1, & \text{for } -d/2<z<+d/2, \\ 0, & \text{for } +d/2<z, \end{Bmatrix}\equiv\mathbf{n}_z\frac{\sigma}{\varepsilon_0}\times\begin{cases} 1, & \text{for } z<-d/2, \\ 0, & \text{for } -d/2<z<+d/2, \\ 1, & \text{for } +d/2<z. \end{cases}$$

Using these values, let us calculate the potential energy U of the field from Eq. (1.72) of the lecture notes, artificially limiting the integration volume to a pillbox of an arbitrary area A, with some fixed thickness $d_0 > d$:

$$\frac{U}{A}=\frac{\varepsilon_0}{2}\left(\frac{\sigma}{\varepsilon_0}\right)^2(d_0-d)\equiv\text{const}-\frac{\sigma^2}{2\varepsilon_0}d,$$

where "const" means a quantity independent of d. Now the vertical force (per unit area) exerted at the top plane may be calculated as $F/A = -\partial(U/A)/\partial d$, giving the same result (*) as the first method. The same is true for the lower plane whose vertical coordinate is (const – d), so the ratio U/A should be differentiated over ($-d$) rather than d.

Problem 1.17. Explore the relationship between the Laplace equation (1.42) and the minimum of the electrostatic field energy (1.65).

Solution: Let us consider a small variation $\delta\phi(\mathbf{r})$ of the electrostatic potential inside a charge-free volume V limited by a closed surface S, such that the variation vanishes at the surface, but is otherwise arbitrary. Let us calculate the corresponding variation of the electric field energy (1.65):

$$\delta U=\delta\left(\frac{\varepsilon_0}{2}\int_V(\nabla\phi)^2d^3r\right)=\frac{\varepsilon_0}{2}\int_V\delta(\nabla\phi\cdot\nabla\phi)d^3r=\varepsilon_0\int_V\nabla\phi\cdot\delta(\nabla\phi)d^3r=\varepsilon_0\int_V\nabla\phi\cdot\nabla(\delta\phi)d^3r\,. \qquad (*)$$

[15] Different elementary charges of the same plane do Coulomb-interact, but the elementary forces between them are directed along the plane, and cancel at summation, due to the 3rd Newton law.

(The last step of the above calculation has exploited the main rule of the variational calculus that the operations of small variation and the usual differentiation are interchangeable.)[16] Now let us work out the resulting 3D integral by parts – just as was done at the derivation of Eq. (1.65) of the lecture notes, i.e. by applying the divergence theorem[17] to the vector function $\mathbf{f} = \delta\phi\nabla\phi$, and then differentiating this product by parts:[18]

$$\oint_S (\delta\phi\nabla\phi)_n d^2r = \int_V \nabla\cdot(\delta\phi\nabla\phi)d^3r \equiv \int_V \nabla(\delta\phi)\cdot\nabla\phi\, d^3r + \int_V \delta\phi\nabla^2\phi\, d^3r. \qquad (**)$$

Due to our condition $\delta\phi|_S = 0$, the surface integral on the left-hand side of this relation has to equal zero, while the first volume integral on the right-hand side equals that in the last form of Eq. (*). Hence, Eqs. (*) and (**) may be combined to give

$$\delta U = -\varepsilon_0 \int_V \delta\phi\nabla^2\phi\, d^3r.$$

Since the variation $\delta\phi$ is arbitrary, this expression shows that the only way for the potential energy of the field to have the lowest possible value (just as it does in any stable equilibrium in classical mechanics), and hence the variation δU to equal zero, is to have the Laplace operator of the field equal zero, i.e. to satisfy the Laplace equation (1.42) in all charge-free spatial regions.

Note that this fact may be used to evaluate approximate (say, just guessed) solutions of the Laplace equation in situations when finding the exact one is difficult. Indeed, if several such *trial solutions* have been suggested, the one giving the lowest energy (1.65) has a good chance of being the closest to the genuine one. Moreover, if we have guessed a family of such approximate solutions depending on a parameter, the best of them may be found by varying this parameter to minimize the corresponding energy (1.65).[19]

Problem 1.18. Prove the following *reciprocity theorem of electrostatics*:[20] if two spatially-confined charge distributions $\rho_1(\mathbf{r})$ and $\rho_2(\mathbf{r})$ create, respectively, electrostatic potentials $\phi_1(\mathbf{r})$ and $\phi_2(\mathbf{r})$, then

$$\int \rho_1(\mathbf{r})\phi_2(\mathbf{r})d^3r = \int \rho_2(\mathbf{r})\phi_1(\mathbf{r})d^3r.$$

Hint: Consider the integral $\int \mathbf{E}_1\cdot\mathbf{E}_2 d^3r$.

Solution: By applying Eq. (1.33) of the lecture notes to $\mathbf{E}_1$(r), let us rewrite the integral mentioned in the *Hint* as

$$\int \mathbf{E}_1\cdot\mathbf{E}_2 d^3r = -\int \nabla\phi_1\cdot\mathbf{E}_2 d^3r.$$

Now we may use the rule of spatial differentiation of a vector-by-scalar function product[21] to continue as follows:

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[16] This rule has a simple geometric meaning – see, e.g., CM Sec. 2.1, in particular Fig. 2.2.

[17] See, e.g., MA Eq. (12.2).

[18] See, e.g., MA Eq. (11.4a).

[19] Such *variational method* is especially popular in quantum mechanics – see, e.g., QM Sec. 2.9 and on.

[20] This is only the simplest one of the whole family of reciprocity theorems in electromagnetism – see, e.g., Sec. 6.8 of the lecture notes.

[21] See, e.g., MA Eq. (11.4a), with $f = \phi_1$ and $\mathbf{g} = \mathbf{E}_2$.

$$-\int \nabla\phi_1 \cdot \mathbf{E}_2 d^3r = \int \phi_1 (\nabla \cdot \mathbf{E}_2) d^3r - \int \nabla(\phi_1 \mathbf{E}_2) d^3r.$$

Next, we may use the inhomogeneous Maxwell equation (1.27) in the first integral on the right-hand side, and the divergence theorem[22] to transform the second integral to that of $(\phi_1 \mathbf{E}_2)_n$ over some very distant closed surface S that may be taken for the limit of our spatial integration. As a result, our expression becomes

$$\frac{1}{\varepsilon_0}\int \phi_1 \rho_2 d^3r - \oint_S (\phi_1 \mathbf{E}_2)_n d^2r .$$

Since per the problem's conditions, the charge (and hence the field) distributions are space-confined, we may select the surface S so distant that the surface integral in the above expression is negligible,[23] and we get

$$\int \mathbf{E}_1 \cdot \mathbf{E}_2 d^3r = \frac{1}{\varepsilon_0}\int \phi_1 \rho_2 d^3r \cdot$$

Now repeating the same calculation with swapped indices, we arrive at the reciprocity theorem.

Note that if some parts of these two charge distributions reside on some surface(s) S, and may be well described by *surface* charge densities $\sigma_1(\mathbf{r})$ and $\sigma_2(\mathbf{r})$ (as is very instrumental, for example, in systems with good conductors, to be discussed in Chapter 2), the reciprocity theorem may be rewritten as

$$\int_V \rho_1(\mathbf{r})\phi_2(\mathbf{r})d^3r + \int_S \sigma_1(\mathbf{r})\phi_2(\mathbf{r})d^2r = \int_V \rho_2(\mathbf{r})\phi_1(\mathbf{r})d^3r + \int_S \sigma_2(\mathbf{r})\phi_1(\mathbf{r})d^2r ,$$

where $\rho_1(\mathbf{r})$ and $\rho_2(\mathbf{r})$ are the remaining "genuinely-volume" parts of the distributions. (In this form, it is sometimes called the "Green's reciprocity theorem".)

Problem 1.19. Calculate the energy of the electrostatic interaction of two spheres, of radii R_1 and R_2, each with a spherically symmetric charge distribution, separated by distance $d \geq R_1 + R_2$.

Solution: According to Eq. (1.55) of the lecture notes, applied sequentially to each of the spheres, the energy U_{int} of their interaction may be calculated in any of two ways:[24]

$$U_{\text{int}} = \int_{\text{sphere 1}} \rho_1(\mathbf{r})\phi_2(\mathbf{r})d^3r , \qquad (*)$$

$$U_{\text{int}} = \int_{\text{sphere 2}} \rho_2(\mathbf{r})\phi_1(\mathbf{r})d^3r . \qquad (**)$$

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[22] If you still need to, see MA Eq. (12.2).

[23] Indeed, we may take S in the shape of a spherical shell, with the radius R much larger than the spatial scales of distributions $\rho_1(\mathbf{r})$ and $\rho_2(\mathbf{r})$. At such large distances, their fields are close to those of point charges, so according to Eqs. (1.6) and (1.35), $\phi_1 \propto Q_1/R$ and $E_2 \propto Q_2/R^2$, so the integral over surface S, of area $A \propto R^2$, is decreasing with R as $1/R$ – or even faster if any of the net charges $Q_{1,2}$ vanishes.

[24] Their equivalence may be strictly confirmed using the reciprocity theorem (see the previous problem) even if the charge distributions $\rho_{1,2}$(r) partly or fully overlap – even though in our current case, they do not.

But as was discussed in Sec. 1.2 of the lecture notes (see Eqs. (1.19)-(1.20)), the electric field, and hence the electrostatic potential ϕ, created by a spherically symmetric charge distribution $\rho(\mathbf{r})$ outside of its boundary, coincides with that of the full charge Q of the sphere concentrated in its center. Applying this fact to, for example, the first sphere, we get

$$\phi_1(\mathbf{r}) = \phi_1^{(\text{point})}(\mathbf{r}) = \frac{Q_1}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}_1|}, \qquad \text{where } Q_1 \equiv \int_{\text{sphere 1}} \rho_1(\mathbf{r}) d^3r, \qquad \text{for } |\mathbf{r}-\mathbf{r}_1| \geq R_1,$$

where $\mathbf{r}_1$ is the center of sphere 1. Since, per the problem's conditions, all points $\mathbf{r}$ inside sphere 2 do satisfy the last condition, we may apply this expression for $\phi_1(\mathbf{r})$ to Eq. (**), getting

$$U_{\text{int}} = \int_{\text{sphere 2}} \rho_2(\mathbf{r}) \phi_1^{(\text{point})}(\mathbf{r}) d^3r.$$

This result means that the interaction energy cannot depend on whether the charge of sphere 1 is distributed as given or concentrated at point $\mathbf{r}_1$, i.e. should be invariant with respect to the replacement

$$\rho_1(\mathbf{r}) \to Q_1 \delta(\mathbf{r}-\mathbf{r}_1).$$

Making this replacement in Eq. (*), we get

$$U_{\text{int}} = Q_1 \phi_2(\mathbf{r}_1). \qquad (***)$$

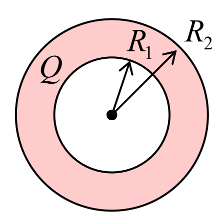
Now let us use the same argument to express the potential created by the spherically symmetric distribution $\rho_2(\mathbf{r})$:

$$\phi_2(\mathbf{r}) = \phi_2^{(\text{point})}(\mathbf{r}) = \frac{Q_2}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}_2|}, \qquad \text{where } Q_2 \equiv \int_{\text{sphere 2}} \rho_2(\mathbf{r}) d^3r, \qquad \text{for } |\mathbf{r}-\mathbf{r}_2| \geq R_2,$$

where $\mathbf{r}_2$ is the position of the center of sphere 2. By applying this formula to point $\mathbf{r}_1$ (which, by the problem's conditions, satisfies the condition $|\mathbf{r} - \mathbf{r}_2| > R_2$) and plugging the result into Eq. (***), we finally get

$$U_{\text{int}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 d}, \qquad \text{with } d \equiv |\mathbf{r}_1 - \mathbf{r}_2|.$$

In plain English, the spheres interact as if their electric charges were concentrated in their centers. Note, however, that this result, by its derivation, is only applicable at $d \geq R_1 + R_2$ when the charge distributions $\rho_1(\mathbf{r})$ and $\rho_2(\mathbf{r})$ do not overlap – even partly.



<u>Problem 1.20</u>. Calculate the electrostatic energy U of a (generally, thick) spherical shell, with charge Q uniformly distributed through its volume – see the figure on the right. Interpret the dependence of U on the inner cavity's radius R_1, at fixed Q and R_2.

Solution: Calculating the only (radial) component E of the electric field $\mathbf{E} = \mathbf{n}_r E(r)$ (for example, by using the Gauss law), we get

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \times \begin{cases} 0, & \text{for } 0 \le r < R_1, \\ \left(r^3 - R_1^3\right)/\left(R_2^3 - R_1^3\right), & \text{for } R_1 < r < R_2, \\ 1, & \text{for } R_2 < r, \end{cases}$$

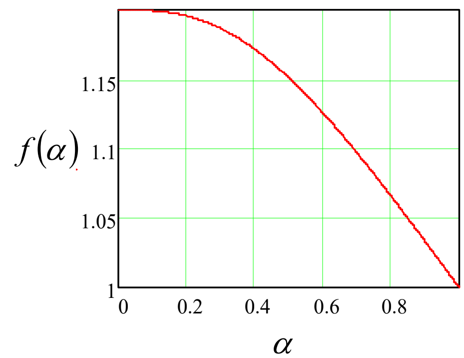
so Eq. (1.65) for the electrostatic energy yields

$$U = \frac{\varepsilon_0}{2}\int E^2(r)dr^3 = \frac{\varepsilon_0}{2}4\pi\int_0^\infty E^2(r)r^2dr = \frac{\varepsilon_0}{2}4\pi\left(\frac{Q}{4\pi\varepsilon_0}\right)^2\left[\int_{R_1}^{R_2}\left(\frac{r^3 - R_1^3}{R_2^3 - R_1^3}\right)^2\frac{dr}{r^2} + \int_{R_2}^{\infty}\frac{dr}{r^2}\right]$$

$$\equiv \frac{Q^2}{8\pi\varepsilon_0 R_2}\left[\int_\alpha^1\left(\frac{\xi^3 - \alpha^3}{1-\alpha^3}\right)^2\frac{d\xi}{\xi^2} + \int_1^\infty\frac{d\xi}{\xi^2}\right] = \frac{Q^2}{8\pi\varepsilon_0 R_2}f(\alpha), \quad \text{with } f(\alpha) \equiv \frac{(1/5) - \alpha^3 + (9/5)\alpha^5 - \alpha^6}{\left(1-\alpha^3\right)^2} + 1,$$

where $\alpha \equiv R_1/R_2 \le 1$.

The figure on the right shows a plot of the function $f(\alpha)$, i.e. of the normalized electrostatic energy as a function of R_1 at fixed Q and R_2. At $\alpha = 0$, i.e. for a solid sphere, the function reaches its maximum $f(0) = 6/5$ (so U is reduced to Eq. (1.66) of the lecture notes, with $R = R_2$). On the other hand, at $\alpha \to 1$, i.e. for a very thin spherical shell, it tends to $f(1) = 1$, giving



$$U = U_{\min} = \frac{Q^2}{8\pi\varepsilon_0 R_2}.$$

This behavior is very natural because the elementary charges of the sphere repulse each other and "try" to go apart as far as possible, in particular by increasing its inner radius.

Chapter 2. Charges and Conductors

Problem 2.1. Calculate the force (per unit area) exerted on the surface of a conductor by an electric field normal to it. Compare the result with the electric field's definition given by Eq. (1.6) of the lecture notes, and comment.

Solution: Let us make a natural assumption that the density $u(\mathbf{r})$ of the electrostatic energy, given by Eq. (1.65) of the lecture notes, at the conductor's surface is a smooth function of the surface's position. Then, at a small displacement δx of the surface, in the direction along the field vector $\mathbf{E}$,[25] the change of the full energy (1.65), per unit area, may be calculated as

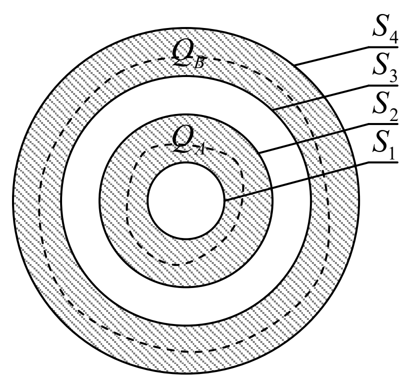
$$\frac{\delta U}{A} = \frac{1}{A}\delta\left(\frac{\varepsilon_0}{2}\int E^2 d^3r\right) = \frac{\varepsilon_0}{2}E^2\frac{1}{A}\delta V = \frac{\varepsilon_0}{2}E^2(-\delta x),$$

where V is the volume outside the conductor (filled with the field). Hence the force exerted by the field is directed out of the conductor (which is therefore *attracted* by the field – of any polarity), and equal to

$$\frac{F}{A} = -\frac{1}{A}\frac{\delta U}{\delta x} = \frac{\varepsilon_0}{2}E^2 \equiv \frac{1}{2}\sigma E, \qquad (*)$$

where σ is the surface charge density – see Eq. (2.3).

The apparent contradiction (the extra factor ½) between this result and the electric field's definition (1.6) reduced to a unit surface area ($F/A = \sigma E$) may be interpreted as follows. The field equals E on one side of the surface charge layer but vanishes on its other side (inside the conductor), so the average field inside the layer is $E_{\text{ave}} = E/2$. Hence the force per unit area may be calculated as $F/A = \sigma E_{\text{ave}} = (\varepsilon_0 E)(E/2)$, in agreement with Eq. (*).



Problem 2.2. Electric charges Q_A and Q_B have been put on two conducting concentric spherical shells – see the figure on the right. What is the full charge of each of the surfaces S_1-S_4?

Solution: Let us start with applying the Gauss law (1.16),

$$\oint_S E_n d^2r = \frac{Q}{\varepsilon_0} \qquad (*)$$

(where Q is the full charge inside the closed surface S), to any surface located inside the inner sphere and containing the surface S_1 inside it – see the inner dashed line in the figure above. As was discussed in Sec. 2.1 of the lecture notes, in statics, the electric field inside any conductor should be zero, so the left-hand side of Eq. (*) for that surface vanishes, indicating that the full charge of surface S_1 equals zero: $Q_1 = 0$. Hence the full charge Q_A of the first sphere should reside completely on its external surface: $Q_2 = Q_A$.

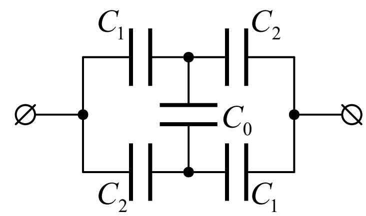
[25] A small shift *normal* to the vector $\mathbf{E}$ evidently cannot change $u(\mathbf{r})$ at all.

Now let us consider a similar Gaussian volume with the boundary inside the outer conducting sphere, i.e. containing surfaces S_1, S_2, and S_3. Similarly, the left-hand side of Eq. (*) equals zero, and the Gauss law gives

$$0 = \frac{Q_1 + Q_2 + Q_3}{\varepsilon_0}.$$

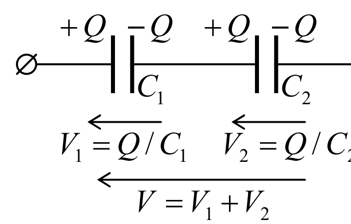
With our prior results for Q_1 and Q_2, this means that $Q_3 = -Q_A$. Hence the charge residing on the external surface S_4 of the same sphere is

$$Q_4 = Q_B - Q_3 = Q_B - (-Q_A) \equiv Q_A + Q_B.$$



Problem 2.3. Calculate the mutual capacitance between the terminals of the lumped-capacitor circuit shown in the figure on the right. Analyze and interpret the result for major particular cases.

Solution: When solving the lumped-capacitor circuit problems by prescribing certain charges to the capacitor plates, it is necessary to remember that the net charges of each isolated conducting "island" of the circuit and of both plates of each capacitor have to equal zero. Indeed, since for the capacitance calculation, we are only interested in the charges induced by the voltage V applied to the circuit, we may take the total charge of each insulated island at $V = 0$ for zero, and because of its isolation, the net charge of the island cannot change when the voltage is applied. On the other hand, the lumped capacitor model is only valid if the electric field outside the capacitors is negligible, and hence the net charge of each capacitor has to be negligibly small – see Fig. 2.5 of the lecture notes.



For example, for the elementary example of two capacitors connected in series (see the figure on the right), with one isolated island, these requirements immediately imply that all plate charges should have the same magnitude. From this charge pattern, and Eq. (2.26) applied to each capacitor, we get $V_1 = Q/C_1$ and $V_2 = Q/C_2$. Since these voltages across each capacitor[26] are, by definition, just the differences between the electrostatic potentials of their plates, at the connection in series, they just add up (see the same figure again), giving $V = V_1 + V_2 = Q(1/C_1 + 1/C_2)$ for the voltage between the external terminals. Now considering the whole circuit as a "black box" between the terminals, we have to define its (mutual) capacitance C as the ratio of the charge Q passed through the terminals at the circuit's charging, to the resulting voltage V between them. So, in this easy case, this definition yields the well-known answer:

$$C \equiv \frac{Q}{V} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1}, \qquad \text{i.e. } \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2},$$

with an evident generalization to the case of an arbitrary number of capacitances connected in series.

The reader has to excuse me for the detailed discussion of this apparently trivial example. Indeed, without a clear understanding of this general approach, the solution of more complex problems, such as our current problem, may be difficult because it cannot be reduced to chains of capacitances connected either in parallel or in series. For this problem, taking into account the capacitor and island

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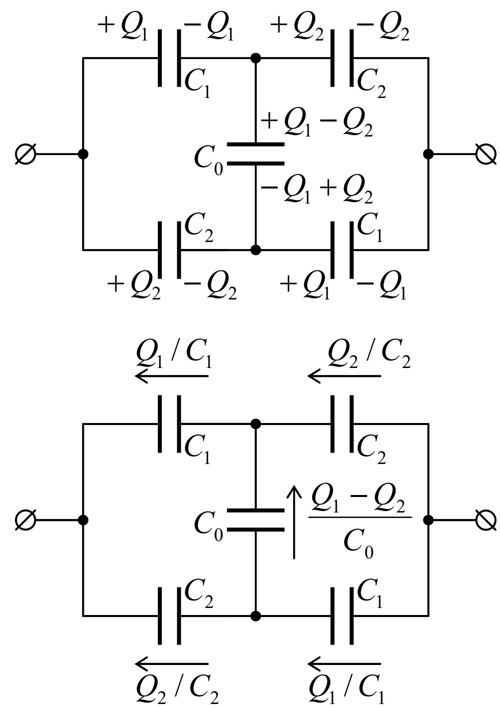
[26] They are sometimes called "voltage drops", though it is more logical to call them "potential drops".



neutrality and the circuit symmetry, we arrive at the general charge distribution shown on the top panel of the figure on the right, with the total inserted charge

$$Q = Q_1 + Q_2, \qquad (*)$$

but with Q_1 and Q_2 still unknown.

Now using Eq. (2.26) to calculate the voltage drop across each capacitor, we arrive at the voltage pattern shown on the bottom panel on the figure on the right, where the arrow directions show the *assumed* voltage polarities, correct if all specified capacitor charges (Q_1, Q_2, and $Q_1 - Q_2$) are positive. For these voltage drops, we may write two relations: the obvious equality for the voltage between the terminals:

$$V = \frac{Q_1}{C_1} + \frac{Q_2}{C_2}, \qquad (**)$$

and the condition that the net voltage drop along any of the closed loops of the circuit is zero:[27]

$$\frac{Q_1}{C_1} - \frac{Q_2}{C_2} + \frac{Q_1 - Q_2}{C_0} = 0. \qquad (***)$$

Solving the system of two linear equations (*) and (***) for $Q_{1,2}$, and plugging the result into Eq. (**), we finally get $V = Q/C$, where

$$C = \frac{2C_1C_2 + (C_1 + C_2)C_0}{C_1 + C_2 + 2C_0}. \qquad (****)$$

Let us start the analysis of this result from the case $C_0 = 0$. In this particular case, Eq. (****) is reduced to

$$C = \frac{2C_1C_2}{C_1 + C_2}, \qquad \text{i.e. } C = 2C', \quad \text{where } \frac{1}{C'} \equiv \frac{1}{C_1} + \frac{1}{C_2}.$$

This result is natural, because in this case the capacitor C_0 cannot accommodate any charge, and may be replaced with an open circuit. As a result, our full system becomes just a *parallel* connection of two similar branches, each being just a connection *in series* of the capacitors C_1 and C_2, with the branch capacitance C'.

Note that the last result (or rather its particular form $C = C_1 = C_2$) is also valid in the case of an arbitrary C_0, but equal C_1 and C_2. Indeed, in this case, the circuit is symmetric with respect to the central horizontal line in any of the figures above, so there is no voltage drop between its internal notes.

[27] In the formal theory of lumped-element circuits, this condition is called the *2nd* (or "loop") *Kirchhoff rule*, with the *1st* (or "node") *Kirchhoff rule* saying that the algebraic sum of the charges/currents flowing into each circuit *node* (wire connection) should equal zero. (For our circuit, the latter rule yields Eq. (*), obtained above from the reasoning similar to that used for the proof of the rule in the general case.) The Kirchhoff rules will be discussed in Chapters 4 and 6 of the lecture notes.

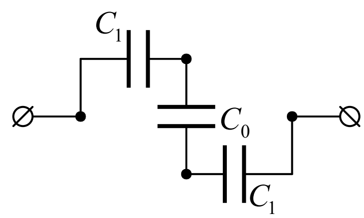
In the opposite limit, $C_0 \to \infty$ (practically meaning that this capacitance is much larger than both C_1 and C_2), Eq. (****) yields

$$C \to \frac{C''}{2}, \qquad \text{where } C'' \equiv C_1 + C_2 .$$

This is also natural, because (as the last figure above shows) a very large capacitance C_0 makes the potential difference between two internal nodes of the circuit negligible, so it becomes just a connection *in series* of two similar groups of capacitors C_1 and C_2, connected *in parallel* within each group.

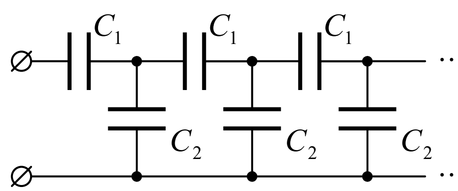
Finally, if one of the branch capacitances (say, C_2) tends to 0, Eq. (****) is reduced to

$$C = \frac{C_1 C_0}{C_1 + 2C_0}, \qquad \text{i.e. } \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_0} + \frac{1}{C_1} .$$

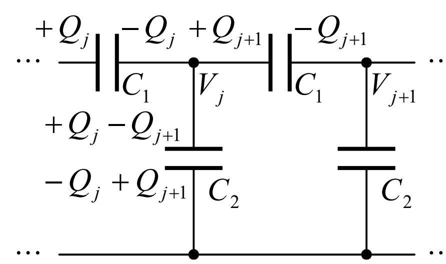


This result becomes clear if we redraw our circuit for this particular case, by replacing both zero capacitors C_2 with open circuits. The resulting system, shown in the figure on the right, is just a connection in series of 3 capacitors: C_1, C_0, and C_1 again, correctly described by the last formula for C. (The result for $C_1 = 0$ is similar, with the obvious index replacement.)

So, the circuit under analysis, despite its relatively simple topology,[28] becomes either the capacitors' connection in series or their connection in parallel in different ultimate cases, but generally it is neither.



Problem 2.4. Calculate the capacitance between the terminals of the semi-infinite lumped-capacitor chain shown in the figure on the right, and find the law of the applied voltage's decay along the system. Analyze and interpret the result.



Solution: Since all links of the chain are similar, it is sufficient to analyze just a couple of them, say with numbers j and $(j + 1)$, where j is the "distance" from the input terminals. Assigning the electric charges to the capacitor plates, with the island and capacitor neutrality rules discussed in detail in the previous problem's solution, we arrive at the charge and voltage distribution pattern shown in the figure on the right. As it shows, we may describe the circuit by Eq. (2.26) spelled out for two different capacitors of the system:

$$V_j - V_{j+1} = \frac{Q_{j+1}}{C_1}, \qquad Q_j - Q_{j+1} = C_2 V_j . \qquad (*)$$

Due to the linearity of these relations, we may expect both variables V and Q to decay exponentially with the "distance" j from the system's terminals:

[28] With a galvanometer instead of the capacitor C_0, such circuits (of either capacitors or resistors) are broadly used for sensitive differential measurements and have deserved a special name, the *Wheatstone bridges*.

$$V_j \propto Q_j \propto e^{-\lambda j}, \qquad \text{so that } \frac{V_{j+1}}{V_j} = \frac{Q_{j+1}}{Q_j} = e^{-\lambda} \equiv \alpha < 1.$$

Indeed, using the last relations to eliminate V_{j+1} and Q_{j+1} from Eqs. (*), we get a system of two linear homogenous equations for the same two variables V_j and Q_j:

$$(1-\alpha)V_j = \frac{\alpha Q_j}{C_1}, \qquad (1-\alpha)Q_j = C_2 V_j. \tag{**}$$

These equations are compatible (and hence the assumed solution is valid for all j) if the determinant of this system equals zero:

$$\begin{vmatrix} 1-\alpha & -\alpha / C_1 \\ -C_2 & 1-\alpha \end{vmatrix} = 0, \qquad \text{i.e. if } \alpha^2 - \left(2 + \frac{C_2}{C_1}\right)\alpha + 1 = 0.$$

The needed root $\alpha < 1$ of this quadratic equation[29] is

$$\alpha = 1 + \frac{C_2}{2C_1} - \left(\frac{C_2}{C_1} + \frac{C_2^2}{4C_1^2}\right)^{1/2}. \tag{***}$$

Plugging it back into any of Eqs. (**), for example the first one, we get the sought-for capacitance between the input terminals:

$$C \equiv \frac{Q_1}{V_0} = \frac{Q_{j+1}}{V_j} = \frac{\alpha Q_j}{V_j} = C_1(1-\alpha) = \left(C_1 C_2 + \frac{C_2^2}{4}\right)^{1/2} - \frac{C_2}{2}. \tag{****}$$

Let us consider what our results (***) and (****) yield in two ultimate cases. If $C_1/C_2 \to 0$, then

$$\alpha \to \frac{C_2}{2C_1} >> 1, \qquad C \to C_1.$$

This result means that the voltage between the input terminals does not propagate noticeably beyond the first capacitor C_1 of the chain, because the next, relatively large capacitance C_2 effectively "grounds" the voltage.

In the opposite limit, $C_1/C_2 \to \infty$, Eq. (***) yields

$$\alpha \to 1 - \left(\frac{C_2}{C_1}\right)^{1/2} \to 1, \qquad \text{so that } \lambda \equiv \ln\frac{1}{\alpha} \to \left(\frac{C_2}{C_1}\right)^{1/2} << 1.$$

According to the definition of λ, this means that the voltage applied to the chain's terminals penetrates deep into the chain, decaying at the "distance" $N \equiv 1/\lambda \approx (C_1/C_2)^{1/2} >> 1$. For the capacitance between the terminals, in this limit, Eq. (****) gives

$$C \to (C_1 C_2)^{1/2} \equiv N C_2.$$

The last form of this result may be interpreted by saying that the input voltage penetrating by "distance" $N >> 1$ into the chain "sees" N capacitors C_2 which are effectively connected in parallel.

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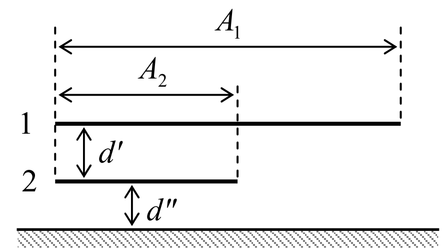
29 The second root $\alpha' = 1/\alpha > 1$ of the equation describes an exponential growth of V_j and Q_j with j, and could be relevant if our chain had a finite length.



Problem 2.5. A system of two thin conducting plates is located over a ground plane as shown in the figure on the right, where A_1 and A_2 are the areas of the indicated plate parts, while d' and d'' are the distances between them. Neglecting the fringe effects, calculate:

(i) the effective capacitance of each plate, and
(ii) their mutual capacitance.

Solutions: Let the plates have potentials ϕ_1 and ϕ_2 (relative to the ground). They create uniform vertical electric fields in each of the three regions of the system. Integrating Eq. (1.33) of the lecture notes, we may readily find that the field in the gap (of thickness d' and area A_2) between the plates is $E' = (\phi_1 - \phi_2)/d'$, that in the gap (of the same area A_2, but thickness d'') between plate 2 and the ground is $E'' = \phi_2/d''$, and, finally, in the clear gap (of thickness $d' + d''$ and area $A_1 - A_2$) between plate 1 and the ground, the field is $E_0 = \phi_1/(d_1 + d_2)$. Now applying the Gauss law (1.16) to a flat pillbox drawn around each plate,[30] we may calculate their total electric charges:

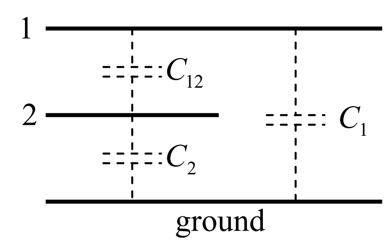
$$\begin{aligned} Q_1 &= \varepsilon_0\left[(A_1 - A_2)E_0 + A_2E'\right] = \varepsilon_0\left[\frac{A_1 - A_2}{d' + d''}\phi_1 + \frac{A_2}{d'}(\phi_1 - \phi_2)\right] \equiv \varepsilon_0\left(\frac{A_1 - A_2}{d' + d''} + \frac{A_2}{d'}\right)\phi_1 - \varepsilon_0\frac{A_2}{d'}\phi_2, \\ Q_2 &= \varepsilon_0 A_2(E'' - E') = \varepsilon_0 A_2\left(\frac{\phi_2}{d''} - \frac{\phi_1 - \phi_2}{d'}\right) \equiv -\varepsilon_0\frac{A_2}{d'}\phi_1 + \varepsilon_0 A_2\left(\frac{1}{d'} + \frac{1}{d''}\right)\phi_2 . \end{aligned} \tag{*}$$

These relations enable us to answer both posed questions.

(i) To calculate the effective capacitance of each plate, we should find its charge when the counterpart plate is grounded.[31] From Eqs. (*) we readily get

$$Q_1\Big|_{\phi_2=0} = \varepsilon_0\left(\frac{A_1 - A_2}{d' + d''} + \frac{A_2}{d'}\right)\phi_1, \qquad \text{i.e. } C_1^{\text{ef}} = \varepsilon_0\left(\frac{A_1 - A_2}{d' + d''} + \frac{A_2}{d'}\right), \tag{**}$$

$$Q_2\Big|_{\phi_1=0} = \varepsilon_0 A_2\left(\frac{1}{d'} + \frac{1}{d''}\right)\phi_2, \qquad \text{i.e. } C_2^{\text{ef}} = \varepsilon_0 A_2\left(\frac{1}{d'} + \frac{1}{d''}\right). \tag{***}$$



The first of these results, Eq. (**), may be readily understood, in the language of the equivalent circuit shown in the figure on the right, as the parallel connection of the "direct" capacitance,

$$C_1 \equiv \varepsilon_0\frac{A_1 - A_2}{d' + d''},$$

of the clear gap between plate 1 and the ground plane, with the direct capacitance between the plates:

[30] Applying the Gauss law to plate 1, we should remember that the field above it has to be vanishingly small because any appreciable field in that region (of volume $\sim A_1^{3/2}$) would be of prohibitive energy cost. Another explanation of the same fact is that all charges of plate 1 are strongly attracted to their opposite-sign images in the ground plane (see Sec. 2.6 of the lecture notes), so may reside only on the lower surface of the plate.

[31] The effective capacitances may be also calculated from the reciprocal capacitance matrix $p_{jj'}$ by using Eq. (2.33) and its counterpart for the second plate, but in our simple case, this approach is less straightforward.

$$C_{12} \equiv \varepsilon_0 \frac{A_2}{d'} .$$

On the other hand, Eq. (***) describes the parallel connection of the same C_{12} with the direct capacitance,

$$C_2 \equiv \varepsilon_0 \frac{A_2}{d''} ,$$

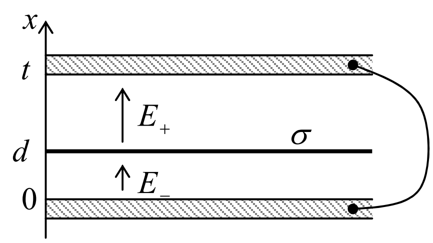
between plate 2 and the ground.

(ii) To calculate the mutual capacitance between the plates, we have to take $Q_2 = -Q_1 \equiv Q$, and calculate $V \equiv \phi_2 - \phi_1$ from Eqs. (*). The result of this simple calculation may be represented in the form,

$$C \equiv \frac{Q}{V} = \varepsilon_0 \left[\frac{A_2}{d'} + \left(\frac{d' + d''}{A_2 - A_1} + \frac{d'}{A_2} \right)^{-1} \right] \equiv C_{12} + \left(\frac{1}{C_1} + \frac{1}{C_2} \right)^{-1} ,$$

and understood from the same equivalent circuit (see the figure above) as the *parallel* connection of C_{12} with the *serial* connection of C_1 and C_2.

These facts illustrate how convenient the language of equivalent circuits is. However, note again that this approach is valid only if the system under analysis may be partitioned into a set of independent field regions represented as lumped circuit elements – see Fig. 2.5 of the lecture notes and its discussion.



Problem 2.6. A wide and thin film carrying a uniformly distributed electric charge of areal density σ is placed inside a similarly wide plane capacitor, whose plates are connected with a wire – see the figure on the right. Neglecting the fringe effects, calculate the surface charges of the plates and the net force exerted on the film (per unit area).

Solution: Let us pursue the most methodical (but not the shortest) way to solve the problem: solving the Poisson equation (1.41) for the electrostatic potential ϕ. Since, due to the problem's symmetry, ϕ may depend on just one Cartesian coordinate (say, x), normal to the planes of the electrode surfaces and the thin film, the equation is reduced to

$$\frac{d^2\phi}{dx^2} = -\frac{\sigma}{\varepsilon_0} \delta(x - d) . \tag{*}$$

Here the origin of x is taken at the bottom plate's surface, so at the top plate's surface, $x = t$ – see the figure above. Since the plates are connected with a conducting wire, in statics, their potentials are equal. Taking this value for zero, we may write the boundary conditions for the function $\phi(x)$ as

$$\phi(0) = \phi(t) = 0 . \tag{**}$$

The solution of this boundary problem is straightforward. Indeed, per Eq. (*), in each of the two gaps between the film and the plates, $d^2\phi/d^2x = 0$, so ϕ has to be a linear function of x:

$$\phi(x) = \begin{cases} c_- - E_- x, & \text{at } 0 < x < d, \\ c_+ - E_+ x, & \text{at } d < x < t, \end{cases}$$

where $E_\pm$ are the corresponding values of the electric field, $E = -d\phi/dx$ – see the figure above. Integrating Eq. (*) over an infinitesimal interval $[d - 0, d + 0]$, we get

$$\left.\frac{d\phi}{dx}\right|_{x=d+0} - \left.\frac{d\phi}{dx}\right|_{x=d-0} \equiv -E_+ + E_- = -\frac{\sigma}{\varepsilon_0}.$$

The boundary conditions (**) give two more equations for the constants $c_\pm$ and $E_\pm$:

$$c_- = 0, \qquad c_+ - E_+ t = 0 .$$

Finally, the potential has to be continuous, so both expressions for $\phi(x)$ have to give the same value at the boundary $x = d$:

$$\phi(d) = c_- - E_- d = c_+ - E_+ d . \qquad (***)$$

Solving this simple system of four equations, we get[32]

$$E_- = -\frac{\sigma}{\varepsilon_0}\frac{t-d}{t}, \qquad E_+ = \frac{\sigma}{\varepsilon_0}\frac{d}{t}. \qquad (****)$$

Due to the fundamental relation (2.3), these formulas immediately yield the surface charges of the capacitor's plates:

$$\sigma_{\text{bottom plate}} = \varepsilon_0 E_- = -\sigma\frac{t-d}{t}, \qquad \sigma_{\text{top plate}} = -\varepsilon_0 E_+ = -\sigma\frac{d}{t}.$$

(As a sanity check, $\sigma_{\text{bottom plate}} + \sigma_{\text{top plate}} + \sigma = 0$, as it should be.[33])

Next, plugging Eqs. (****) for the fields $E_\pm$ into Eq. (1.65) of the lecture notes, we may calculate the electrostatic potential energy of the system (per unit area):

$$\frac{U}{A} = \frac{\varepsilon_0}{2}\left[E_+^2(t-d) + E_-^2 d\right] = \frac{\sigma^2}{2\varepsilon_0}\left[\left(\frac{d}{t}\right)^2(t-d) + \left(\frac{t-d}{t}\right)^2 d\right] \equiv \frac{\sigma^2}{2\varepsilon_0}\frac{(t-d)d}{t}.$$

Now the force exerted on the film (also per unit area) may be found just as the (minus) gradient of its potential energy – see, e.g., Eq. (1.32). In our geometry, the only nonvanishing component of the gradient is vertical, so $\mathbf{F} = F\mathbf{n}_x$, with[34]

$$\frac{F}{A} = -\frac{\partial(U/A)}{\partial d} = \frac{\sigma^2}{2\varepsilon_0}\frac{t-2d}{t}.$$

This result shows that the film is attracted to the nearest plate, so at the middle between the plates, at $d = t/2$, the net force vanishes, i.e. the film is in equilibrium. Since at this point (and actually at any position d)

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32 Alternatively (and easier), these relations for the fields $E_\pm$ may be obtained by applying the Gauss theorem to three pillboxes with various positions of their lids – see the derivation of Eq. (1.23) in the lecture notes. This simple additional exercise is highly recommended to the reader.

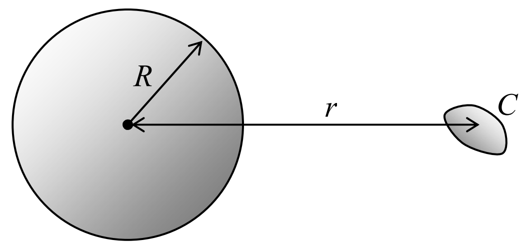
33 Even if the net electric charge of the capacitor's plates is different from zero, it resides on their exterior surfaces – cf. the solution of Problem 1.

34 A shorter but less obvious way to obtain the same result is to say that the effective electric field E_{ef} acting on the film (so that $F/A = E_{\text{ef}}\sigma$) is equal to the average of the fields on its two sides: $E_{\text{ef}} = (E_+ + E_-)/2$.

$$\frac{\partial^2 U}{\partial d^2} = -\frac{\sigma^2}{\varepsilon_0 t} < 0,$$

this equilibrium is unstable.[35]

Problem 2.7. A very small conductor (possibly, of an irregular shape) with self-capacitance C is located at distance r from the center of a conducting sphere of radius R – see the figure on the right. In the first approximation in C, find the reciprocal capacitance matrix of the system. Use the matrix to calculate its potential energy and the force of the conductor interaction for two cases:



(i) the conductor *charges* Q are equal, and

(ii) the conductor *potentials* ϕ are kept equal.

Solution: Marking the variables describing the sphere with index "s", and those describing the small conductor with index "c", we may rewrite the definition of the reciprocal capacitance matrix given by Eq. (2.19) of the lecture notes, with an account of Eq. (2.20), as

$$\begin{aligned} \phi_\mathrm{s} &= p_\mathrm{s} Q_\mathrm{s} + p Q_\mathrm{c}, \\ \phi_\mathrm{c} &= p Q_\mathrm{s} + p_\mathrm{c} Q_\mathrm{c}. \end{aligned} \qquad (*)$$

Using the same argumentation as at the derivation of Eq. (2.29), we may argue that the relatively weak interaction of these two conductors, described by the off-diagonal matrix element p, cannot affect its diagonal elements p_s and p_c substantially, so we may write

$$p_\mathrm{s} \approx \frac{1}{C_\mathrm{s}} = \frac{1}{4\pi\varepsilon_0 R}, \qquad p_\mathrm{c} \approx \frac{1}{C} \equiv \frac{1}{4\pi\varepsilon_0 a},$$

where Eq. (2.17) and the estimate (2.18) are used to define the small conductor's size parameter a, with $a << r, R$. Also, as was argued in Sec. 2.2, if the small conductor is electroneutral ($Q_\mathrm{c} = 0$), its potential has to be approximately equal to that created at its location by other charges of the system; in our case, according to Eq. (1.35), this means that

$$\phi_\mathrm{c}\Big|_{Q_\mathrm{c}=0} \equiv p Q_\mathrm{s} \approx \frac{Q_\mathrm{s}}{4\pi\varepsilon_0 r}, \quad \text{i.e.} \quad p \approx \frac{1}{4\pi\varepsilon_0 r}.$$

Now that we have the reciprocal capacitance matrix established, we may readily answer all posed questions.

(i) If $Q_\mathrm{s} = Q_\mathrm{c} = Q$, then Eq. (2.24) yields the following potential energy of the system:

$$U = Q^2\left(\frac{p_\mathrm{s}}{2} + p + \frac{p_\mathrm{c}}{2}\right) \approx \frac{Q^2}{4\pi\varepsilon_0}\left(\frac{1}{2R} + \frac{1}{r} + \frac{1}{2a}\right).$$

[35] If the last point is not absolutely clear, please revisit the discussion of the fixed point stability in CM Sec. 3.2.

Of the three terms in the parentheses, only the middle one depends on the distance r between the sphere and the small conductor, and hence determines the force of their interaction (repulsion):

$$F = -\frac{\partial U}{\partial r} \approx \frac{Q^2}{4\pi\varepsilon_0 r^2}.$$

So, quite naturally, in our limit $a << r$, R, the conductors interact as point charges.

(ii) If $\phi_\text{s} = \phi_\text{c} = \phi$, then Eq. (*), together with the matrix elements listed above, gives the following simple system of two linear equations for the two charges of the conductors:

$$\phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_s}{R} + \frac{Q_\text{c}}{r}\right), \qquad \phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_\text{s}}{r} + \frac{Q_\text{c}}{a}\right).$$

Solving it, we get

$$Q_s = 4\pi\varepsilon_0\phi\frac{Rr(r-a)}{r^2 - Ra}, \qquad Q_\text{c} = 4\pi\varepsilon_0\phi\frac{ar(r-R)}{r^2 - Ra}, \tag{**}$$

From here, we can immediately calculate the repulsion force:

$$|F| = \frac{Q_\text{s}Q_\text{c}}{4\pi\varepsilon_0 r^2} = 4\pi\varepsilon_0\phi^2\frac{Rar(r-a)(r-R)}{\left(r^2 - Ra\right)^2} \approx 4\pi\varepsilon_0\phi^2\frac{Ra}{r^2}\left(1 - \frac{R}{r}\right). \tag{***}$$

where the last approximation is valid if a is smaller than not only r and R but also than r^2/R. [36]

With the potential energy, we have to be more careful than in Task (i), because Eq. (2.24) is only valid for two independently-fixed charges which, in particular, would not depend on the distance r, as they do for our system – see Eq. (**) again. The simplest way to calculate U for our current case of equal potentials is to use the already found charges Q_s and Q_c to calculate the self-capacitance of our total system, keeping only the terms of the 0th and 1st order in small a:

$$C_\Sigma \equiv \frac{Q_\text{s} + Q_\text{c}}{\phi} = 4\pi\varepsilon_0\frac{Rr(r-a) + ar(r-R)}{r^2 - Ra} \approx 4\pi\varepsilon_0\left[R + a\left(1 - \frac{R}{r}\right)^2\right],$$

and then use the last of Eqs. (2.15) to find the energy:

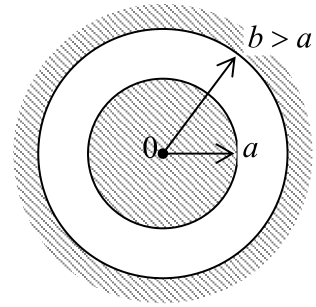
$$U \equiv \frac{C_\Sigma}{2}\phi^2 \approx 2\pi\varepsilon_0\phi^2\left[R + a\left(1 - \frac{R}{r}\right)^2\right].$$

With our assumptions $a << r$, R, r^2/R, the second term in the square brackets is much smaller than the first one, and for some purposes may be neglected, but since only this term depends on the distance r between the sphere and the small conductor, we need to keep it if we want to use this expression for the sanity check of the above result for the repulsion force:

$$F = -\frac{\partial U}{\partial r} = 2\pi\varepsilon_0\phi^2 a\frac{\partial}{\partial r}\frac{(r-R)^2}{r^2} \equiv 2\pi\varepsilon_0\phi^2 a\frac{\partial}{\partial r}\left(1 - 2\frac{R}{r} + \frac{R^2}{r^2}\right) = 2\pi\varepsilon_0\phi^2 a \times \frac{2R}{r^2}\left(1 - \frac{R}{r}\right),$$

i.e. exactly Eq. (***).

[36] It is curious that the force is not a monotonic function of the distance r, but has a maximum at $r = (3/2)R$.



Problem 2.8. Use the Gauss law to calculate the mutual capacitance of the following two-electrode systems, both with the cross-section shown in Fig. 2.7 of the lecture notes (reproduced on the right):

(i) a conducting sphere in the center of a spherical cavity inside another conductor, and

(ii) a long conducting round cylinder on the axis of a cylindrical cavity inside another conductor, i.e. a coaxial cable. (In this case, we speak about the capacitance per unit length).

Compare the results with those obtained in Sec. 2.2 using the Laplace equation.

Solutions:

(i) Applying the Gauss Law to a sphere of a radius r in the range $a < r < b$, we get the following result for the magnitude of the electric field $\mathbf{E} = E(r)\mathbf{n}_r$:

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2},$$

where Q is the total charge of the inner conductor. Integrating this result along the radius, we find that the voltage $V \equiv \phi(b) - \phi(a)$ between the electrodes is

$$V \equiv \int_a^b E(r)dr = \frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right),$$

so the mutual capacitance

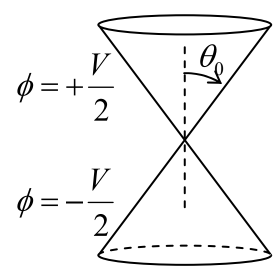
$$C \equiv \frac{Q}{V} = 4\pi\varepsilon_0\left(\frac{1}{a} - \frac{1}{b}\right)^{-1} = 4\pi\varepsilon_0\frac{ab}{b-a},$$

in full agreement with Eq. (2.56) of the lecture notes.

(ii) An absolutely similar calculation, but applied to a cylinder of radius ρ, with charge $\lambda \equiv Q/l$ per unit length, yields

$$E(r) = \frac{\lambda}{2\pi\varepsilon_0\rho}, \qquad \frac{C}{l} = \frac{2\pi\varepsilon_0}{\ln(b/a)},$$

in full agreement with Eq. (2.49).



Problem 2.9. Calculate the electrostatic potential's distribution around two barely separated conductors in the form of coaxial round cones (see the figure on the right), with voltage V applied between them. Compare the result with that of a similar 2D problem, with the cones replaced with plane-face wedges. Can you calculate the mutual capacitances between the conductors in these systems? If not, can you estimate them?

Solution: In the spherical coordinates $\{r, \theta, \varphi\}$, with the z-axis directed along the common symmetry axis of the cones, the conductors' surfaces are described by the relations $\theta = \theta_0$ and $\theta = \pi - \theta_0$, so the values $\pm V/2$ of the electrostatic potential at these surfaces do not depend on the coordinates r and φ. So, it is natural to look for the solution of this boundary problem in the form $\phi = \phi(\theta)$. In this case, the Laplace equation is reduced to[37]

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\phi}{d\theta}\right)=0, \qquad \text{for } 0<\theta<\pi .$$

Integrating this equation sequentially twice, we get

$$\sin\theta\frac{d\phi}{d\theta}=c_1, \qquad \text{i.e. } d\phi=c_1\frac{d\theta}{\sin\theta}, \qquad \text{so } \phi(\theta)=c_1\int\frac{d\theta}{\sin\theta}\equiv c_1\int\frac{\sin\theta d\theta}{\sin^2\theta}=-c_1\int\frac{d\xi}{1-\xi^2},$$

where $\xi \equiv \cos\theta$. The last integral is easy,[38] and we get

$$\phi(\theta)=\frac{c_1}{2}\ln\frac{1-\xi}{1+\xi}+c_2\equiv\frac{c_1}{2}\ln\frac{1-\cos\theta}{1+\cos\theta}+c_2\equiv c_1\ln\tan\frac{\theta}{2}+c_2 .$$

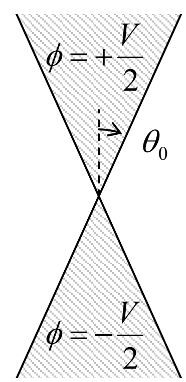
With the proper choice of the two integration constants $c_{1,2}$, we may indeed satisfy the boundary conditions on the surfaces of both conductors:

$$\phi(\theta_0)\equiv c_1\ln\tan\frac{\theta_0}{2}+c_2=+\frac{V}{2}, \qquad \phi(\pi-\theta_0)\equiv c_1\ln\tan\frac{\pi-\theta_0}{2}+c_2\equiv -c_1\ln\tan\frac{\theta_0}{2}+c_2=-\frac{V}{2}.$$

Solving this system of two equations for the constants $c_{1,2}$, we finally get the potential distribution:

$$\phi(\theta)\equiv\frac{V}{2}\frac{\ln\tan(\theta/2)}{\ln\tan(\theta_0/2)}, \qquad \text{for } \theta_0\le\theta\le\pi-\theta_0 . \tag{*}$$

This is *a* solution, and hence *the* (unique) solution of this boundary problem.



A similar 2D problem, on the potential distribution between two conducting cylindrical wedges, with the cross-section shown in the figure on the right, may be solved using the polar coordinates. Using the same argumentation as in the above cone problem, we may expect the electrostatic potential to be the function of the polar angle alone. (For the purposes of comparison with the just solved cone problem, I will denote this angle, referred to the polar axis, by the same letter θ.) Then the 2D Laplace equation is reduced to a very simple form,[39]

$$\frac{d^2\phi}{d\theta^2}=0,$$

with the linear solution

$$\phi(\theta)=c_1\theta+c_2 .$$

[37] See, e.g., MA Eq. (10.9) with $\partial/\partial r = \partial/\partial\varphi = 0$.

[38] As a math reminder, it may be worked out by representing the function under the integral as the sum of two simple singularities ("single poles"):

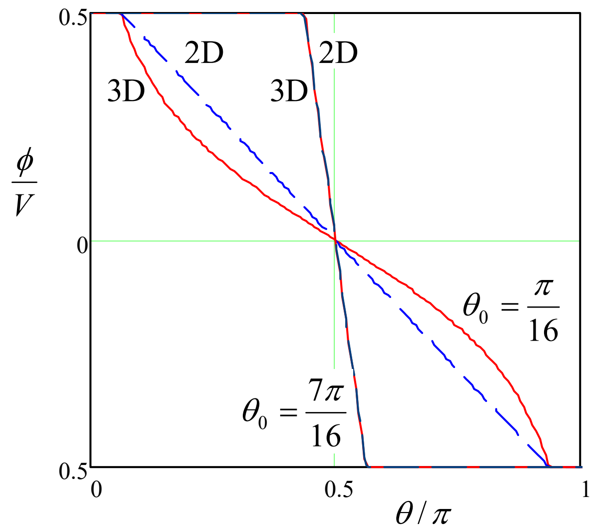
$$\frac{1}{1-\xi^2}\equiv\frac{1}{2(1-\xi)}+\frac{1}{2(1+\xi)}.$$

[39] See, e.g., MA Eq. (10.3) with $\partial/\partial\rho = \partial/\partial z = 0$, and $\varphi\to\theta$.

Selecting the constants $c_{1,2}$ to satisfy the same boundary conditions, $\phi(\theta_0) = +V/2$, $\phi(\pi - \theta_0) = -V/2$, we get

$$\phi(\theta) \equiv \frac{V}{2}\frac{\pi - 2\theta}{\pi - 2\theta_0}, \qquad \text{for } \theta_0 \leq \theta \leq \pi - \theta_0 . \qquad (**)$$

The plots of the 3D function (*) and the 2D function (**), each for the same two representative values of the angle θ_0, are shown in the figure on the right. They show that the 3D function's gradient (and hence the electric field) is more localized near the conducting electrodes, though for larger values of θ_0, approaching $\pi/2$, the difference is insignificant, i.e. the 3D function $\phi(\theta)$ is also virtually linear.



Finally, to calculate the mutual capacitance between the electrodes (for any dimensionality), we need to calculate the surface charge density on, say, the top electrode:

$$\sigma = \varepsilon_0 E_n \Big|_{\theta=\theta_0} = -\varepsilon_0 \frac{\partial \phi}{\partial n}\Big|_{\theta=\theta_0},$$

where n is the linear coordinate normal to the surface. In both the 3D case and the 2D case, this coordinate is proportional to the distance from the system's center, so σ is inversely proportional to this distance.[40] For example, in the 3D case of two conducting cones,[41]

$$\sigma = \frac{c}{r}, \qquad \text{with } c \equiv -\varepsilon_0 \frac{d\phi(\theta)}{d\theta}\Big|_{\theta=\theta_0}.$$

Even without calculating the coefficient c, we see that the integral determining the total charge Q of the upper cone,

$$Q = \int_S \sigma d^2 r = 2\pi \int_0^\infty \sigma r dr = 2\pi c \int_0^\infty dr,$$

strongly diverges at the upper limit. Physically, this means that to calculate a finite value of the system's mutual capacitance $C = Q/V$, we need to specify how exactly their spatial extent is limited.

The 2D system (of two conducting wedges) features a similar divergence not only at large but also at small distances, though in both cases, rather mildly. Indeed, for it:[42]

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[40] Note that the $1/r$-type divergence of the electric field and the surface charge density near the systems' centers is quantitatively different from the one discussed in Sec. 2.6 of the lecture notes for a single conducting corner/wedge – see Eq. (2.123). The difference is due to the fact that in the latter case, the field was due to some remote sources, rather than the oppositely biased counterpart wedge.

[41] See, e.g., MA Eq. (10.8) with $\partial/\partial r = \partial/\partial\varphi = 0$.

[42] See, e.g., MA Eq. (10.2) with $\partial/\partial\rho = \partial/\partial z = 0$ and $\varphi \to \theta$.

$$\sigma = \frac{c}{\rho}, \qquad \text{with } c \equiv -\varepsilon_0 \frac{d\phi(\theta)}{d\theta}\bigg|_{\theta=\theta_0} = \varepsilon_0 \frac{V}{\pi - 2\theta_0},$$

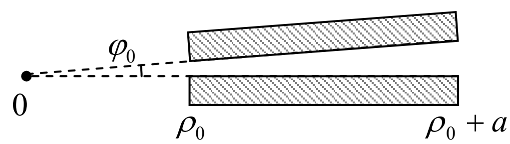
where ρ is the 2D radius in the cross-section's plane, so the charge per unit length of the system,

$$\frac{Q}{l} = \frac{1}{l}\int_S \sigma d^2 r = 2\int_0^\infty \sigma d\rho = 2c\int_0^\infty \frac{d\rho}{\rho} = \frac{2\varepsilon_0 V}{\pi - 2\theta_0}\int_0^\infty \frac{d\rho}{\rho} \to \infty .$$

So, the calculation of a finite mutual capacitance between the wedges requires a specification of not only their exact shape at large distances but also of the exact geometry of the gap between them. However, due to the weak (logarithmic) character of this singularity, a fair estimate of the capacitance may be given as

$$\frac{C}{l} \approx \frac{2\varepsilon_0}{\pi - 2\theta_0} \ln\frac{\rho_{\max}}{\rho_{\min}},$$

where the two values of ρ are, respectively, the scales of the wedges' size and of the gap's width.



Problem 2.10. Calculate the mutual capacitance between two rectangular planar electrodes of area $A = a \times l$, with a very small angle φ_0 between them – see the figure on the right.

Solution: As was discussed in Sec. 2.6 of the lecture notes, the variable separation in polar coordinates shows that the Laplace equation has a partial solution expressed by the second of Eqs. (2.110):

$$\phi = c_0 + s_0 \varphi .$$

If the polar coordinates' center coincides with the point 0 in which inner surface planes cross (see the figure above), i.e. if the surfaces correspond to fixed values of the angle φ (which differ from each other by φ_0), the boundary conditions on these surfaces are also satisfied by this solution, provided that $s_0\varphi_0 = V$, where V is the voltage between the electrodes, so the following potential distribution,

$$\phi = V\frac{\varphi}{\varphi_0} + \text{const}, \qquad (*)$$

is the solution of our boundary problem, provided that

$$(\rho_0 + a)\varphi_0 << a, l,$$

so that the fringe effects are minor, and boundary conditions on outer surfaces are not important.

Now the inner surface charge densities may be calculated from the basic Eq. (2.3):

$$\sigma = -\varepsilon_0 \frac{\partial \phi}{\partial n}.$$

At the surfaces of constant φ, $\partial/\partial n = \mp(1/\rho)\partial/\partial\varphi$, where ρ is the distance from the point 0, and the upper/lower sign corresponds to the upper/lower plate, so

$$\sigma_{\pm} = \pm\varepsilon_0 \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi} = \pm\varepsilon_0 \frac{1}{\rho}\frac{V}{\varphi_0} \ .$$

Integrating these densities over the plate surfaces, we get their full charges:

$$Q_{\pm} = \int \sigma_{\pm} d^2 r = l \int_{\rho_0}^{\rho_0+a} \sigma_{\pm} d\rho = \pm\varepsilon_0 \frac{V}{\varphi_0} l \int_{\rho_0}^{\rho_0+a} \frac{d\rho}{\rho} = \pm\varepsilon_0 \frac{V}{\varphi_0} l \ln\frac{\rho_0 + a}{\rho_0},$$

so the mutual capacitance

$$C \equiv \frac{Q_+}{V} = \varepsilon_0 \frac{l}{\varphi_0} \ln\frac{\rho_0 + a}{\rho_0} \equiv \varepsilon_0 \frac{l}{\varphi_0} \ln\left(1 + \frac{a}{\rho_0}\right). \qquad (**)$$

In the limit of a very short capacitor, with $a << \rho_0$ (but still $\varphi_0 << a/\rho_0$, l/ρ_0), whose plates are essentially parallel, this expression may be approximated as

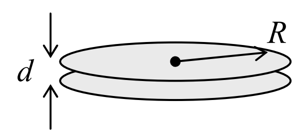
$$C \approx \varepsilon_0 \frac{l}{\varphi_0}\frac{a}{\rho_0} = \varepsilon_0 \frac{A}{d},$$

where $d \equiv \rho_0\varphi_0$ is the distance between the plates, thus reducing our result to the familiar Eq. (2.28).

On the other hand, at $\rho_0 << a$, the potential distribution (*) and hence Eq. (**), in the form

$$C \approx \varepsilon_0 \frac{l}{\varphi_0} \ln\left(\frac{a}{\rho_0}\right),$$

are valid for an arbitrary opening angle φ_0 – cf. the solution of the previous problem for the 2D system.



Problem 2.11. Using the results for a single thin round disk, obtained in Sec. 2.4 of the lecture notes, consider a system of two such disks at a small distance $d << R$ from each other – see the figure on the right. In particular, calculate:

(i) the reciprocal capacitance matrix of the system,
(ii) the mutual capacitance between the disks,
(iii) the partial capacitance of one disk, and
(iv) the effective capacitance of one disk

– all in the first nonvanishing approximation in $d/R << 1$. Compare the results (ii)-(iv) and interpret their similarities and differences.

Solutions:

(i) Due to the symmetry of this system with respect to the disk swap and the common property expressed by Eq. (2.20) of the lecture notes, the general linear relations (2.19) are reduced to

$$\begin{aligned}\phi_1 &= p' Q_1 + p Q_2, \\ \phi_2 &= p Q_1 + p' Q_2 .\end{aligned} \qquad (*)$$

At $d << R$, approximate expressions for the remaining two coefficients $\mathscr{p}'$ and $\mathscr{p}$ may be found from the two particular problems already solved in the lecture notes. Indeed, if both disks carry equal charges of the same sign: $Q_1 = Q_2 = Q/2$, then there is virtually no field inside the gap separating them, while the field outside is virtually the same as that of a single disk charged with charge Q. According to Eq. (2.69), in this case

$$\phi_1 = \phi_2 = \frac{Q}{8\varepsilon_0 R}.$$

On the other hand, in this case, Eqs. (*) give

$$\phi_1 = \phi_2 = \frac{\mathscr{p} + \mathscr{p}'}{2} Q.$$

Comparing these two expressions, we get

$$\mathscr{p} + \mathscr{p}' = \frac{1}{4\varepsilon_0 R}. \qquad (**)$$

On the other hand, if the disk charges are equal but opposite, say $Q_1 = -Q_2 = Q$, then the system is nothing more than a plane capacitor, with a uniform field inside the gap and a negligible field outside it, so according to Eq. (2.28),

$$\phi_1 = -\phi_2 = \frac{V}{2} = \frac{Q}{2C} = \frac{Qd}{2\varepsilon_0 A} = \frac{Qd}{2\pi\varepsilon_0 R^2}.$$

Comparing this result with the prediction of Eqs. (*) for this case,

$$\phi_1 = -\phi_2 = (\mathscr{p}' - \mathscr{p})Q,$$

we get one more equation for $\mathscr{p}$ and $\mathscr{p}'$:

$$\mathscr{p}' - \mathscr{p} = \frac{d}{2\pi\varepsilon_0 R^2} << \mathscr{p} + \mathscr{p}'. \qquad (***)$$

Solving the system of two linear equations (**) and (***), we get

$$\mathscr{p}' = \frac{1}{8\varepsilon_0 R} + \frac{d}{4\pi\varepsilon_0 R^2} \equiv \frac{1}{8\varepsilon_0 R}\left(1 + \frac{2d}{\pi R}\right), \qquad \mathscr{p} = \frac{1}{8\varepsilon_0 R} - \frac{d}{4\pi\varepsilon_0 R^2} \equiv \frac{1}{8\varepsilon_0 R}\left(1 - \frac{2d}{\pi R}\right).$$

(ii) With these expressions, the *mutual* capacitance, defined by Eq. (2.26) with $\mathscr{p}_1 = \mathscr{p}_2 = \mathscr{p}'$,

$$C \equiv \frac{1}{2(\mathscr{p}' - \mathscr{p})},$$

becomes

$$C = \frac{\pi\varepsilon_0 R^2}{d}$$

– admittedly, the expression already used above.

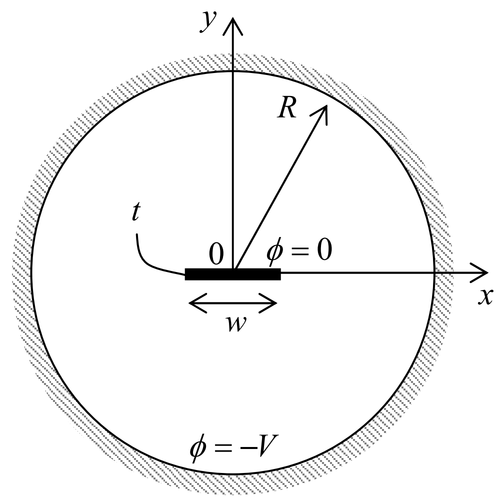
(iii) Both *partial* capacitances of the disks, generally defined as $C_1 = 1/\mathscr{p}_1$ and $C_2 = 1/\mathscr{p}_2$, in our symmetric case are equal to $1/\mathscr{p}'$, and in the first approximation, are independent of $d << R$:

$$C_1 = C_2 \approx 8\varepsilon_0 R.$$

(iv) On the contrary, the *effective* capacitances of the disks, which may be calculated from Eq. (2.34), are much larger and inversely proportional to the distance between the disks:

$$C_{1,2}^{ef} \approx C = \frac{\pi\varepsilon_0 R^2}{d} >> C_{1,2}.$$

This big difference between the partial and effective capacitance (typical for all strongly coupled systems of conductors) is very natural. If one disk is charged and the second one is not, the latter disk acquires almost the whole potential of the charged counterpart,[43] and virtually does not affect the field distribution in space. However, if the second disk is grounded (i.e. set to $\phi = 0$), it kills the electric field everywhere besides the narrow gap between the charged disk and the effective ground.



Problem 2.12.* Calculate the mutual capacitance (per unit length) between two cylindrical conductors forming a system with the cross-section shown in the figure on the right, in the limit $t << w << R$.

Hint: You may like to use the elliptic coordinates mentioned in Sec. 2.4 of the lecture notes. They are defined by the following equality:

$$x + iy = c\cosh(\mu + i\nu). \qquad (*)$$

where c is a constant.

Solution: Since the complex function $\boldsymbol{z} = c\cosh \boldsymbol{w}$ defining these orthogonal coordinates is analytic, the Laplace equation in them is simple – see Sec. 2.4 of the lecture notes:[44]

$$\left(\frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2}\right)\phi = 0. \qquad (**)$$

According to Eq. (*), the lines of constant μ on the [x, y] plane are ellipses with horizontal and vertical semi-axes $c\cosh\mu$ and $c\sinh\mu$, respectively. At $\mu \to 0$, the ellipse degenerates into a straight horizontal segment $-c < x < +c$, while at $\mu >> 1$, the ellipse is virtually a circle of the radius $\rho = (c/2)e^{\mu}$. As a result, if we select the axes x and y as shown in the figure above, and take $c = w/2$, the boundary conditions on the conductor surfaces may be satisfied, at $t << w << R$, by a potential distribution $\phi(\mu)$ independent of ν:

$$\phi(0) = 0, \qquad \phi\left(\ln\frac{4R}{w}\right) = -V, \qquad (***)$$

where V is the voltage between the conductors. Hence the boundary problem is satisfied with a function $\phi(\mu)$, provided that it obeys the 1D Laplace equation following from Eq. (**):

[43] Electronic engineers call such electrodes "floating".

[44] Just for the reader's reference, the Laplace operator in these coordinates is

$$\nabla^2 = \frac{1}{c^2(\cosh^2\mu - \cos^2\nu)}\left(\frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2}\right),$$

in compliance with Eq. (**).

$$\frac{d^2\phi}{d\mu^2} = 0.$$

Per this equation, $\phi\,(\mu)$ is just a linear function $c_1\mu + c_2$. Selecting the two constants $c_{1,2}$ to satisfy the boundary conditions (***), we get

$$\phi = -V\frac{\mu}{\ln(4R/w)}.$$

To calculate the surface charge of the central conductor by using the general Eq. (2.3) of the lecture notes, we need to calculate the (normal) electric field at its surface ($\mu \to 0$), where Eq. (*), with $c = w/2$, yields $x = (w/2)\cos\beta$ and $y \approx (w/2)\mu \sin\beta$:

$$E_n = -\left.\frac{\partial\phi}{\partial y}\right|_{\substack{y=+0,\\ -w/2<x<+w/2}} = \frac{V}{(w/2)\ln(4R/w)\sin\beta},$$

so the two-surface charge density

$$\sigma = 2\varepsilon_0 E_n = 2\varepsilon_0\frac{V}{(w/2)\ln(4R/w)\sin\beta} \equiv 2\varepsilon_0\frac{V}{(w/2)\ln(4R/w)\left[1-(2x/w)^2\right]^{1/2}}.$$

Integrating this distribution over the strip's width, we get its total charge (per unit length)

$$\frac{Q}{l} = \int_{-w/2}^{+w/2}\sigma dx = \frac{2\varepsilon_0 V}{(w/2)\ln(4R/w)}\int_{-w/2}^{+w/2}\frac{dx}{\left[1-(2x/w)^2\right]^{1/2}} = \frac{2\pi\varepsilon_0}{\ln(4R/w)}V,$$

so the mutual capacitance per unit length is

$$\frac{C}{l} \equiv \frac{Q/l}{V} = \frac{2\pi\varepsilon_0}{\ln(4R/w)}.$$

Comparing this result with Eq. (2.49), we see that the only difference between the capacitance of this system and that of the round coaxial cable with $a = R$ and $b = w$ is minor: just a different numerical factor of the order of 1 under the logarithm of a large argument.

Problem 2.13. Calculate the mutual capacitance (per unit length) between two similar, long, parallel wires, each with a round cross-section of radius R, whose axes are separated by distance $d > 2R$. Explore and interpret the result in the limits $R \to 0$ and $R \to 2d$.

Hint: You may like to use the 2D orthogonal *bipolar coordinates* $\{\tau, \sigma\}$ defined by the following relations with the Cartesian coordinates $\{x, y\}$:

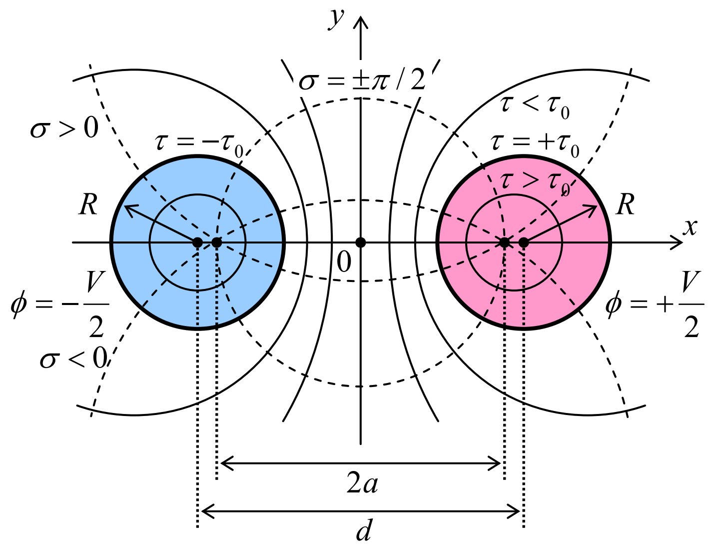
$$x = a\frac{\sinh\tau}{\cosh\tau - \cos\sigma}, \qquad y = a\frac{\sin\sigma}{\cosh\tau - \cos\sigma}, \qquad \text{with } -\infty < \tau < +\infty, \quad -\pi \le \sigma \le +\pi.$$

In these coordinates, the Laplace operator is

$$\nabla^2 = \frac{1}{a^2}\left(\cosh\tau - \cos\sigma\right)\left(\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\sigma^2}\right).$$

Solution: Using the given definition of the bipolar coordinates, it is elementary to prove that the lines of constant τ are circles of radii $\rho_\tau = a/\sinh\tau$, centered at the points with $x = x_\tau \equiv a\coth\tau$ and $y = 0$, and the lines of constant σ are also circles but with the radii $\rho_\sigma = a/\sin\sigma$, centered at the points with $x = 0$ and $y = y_\sigma = a\,\text{cotan}\,\sigma$ – see, respectively the solid and dashed lines in the figure below. All circles of constant σ pass through the *focal points* $x = \pm a$ and are orthogonal to the τ-circles at all points. As a result, if we use the coordinate system shown in this figure, and select the parameters a and τ_0 to satisfy the following relations:

$$R = \frac{a}{\sinh \tau_0}, \quad \frac{d}{2} = a \coth \tau_0,$$



then the two τ-circles with $\tau = \pm\tau_0$ coincide with the circular borders of the cross-sections of our conductors – see the colored circles in the same figure. Solving this simple system of equations, we get

$$a = \left[\left(\frac{d}{2}\right)^2 - R^2\right]^{1/2} \geq \frac{d}{2}, \quad \tau_0 = \cosh^{-1}\frac{d}{2R}.$$

With this choice, the boundary conditions for the electrostatic potential ϕ as a function of the bipolar coordinates τ and σ become very simple, especially if we assume that our system of two conductors is voltage-biased in the symmetric way shown in the figure above:

$$\phi(\pm\tau_0, \sigma) = \pm\frac{V}{2}, \qquad \text{for any } \sigma.$$

Since, per the expression for ∇^2 provided in the *Hint*, the Laplace equation in these coordinates is also very simple:[45]

$$\frac{\partial^2\phi}{\partial\tau^2} + \frac{\partial^2\phi}{\partial\sigma^2} = 0,$$

the solution of our boundary problem is obviously given by a linear function of τ alone, namely

$$\phi = V\frac{\tau}{2\tau_0}.$$

What remains is to calculate the electric charge (per unit length), of one (say, the right) wire, by using the basic Eq. (2.3):

$$\frac{Q}{l} = \varepsilon_0 \oint_R E_n dr, \qquad (*)$$

where the integral is over the boundary of the wire's cross-section, i.e. over the circle with $\tau = \tau_0 =$ const. Generally, this may be done directly, by expressing E_n as $-\partial\phi/\partial n$ and calculating the differential dn as a function of $d\tau$ on this circle, but that must be done very carefully because the change $d\tau$ affects

[45] This should be not too surprising, because the above relations between the bipolar and Cartesian coordinates may be represented by an analytic complex function: $\boldsymbol{z} \equiv ia\,\text{cotan}(\boldsymbol{w}/2)$, where $\boldsymbol{z} \equiv x + iy$ and $\boldsymbol{w} \equiv \sigma + i\tau$ – see the discussion of conformal mapping in Sec. 2.4 of the lecture notes.

not only ρ_τ but also x_τ. A simpler approach is to notice that the integral on the right-hand side of Eq. (*) is the full flux of the electric field through the wire's surface. Since the free-space region between this surface and the symmetry plane, $x = 0$ (on which $\tau = 0$ as well) has no electric charges, per the Gauss law, the incoming flux has to be equal to the outcoming flux through that plane. Hence we may write

$$\frac{Q}{l} = -\varepsilon_0 \int_{-\infty}^{+\infty} E_x\big|_{x=0} dy = \varepsilon_0 \int_{-\infty}^{+\infty} \frac{\partial \phi}{\partial x}\Big|_{x=0} dy .$$

The advantage of this expression over Eq. (*) is that at $x \to 0$ (and hence $\tau \to 0$), the relations between the bipolar and Cartesian coordinates simplify:

$$x \to a\frac{\tau}{1-\cos\sigma}, \qquad y \to a\frac{\sin\sigma}{1-\cos\sigma},$$

so

$$\frac{Q}{l} = \varepsilon_0 \int_{-\infty}^{+\infty} \left[\partial\phi \Big/ \partial\left(a\frac{\tau}{1-\cos\sigma}\right)\right]_{\sigma=\text{const}} d\left(a\frac{\sin\sigma}{1-\cos\sigma}\right) = \varepsilon_0 \frac{V}{2\tau_0}\int_{-\pi}^{+\pi} d\sigma = \pi\varepsilon_0 \frac{V}{\tau_0},$$

giving us the following final expression for the mutual capacitance between the wires (also per unit length):

$$\frac{C}{l} \equiv \frac{Q}{lV} = \frac{\pi\varepsilon_0}{\tau_0} \equiv \frac{\pi\varepsilon_0}{\cosh^{-1}(d/2R)}.$$

According to this result,

$$\frac{C}{l} \approx \pi\varepsilon_0 \times \begin{cases} 1/\ln(d/R), & \text{for } R << d, \\ (R/t)^{1/2}, & \text{for } t \equiv d - 2R << d. \end{cases} \qquad (**)$$

The first of these expressions may be derived in the simple way discussed for the two spheres in Sec. 2.2 of the lecture notes – see Fig. 2.4 and the accompanying text. Namely, assuming that the charge Q of the right wire is known, and neglecting the left wire effects, we may readily calculate (say, using the Gauss law) its electric field at an arbitrary point $\boldsymbol{\rho}$ of the $[x, y]$ plane:

$$\mathbf{E}_\text{R} = \frac{Q}{l}\frac{\boldsymbol{\rho} - (d/2)\mathbf{n}_x}{2\pi\varepsilon_0[\boldsymbol{\rho} - (d/2)\mathbf{n}_x]^2}.$$

Similarly, the field of the left wire (carrying the opposite charge, $-Q/l$ per unit length) is

$$\mathbf{E}_\text{L} = -\frac{Q}{l}\frac{\boldsymbol{\rho} + (d/2)\mathbf{n}_x}{2\pi\varepsilon_0[\boldsymbol{\rho} + (d/2)\mathbf{n}_x]^2},$$

so on the x-axis,

$$\mathbf{E}_\text{L} + \mathbf{E}_\text{R} = \mathbf{n}_x \frac{Q}{2\pi\varepsilon_0 l}\left(-\frac{1}{x - d/2} + \frac{1}{x + d/2}\right).$$

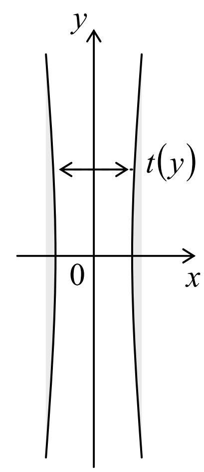
Now we can calculate voltage V between the wires approximately by integrating the net field $E \approx E_\text{L} + E_\text{R}$ along the free-space segment of the x-axis between the wires:

$$V \approx \phi\big|_{x=+d/2-R} - \phi\big|_{x=-d/2+R} = -\int_{-d/2+R}^{+d/2-R} (E_L + E_R)_x \, dx = \frac{Q}{2\pi\varepsilon_0 l} \int_{-d/2+R}^{+d/2-R} \left(-\frac{1}{x-d/2} + \frac{1}{x+d/2} \right) dx$$

$$= -\frac{Q}{2\pi\varepsilon_0 l} \left(\ln\frac{R}{d-R} - \ln\frac{d-R}{R} \right) \approx \frac{Q}{\pi\varepsilon_0 l} \ln\frac{d}{R}, \qquad \text{for } R << d,$$

thus confirming the first of Eqs. (**) and providing its interpretation.

In the second limit, $2R \to d$, the wires are so close to each other (see the figure on the right) that each elementary fragment of area $dA = l \times dy$ may be considered as a plane capacitor of thickness



$$t(y) = d - 2(R^2 - y^2)^{1/2} \approx d - 2R + \frac{y^2}{R} \equiv t + \frac{y^2}{R}, \quad \text{with } t \equiv t(0) = d - 2R,$$

whose capacitance is approximately obeying Eq. (2.28) of the lecture notes: $dC = \varepsilon_0 dA/t(y)$. The summation of all such elementary capacitances, effectively connected in parallel, gives the integral:[46]

$$\frac{C}{l} = \int \frac{dC}{l} = \varepsilon_0 \int_{-\infty}^{+\infty} \frac{dy}{t(y)} = \varepsilon_0 \int_{-\infty}^{+\infty} \frac{dy}{t + y^2/R} = \pi\varepsilon_0 \left(\frac{R}{t} \right)^{1/2}, \qquad \text{for } t << d,$$

thus confirming (and explaining) the second of Eqs. (**). This result is of significant practical importance, in particular, for the so-called twisted-pair transmission lines – to be discussed in Section 7.5 of this course.

Note that the bipolar coordinates allow a straightforward generalization of our result to the system of two wires with different radii (say, R_1 and R_2):

$$\frac{C}{l} = \frac{2\pi\varepsilon_0}{\cosh^{-1}\left[\left(d^2 - R_1^2 - R_2^2 \right) / 2R_1 R_2 \right]}.$$

So, these coordinates are very useful. The same is true for their two simple 3D generalizations, the so-called *toroidal coordinates* and *bispherical coordinates*, that may be obtained by additional rotations about, respectively, the τ-axis and the σ-axis. The reader is encouraged to explore these options.

Problem 2.14. Formulate the 2D electrostatic problems that may be simply solved using each of the following analytic functions of the complex variable $\boldsymbol{z} \equiv x + iy$:

(i) $\boldsymbol{w} = \ln \boldsymbol{z}$,
(ii) $\boldsymbol{w} = \boldsymbol{z}^{1/2}$,
(iii) $\boldsymbol{w} = \boldsymbol{z} + 1/\boldsymbol{z}$,

and solve these problems.

[46] See, e.g., MA Eq. (6.5a). Note that our integral is converging at $|y| \sim (Rt)^{1/2} << R$, thus giving *a posteriori* justification for the used approximation for the function $t(y)$.

Solutions: For a 2D boundary problem of electrostatics to be readily solvable with a conformal mapping, the borders of the conductor's cross-sections should coincide with lines of either constant real or constant imaginary part of the complex function $\boldsymbol{w}$.

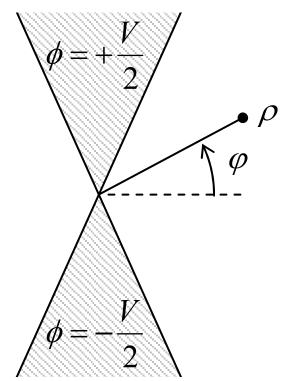
(i) Plugging the definitions $\boldsymbol{w} \equiv u + iv$ and $\boldsymbol{z} \equiv x + iy$ into the given equality $\boldsymbol{w} = \ln \boldsymbol{z}$, and separating the real and imaginary parts, we get

$$u = \ln\left(x^2 + y^2\right)^{1/2}, \quad v = \tan^{-1}\frac{y}{x}.$$

This means that on the $[x, y]$ plane, the lines of constant u are concentric circles, while those of equal v are straight lines extending from the origin to infinity. As a result, for this conformal map, two examples of easily solvable boundary problems are:

Problem A. Calculate the field distribution between two concentric cylinders of radii a and b, held at different potentials. (This is exactly the coaxial cable problem already solved in Sec. 2.3 of the lecture notes).

Problem B. A similar problem for two cylindrical wedges, barely separated at the origin – see the figure on the right. (This is exactly the second part of Problem 9.)



The solution of Problem A: since the analytic function $u(x, y)$ satisfies the Laplace equation, and the conductors' surfaces are equipotential, any line of constant u should be equipotential, so the potential ϕ may depend only on u. Hence the Laplace equation is reduced to $d^2\phi/d^2u = 0$; its solution is a linear function of u:

$$\phi = c_1 u + c_2 = c_1 \ln\left(x^2 + y^2\right)^{1/2} + c_2 = c_1 \ln \rho + c_2,$$

where constants $c_{1,2}$ may be readily calculated from the potentials fixed on the concentric conducting cylinders. This is of course just the result given by Eq. (2.43), which was obtained in Sec. 2.3 of the lecture notes by using a more general approach.

For Problem B, a similar argumentation yields the same result as was obtained in the solution of Problem 9 (with the notation replacement $\theta \to \pi/2 - \varphi$):

$$\phi = c_1 v + c_2 = c_1 \tan^{-1}\frac{y}{x} + c_2 \equiv c_1\varphi + c_2,$$

which is valid in the free space between the conducting wedges.

(ii) In the same fashion, separating the real and imaginary parts of the function $\boldsymbol{w} = \boldsymbol{z}^{1/2}$, we get

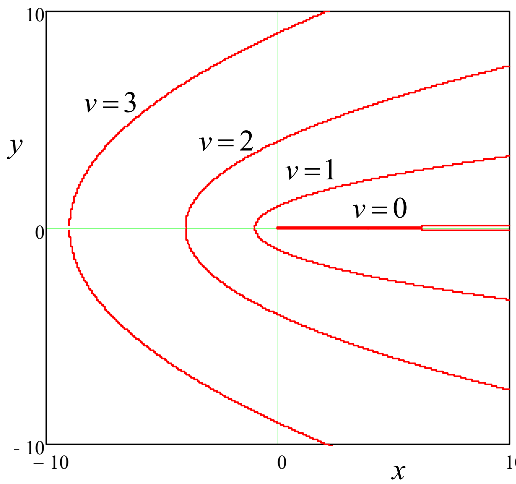
$$u = \pm\left\{\frac{1}{2}\left[\left(x^2 + y^2\right)^{1/2} + x\right]\right\}^{1/2}, \qquad v = \pm\left\{\frac{1}{2}\left[\left(x^2 + y^2\right)^{1/2} - x\right]\right\}^{1/2}, \qquad (*)$$

where the signs in both formulas should be changed simultaneously.[47] As the second of these relations shows, the lines of constant v on the $[x, y]$ plane are parabolas:

[47] As evident from Fig. 2.9b of the lecture notes, which shows the reciprocal map $\boldsymbol{z} = \boldsymbol{w}^{1/2}$, in our current problem, each point $\{x, y\}$ is mapped on two points $\{u, v\}$.

$$x = \frac{y^2}{4v^2} - v^2,$$

in the limit $v \to 0$ giving the straight segment $0 < x < +\infty$ – see the figure on the right. (As the first of Eqs. (*) shows, the lines of constant u are similar, with the opposite direction of parabola bending.)



This mapping may be used, for example, to find the field distribution around the edge of a thin conducting sheet occupying the half-plane $x > 0, y = 0$. For this problem, $d^2\phi/dv^2 = 0$, with the obvious solution

$$\phi = c_1 v + c_2 \equiv c_1 \left\{ \frac{1}{2}\left[\left(x^2 + y^2\right)^{1/2} - x \right] \right\}^{1/2} + c_2 .$$

The condition of a fixed potential (say, $\phi = 0$) of the conducting half-plane corresponding to $v = 0$ gives $c_2 = 0$, so finally

$$\phi = c_1 \left\{ \frac{1}{2}\left[\left(x^2 + y^2\right)^{1/2} - x \right] \right\}^{1/2}, \qquad (**)$$

where the constant c_1 is determined by the potential at the (distant) external electrode creating the electric field. This means that the figure above shows the equipotential lines corresponding to $\phi = c_1 v$.

Near the surface of the half-plane conductor ($x > 0$, $|y| << x$), Eq. (**) yields

$$\phi = c_1 \frac{|y|}{2x},$$

so both the electric field at the sheet surfaces,

$$E_n = -\frac{\partial \phi}{\partial y}\Big|_{y=0} = \pm \frac{c_1}{2x},$$

and the surface charge density $\sigma = \varepsilon_0 E_n$ have the same strong (non-integrable) divergences at the edge, as in Problem B of Task (i), discussed above. (See the discussion in the model solution of Problem 9.)

(iii) The given complex function,

$$\boldsymbol{w} \equiv u + iv = \boldsymbol{z} + \frac{1}{\boldsymbol{z}} = x + iy + \frac{1}{x + iy} \equiv x\left(1 + \frac{1}{x^2 + y^2}\right) + iy\left(1 - \frac{1}{x^2 + y^2}\right), \qquad (***)$$

maps the left and right parts of the x-axis (with, respectively, $x \le 0$ and $0 \le x$) on the corresponding "outer" parts (with $|u| \ge 1$) of the u-axis:

$$\boldsymbol{w}\Big|_{y=0} \equiv x + \frac{1}{x}, \quad \text{so} \quad \begin{cases} +1 \le u < +\infty, & \text{for } 0 \le x < +\infty, \\ -\infty < u \le -1, & \text{for } -\infty < x \le -1, \end{cases}$$

while in both cases, $v = 0$. The remaining "inner" part of the u-axis (with $|u| \le 1$) is covered by mapping of the unit-radius circle on the $[x, y]$ plane. Indeed, taking the complex argument $\boldsymbol{z}$ in the polar form $\rho\exp\{i\varphi\}$, for the particular value $\rho = 1$, we get

$$\boldsymbol{w}\Big|_{\rho=1} = \exp\{i\varphi\} + \exp\{-i\varphi\} \equiv 2\cos\varphi, \qquad \text{so } u = 2\cos\varphi, \quad v = 0 .$$

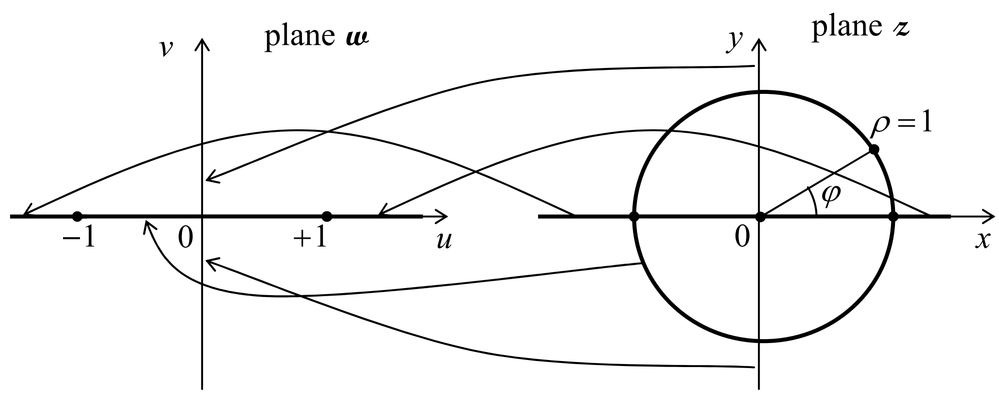
Also, both vertical semi-axes $\boldsymbol{z} = iy$ are mapped onto the whole v-axis:

$$\boldsymbol{w}\Big|_{x=0} \equiv iv = i\left(y - \frac{1}{y}\right), \qquad \text{so } v = y - \frac{1}{y}, \quad u = 0 .$$

Finally, for all very distant points, the mapping is identical:

$$\boldsymbol{w} \to \boldsymbol{z}, \quad \text{in particular } v \to y, \quad \text{for } |\boldsymbol{z}| \equiv \left(x^2 + y^2\right)^{1/2} >> 1 .$$

The figure below summarizes the mapping.



Now let us consider the following boundary problem: a function ϕ satisfies the 2D Laplace equation on the whole $[u, v]$ plane, with the following boundary condition imposed at large distances:

$$\phi \to cv, \qquad \text{for } v \to \pm\infty ,$$

where c is a constant. Its evident solution, valid on the whole plane, is also $\phi = cv$, in particular giving $\phi = 0$ on the whole horizontal axis $v = 0$. Plugging in the expression for the function $v(x, y)$ following from Eq. (***), we see that the resulting function,

$$\phi = cy\left(1 - \frac{1}{x^2 + y^2}\right), \tag{****}$$

gives the solution of the 2D Laplace equation on the $[x, y]$ plane, with the corresponding boundary conditions:

$$\phi \to cy, \qquad \text{for } y \to \pm\infty;$$

$$\phi = 0, \qquad \text{for } y = 0 \text{ and } \rho \equiv \left(x^2 + y^2\right)^{1/2} = 1.$$

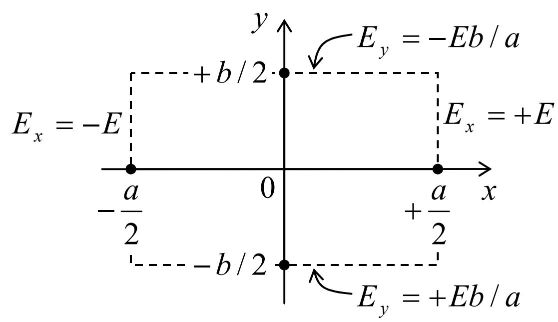
But with the proper choice of the constant ($c = -E_0$), and rescaling of x and y to an arbitrary radius R, this is just the conducting-cylinder problem solved by another (variable-separation) method in Sec. 2.6 of the lecture notes. Indeed, the functional form of Eq. (****) coincides with Eq. (2.117), with the x and y swapped to conform with the coordinate choice accepted in that solution – see Fig. 2.15.

So, for this particular function, just for function (i), the conformal mapping does not give us any new results. However, in physics, having discretion in the choice of the solution methods is not only aesthetically pleasing but sometimes practically fruitful, because different approaches may suggest generalizations to different groups of more complex problems.

Problem 2.15. On each side of a cylindrical volume with a rectangular cross-section $a\times b$, with no electric charges inside it, the electric field's component normal to the side's plane is constant, and also equal and opposite to that on the opposite side. Calculate the distribution of the electric potential inside the volume, provided that the magnitude of the normal components on the sides of length b equals E. Suggest a practicable method to implement such potential distribution.

Solution: First of all, let us calculate the normal component E' of the field on the other pair of walls. Since the volume is free of charge, the Gauss law (1.16) applied to its unit length yields

$$-2Eb + 2E'a = 0, \qquad \text{i.e. } E' = E\frac{b}{a}.$$



Hence, in suitable Cartesian coordinates (see the figure on the right), we get the following 2D boundary problem for the distribution of the electrostatic potential $\phi(x, y)$:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 0, \quad \text{for } |x| < \frac{a}{2}, |y| < \frac{b}{2};$$

$$\left.\frac{\partial \phi}{\partial x}\right|_{x=\pm a/2,\, |y|<b/2} = \mp E, \quad \left.\frac{\partial \phi}{\partial y}\right|_{y=\pm b/2,\, |x|<a/2} = \pm E\frac{b}{a}.$$

There are two ways to solve this problem. The regular ("pedestrian":-) way is to use the standard variable separation method,[48] with due respect to the evident symmetry of our system (in the above coordinates, taking $\phi(0, 0) = 0$, we have $\phi(-x, y) = \phi(x, -y) = \phi(x, y)$, so $\partial\phi/\partial x = 0$ at $y = 0$, and $\partial\phi/\partial y = 0$ at $x = 0$), and then use the summation formula[49]

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \pi n\xi = \frac{\pi^2}{4}\left(\xi^2 - \frac{1}{3}\right), \qquad \text{for } |\xi| \leq 1,$$

to bring the result to the following very simple form (in particular, independent of b):

$$\phi = E\frac{x^2 - y^2}{a}. \qquad (*)$$

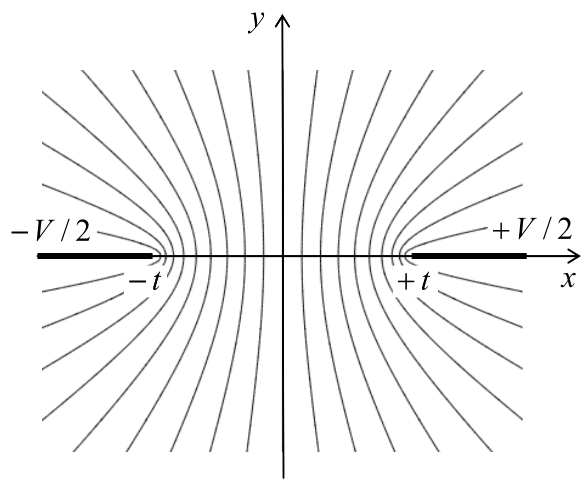
Though this exercise is highly recommended to the readers, they could notice that our boundary problem is satisfied by the field that was already calculated in Sec. 2.4 of the lecture notes – see Eqs. (2.76)-(2.77) for the case $V = 2Ea$. This means, in particular, that the corresponding potential distribution (*) may be created in practice, for example, by using the quadrupole electrostatic lens shown in Fig. 2.9a.

Problem 2.16. Complete the solution of the problem shown in Fig. 2.12 of the lecture notes (reproduced below), by calculating the distribution of the surface charge on the semi-planes. Can you calculate the mutual capacitance between the semi-planes (per unit length of the system)? If not, can you estimate it?

---

[48] See Sec. 2.5 of the lecture notes.

[49] It may be readily proved by the Fourier expansion of its right-hand side on the interval $-1 \leq \xi \leq +1$, repeated with period $\Delta\xi = 2$ over all the ξ-axis.



Solution: Near the electrode's surface ($y \to 0$, $x^2 > t^2$), Eq. (2.83) of the lecture notes yields

$$\phi \to \frac{V}{\pi}\sin^{-1}\left\{1 - \frac{y^2}{4x}\left[\frac{1}{x+t} + \frac{1}{x-t}\right]\right\}$$
$$\to \frac{V}{2} \pm \frac{Vy}{\pi(x^2 - t^2)^{1/2}}.$$

From here, the only non-vanishing (vertical) component of the field is

$$E_n\Big|_{y=\pm 0} = \mp \frac{\partial \phi}{\partial y}\Big|_{y=\pm 0} = \mp \frac{V}{\pi\left(x^2 - t^2\right)^{1/2}},$$

so, according to Eq. (1.24) of the lecture notes, the full (double-side) surface charge density is

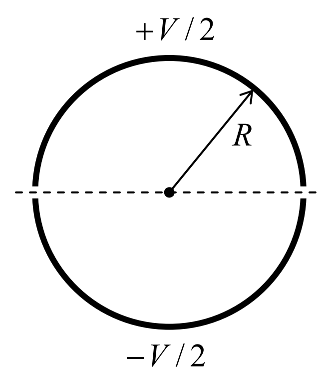
$$\sigma = 2\varepsilon_0 E_n\Big|_{y=\pm 0} \frac{2\varepsilon_0 V}{\pi\left(x^2 - t^2\right)^{1/2}}.$$

Its divergence at $x \to t$ is not too strong (it is integrable), just as in the thin-disk problem solved in Sec. 2.4, but in contrast to that problem, the total charge of each semi-plane (per unit length) is still infinite, because the corresponding integral,

$$\frac{Q}{l} = \int_t^\infty \sigma\, dx = \frac{2\varepsilon_0 V}{\pi}\int_t^\infty \frac{dx}{\left(x^2 - t^2\right)^{1/2}} = \frac{2\varepsilon_0 V}{\pi}\ln\left[x + \left(x^2 - t^2\right)^{1/2}\right]_{x=t}^{x\to\infty}, \qquad (*)$$

diverges at the *upper* limit, if only very slowly (logarithmically). Hence in order to calculate the mutual capacitance per unit length, $C/l \equiv Q/lV$, one needs to make one more step from this idealized model toward reality – for example, take into account a large but finite width w of each semi-plate. Even without solving the resulting problem exactly, we can use Eq. (*) to make a semi-quantitative prediction of the result by truncating the integral at $x = w >> t$:

$$\frac{C}{l} = \frac{Q}{lV} \approx \frac{2\varepsilon_0}{\pi}\ln\left[x + \left(x^2 - t^2\right)^{1/2}\right]_{x=t}^{x=w>>t} \approx \frac{2\varepsilon_0}{\pi}\ln\frac{2w}{t}.$$



Problem 2.17. A straight, long, thin, round-cylindrical conducting pipe has been cut, along its axis, into two equal parts – see the figure on the right.

(i) Use the conformal mapping method to calculate the distributions of the electrostatic potential created by voltage V applied between the two parts, both outside and inside the pipe, and of the surface charge.

(ii)* Calculate the mutual capacitance between the pipe's halves (per unit length), taking into account a small cut width $2t << R$.

Hints: In Task (i), you may like to use the following complex function:

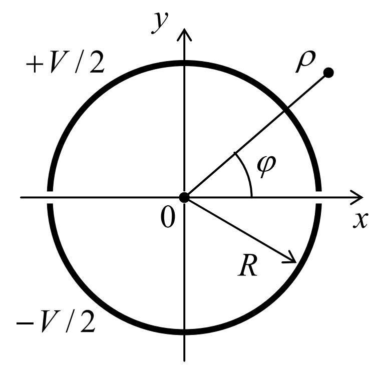
$$\boldsymbol{w} = \ln\frac{R + \boldsymbol{z}}{R - \boldsymbol{z}},$$

while in Task (ii), you may use the solution of the previous problem.

Solutions:

(i) Let us consider the plane of the pipe's cross-section as the plane of the following complex variable:

$$\mathbf{z} = x + iy = \rho e^{i\varphi},$$



where $\{x, y\}$ are the Cartesian coordinates, and $\{\rho, \varphi\}$ are the polar coordinates, introduced as shown in the figure on the right – so, $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$. Let us analyze the conformal mapping performed by the function given in the first *Hint*. For the points on the pipe's walls ($\rho = R$, i.e. $\mathbf{z} = Re^{i\varphi}$):

$$\mathbf{w} = \ln\frac{R + Re^{i\varphi}}{R - Re^{i\varphi}} \equiv \ln\frac{1 + e^{i\varphi}}{1 - e^{i\varphi}} \equiv \ln\frac{e^{-i\varphi/2} + e^{i\varphi/2}}{e^{-i\varphi/2} - e^{i\varphi/2}} = \ln\frac{2\cos(\varphi/2)}{-2i\sin(\varphi/2)} = -\ln\left(-i\tan\frac{\varphi}{2}\right),$$

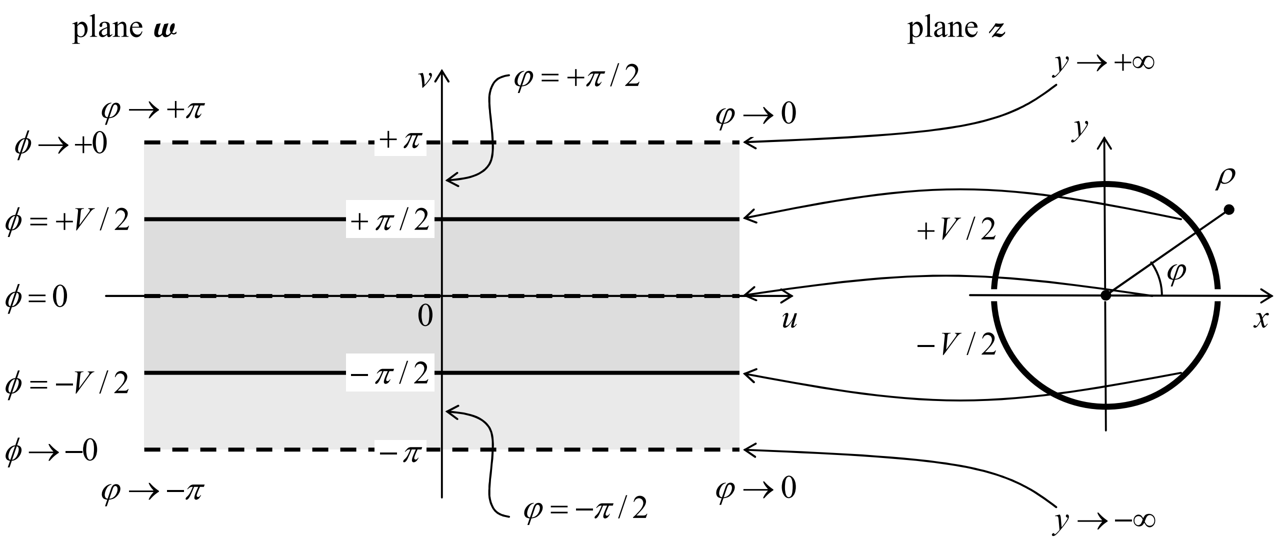
so, by using the standard notation $\mathbf{w} = u + iv$ (where u and v are real), we get

$$e^{-\mathbf{w}} \equiv e^{-u}e^{-iv} \equiv e^{-u}(\cos v - i\sin v) = -i\tan\frac{\varphi}{2}.$$

Since the right-hand side of this equality is purely imaginary, so should be the expression before it, which is only possible if $\cos v = 0$, i.e. at the horizontal lines $\sin v = \pm 1$, e.g., $v = \pm\pi/2$.[50] On these two lines, the above relation yields, respectively,

$$\pm e^{-u} = \tan\frac{\varphi}{2}.$$

Since u is real by definition, e^{-u} may be only positive, so as φ is increased from 0 to $+\pi$ (i.e. as we move, on the $\mathbf{z}$-plane, along the pipe's top wall), the corresponding point on the plane $\mathbf{w}$ moves from the right to left along the horizontal line $v = +\pi/2$. Similarly, as φ is decreased from 0 to $-\pi$, the point in plane $\mathbf{w}$ moves, in the same horizontal direction, along the line $v = -\pi/2$ – see the figure below.



[50] Due to the 2π-periodicity of $\mathbf{z}$ as a function of φ, and the function $\mathbf{w}(\mathbf{z})$ being logarithmic, the whole mapping has the 2π-periodicity along the v-axis, so we may limit our analysis to the segment $-\pi \le v \le +\pi$ shown in the figure.

Absolutely similar analyses show that the function $\boldsymbol{w}$ also maps (see the figure above):

– the horizontal axis of the $\boldsymbol{z}$-plane ($y = 0$): onto that ($v = 0$) of the plane $\boldsymbol{w}$,

– the vertical half-axes of the y-axis (with $\varphi = \pm\pi/2$): onto the respective segments of the u-axis, $0 \leq v \leq +\pi$ and $-\pi \leq v \leq 0$, and

– any horizontal segment of the $\boldsymbol{z}$-plane, with $|y| >> |x \pm R|$: onto a horizontal segment on plane $\boldsymbol{w}$, with v approaching $\pi\text{sgn}(y)$. (Indeed, if $y \to \pm\infty$, then

$$e^{\boldsymbol{w}} \equiv e^u(\cos v + i\sin v) = \frac{R + x + iy}{R - x - iy} \equiv -\frac{1 + (x+R)/iy}{1 + (x - R)/iy} \to -1 + 2i\frac{R}{y},$$

so $\cos v \to -1$, while $\text{sgn}\, v = \text{sgn}\, y$.)

To summarize, the function $\boldsymbol{w}$ maps the pipe's interior on the horizontal strip $-\pi/2 \leq v \leq +\pi/2$ (shaded darker in the figure above), and two parts of its exterior, separated by the x-axis, onto two strips: $+\pi/2 \leq v \leq +\pi$ and $-\pi \leq v \leq -\pi/2$ (shaded lighter).

Now we may return to our boundary problem. Since all the boundary conditions imposed on the electrostatic potential ϕ, and the unnecessary but convenient requirement $\phi = 0$ at $\rho = 0$, are independent of u (see the labels on the left side of the figure above), and the function $\boldsymbol{w}$ is analytic, a linear function $\phi(v) = c_1 v + c_2$ satisfies the Laplace equation inside each strip and, at the appropriate choice of the coefficients $c_{1,2}$ (specific for each strip), the boundary conditions as well. With the proper selection of $c_{1,2}$, we get

$$\phi = \frac{V}{\pi} \times \begin{cases} (+\pi - v), & \text{for } +\pi/2 < v < +\pi, \\ v, & \text{for } -\pi/2 < v < +\pi/2, \\ (-\pi - v), & \text{for } -\pi < v < -\pi/2, \end{cases} \qquad (*)$$

(It is straightforward to use the left-side labels in the figure above to verify that this solution indeed satisfies all the boundary conditions.)

What remains is to express v via the coordinates on the $\boldsymbol{z}$-plane. For that, we may use the fact that if $\boldsymbol{w} = \ln\mathcal{F} \equiv \ln(|\mathcal{F}|\exp\{i \arg(\mathcal{F})\}) \equiv \ln(|\mathcal{F}|) + i \arg(\mathcal{F})$, then $v \equiv \text{Im}(\boldsymbol{w}) = \arg(\mathcal{F})$. In our case, we can use this identity with $\mathcal{F} = (R + \boldsymbol{z})/(R - \boldsymbol{z})$, so

$$v \equiv \text{Im}\,\boldsymbol{w} = \arg\frac{R + x + iy}{R - x - iy} \equiv \arg\frac{(R + x + iy)(R - x + iy)}{(R - x - iy)(R - x + iy)} = \arg\frac{R^2 - x^2 - y^2 + 2iRy}{(R - x)^2 + y^2}.$$

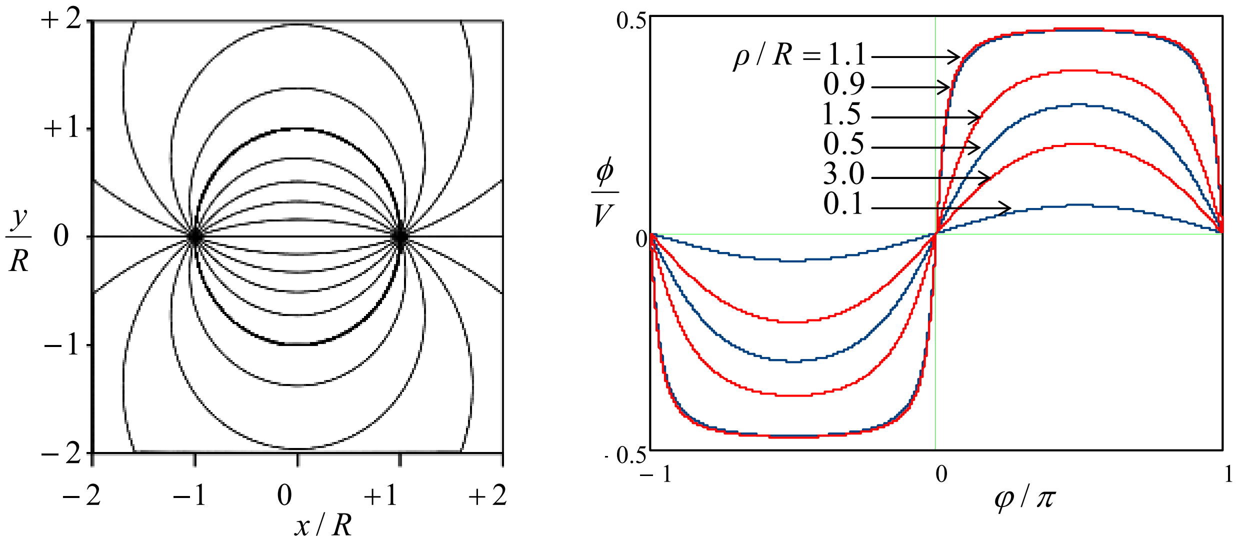
The denominator of the last fraction is, by construction, real, and does not affect its argument, so

$$\tan v = \frac{2Ry}{R^2 - x^2 - y^2} \equiv \frac{2R\rho\sin\varphi}{R^2 - \rho^2},$$

and taking into account the π-periodicity of the tangent function, all Eqs. (*) may be rewritten in a single simple form:

$$\phi(\rho, \varphi) = \frac{V}{\pi}\tan^{-1}\frac{2R\rho\sin\varphi}{\left|R^2 - \rho^2\right|}. \qquad (**)$$

The left panel of the figure below shows the general pattern of the equipotential surfaces (or rather their cross-sections by the plane normal to the pipe's axis) given by this formula. Close to the pipe, the surfaces naturally align to it; they merge in the gaps between the half-pipes, where the electric field is concentrated.



To illustrate this behavior in more detail, the right panel of the figure below shows the potential ϕ as a function of φ for several fixed values of ρ, both larger than R (red lines) and smaller than R (blue lines). These plots show, in particular,[51] that small deviations from the pipe's surface to either side lead to similar deviations of the potential from its piece-constant form at the surface. This fact implies that the magnitude of the electric field on the pipe's surface, $|E_n| = \partial\phi/\partial\rho$ at $\rho = R$, and hence the surface charge density $\sigma = \varepsilon_0 E_n$ is the same, for the same φ, on both (the external and the internal) surfaces of the pipe. Indeed, a direct differentiation of Eq. (**) over ρ gives the same result:

$$\sigma(\varphi) = \frac{\varepsilon_0 V}{\pi R \sin\varphi}. \qquad (***)$$

(ii) The charge density (***) exhibits strong (non-integrable) divergences at $\varphi \to 0$ and $\varphi \to \pm\pi$, i.e. near the gap between the two halves of the pipe; for example, at $\varphi \to 0$,

$$\sigma(\varphi) \to \frac{\varepsilon_0 V}{\pi R \varphi} = \frac{\varepsilon_0 V}{\pi\, y}, \qquad \text{for } |y| << R,$$

so the total charge Q of each half of the pipe, and hence the capacitance $C = Q/V$ of the system (even per unit length along the z-axis) are formally infinite. This is an artifact of neglecting the cut's width $2t$ in the above calculation. In the limit $2t << R$, finite Q and C may be found by comparing the above solution with that of the previous problem (on the field distribution between two planar, thin electrodes separated by a cut of width $2t$), which overlap in a vicinity of the gap.

Indeed, with the proper notation replacement $x \to y$, the solution of the previous problem gives the following result for the single-side charge density:

$$\sigma(\varphi) = \operatorname{sgn}(y) \frac{\varepsilon_0 V}{\pi \left(y^2 - t^2\right)^{1/2}}, \qquad \text{for } |y| > t, \qquad (****)$$

[51] For an analysis of the potential's behavior far outside of the pipe, see the model solution of the next problem.

which gives the same result as Eq. (***) within a relatively broad range of angles:

$$\frac{t}{R} << \varphi << 1.$$

Hence Eq. (****) may be used instead of Eq. (***) at $|y| \equiv |R\varphi| << R$. As a result, the total charge of the upper half of the pipe (per unit length), may be calculated by the separation of the corresponding integral into two parts:[52]

$$\frac{Q}{l} = 2\int_{\sin\varphi>0} \sigma(\varphi)Rd\varphi = 4R\int_{t/R}^{\pi/2}\sigma(\varphi)d\varphi = 4R\left(\int_{t/R}^{\varphi_0}\sigma(\varphi)d\varphi + \int_{\varphi_0}^{\pi/2}\sigma(\varphi)d\varphi\right),$$

where the separation boundary φ_0 is arbitrary within the solutions' overlap region. The inequality $t/R << \varphi_0$ allows us to use Eq. (****) in the first integral, while the inequality $\varphi_0 << 1$ makes the approximation $y = R\varphi$ valid in it. On the other hand, due to the former relation, we may use the uncorrected expression (***) for the charge density in the second integral:

$$\frac{Q}{l} = 4R\frac{\varepsilon_0 V}{\pi}\left(\int_{t/R}^{\varphi_0}\frac{d\varphi}{\left(R^2\varphi^2 - t^2\right)^{1/2}} + \frac{1}{R}\int_{\varphi_0}^{\pi/2}\frac{d\varphi}{\sin\varphi}\right).$$

The remaining calculations are simple. Using variable replacements $\varphi \equiv (t/R)\cosh\xi$, so $d\varphi = (t/R)\sinh\xi d\xi$ in the first integral, and $\zeta \equiv \cos\varphi$, so $d\varphi/\sin\varphi = -d\zeta/\sin^2\varphi \equiv -d\zeta/(1-\zeta^2)$ in the second one, we get

$$\frac{Q}{l} = \frac{4\varepsilon_0 V}{\pi}\left(\int_{\varphi=t/R}^{\varphi=\varphi_0}\frac{\sinh\xi d\xi}{\left(\cosh^2\xi - 1\right)^{1/2}} - \int_{\varphi=\varphi_0}^{\varphi=\pi/2}\frac{d\zeta}{1-\zeta^2}\right) \equiv \frac{4\varepsilon_0 V}{\pi}\left[\int_{\varphi=t/R}^{\varphi=\varphi_0}d\xi - \frac{1}{2}\int_{\varphi=\varphi_0}^{\varphi=\pi/2}\left(\frac{1}{1-\zeta} + \frac{1}{1+\zeta}\right)d\zeta\right]$$

$$= \frac{4\varepsilon_0 V}{\pi}\left(\xi\Big|_{\varphi=t/R}^{\varphi=\varphi_0} - \frac{1}{2}\ln\frac{1+\zeta}{1-\zeta}\Big|_{\varphi=\varphi_0}^{\varphi=\pi/2}\right) \equiv \frac{4\varepsilon_0 V}{\pi}\left(\cosh^{-1}\frac{R\varphi_0}{t} + \frac{1}{2}\ln\frac{1+\cos\varphi_0}{1-\cos\varphi_0}\right).$$

Due to the condition $\varphi_0 >> t/R$, $\cosh^{-1}(R\varphi_0/t)$ may be approximated as $\ln(2R\varphi_0/t) \equiv \ln(4R/t) + \ln(\varphi_0/2)$, while due to the condition $\varphi_0 << 1$, the logarithm in the last term reduces to $\ln(4/\varphi_0^2) \equiv -2\ln(\varphi_0/2)$. As a result, the terms with φ_0 cancel (as they should, due to the qualified-arbitrary choice of this separation boundary), and we finally get

$$\frac{Q}{l} = \frac{4\varepsilon_0 V}{\pi}\ln\frac{4R}{t}, \qquad \text{i.e. } \frac{C}{l} = \frac{4\varepsilon_0}{\pi}\ln\frac{4R}{t}.$$

My earnest advice to the reader is to review the used trick of matching ("*stitching*") two approximate solutions that share a broad overlap region because it is used in many problems of not only physics but also other quantitative sciences and engineering.

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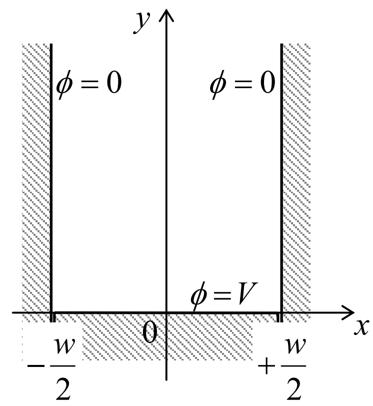
[52] The front factor of 2 takes into account equal densities $\sigma(\varphi)$ of the charge at the inner and outer surface, while the next step uses the obvious mirror symmetry of σ with respect to the vertical axis $x = 0$ ($\varphi = \pm\pi/2$).



Problem 2.18. A gap of constant width w between two grounded conducting semi-spaces is closed, from one side, with a conducting plunger biased with voltage V, so that the cross-section of the system looks like the figure on the right shows. Use the variable separation method to calculate the distribution of the electrostatic potential within the gap.

Solution: Let us use the convenient Cartesian coordinates shown in the figure on the right. In this case, the variable separation may be carried out similarly to how this was done in Sec. 2.5 of the lecture notes for a rectangular box, with just a few changes. First, now we are dealing with a 2D rather than 3D problem, so Eq. (2.85) does not need the factor $Z(z)$. Second, with our current choice of the origin, the solutions $X(x)$ of Eq. (2.89), that satisfy the boundary condition $\phi = 0$ on the gap's walls, are not the sine functions but rather

$$X(x) = \cos kx, \quad \text{with } k = \frac{\pi}{w}(2n-1), \qquad \text{where } n = 1, 2, 3\ldots. \qquad (*)$$

Finally, the corresponding function $Y(y)$ satisfying both the Laplace equation and the condition $\phi \to 0$ at $y \to \infty$, is $\exp\{-ky\}$ with the same k as in Eq. (*), so the solution of the boundary problem is given by the following series:

$$\phi(x,y) = \sum_{n=1}^{\infty} c_n \cos\frac{\pi(2n-1)x}{w} \exp\left\{\frac{\pi(2n-1)y}{w}\right\}, \qquad \text{for } -\frac{w}{2} \le x \le +\frac{w}{2},\ 0 \le y.$$

The coefficients c_n should be found from the still unused boundary condition $\phi(x, 0) = V$:

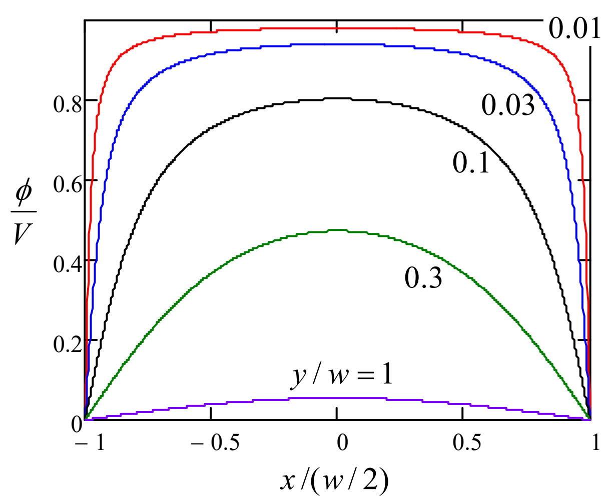
$$V = \sum_{n=1}^{\infty} c_n \cos\frac{\pi(2n-1)x}{w}, \qquad \text{for } -\frac{w}{2} \le x \le +\frac{w}{2}.$$

Multiplying both sides of this equation by $\cos[\pi(2n' - 1)x/w]$ and integrating the result over x from $-w/2$ to $+w/2$, we get

$$V \int_{-w/2}^{+w/2} \cos\frac{\pi(2n'-1)x}{w} dx = c_n \int_{-w/2}^{+w/2} \cos\frac{\pi(2n'-1)x}{w} \cos\frac{\pi(2n-1)x}{w} dx.$$

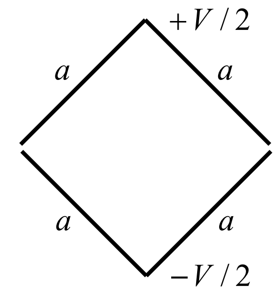
Now, elementary integrations yield $c_n = 4V(-1)^{n-1}/\pi(2n-1)$, so, finally,

$$\phi = \frac{4V}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos\frac{\pi(2n-1)x}{w} \exp\left\{-\frac{\pi(2n-1)y}{w}\right\}.$$



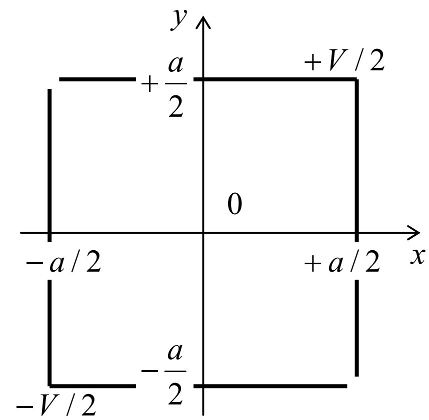
The figure on the right shows the calculated potential as a function of x, for several values of y. As might be expected, near the voltage-biased plunger, the electrostatic potential closely follows its value V, besides small regions near the side walls where it transitions to their (zero) value of ϕ. However, far from the plunger, its effect rapidly decreases, and the potential drops exponentially, as described by the first term of the series, which dominates at large y:

$$\phi \approx \frac{4V}{\pi}\cos\frac{\pi x}{w}\exp\left\{-\frac{\pi y}{w}\right\}, \qquad \text{for } y >> w.$$



Problem 2.19. Use the variable separation method to calculate the electrostatic potential's distribution inside a very long thin-wall metallic box with a quadratic cross-section, cut and voltage-biased as shown in the figure on the right. (Assume that the cut's width is negligibly small.)

Solution: The easiest way to solve this problem is to avoid using the Cartesian coordinates implied by the figure in the assignment, with one of the coordinate axes directed along the cut, but instead let the coordinate axes pass through the middle of each wall. The figure on the right shows the resulting configuration, turned clockwise by $\pi/4$ to make the axes horizontal and vertical – just to comply with tradition and make the forthcoming formulas more intuitive.



The second useful move is to use the linear superposition principle to break the desired potential into two parts, with its components satisfying different boundary conditions:

$$\phi = \phi_\mathrm{h} + \phi_\mathrm{v},$$

where

$$\phi_\mathrm{h} = \begin{cases} \pm V/2, & \text{for } x = \pm a/2, \\ 0, & \text{for } y = \pm a/2, \end{cases} \qquad \phi_\mathrm{v} = \begin{cases} 0, & \text{for } x = \pm a/2, \\ \pm V/2, & \text{for } y = \pm a/2. \end{cases} \qquad (*)$$

The advantage of this separation is that the "horizontal" potential ϕ_h is evidently an odd function of x and an even function of y, while the "vertical" potential has the opposite symmetry. Let us start with the former of these functions, carrying out the variable separation as for the 3D box in Sec. 2.5 of the lecture notes and in the previous problem, but taking into account that the zero boundary conditions at $y = \pm a/2$ make it more natural to use trigonometric functions along this coordinate:

$$\phi_\mathrm{h} = \sum_{n=1}^{\infty} c_n \sinh\frac{(2n-1)\pi x}{a}\cos\frac{(2n-1)\pi y}{a}, \qquad (**)$$

immediately satisfying these conditions and both mirror symmetries of the function. As usual for the variable separation method, the coefficients c_n may be found from the requirement that this solution describes the fixed potential $+V/2$ on the wall with $x = +a/2$:[53]

$$\sum_{n=1}^{\infty} c_n \sinh\frac{(2n-1)\pi}{2}\cos\frac{(2n-1)\pi y}{a} = \frac{V}{2}.$$

Multiplying both sides of this equation by $\cos(2n' - 1)\pi y/a$ and then integrating them over the segment $-a/2 \leq y \leq +a/2$, we get

[53] The boundary condition on the opposite wall, with $x = -a/2$ will be then satisfied automatically, due to the antisymmetry of the function (*) with respect to the inversion $x \to -x$.

$$c_n \sinh\frac{(2n-1)\pi}{2}\int_{-a/2}^{+a/2}\cos^2\frac{(2n-1)\pi y}{a}dy = \frac{V}{2}\int_{-a/2}^{+a/2}\cos\frac{(2n-1)\pi y}{a}dy, \qquad \text{giving } c_n = \frac{2(-1)^n V}{\pi n \sinh(n-½)\pi}..$$

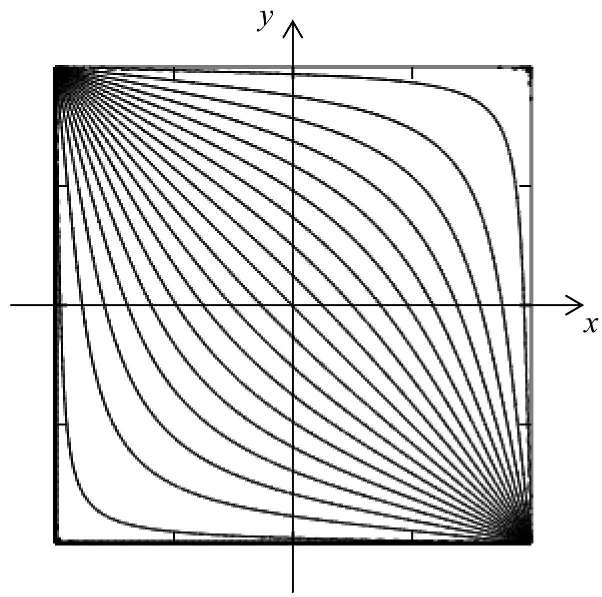
As a result, Eq. (**) becomes

$$\phi_\text{h} = \frac{2V}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)\sinh(n-½)\pi}\sinh\frac{(2n-1)\pi x}{a}\cos\frac{(2n-1)\pi y}{a}. \qquad (***)$$

Now we may notice that since at the swap $x \leftrightarrow y$, the boundary conditions (*) are also swapped, the same has to be true for the functions ϕ_h and ϕ_v, so for the latter function we may use Eq. (***) with this replacement, and the full solution of our problem becomes

$$\phi_\text{h} = \frac{2V}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)\sinh(n-½)\pi}\left[\sinh\frac{(2n-1)\pi x}{a}\cos\frac{(2n-1)\pi y}{a} + \cos\frac{(2n-1)\pi x}{a}\sinh\frac{(2n-1)\pi y}{a}\right].$$

The figure on the right shows the equipotential surfaces described by this result, drawn with equal steps $\Delta\phi = V/20$. They naturally concentrate near the cut gaps where the electric field increases becoming formally infinite in these two corners of the cross-section.

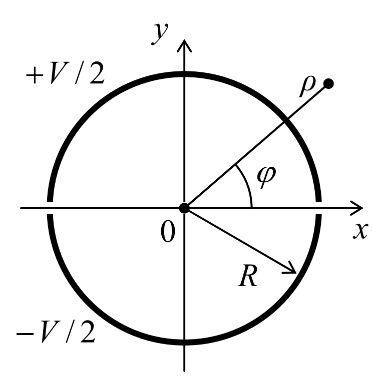


If for some reason we need to express the potential as a function of the coordinates (say, $\{x', y'\}$) implied by the problem's assignment, we may always use the above solution with the substitutions

$$x = \frac{x'+y'}{\sqrt{2}}, \qquad y = \frac{-x'+y'}{\sqrt{2}},$$

corresponding to the back (counterclockwise) rotation of the coordinate axes. (If you want just to visualize the solution, it is easier just to turn your head by $\pi/4$ clockwise, and have one more look at the figure on the right :-)

Problem 2.20. Solve Problem 17(i) by using the variable separation method, and compare the results.



Solution: Introducing the polar coordinates as shown in the figure on the right (i.e. just as in the model solution of Problem 17) and guided by Eq. (2.112) of the lecture notes and the symmetry of the problem, we may look for the solution of the outer problem (for $\rho \geq R$) in the form

$$\phi(\rho,\varphi) = \sum_{n=1}^{\infty}\frac{b_n}{\rho^n}\sin n\varphi.$$

Indeed, the inconsequential coefficient a_0 in Eq. (2.112) may be taken equal to zero for convenience. The logarithmic term in that expression, proportional to b_0, would describe the effect of the net charge of the pipe, which we obviously do not have. The terms proportional to a_n with $n > 0$ would diverge at $\rho \to \infty$ and thus should be excluded, and the terms proportional to $\cos n\varphi$ would give even-function contributions into the potential, which in our problem should be an odd function of φ.

The boundary conditions at $\rho = R$ yield the following system of equations for the coefficients b_n:

$$\sum_{n=1}^{\infty} \frac{b_n}{R^n} \sin n\varphi = \phi(R,\varphi) = \frac{V}{2} \times \begin{cases} +1, & \text{for } 0 < \varphi < \pi, \\ -1, & \text{for } \pi < \varphi < 2\pi. \end{cases}$$

Solving this system in the way usual for the Fourier series, i.e. by multiplying both parts of the equation by $\sin n'\varphi$ with an arbitrary $n' > 0$, integrating them over any 2π-long interval (for example, $0 \le \varphi \le 2\pi$), and using the orthogonality of the functions $\sin n\varphi$ with different n:

$$\int_0^{2\pi} \sin n'\varphi \sin n\varphi \, d\varphi = \pi \delta_{nn'}, \qquad \text{for } n \neq 0,$$

we obtain

$$b_n = \frac{V}{\pi} R^n \int_0^{\pi} \sin n\varphi \, d\varphi = \frac{V}{\pi n} R^n \left(1 - \cos n\pi\right) = \frac{V}{\pi n} R^n \times \begin{cases} 0, & \text{for } n \text{ even}, \\ 2, & \text{for } n \text{ odd}, \end{cases}$$

so denoting the odd values $n = 1, 3, \ldots$ as $2m - 1$ (with $m = 1, 2, \ldots$), we finally get

$$\phi(\rho,\varphi) = \frac{2V}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \left(\frac{R}{\rho}\right)^{2m-1} \sin(2m-1)\varphi, \qquad \text{for } \rho \ge R. \qquad (*)$$

The solution of the internal problem (for $\rho \le R$) is similar, besides that instead of terms proportional to b_n/ρ^n in Eq. (2.112), we need to keep the terms $a_n\rho^n$ that do not diverge at $\rho \to 0$:

$$\phi(\rho,\varphi) = \sum_{n=1}^{\infty} a_n \rho^n \sin n\varphi.$$

Since the boundary conditions at $\rho = R$ are similar for both problems, the above result for b_n is valid for the normalized coefficients a_n as well (with the replacement $b_n/R^n \to a_nR^n$), so the final result is very similar to Eq. (*):

$$\phi(\rho,\varphi) = \frac{2V}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \left(\frac{\rho}{R}\right)^{2m-1} \sin(2m-1)\varphi, \quad \text{for } \rho \le R. \qquad (**)$$

Numerical plots of Eqs. (*) and (**) show that they yield results identical to the explicit expressions obtained in the solution of Problem 17(i).[54] Moreover, these series are equally (if not more) convenient for the analysis of potential behavior far from the pipe's walls. Indeed, as Eqs. (*) and (**) show, at both $\rho >> R$ and $\rho << R$, the series terms decay fast with n, so potential is determined by their first terms, with $m = 1$, i.e. is proportional to $\sin\varphi$. In the former case,

$$\phi(\rho,\varphi) = \frac{2VR}{\pi\rho} \sin\varphi, \quad \text{for } \rho >> R,$$

[54] Their exact equivalence may be also proven analytically by the Fourier expansion of Eqs. (**) of the model solution of Problem 17(i) – an optional exercise highly recommended to the reader.

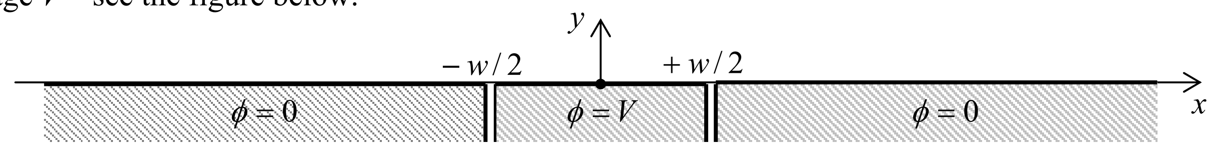
the potential corresponds to the electric field of magnitude $E \propto 1/\rho^2$. As will be discussed in Chapter 3 of this course, such a field may be interpreted as the one created by a straight line of similar electric dipoles. On the other hand, near the cylinder's axis, we get,

$$\phi(\rho,\varphi) = \frac{2V}{\pi R}\rho\sin\varphi, \quad \text{for } \rho << R, \tag{***}$$

meaning that the electric field here is uniform. Indeed, since the product $\rho\sin\varphi$ is just y (see the figure above), Eq. (***) gives

$$\phi = \frac{2V}{\pi R}y, \qquad \text{so that } \mathbf{E} = -\nabla\phi = -\frac{2V}{\pi R}\mathbf{n}_y = \text{const.}$$

Problem 2.21. Use the variable separation method to calculate the potential distribution above the plane surface of a conductor with a strip of width w singled out with very thin cuts and biased with voltage V – see the figure below.



Solution: Let us introduce the Cartesian coordinates as shown in the figure above, and separate the variables similarly to Eq. (2.85) of the lecture notes, besides that in this 2D problem, the partial functions $Z(z)$ are evidently constant and may be taken for 1:

$$\phi_k = X_k(x)Y_k(y),$$

so Eq. (2.87) may be rewritten as

$$\frac{1}{X_k}\frac{d^2X_k}{dx^2} = -\frac{1}{Y_k}\frac{d^2Y_k}{dy^2} = -k^2 = \text{const}.$$

This notation for the separation constant is convenient in our current case because since the electrostatic potential has to tend to zero at $y \to \infty$, only exponentially decaying solutions of the above equation for $Y_k(y)$,

$$Y_k(y) = e^{-ky},$$

with $k > 0$, are acceptable.[55] With that, the possible solutions of the above equation for the functions $X_k(x)$ are $\cos kx$ and $\sin kx$. Taking into account the mirror symmetry of the system with respect to the plane $x = 0$, only the first of them are acceptable:

$$X_k(x) = \cos kx.$$

Now comes the largest difference between this problem and the one discussed in Sec. 2.5(i) of the notes: due to the absence of any period of the system in any direction,[56] the spectrum of possible

[55] Strictly speaking, complex values of k, with Re$k > 0$, are also acceptable, but as will be shown below, the full solution of the problem may be constructed without involving imaginary parts of k.

[56] The problem analyzed in the lecture notes (see Fig. 2.13) may be considered a periodic one, with the rectangular volume repeated infinitely in all three directions.

values of k is continuous rather than discrete. As a result, the sum (2.84) now becomes an integral over k; and with our current eigenfunctions $X_k(x)$ and $Y_k(y)$, it is

$$\phi(x,y) = \int_0^\infty c_k \cos kx\, e^{-ky} dk\,. \qquad (*)$$

The continuous set of coefficients the c_k (essentially, a continuous function of k) may be found from the boundary condition at $y = 0$ and (due to the problem's symmetry) $x \geq 0$:

$$\int_0^\infty c_k \cos kx\, dk = \phi(x,0) = \begin{cases} V, & \text{for } 0 \leq x < w/2, \\ 0, & \text{for } w/2 < x < \infty. \end{cases} \qquad (**)$$

Since this is just the expansion of the function on the right-hand side into the Fourier integral, the way of calculating c_k is well known: we need to multiply both parts of Eq. (**) by $\cos k'x$ with an arbitrary k', and then integrate the result over x. Changing the order of integration on the left-hand side, we get

$$\int_0^\infty c_k dk \int_0^\infty \cos k'x \cos kx\, dx = \int_0^\infty \phi(x,0)\cos k'x\, dx \equiv V\int_0^{w/2} \cos k'x\, dx\,. \qquad (***)$$

The integral over x on the left-hand side is just a sum of two delta functions,[57]

$$\int_0^\infty \cos k'x \cos kx\, dx \equiv \frac{1}{4}\mathrm{Re}\int_{-\infty}^{+\infty} \left[\exp\{i(k-k')x\} + \exp\{i(k+k')x\}\right]dx = \frac{\pi}{2}\left[\delta(k-k') + \delta(k+k')\right],$$

while that on the right-hand side is elementary:

$$\int_0^{w/2} \cos k'x\, dx = \frac{1}{k'}\sin\frac{k'w}{2},$$

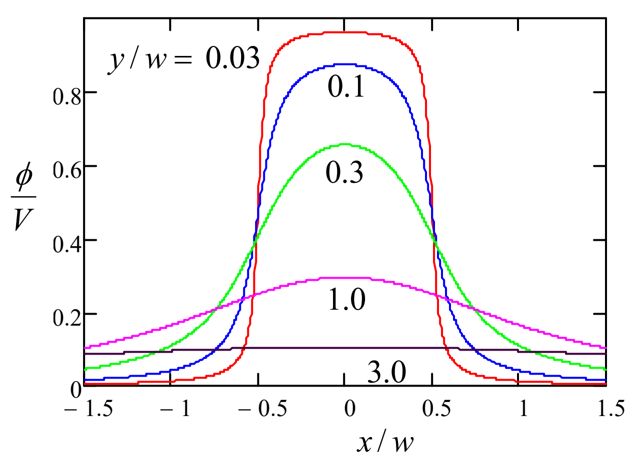
so for any sign of k', Eq. (**) reduces to

$$\frac{\pi}{2}c_{k'} = \frac{V}{k'}\sin\frac{k'w}{2}.$$

Plugging the resulting expression for $c_{k'}$ (with the index k' replaced with k) into Eq. (*), we get

$$\phi(x,y) = \frac{2V}{\pi}\int_0^\infty \sin\frac{kw}{2} \cos kx\; e^{-ky}\, \frac{dk}{k}. \qquad (****)$$

In this particular case, the integral over k may be worked out analytically[58] but generally, such an integral form of the result is typical for variable-separation solutions. Plots in the figure on the right show the potential ϕ as a function of x, for several fixed values of y. We can see that close to the conductor's surface, the voltage-induced potential bump is almost rectangular, with a width of $\Delta x \approx w$ and a height approaching V, but at large distances from the surface, the bump is lower and more spread out, with a width Δx of the order of $y >> w$.



[57] See, e.g., MA Eq. (14.4).
[58] See, e.g., the model solution of Problem 42.

Problem 2.22. The previous problem is now modified: the cut-out and voltage-biased part of the conducting plane is now not a strip, but a square with side w. Calculate the potential distribution above the conductor's surface.

Solution: This problem is conceptually very similar to the previous one, besides that now the potential's distribution is three-dimensional (just like in the problem solved in Sec. 2.5 of the lecture notes), so its solution is essentially a synthesis of the two known solutions, and the explanations below will be very short. Let us use the Cartesian coordinates with the axes x and y in the plane of the conductor's surface and parallel to the cut square's sides, and the z-axis directed normally to the plane, toward the free space, with its origin in the square's center. Then the variable separation yields

$$\phi(x,y,z)=\int_0^\infty d\alpha\int_0^\infty d\beta\, c_{\alpha\beta}\cos\alpha x\cos\beta y\, e^{-\gamma z}, \qquad (*)$$

where the separation constants are related by Eq. (2.88) of the lecture notes:

$$\gamma=\left(\alpha^2+\beta^2\right)^{1/2}>0\,.$$

The boundary condition at the conductor's surface is

$$\int_0^\infty d\alpha\int_0^\infty d\beta\, c_{\alpha\beta}\cos\alpha x\cos\beta y=\phi(x,y,0)=\begin{cases}V, & \text{for}\,|x|<w/2\ \text{and}\ |y|<w/2,\\ 0, & \text{otherwise}.\end{cases}$$

It is clearly satisfied by the product $c_{\alpha\beta}=Vf(\alpha)f(\beta)$, where the function f obeys the condition

$$\int_0^\infty f(\xi)\cos\xi s\,d\xi=\begin{cases}1, & \text{for}\ 0<s<w/2,\\ 0, & \text{for}\ w/2<s<\infty.\end{cases}$$

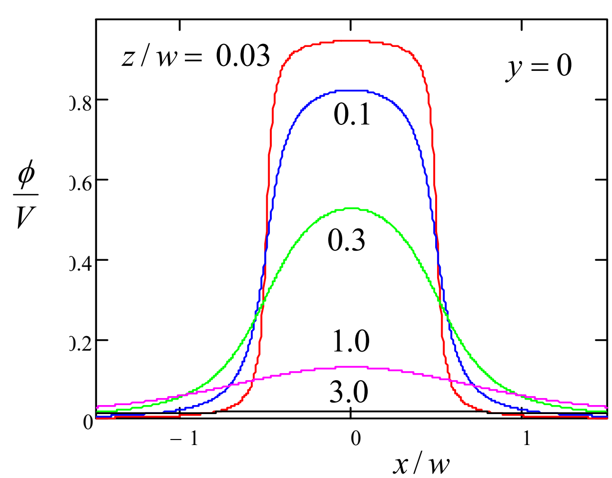
But such a function was already calculated in the previous problem:

$$f(\xi)=\frac{2}{\pi\xi}\sin\frac{\xi w}{2},$$

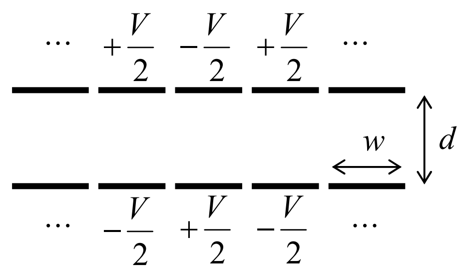
so, finally, the solution (*) may be spelled out as

$$\phi(x,y,z)=\frac{4V}{\pi^2}\int_0^\infty\sin\frac{\alpha w}{2}\cos\alpha x\frac{d\alpha}{\alpha}\int_0^\infty\sin\frac{\beta w}{2}\cos\beta y\frac{d\beta}{\beta}\exp\left\{-\left(\alpha^2+\beta^2\right)^{1/2}z\right\}.$$

This result is plotted, in the figure on the right, as a function of one of the horizontal coordinates (x), for the counterpart horizontal coordinate (y) fixed at zero, and the same values of the vertical coordinate z as were used for the similar plots in the model solution of the previous problem. (Note the difference in the vertical coordinate's notation.)



The comparison of these two figures shows that, quite naturally, the voltage-induced potential bump decays, with height, faster (at $z >> w$, much faster) in our current case of a voltage-biased square than in the previous case of a biased infinitely long strip.



Problem 2.23. Each electrode of a large plane capacitor is cut into parallel long strips of equal width w, with very narrow gaps between them. These strips are kept at alternating potentials, as shown in the figure on the right. Use the variable separation method to calculate the electrostatic potential distribution in space, and explore the limit $w << d$.

Solution: Due to the symmetry of the problem, the electrostatic potential should vanish at:

(i) the horizontal symmetry plane in the middle between the planes, and
(ii) any vertical plane passing between the strips.

Selecting these planes (or rather their traces on the plane of the drawing) for the coordinate axes, we see that each of the functions $X(x)$ and $Y(y)$ forming any particular solution $\phi_k = XY$ has to be odd. Since the functions $X(x)$ also have to be periodic, with the period $\Delta x = 2w$, they may be taken in the form of $\sin kx$ with $k = \pi n/w$, where $n = 1, 2, 3\ldots$. Hence, in order to satisfy the Laplace equation, the corresponding functions $Y(y)$ have to equal $\sinh \alpha y$, so the full solution takes the form

$$\phi(x,y) = \sum_{n=1}^{\infty} \phi_n \sin\frac{\pi n x}{w} \sinh\frac{\pi n y}{w}. \qquad (*)$$

The coefficients ϕ_n should be found from the boundary conditions on the electrodes. Since Eq. (*) already ensures the proper symmetry and periodicity of the solution, it is sufficient to require that it fits the boundary value on just one strip (say, $0 < x < w$, at $y = +d/2$, where $\phi = +V/2$):

$$\frac{V}{2} = \sum_{n=1}^{\infty} \phi_n \sin\frac{\pi n x}{w} \sinh\frac{\pi n d}{2w}.$$

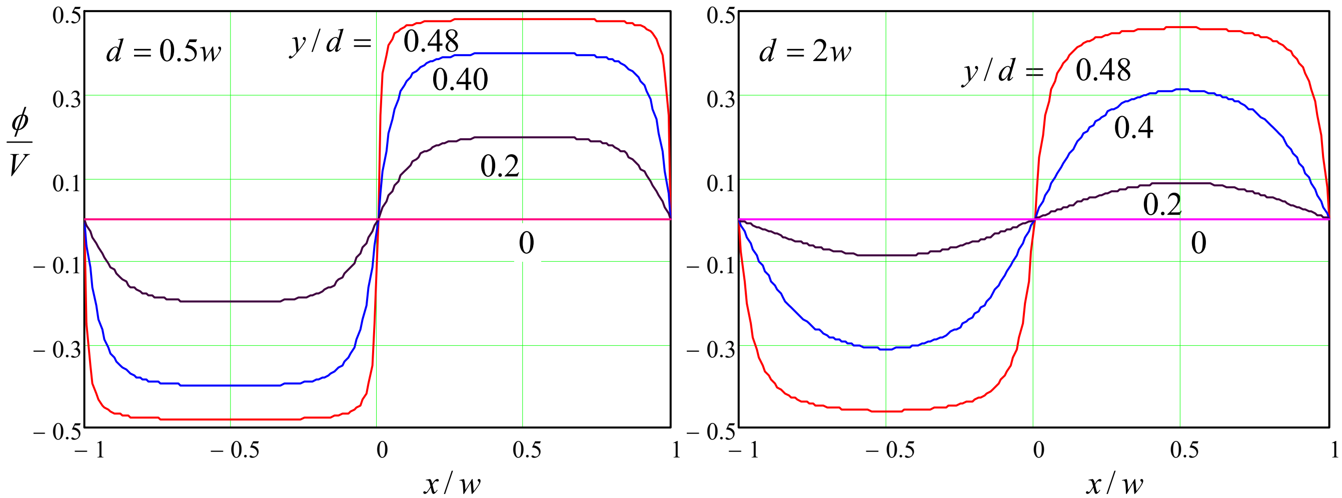
Applying the reciprocal Fourier transform formula (or just multiplying both sides of the last equation by $\sin(\pi x n'/w)$ and integrating the result over x from 0 to w), we readily get

$$\phi_n = \begin{cases} \dfrac{2V}{\pi n}\left(\sinh\dfrac{\pi n d}{2w}\right)^{-1}, & \text{for } n \text{ odd,} \\ 0, & \text{for } n \text{ even.} \end{cases}$$

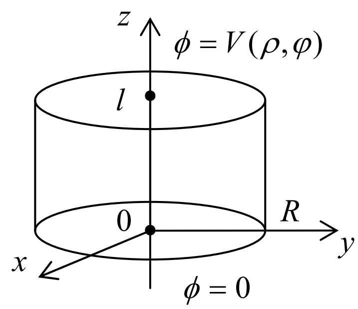
The plots in the figure below show the potential's distribution between the plates for two values of the ratio d/w and for three distances y from the central plane. For any d, close to the electrodes, the potential follows the square-wave profile dictated by the applied voltages. However, if $d >> w$, this is true only very close to their surfaces, while in most of the volume, the potential is relatively small and close to the simple sine function given by the first term of the series (*):

$$\phi(x,y) \approx \frac{2V}{\pi} \frac{\sinh(\pi y/w)}{\sinh(\pi d/2w)} \sin\frac{\pi x}{w}.$$

The solution in the outer regions (at $|y| > d/2$) is very similar, with the multiplier $\sinh(\pi n y/w)$ replaced with $\operatorname{sgn}(y)\exp\{-\pi n|y|/w\}$.



<u>Problem 2.24</u>. Complete the cylinder problem started in Sec. 2.7 of the lecture notes (see Fig. 2.17, reproduced on the right), for the cases when the top lid's voltage is fixed as follows:



(i) $V = V_0 J_1(\xi_{11}\rho/R)\sin\varphi$, where $\xi_{11} \approx 3.832$ is the first root of the Bessel function $J_1(\xi)$;

(ii) $V = V_0 =$ const.

For both cases, calculate the electric field at the centers of the lower and upper lids. (For Task (ii), an answer including series and/or integrals is acceptable.)

Solution: The general solution to this problem was found in Sec. 2.7 of the lecture notes – see Eq. (2.139):

$$\phi(\rho,\varphi,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\frac{\xi_{nm}\rho}{R}\right)(c_{nm}\cos n\varphi + s_{nm}\sin n\varphi)\sinh\frac{\xi_{nm}z}{R}, \qquad (*)$$

where ξ_{nm} is the m^{th} root of the Bessel function $J_n(\xi)$. This solution already satisfies not only the Laplace equation inside the cylinder but also the boundary conditions on its sidewall and bottom lid; hence the coefficients c_{nm} and s_{nm} may be found from the boundary condition on the top lid:

$$\phi(\rho,\varphi,l) = V(\rho,\varphi). \qquad (**)$$

(i) In this case, the function $V(\rho, \varphi)$ has only one azimuthal harmonic proportional to $\sin\varphi$, and only one radial harmonic, proportional to $J_1(\xi_{11}\rho/R)$, so only one of the coefficients c_{nm}, s_{nm} is different from zero:

$$s_{11}\sinh\frac{\xi_{11}l}{R} = V_0,$$

and Eq. (*) reduces to a simple solution:

$$\phi = V_0 J_1\left(\frac{\xi_{11}\rho}{R}\right)\frac{\sinh(\xi_{11}z/R)}{\sinh(\xi_{11}l/R)}\sin\varphi.$$

Since $J_1(0) = 0$, the potential equals zero along the symmetry axis of the system, so in this case there is no normal electric field $E_n = -\partial\phi/\partial z$ in the center of either lid. However, since the potential changes within the top lid's plane,

$$V \to V_0 \lim_{\rho\to 0} J_1\left(\frac{\xi_{11}\rho}{R}\right)\sin\varphi = V_0 \frac{\xi_{11}\rho}{2R}\sin\varphi \equiv V_0 \frac{\xi_{11}}{2R} y\ ,$$

the field in its center does have a horizontal component directed along the y-axis:

$$E_y\Big|_{\rho=0} = -V_0 \frac{\xi_{11}}{2R} \approx -1.916\frac{V_0}{R}\ .$$

(ii) Here, the top lid's potential does not depend on the angle φ, and hence only one angular function (with $n = 0$) fits this boundary condition, so Eq. (**) reduces to an axially-symmetric form:

$$V_0 = \sum_{m=1}^{\infty} c_{0m} J_0\left(\frac{\xi_{0m}\rho}{R}\right)\sinh\frac{\xi_{0m} l}{R}\ .$$

Multiplying both parts of this equation by $\rho J_0(x_{0m}\rho/R)$, integrating the result over ρ from 0 to R, and using Eq. (2.141) of the lecture notes, we get

$$c_{0m} = \frac{2V_0}{R^2 J_1^2(\xi_{0m})\sinh(\xi_{0m} l/R)}\int_0^R J_0\left(\xi_{0m}\frac{\rho}{R}\right)\rho\, d\rho\ .$$

(Actually, the last integral may be worked out analytically, giving a more explicit result:

$$c_{0m} = \frac{2V_0}{\xi_{0m} J_1(\xi_{0m})\sinh(\xi_{0m} l/R)}\ ,$$

but I did not require my SBU students to accomplish this astonishing mathematical feat :-)

According to Eq. (*), and due to the fact that $J_n(0) = 0$ unless $n = 0$ (see, e.g., Fig. 2.18 in the lecture notes), the potential's distribution along the symmetry axis of the cylinder is contributed only by the terms with $n = 0$:

$$\phi(z) = \sum_{m=1}^{\infty} c_{0m}\sinh\frac{\xi_{0m} z}{R}\ ,$$

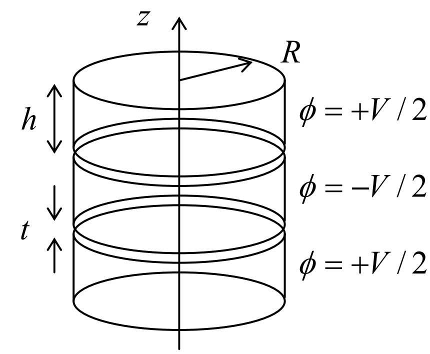
so the (purely vertical) field in the center of the bottom lid (located at $z = 0$) is

$$E_n\Big|_{\text{bottom}} = E_z\Big|_{z=0} = -\frac{d\phi(z)}{dz}\Big|_{z=0} = -\sum_{m=1}^{\infty} c_{0m}\frac{\xi_{0m}}{R}, \qquad (***)$$

while the field at the top lid ($z = l$) is

$$E_n\Big|_{\text{top}} = -E_z\Big|_{z=l} = \frac{d\phi(z)}{dz}\Big|_{z=l} = \sum_{m=1}^{\infty} c_{0m}\frac{\xi_{0m}}{R}\cosh\frac{\xi_{0m} l}{R}.$$

For any $l > 0$, each term of the last series is larger than the corresponding term of Eq. (***), so the field's magnitude at the bottom lid is always lower – physically, this fact is due to the screening effect of the cylinder's sidewall. However, at $l/R \to 0$, $\cosh(\xi_{0m} l/R) \to 1$ for all leading terms in the series, and this difference is negligible.



Problem 2.25. For the infinitely long periodic system sketched in Fig. 2.21 of the lecture notes (reproduced on the right), assuming that $t << h, R$:

(i) calculate and sketch the electrostatic potential's distribution inside the system for various values of the ratio R/h, and

(ii) simplify the results for the limit $R/h \to 0$.

Solutions:

(i) Due to the problem's $2h$-periodicity along the z-axis, its particular solutions should be proportional to linear combinations of $\sin kz$ and $\cos kz$, with $2kh = 2\pi m$, where $m = 1, 2, 3\ldots$, so the spectrum of possible values of the parameter k is restricted to $k = \pi m/h$. Moreover, if the origin of the z-axis is selected at one of the gaps between the rings, the solution should be an odd function of z, so for the internal problem ($\rho \leq R$) we may write

$$\phi(\rho, z) = \sum_{m=1}^{\infty} c_m I_0\left(\frac{\pi m \rho}{h}\right) \sin\frac{\pi m z}{h}.$$

This particular modified Bessel function of the first kind has zero index ν due to the evident axial symmetry of the problem – see Eq. (2.128); the function's argument results from the discreteness of the variable separation constant k: $\xi \equiv k\rho = \pi m \rho/h$. Finally, we had to drop the modified Bessel functions of the second kind from our solution because they diverge at $\rho \to 0$ (see the right panel of Fig. 2.22) and thus cannot be used to represent the finite electrostatic potential inside the system.[59]

The coefficients c_m in the above series should be found from the boundary condition on the conducting rings; since the proper periodicity is already incorporated into our solution, it is sufficient to require that

$$\phi(R, z) \equiv \sum_{m=1}^{\infty} c_m I_0\left(\frac{\pi m R}{h}\right) \sin\frac{\pi m z}{h} = \frac{V}{2}, \quad \text{for } 0 \leq z \leq h.$$

Multiplying both sides of this expression by $\sin(\pi m' z/h)$ and integrating the result from 0 to h, we get

$$c_m = \frac{V}{\pi m}\left[I_0\left(\frac{\pi m R}{h}\right)\right]^{-1} \times \begin{cases} 1, & \text{for } m \text{ odd}, \\ 0, & \text{for } m \text{ even}, \end{cases}$$

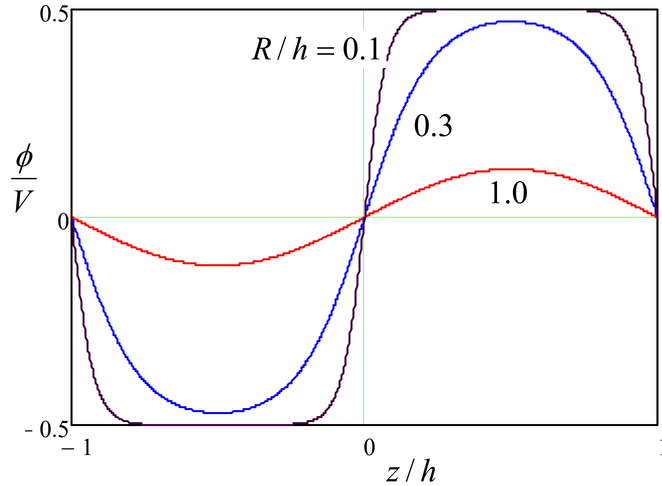
so with the substitution $m = 2n - 1$ (where $n = 1, 2, 3,\ldots$), we get

$$\phi(\rho, z) = \frac{2V}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\frac{\pi(2n-1)z}{h} \frac{I_0[\pi(2n-1)\rho/h]}{I_0[\pi(2n-1)R/h]}. \qquad (*)$$

The figure below shows the plots of this result for the cylinder's axis ($\rho = 0$), on one period of its z-dependence, for several values of the ratio R/h. At $R >> h$, the result is dominated by the first term of the series (with $m = 1$), which is proportional to $1/I_0(\pi R/h)$, so at $R >> h$, according to Eq. (2.158), the potential's variation amplitude equals $(8R/h)^{1/2} V \exp\{-\pi R/h\} << V$.

[59] Note that these functions have to be used in the solution of the *external* problem ($\rho \geq R$), which is similar to that discussed above, besides the replacement of $I_0(\pi m \rho/h)$ with $K_0(\pi m \rho/h)$.

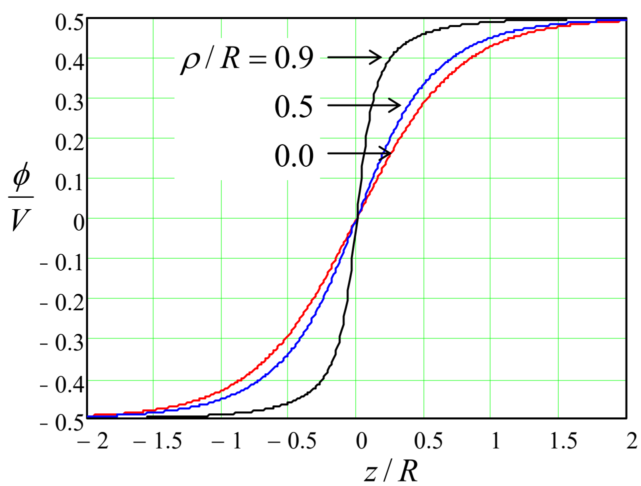
On the contrary, at $R << h$, the potential follows that of the closest ring electrode at most of the axis, besides narrow ($\Delta z \sim R$) intervals centered to gaps between the electrodes. The exact behavior of the potential in these intervals (for example, in the close vicinity of the origin, $|z| \sim R << h$) is of considerable practical interest, because it is also pertinent to a system of just two long coaxial cylinders, with voltage V between them, which may serve as an effective electron lens.[60]



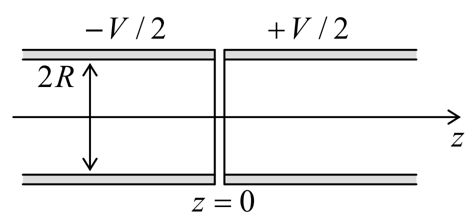
(ii) In order to simplify Eq. (*) in the limit $R/h \to 0$, we may notice that in the range $|z| \sim R$, each term of the sum is much smaller than 1, so it may be well approximated with an integral:

$$\phi(\rho, z) \to \frac{2V}{\pi}\int_0^\infty \frac{1}{2n-1}\sin\frac{\pi(2n-1)z}{h}\frac{I_0(\pi(2n-1)\rho/h)}{I_0(\pi(2n-1)R/h)}dn \equiv \frac{V}{\pi}\int_0^\infty \frac{\sin kz}{k}\frac{I_0(k\rho)}{I_0(kR)}dk, \qquad \text{for } |z| \sim R << h,$$

where $k \equiv \pi(2n-1)/h$. This integral converges both at the lower limit (because at $k \to 0$, $\sin(kz)/k \to z$), and at the upper limit (because at $kR \to \infty$, the function $I_0(kR)$ grows exponentially – see, e.g., Eq. (2.158) and/or the left panel of Fig. 2.22), and is convenient both for numerical calculations (for example, see the figure on the right) and for asymptotic behavior analyses, because it does not depend on h.



Problem 2.26. A long round cylindrical conducting pipe is split, with a very narrow cut normal to its axis, into two parts that are voltage-biased as the figure on the right shows. Use two different approaches to calculate the force exerted by the resulting field upon a charged particle flying along the pipe close to its axis. Can the system work as an electrostatic lens?

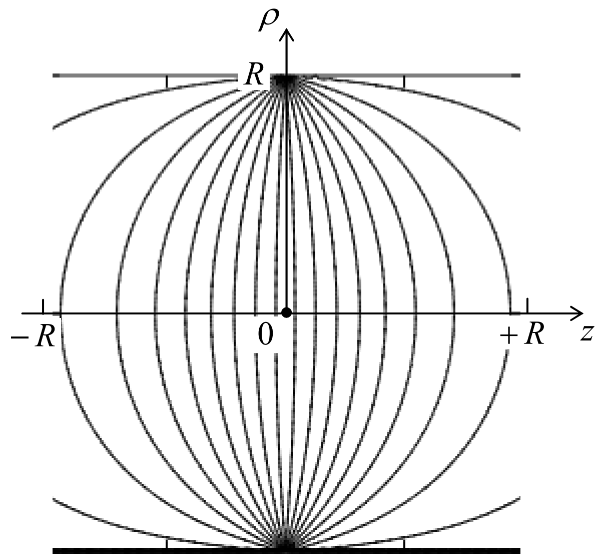


Solution:

Approach 1. First, we may notice that the distribution of the electrostatic potential everywhere inside the pipe is given by the solution of the previous problem in the proper limit $h/R \to \infty$:

[60] The convenience of such a lens in comparison with the one discussed in Problem 1.9 is that in it, the accelerating electric field is confined to a limited length of $\Delta z \sim R$.

$$\phi(\rho, z) = \frac{V}{\pi} \int_0^\infty \frac{\sin kz}{k} \frac{I_0(k\rho)}{I_0(kR)} dk, \qquad (*)$$



where ρ is the distance from the pipe's axis, and z is the distance along the axis, referred to the cut's plane – see the figure in the assignment. The figure on the right shows the corresponding equipotential surfaces drawn with equal steps $V/20$; note that the field near the pipe's axis is spread by distances $\Delta z \sim R$ from the cut's plane $z = 0$. (See also the last figure in the model solution of the previous problem.)

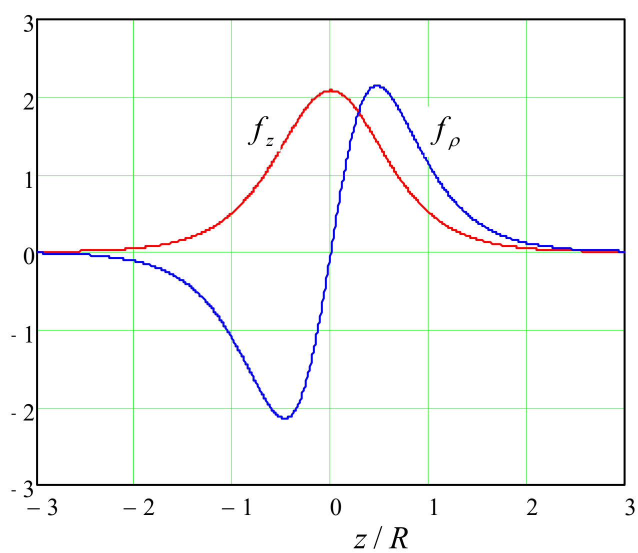
Since the integral in Eq. (*) converges at $k_{\max} \sim 1/\min[R, |z|]$, for the points close to the axis (with $\rho \ll R$), we may expand the modified Bessel function $I_0(k\rho)$ in the Taylor series in its small argument and truncate it to the leading terms. Per the first of Eqs. (2.159) of the lecture notes, $I_0(\xi) = J_0(i\xi)$, so we may use Eq. (2.133) to get

$$I_0(\xi) \approx 1 - \left(\frac{i\xi}{2}\right)^2 \equiv 1 + \frac{\xi^2}{4}, \qquad \text{for } \xi \ll 1.$$

This accuracy is sufficient to calculate both nonvanishing components (longitudinal and radial) of the force $\mathbf{F} = -q\nabla\phi$ exerted by this field on a particle of charge q:[61]

$$F_z = -q\frac{\partial \phi}{\partial z} \approx -\frac{qV}{\pi}\int_0^\infty \frac{\cos kz}{I_0(kR)} dk \equiv -\frac{qV}{\pi R} f_z\left(\frac{z}{R}\right), \quad \text{with } f_z(\xi) \equiv \int_0^\infty \frac{\cos \kappa\xi}{I_0(\kappa)} d\kappa,$$

$$F_\rho = -q\frac{\partial \phi}{\partial \rho} \approx -q\frac{V}{\pi}\int_0^\infty \frac{\sin kz}{k}\frac{1}{I_0(kR)}\frac{\partial}{\partial \rho}\left[1 + \frac{(k\rho)^2}{4}\right] dk = -\frac{qV}{\pi R}\frac{\rho}{2R} f_\rho\left(\frac{z}{R}\right), \quad \text{with } f_\rho(\xi) \equiv \int_0^\infty \frac{\sin \kappa\xi}{I_0(\kappa)} \kappa d\kappa.$$



The figure on the right shows the integrals participating in these formulas as functions of the normalized distance $\xi \equiv z/R$ from the cut. Not surprisingly, it shows that both forces are substantial only at distances $z \sim R$ near the cut and that the longitudinal field is an even function of z, while the radial field is its odd function.[62] As a result, if the initial longitudinal velocity of the particle is so high that its acceleration or deceleration (depending on the sign of the qV product) by the longitudinal field is negligible, it does not acquire an appreciable net radial velocity v_ρ necessary for focusing. Note that this *potential-*

[61] See MA Eq. (10.2) with $\partial/\partial\varphi = 0$.

[62] Note that the radial force is also much smaller, due to the pre-integral factor $\rho/2R \ll 1$.

step situation is very different from the *field-step* arrangement that was the subject of Problem 1.9, in which the induced radial velocity and hence the focusing are direct effects.

However, the focusing still happens even in the system under analysis, due to the following secondary effect. Due to the longitudinal acceleration/deceleration of the charged particle by this field, the time periods of its passage through the negative and positive "bumps" of the radial force F_ρ (see the figure above again) become different, so their *time* (rather than *length*) integrals become different, giving the particle a non-zero net momentum $p_\rho = \int F_\rho dt$ and hence a non-zero final velocity $v_\rho = p_\rho/m$ in the radial direction. Since this velocity has the same proportionality to the initial distance ρ of the particle from the axis as the force F_ρ itself, the time $t = \rho/v_\rho$ and hence the length $f = v_z t$ of its crossing the axis are independent of ρ, so a parallel beam of such particles with different initial values of $\rho << R$ is focused at one point $z = f$. The calculation of this focal distance in the first approximation in small $1/v_z$ is conceptually straightforward but leads to bulky formulas, and I have to leave this task for an appropriate special-topic course.

Approach 2. The structure of Eq. (*) used in the previous approach is essentially an artifact of the z-periodicity of the system considered in the previous problem. For our current system, it is more natural to use the expansion in a series over the Bessel functions $J_n(\xi)$, as this was done at the beginning of Sec. 2.7 of the lecture notes for the system shown in Fig. 2.17. However, at that problem's solution, we have seen that such expansions are very convenient for the functions turning to zero at $\rho = R$. For our current problem, we may introduce such a function as follows:

$$\tilde{\phi} \equiv \begin{cases} \phi - V/2, & \text{for } z > 0, \\ \phi + V/2, & \text{for } z < 0, \end{cases} \qquad \text{so that } \tilde{\phi} = 0 \text{ for } \begin{cases} \rho = R, & \text{at any } z \neq 0, \\ z \to \pm\infty, & \text{at any } \rho \leq R. \end{cases}$$

One more boundary condition for this function follows from the continuity of the full potential at $z = 0$:

$$\phi\big|_{z=+0} - \phi\big|_{z=-0} \equiv \left(\tilde{\phi}\big|_{z=+0} + V/2\right) - \left(\tilde{\phi}\big|_{z=-0} - V/2\right) = 0, \quad \text{so} \quad \tilde{\phi}\big|_{z=+0} - \tilde{\phi}\big|_{z=-0} = -V.$$

The (minor) inconvenience of this approach is that since the function $\tilde{\phi}$ so defined experiences a jump equal to V at the plane $z = 0$, it does not satisfy the Laplace equation exactly on that plane. However, this drawback is compensated by the evident antisymmetry of this function: $\tilde{\phi}(\rho,-z) = -\tilde{\phi}(\rho,z)$, so, in particular, the above boundary condition yields

$$\tilde{\phi}\big|_{z=+0} = -\frac{V}{2}, \tag{**}$$

and it is sufficient to solve this boundary problem only for $z \geq 0$.

Separating the variables ρ and z may be done as in Sec. 2.4 but with the appropriate choice for the solution of Eq. (2.126) and with the due account of the axial symmetry of our current problem ($\nu = n = 0$). So, instead of the double series (2.139), we get a single series:

$$\tilde{\phi}\big|_{z\geq 0} = \sum_{m=1}^{\infty} c_m J_0\left(\xi_{0m}\frac{\rho}{R}\right)\exp\left\{-\xi_{0m}\frac{z}{R}\right\},$$

where ξ_{0m} is the m^{th} zero of the function $J_0(\xi)$ – see the upper row in Table 2.1 of the lecture notes. This solution, with arbitrary coefficients c_m, satisfies the Laplace equation and all boundary conditions besides Eq. (**); to satisfy it, we need to have

$$\sum_{m=1}^{\infty} c_m J_0\left(\xi_{0m}\frac{\rho}{R}\right) = -\frac{V}{2}.$$

Let us multiply both sides of the last equality by $(\rho/R)J_0(\xi_{0m'}\rho/R)$, integrate the result over the interval $0 \leq s \equiv \rho/R \leq 1$, and then use the orthonormality condition (2.141) with $n = 0$. For the only nonvanishing term of the expansion, we get

$$c_m \frac{1}{2}\left[J_1(\xi_{0m})\right]^2 = -\frac{V}{2}\int_0^1 J_0(\xi_{0m}s)sds.$$

The integral on the right-hand side of this equation may be readily worked out using the recurrence relation (2.143) with $n = 1$:

$$\xi J_0(\xi) = \frac{d}{d\xi}\left[\xi J_1(\xi)\right], \qquad \text{so } sJ_0(s\xi_{0m}) = \frac{1}{\xi_{0m}}\frac{d}{ds}\left[sJ_1(\xi_{0m}s)\right].$$

As a result, we get $c_m = -V/\xi_{0m}J_1(\xi_{0m})$, and with the account of the definition and the symmetry of $\tilde{\phi}$, the following final solution:

$$\phi = V\,\mathrm{sgn}(z)\left[\frac{1}{2} - \sum_{m=1}^{\infty}\frac{J_0(\xi_{0m}\rho/R)\exp\{-\xi_{0m}|z|/R\}}{\xi_{0m}J_1(\xi_{0m})}\right]. \qquad (***)$$

Though this formula looks very different from Eq. (*), it gives exactly the same results at each point! Such duality (when it can be found) is very useful in physics, because it may enable using the form best suitable for our needs. In our particular case, Eq. (*) may be preferable for numerical computation at an arbitrary point, because it does not require the calculation of "all" (in practice, a dozen or so) zero points ξ_{0m}. On the other hand, Eq. (***) reveals the asymptotic behaviors of the potential much better. In particular, it shows that at large distances from the cut, the difference between the potential on the pipe's symmetry axis and on its walls scales as $\exp\{-\xi_{01}|z|/R\} \approx \exp\{-2.405|z|/R\}$. That formula also allows the calculation of the forces F_ρ and F_z in the form of series rather than the integrals obtained in Approach 1.[63] However, these formulas also give the same final results as our first approach (see the plots above) and of course, lead to the same conclusions concerning the charged particle focusing.

Problem 2.27. Use the variable separation method to calculate the potential distribution inside and outside of a thin spherical shell of radius R, with a fixed potential distribution on it: $\phi(R,\theta,\varphi) = V_0 \sin\theta\cos\varphi$.

Solution: According to Eq. (2.182) of the lecture notes, the surface potential is proportional to the product $\mathcal{P}_1^{\,1}(\cos\theta)\mathcal{F}_1(\varphi)$, i.e. to a single spherical function with $l = 1$ and $n = 1$. Per Eq. (2.184), inside the sphere, the particular solution of the Laplace equation, proportional to this function, can use only radial functions $a_l r^l$, which do not diverge at $r \to 0$, while for the outer problem, we may only use functions b_l/r^{l+1}, which do not diverge at $r \to \infty$. As a result, the general axially symmetric solution (2.172) of the Laplace equation in the spherical coordinates is reduced, in our case, to

[63] This simple calculation is a simple but useful additional exercise, highly recommended to the reader.

$$\phi(r,\theta,\varphi) = \sin\theta\cos\varphi \times \begin{cases} a_1 r, & \text{for } r \le R, \\ \dfrac{b_1}{r^2}, & \text{for } R \le r. \end{cases}$$

Finding the constants a_1 and b_1 from the boundary conditions on the shell ($r = R$), we get

$$\phi(r,\theta,\varphi) = V_0 \sin\theta\cos\varphi \times \begin{cases} (r/R), & \text{for } r \le R, \\ (R/r)^2, & \text{for } R \le r. \end{cases}$$

Note that the first of these results may be very simply expressed in Cartesian coordinates,

$$\phi = \frac{V_0}{R}x, \quad \text{for } r \le R,$$

so the electric field inside the sphere is uniform: $\mathbf{E} = -(V_0/R)\mathbf{n}_x$.

Problem 2.28. A thin spherical shell carries an electric charge with areal density $\sigma = \sigma_0\cos\theta$. Calculate the spatial distributions of the electrostatic potential and the electric field, both inside and outside the shell.

Solution: Taking into account the axial symmetry of the problem, the potential distributions both inside (ϕ_{in}) and outside (ϕ_{out}) of the shell may be represented as the series given by Eq. (2.172) of the lecture notes, with only the terms that do not diverge in the corresponding space region:

$$\phi_{\text{in}} = \sum_{l=0}^{\infty} a_l r^l \mathcal{P}_l(\cos\theta), \qquad \phi_{\text{out}} = a_0' + \sum_{l=0}^{\infty} \frac{b_l}{r^{l+1}} \mathcal{P}_l(\cos\theta).$$

We may select the arbitrary constant a_0' to equal zero (conveniently corresponding to $\phi \to 0$ at $r \to \infty$), while the coefficients a_0 and b_0 equal zero because the corresponding terms would describe the Coulomb field of shell's net charge $Q = \int \sigma d^2r$ (which our particular charge distribution does not have), so

$$\phi_{\text{in}} = \sum_{l=1}^{\infty} a_l r^l \mathcal{P}_l(\cos\theta), \qquad \phi_{\text{out}} = \sum_{l=1}^{\infty} \frac{b_l}{r^{l+1}} \mathcal{P}_l(\cos\theta).$$

The internal and external fields are related by two boundary conditions. The first of them is potential's continuity (necessary to avoid an infinite electric field), giving us a set of homogenous linear relations

$$a_l R^l = \frac{b_l}{R^{l+1}}, \quad \text{for } l = 1, 2, ... \qquad (*)$$

The second boundary condition may be obtained by applying the Gauss law to a small, flat pillbox drawn around a small part of the shell:

$$\left[\frac{\partial\phi_{\text{out}}}{\partial r} - \frac{\partial\phi_{\text{in}}}{\partial r}\right]_{r=R} = -\frac{\sigma}{\varepsilon_0} \equiv -\frac{\sigma_0}{\varepsilon_0}\cos\theta .$$

For all Legendre polynomials but the first one (with $l = 1$), this condition also gives homogeneous linear relations between a_l and b_l. Such a system of two homogeneous equations is satisfied by the trivial solution $a_l = b_l = 0$, and since the solution of this electrostatic problem is unique, this is *the* genuine solution. The only exception is the relation for $l = 1$, with $\mathcal{P}_l(\cos\theta) \equiv \cos\theta$, which does have a finite right-hand side:

$$-2\frac{b_1}{R^3} - a_1 = -\frac{\sigma_0}{\varepsilon_0}.$$

Solving this equation together with Eq. (*) for $l = 1$, namely $a_1R = b_1/R^2$, we get

$$a_1 = \frac{\sigma_0}{3\varepsilon_0}, \quad b_1 = \frac{\sigma_0 R^3}{3\varepsilon_0}$$

so, finally,

$$\phi_{\text{in}} = \frac{\sigma_0}{3\varepsilon_0} r\cos\theta, \quad \phi_{\text{out}} = \frac{\sigma_0 R}{3\varepsilon_0 r^2}\cos\theta.$$

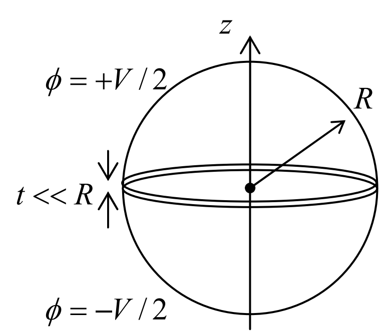
The first result describes a uniform electric field inside the shell:

$$\mathbf{E}_{\text{in}} = -\frac{\sigma_0}{3\varepsilon_0}\mathbf{n}_z,$$

which the second one, a dipole field, corresponding to the dipole moment $\mathbf{p} = (4\pi/3)\sigma_0 R\mathbf{n}_z$, outside it:

$$\mathbf{E}_{\text{out}} = \frac{\sigma_0}{3\varepsilon_0}\left(\mathbf{n}_r \frac{2\cos\theta}{r^3} + \mathbf{n}_\theta \frac{\sin\theta}{r^3}\right).$$

These results are particular manifestations of the general relations to be discussed in Sec. 3.1 of the lecture notes: of Eqs. (3.24) and (3.13), respectively.



Problem 2.29. Use the variable separation method to solve the problem already considered in Sec. 2.10 of the lecture notes: calculate the potential distribution both inside and outside of a thin spherical shell of radius R, separated with a very thin cut along the central plane $z = 0$ into two halves, with voltage V applied between them – see Fig. 2.32, partly reproduced on the right. Analyze the solution; in particular, compare the field at the z-axis, for $z > R$, with Eq. (2.218).

Hint: You may like to use the following integral of a Legendre polynomial with an odd index $l = 2n - 1 = 1, 3, 5, \ldots$:[64]

$$I_n \equiv \int_0^1 \mathcal{P}_{2n-1}(\xi)d\xi = \frac{1}{n!}\cdot\left(\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\ldots\left(\frac{3}{2} - n\right) \equiv (-1)^{n-1}\frac{(2n-3)!!}{2n(2n-2)!!}.$$

Solution: Due to the axial symmetry of the problem, we may look for the solution of the Laplace equation in the form given by Eq. (2.172) of the lecture notes. Moreover, due to the symmetry of the applied potentials, in that series, we may take $a_0 = b_0 = 0$, so the term with $l = 0$ may be dropped, giving

$$\phi(r,\theta) = \sum_{l=1}^{\infty}\left(a_l r^l + \frac{b_l}{r^{l+1}}\right)\mathcal{P}_l(\cos\theta).$$

[64] As a reminder, the *double factorial* (also called "semifactorial") operator (!!) is similar to the usual factorial operator (!), but with the product limited to numbers of the same parity as its argument – in our particular case, of odd numbers in the numerator and even numbers in the denominator.

For the internal problem (potential at $r \leq R$) we have to set all coefficients b_l to zero because, otherwise, the corresponding terms would diverge at $r \to 0$. For the coefficients a_n, the boundary conditions at $r = R$ yield the following system of equations:

$$\sum_{l=1}^{\infty} a_l R^l \mathcal{P}_l(\cos\theta) = \phi(R,\theta) \equiv \frac{V}{2}\operatorname{sgn}\left(\frac{\pi}{2}-\theta\right), \qquad \text{for } r \leq R. \qquad (*)$$

On the contrary, for the external problem ($r \geq R$), these are the coefficients a_n that have to vanish to avoid the divergence at $r \to \infty$, while the coefficients b_n have to be found from a system of equations similar to Eq. (*):

$$\sum_{l=1}^{\infty} \frac{b_l}{R^{l+1}} \mathcal{P}_l(\cos\theta) = \phi(R,\theta) \equiv \frac{V}{2}\operatorname{sgn}\left(\frac{\pi}{2}-\theta\right), \qquad \text{for } r \geq R. \qquad (**)$$

Due to this similarity, both systems may be solved similarly: by multiplying both sides by the product $\sin\theta\, \mathcal{P}_{l'}(\cos\theta)$ with an arbitrary index $l' > 0$, integrating the results over the segment $0 \leq \theta \leq \pi$, and then using the orthonormality condition (2.171). The results are, naturally, also similar:

$$a_l R^l \Big|_{r\leq R} = \frac{b_l}{R^{l+1}}\Big|_{r\geq R} = \frac{V}{2}\frac{2l+1}{2}\int_{-1}^{+1} \operatorname{sgn}(\xi)\mathcal{P}_l(\xi)\,d\xi,$$

where $\xi \equiv \cos\theta$. Since, according to Eq. (2.169) (see also Fig. 2.23) of the lecture notes, the Legendre polynomials with even indices are symmetric functions of their argument, while those with odd index are antisymmetric, the last integral vanishes for even l, while for odd $l = 2n - 1$ (where $n = 1, 2, \ldots$), it may be rewritten as

$$a_{2n-1}R^{2n-1}\Big|_{r\leq R} = \frac{b_{2n-1}}{R^{2n}}\Big|_{r\geq R} = V\left(2n-\frac{1}{2}\right)\int_0^{+1}\mathcal{P}_{2n-1}(\xi)\,d\xi \equiv V\left(2n-\frac{1}{2}\right)I_n,$$

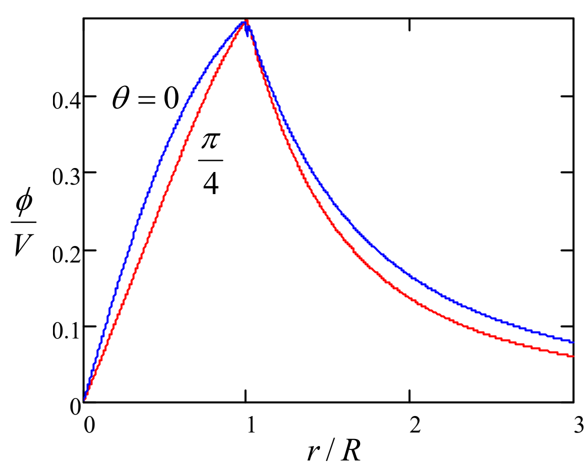
where I_n are the numbers specified in the *Hint*. Since these numbers drop very rapidly with the growth of n and also alternate their signs, the series giving the potential distribution inside the shell,

$$\phi(r,\theta) = \sum_{n=1}^{\infty} a_{2n-1} r^{2n-1}\mathcal{P}_{2n-1}(\cos\theta) = V\sum_{n=1}^{\infty}\left(2n-\frac{1}{2}\right)\left(\frac{r}{R}\right)^{2n-1} I_n \mathcal{P}_{2n-1}(\cos\theta), \qquad \text{for } r \leq R, \qquad (***)$$

and outside it,

$$\phi(r,\theta) = \sum_{n=1}^{\infty} \frac{b_{2n-1}}{r^{2n}}\mathcal{P}_l(\cos\theta) = V\sum_{n=1}^{\infty}\left(2n-\frac{1}{2}\right)\left(\frac{R}{r}\right)^{2n} I_n \mathcal{P}_{2n-1}(\cos\theta), \qquad \text{for } r \geq R, \qquad (****)$$

converge reasonably fast even for the points rather close to the shell ($r \approx R$).



For example, the figure on the right shows the resulting potential distribution along the radius r, plotted for two values of the polar angle θ. For $\theta = 0$ (i.e. along the symmetry axis z), and $r \geq R$, the result exactly coincides with that given by Eq. (2.218) of the lecture notes, obtained by the Green's function method.

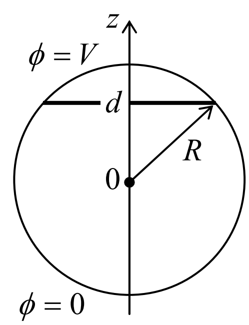
Besides giving the result for any point of the system, Eqs. (***) and (****) are very convenient for obtaining analytical asymptotic expressions for the important cases $r << R$ (the shell center's vicinity) and $r >> R$ (points far from the pipe). Indeed, in both cases, good approximations are given by the first terms of these series (with $n = 1$), so since $I_1 = ½$, we get

$$\phi(r,\theta) \to \frac{3}{4} V \cos\theta \times \begin{cases} (r/R), & \text{for } r << R, \\ (R/r)^2, & \text{for } r >> R. \end{cases}$$

The first of these expressions yields the electric field near the shell's center,

$$\mathbf{E}\big|_{r=0} = -\nabla\phi\big|_{r<R} = -\frac{3}{4}\frac{V}{R}\mathbf{n}_z,$$

while the second one generalizes Eq. (2.219) of the lecture notes for arbitrary values of the polar angle.



Problem 2.30. Calculate, up to terms $O(1/r^2)$, the long-range electric field induced by a split and voltage-biased conducting sphere – similar to that discussed in Sec. 2.10 of the lecture notes (see Fig. 2.32) and in the previous problem, but with the cut's plane at an arbitrary distance $d < R$ from the center – see the figure on the right.

Solution: Applying the variable separation method just as was done in the model solution of the previous problem, for the field outside of the sphere[65] we may write a similar expression:

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \frac{b_l}{r^{l+1}} \mathcal{P}_l(\cos\theta), \qquad \text{for } r \geq R, \qquad (*)$$

and calculate the expansion coefficients b_l in a similar way, from the following evident generalization of Eq. (**) of that solution:

$$\sum_{l=0}^{\infty} \frac{b_l}{R^{l+1}} \mathcal{P}_l(\cos\theta) = \phi(R,\theta) = V \times \begin{cases} 0, & \text{for } -1 < \cos\theta < d/R, \\ 1, & \text{for } d/R < \cos\theta < +1. \end{cases}$$

As in that problem, multiplying both parts of this equation by $P_{l'}(\cos\theta)$ and integrating the result over the surface of the sphere, we may use the orthonormality condition (2.171) to get

$$\frac{b_l}{R^{l+1}} \frac{2}{2l+1} = \int_{-1}^{+1} \mathcal{P}_l(\xi)\phi(R, \cos^{-1}\xi)d\xi = V \int_{d/R}^{+1} \mathcal{P}_l(\xi)d\xi. \qquad (**)$$

(Here $\xi \equiv \cos\theta$, so $d/R = \cos\theta_0$, where θ_0 is the polar angle of the sphere surface's cut line.)

By using Rodrigues' formula (2.169), the last integral may be expressed via the Legendre polynomials of d/R. However, we are only asked about the field at large distances, $r >> R$. This field, with the requested accuracy $O(1/r^2)$, is given by the first two terms of the expansion (*):

$$\phi(r,\theta) \approx \frac{b_0}{r}\mathcal{P}_0(\cos\theta) + \frac{b_1}{r^2}\mathcal{P}_1(\cos\theta) \equiv \frac{b_0}{r} + \frac{b_1}{r^2}\cos\theta, \qquad \text{for } r >> R, \qquad (***)$$

[65] For the field in this region, it obviously does not matter whether the conducting sphere is solid, or it is a spherical shell.

so we only need to calculate two coefficients, b_0 and b_1. For them, Eq. (**) yields simple results:

$$b_0 = \frac{1}{2} RV \int_{d/R}^{+1} d\xi = \frac{1}{2} V(R-d), \qquad b_1 = \frac{3}{2} R^2 V \int_{d/R}^{+1} \xi d\xi = \frac{3}{4} V\left(R^2 - d^2\right),$$

so Eq. (***) gives

$$\phi(r,\theta) \to \frac{1}{2} V \frac{R-d}{r} + \frac{3}{4} V \frac{R^2 - d^2}{r^2} \cos\theta, \qquad \text{for } r >> R.$$

At $d = 0$ (i.e. at the cut's plane passing through the sphere's center), the first term becomes $(1/2)VR/r$ and is appropriately twice less than that of an uncut sphere kept at potential V, while the second term coincides with Eq. (2.219) of the lecture notes, and with the solution of the previous problem. On the other hand, at $d \to R$ (i.e. in the limit on an uncut, grounded sphere), both terms vanish – as they should.

Note also that all the above calculations, and hence their results, are valid for any sign of d, with the only limitation $-R \leq d \leq +R$.

Problem 2.31. Calculate the field distribution in the simple electrostatic lens that was the subject of Problem 1.9, provided that the separation of the two field regions is provided by a thin conducting membrane, with a round hole of radius R.

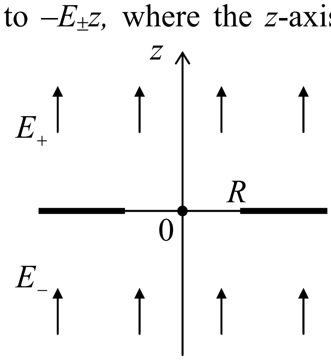
Hint: You may like to use the fact that the general axially symmetric solution of the Laplace equation in the oblate ellipsoidal coordinates (see Eqs. (2.59)-(2.60) of the lecture notes) may be represented in the following variable-separation form:

$$\phi = \sum_{n=0}^{\infty} \left[p_n \mathcal{P}_n(i \sinh\alpha) + q_n \mathcal{Q}_n(i \sinh\alpha) \right] \mathcal{P}_n(\cos\beta),$$

where p_n and q_n are constants, $\mathcal{P}_n$ are the Legendre polynomials (2.169), which are sometimes called the *Legendre functions of the first kind*, while $\mathcal{Q}_n$ are the *Legendre functions of the second kind* (briefly mentioned, in a different context, in Sec. 2.8) that may be defined by the following recurrence relations:

$$\mathcal{Q}_0(\xi) = \frac{1}{2} \ln \frac{1+\xi}{1-\xi}, \qquad \mathcal{Q}_1(\xi) = \frac{\xi}{2} \mathcal{Q}_0(\xi) - 1, \qquad \mathcal{Q}_{n>2}(\xi) = \frac{2n-1}{n} \xi \mathcal{Q}_{n-1}(\xi) - \frac{n-1}{n} \mathcal{Q}_{n-2}(\xi).$$

Solution: We may expect that far from the hole its effects are negligible, and hence the electrostatic potential tends to that of the corresponding external field, i.e. to $-E_\pm z$, where the z-axis coincides with the symmetry axis of the system, with $z = 0$ at the middle of the hole – see the figure on the right. Hence if we take the conducting membrane's potential for zero, and represent the full solution in the form



$$\phi(\mathbf{r}) = \begin{cases} -E_+ z + \tilde{\phi}_+(\mathbf{r}), & \text{for } z > 0, \\ -E_- z + \tilde{\phi}_-(\mathbf{r}), & \text{for } z < 0, \end{cases}$$

then the hole-induced perturbations $\tilde{\phi}_\pm$ have to satisfy the Laplace equation (each one, in the corresponding semi-space) and the following boundary conditions:

$$\left(\tilde{\phi}_+ - \tilde{\phi}_-\right)_{\substack{z=0\\ r\leq R}} = 0, \qquad \left(\frac{\partial \tilde{\phi}_+}{\partial z} - \frac{\partial \tilde{\phi}_-}{\partial z}\right)_{\substack{z=0\\ r\leq R}} \equiv E_+ - E_-, \qquad \left(\tilde{\phi}_\pm\right)_{r>>R} \to 0. \qquad (*)$$

(The second of these equalities is the requirement for the z-component of the full electric field to be continuous inside the hole, where there are no electric charges; the continuity of the in-plane component of the field is already ensured by the first of these conditions.)

Since at $z \to 0$ and $r < R$, the coordinate α of the oblate ellipsoidal coordinates (2.59) tends to zero as well, we may approximate z as $R\alpha\cos\beta$, so $(\partial/\partial z)_{z=0} = (1/R\cos\beta)(\partial/\partial\alpha)_{\alpha=0}$. As a result, the boundary conditions (*) may be recast as

$$\left(\tilde{\phi}_+ - \tilde{\phi}_-\right)_{\alpha=0} = 0, \qquad \left(\frac{\partial \tilde{\phi}_+}{\partial \alpha} + \frac{\partial \tilde{\phi}_-}{\partial \alpha}\right)_{\alpha=0} \equiv \left(E_+ - E_-\right)R\left|\cos\beta\right|, \qquad \left(\tilde{\phi}_\pm\right)_{\alpha\to\infty} \to 0. \qquad (**)$$

(The change of the sign at the second of the derivatives and also the modulus sign are due to the fact that at the same point of the plane $\alpha = 0$, the factor $\cos\beta$ has opposite signs for the functions $\tilde{\phi}_\pm$ because they are defined for opposite signs of $z \propto \cos\beta$.) We see that these conditions, which should be satisfied for all values of β, include this variable only via one function, $\cos\beta$. But according to Eqs. (2.169)-(2.170), this is just a specific Legendre polynomial $\mathcal{P}_1(\cos\beta)$, which is orthogonal to all other $\mathcal{P}_n(\cos\beta)$. Hence, in the series mentioned in the *Hint*, all coefficients p_n and q_n with $n \neq 1$ should equal zero, so

$$\begin{aligned}
\tilde{\phi}_\pm &= \left[(p_1)_\pm \mathcal{P}_1(i\sinh\alpha) + (q_1)_\pm \mathcal{Q}_1(i\sinh\alpha)\right]\mathcal{P}_1(\cos\beta) \\
&\equiv \left[(p_1)_\pm i\sinh\alpha + (q_1)_\pm\left(\frac{i\sinh\alpha}{2}\ln\frac{1+i\sinh\alpha}{1-i\sinh\alpha} - 1\right)\right]\cos\beta \\
&= \left\{(p_1)_\pm i\sinh\alpha + (q_1)_\pm\left[\sinh\alpha\left(\tan^{-1}\frac{1}{\sinh\alpha} - \frac{\pi}{2}\right) - 1\right]\right\}\cos\beta.
\end{aligned}$$

(The last step has used the fact that the modulus of the complex function under the logarithm equals 1, so the logarithm is equal to i multiplied by the function's phase.)

At $\alpha \to \infty$, each term of the sum in the last expressions diverges as $\sinh\alpha$, i.e. does not satisfy the last of the boundary conditions (**); however, if we take $(p_1)_\pm/(q_1)_\pm = i\pi/2$, these divergences cancel and the resulting solutions,

$$\tilde{\phi}_\pm = (q_1)_\pm\left\{\frac{\pi}{2}\sinh\alpha + \sinh\alpha\left(\tan^{-1}\frac{1}{\sinh\alpha} - \frac{\pi}{2}\right) - 1\right\}\cos\beta \equiv (q_1)_\pm\left(\sinh\alpha\,\tan^{-1}\frac{1}{\sinh\alpha} - 1\right)\cos\beta,$$

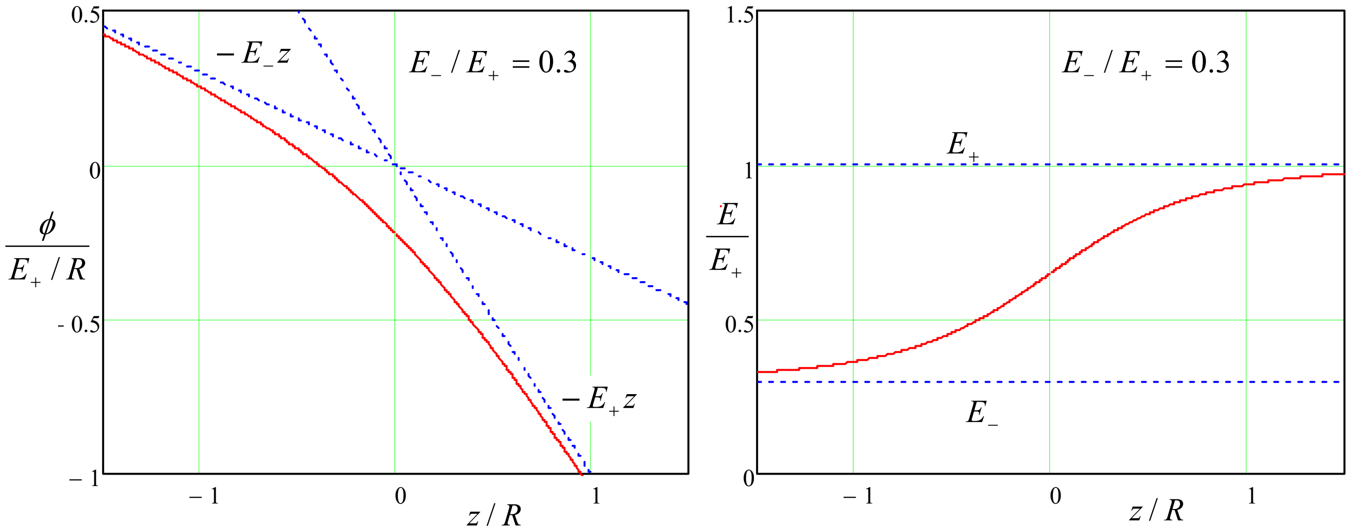
tend to zero as $(\sinh\alpha)^{-2}$ at $\alpha \to \infty$. With this, the two first boundary conditions (**) give a simple system of two linear equations for the two coefficients $(q_1)_\pm$. Solving them, we get the following final solution:[66]

$$\phi = \frac{R}{\pi}\left(E_+ - E_-\right)\left(\sinh\alpha\,\tan^{-1}\frac{1}{\sinh\alpha} - 1\right)\left|\cos\beta\right| - \begin{cases} E_+ z, & \text{for } z > 0, \\ E_- z, & \text{for } z < 0. \end{cases}$$

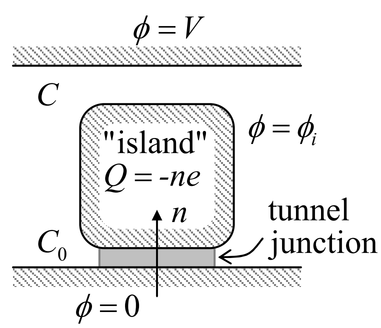
[66] It is very instructive to compare this compact solution with the much more cumbersome derivation of the same result, using the cylindrical-coordinate expansion, described in Sec. 3.13 of Jackson's *Classical Electrodynamics*.

The two panels of the figure below show this distribution of the electrostatic potential and the corresponding distribution of the electric field along the symmetry axis of the lens (where $\sinh\alpha = z/R$, $|\cos\beta| = 1$), for a particular ratio of the external field strengths. Naturally, the plots describe a smooth crossover, on a length of the order of R, between the two values of the external field (shown by the dotted lines). In particular, at the center of the system ($\alpha = \beta = 0$),

$$E = \frac{E_+ + E_-}{2}, \quad \phi = \frac{R}{\pi}(E_- - E_+).$$



Note, however, that the field outside of the hole, i.e. at $\alpha > 0$, and $z = 0$ (i.e. $\beta = \pm\pi/2$, $\cos\beta = 0$), experiences a jump, due to the membrane's surface charge induced by the field. At $\alpha >> 1$, i.e. far from the hole, this jump tends to $E_+ - E_-$.



Problem 2.32. A small conductor (in this context, usually called a *single-electron island*) is placed between two conducting electrodes, with voltage V applied between them. The gap between the island and one of the electrodes is so narrow that electrons may tunnel quantum-mechanically through this "junction" – see the figure on the right. Neglecting thermal excitation effects, calculate the equilibrium charge of the island as a function of V.

Hint: To solve this problem, you do not need to know much about the quantum-mechanical tunneling between conductors, besides that such tunneling of an electron, together with energy relaxation of the resulting excitations, may be considered a single inelastic (energy-dissipating) event.[67] At negligible thermal excitations, such an event takes place only if it decreases the total potential energy of the system.

[67] Strictly speaking, this statement, implying negligible quantum-mechanical coherence of the tunneling events, is correct only if the junction transparency is so low that its effective electric resistance R is much higher than the fundamental quantum unit of resistance, $R_Q \equiv \pi\hbar/2e^2 \approx 6.5$ kΩ – see, e.g., QM Sec. 3.2. However, this condition is satisfied in most experimental tunnel junctions.

Solution: Let us assume that the electric field induced by the island's charge Q does not extend beyond the space between the two electrodes. Then, regardless of the island's geometry, according to the linear superposition principle, the electrostatic potential ϕ_{is} of the island has to be a linear function of Q and the applied voltage V. For the convenience of what follows, we may write this linear relation in the form

$$Q = C_0\phi_{\text{is}} + C(\phi_{\text{is}} - V), \qquad \text{i.e. } \phi_{\text{is}} = \frac{Q}{C_\Sigma} + \phi_{\text{ext}}, \qquad \text{with } \phi_{\text{ext}} \equiv \frac{C}{C_\Sigma}V \quad \text{and } C_\Sigma \equiv C + C_0, \qquad (*)$$

where the coefficients C_0 and C have the physical sense of the mutual capacitances between the island and the system's electrodes (see the figure above), so C_Σ of the full self-capacitance of the island at $V = 0$. (Note that Eq. (*) is valid even if the electric fields associated with the capacitances C_0 and C are not spatially separated in full, and hence the lumped capacitor model cannot be used from the very beginning – cf. Fig. 2.5 of the lecture notes.)

Since the first term in the potential ϕ_{is} given by Eq. (*) is proportional to Q, the corresponding electrostatic energy has to be calculated by using Eq. (1.61) of the lecture notes. On the other hand, the energy associated with the externally-induced potential ϕ_{ext}, which does not depend on Q, is described by Eq. (1.54). As a result, the total electrostatic energy of the system (or, more exactly, its part dependent on Q) is

$$U(Q) = \frac{Q^2}{2C_\Sigma} + Q\phi_{\text{ext}}. \qquad (**)$$

In order to use this result, it is convenient to introduce the notion of "external charge",

$$Q_{\text{ext}} \equiv \phi_{\text{ext}}C_\Sigma \equiv CV,$$

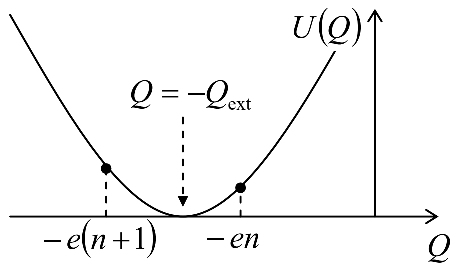
because in this notation, Eq. (**) takes a very simple form:

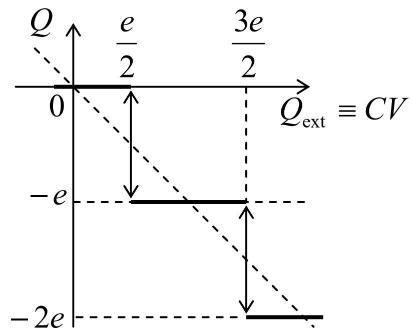
$$U(Q) = \frac{Q^2}{2C_\Sigma} + \frac{QQ_{\text{ext}}}{C_\Sigma} \equiv \frac{(Q + Q_{\text{ext}})^2}{2C_\Sigma} + \text{const}. \qquad (**)$$

The plot of this expression as a function of Q is a quadratic parabola with the minimum at $Q = -Q_{\text{ext}}$ (see the left panel of the figure below) and if Q could take any real value, the system's static equilibrium would follow this minimum. However, in reality, Q is a multiple of the fundamental charge $e \approx 1.6\times 10^{-19}$ C: $Q = -ne$, where n is the number of electrons acquired by the island since it had been electrically neutral.[68] As a result, the system's equilibrium corresponds to the value $Q = -en$ closest to $-Q_{\text{ext}}$, and the change of n by ± 1 (via tunneling of a single electron into or from the island) takes place when Q_{ext} crosses any of the critical points $e(n + ½)$ – see the right panel in the figure below.

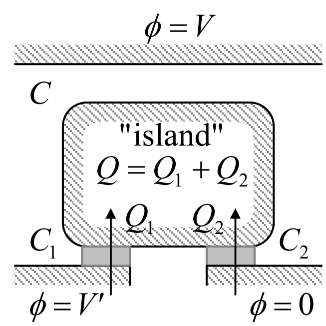
The simplest interpretation of this *Coulomb staircase* pattern is that at small C_Σ, the n extra electrons in the island, whose negative charge is not compensated by the positive charge of the atomic lattice, repel each other strongly, so the island has only as many of them as necessary to compensate the externally-induced polarization charge Q_{ext}.

[68] It is crucial that this discreteness does *not* pertain to Q_{ext}, which is just a normalized external voltage and hence is *not* quantized – at least on this scale. In other words, Q_{ext} is just the charge of polarization of the island by voltage V, which may take any real values – see the discussion in Sec. 2.9 of the lecture notes.





The most counter-intuitive aspect of this effect is that it may take place in rather macroscopic conducting islands – with a size up to a few microns, with zillions (e.g., $\sim 10^{10}$) of background electrons. However, for that, the temperature has to be low enough, to make the energy scale of the masking thermal fluctuations, k_BT, much smaller than the relevant scale of U, namely $e^2/2C_\Sigma$. As a result, the Coulomb staircase (or rather its footprint on the rf impedance of the system) was observed experimentally and explained theoretically only in the 1970s.[69] Presently, this system, called the *single-electron box*, is considered a generic device, from which quite a few other single-electron devices may be derived – see, e.g., the next problem.



Problem 2.33. The system discussed in the previous problem is now generalized as the figure on the right shows. If the voltage V' applied between the two bottom electrodes is sufficiently large, electrons can successively tunnel through two junctions of this system (called the *single-electron transistor*), carrying some dc current between these electrodes. Neglecting thermal excitations, calculate the region of voltages V and V' where such a current is fully suppressed (*Coulomb-blocked*).

Solution: Repeating the argumentation of the previous problem, for the island's electrostatic potential we may write a similar expression:

$$\phi_{\text{is}} = \frac{Q}{C_\Sigma} + \phi_{\text{ext}} + \phi_{\text{ext}}', \qquad \text{with } \phi_{\text{ext}} \equiv \frac{C}{C_\Sigma}V, \quad \phi_{\text{ext}}' \equiv \frac{C_1}{C_\Sigma}V', \quad \text{and } C_\Sigma \equiv C_1 + C_2 + C, \quad (*)$$

where the constants C_1, C_2, and C have the sense of mutual capacitances between the island and the corresponding electrodes of the system – see the figure above.

Now, writing the expression for the system's potential energy, we may follow the argumentation of the model solution of the previous problem, but should take into account that Eq. (1.54) of the lecture notes, used in that solution, is valid only if the charge q_k has been moved into the system under analysis from a region with negligible potential. If that initial potential $\phi_{k\text{-ini}}$ is different from zero, then we should generalize that relation as

$$U_{\text{ext}} = \sum_k q_k \left[\phi_{\text{ext}}(\mathbf{r}_k) - (\phi_k)_{\text{ini}}\right].$$

[69] J. Lambe and R. Jaklevic, *Phys. Rev. Lett.* **22**, 1371 (1976).

Indeed, the expression in the square brackets is just the work of external forces, and hence the potential energy's increase, at a transfer of a unit charge between the points with potentials $(\phi_k)_{\text{ini}}$ and $\phi_{\text{ext}}(\mathbf{r}_k)$. In our current problem, ϕ_{ext} is actually the sum $(\phi_{\text{ext}} + \phi_{\text{ext}}')$ defined by Eq. (*),$(\phi_k)_{\text{ini}} = V'$ for the charge Q_1 brought into the island by the electrons tunneling through the left junction, and $(\phi_k)_{\text{ini}} = 0$ for the charge Q_2 brought in through the right junction. (The total charge of the island is evidently $Q = Q_1 + Q_2$, see the figure above.) As a result, for the total electrostatic energy of the system, we get

$$U(Q_1, Q_2) = \frac{Q^2}{2C_\Sigma} + Q_1(\phi_{\text{ext}} + \phi_{\text{ext}}' - V') + Q_2(\phi_{\text{ext}} + \phi_{\text{ext}}'),$$

Now by using Eqs. (*) for ϕ_{ext} and ϕ_{ext}', we may rewrite this expression in the form

$$U(Q_1, Q_2) = \frac{(Q_1 + Q_2 + Q_{\text{ext}})^2}{2C_\Sigma} + \left(\frac{C_1}{C_\Sigma}Q_2 - \frac{C_2 + C}{C_\Sigma}Q_1\right)V' + \text{const}, \qquad \text{where } Q_{\text{ext}} \equiv CV, \quad (**)$$

which is the proper generalization of Eq. (**) in the model solution of the previous problem.

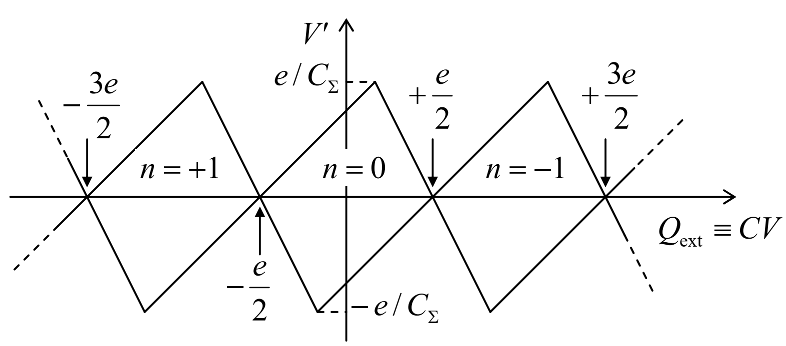
In order to use this expression for finding the dc current threshold, we should take into account the discreteness of the electron charges passed through the junctions,

$$Q_1 = -n_1 e, \quad Q_2 = -n_2 e,$$

and compare $U(-en_1, -en_2)$ with all four values of energy, $U(-en_1 \pm e, -en_2)$ and $U(-en_1, -en_2 \pm e)$, resulting from tunneling of an additional electron through either junction, in either direction. If any of these events leads to an increase of U, the state is stable, and dc current cannot flow through the system. An elementary calculation yields four stability conditions, which may be summarized as follows:

$$\left(n - \frac{1}{2}\right)e < \begin{Bmatrix} Q_{\text{ext}} + C_1 V' \\ Q_{\text{ext}} - (C_2 + C)V' \end{Bmatrix} < \left(n + \frac{1}{2}\right)e,$$

where $n \equiv n_1 + n_2$, i.e. of the total number of charge-uncompensated electrons in the island. This result may be understood easier by plotting the boundaries for these conditions on the plane of applied voltages V and V' – see the figure below. Only if the point $\{V, V'\}$ is inside one of the so-called *Coulomb diamonds* of this periodic diagram (which continues infinitely to both sides of the horizontal axis),[70] with the indicated value of n, the charge state is stable, with no persistent sequential tunneling of electrons. This stable state is said to be due to the *Coulomb blockade* of current. Note that the largest magnitude of the blockade threshold is e/C_Σ, clearly indicating its single-electron nature.



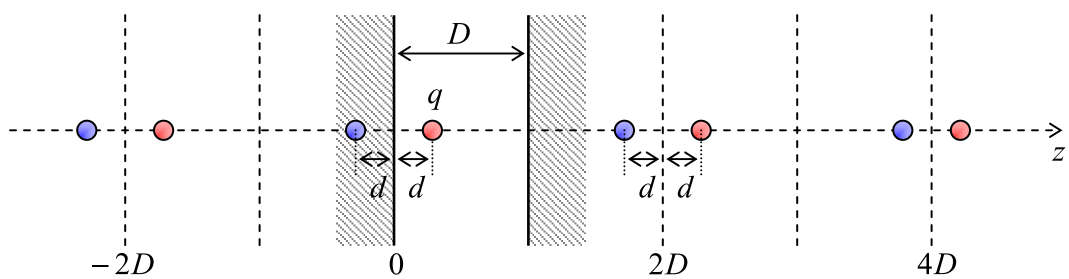
[70] In the particular case $V' = 0$, this pattern is reduced to the Coulomb staircase analyzed in the previous problem.

This result means that the gate voltage V may control the average ("dc") current in this system, so it may be indeed used as a transistor, based just on the Coulomb repulsion of electrons rather than on any specific material properties. Invented in 1985, and experimentally demonstrated for the first time in 1987, single-electron transistors are used in experimental physics, mostly for sensitive electric measurements at low temperatures. The transistor may be further modified to obtain other Coulomb-blockade devices, in particular, fundamental dc current standards and nonvolatile memory cells with digital bits stored in the form of single electrons. Unfortunately, the tough temperature-to-size trade-offs, and uncontrollable blockade threshold offsets by randomly located charged impurities have so far prevented the practical use of these devices in digital electronics.[71]

Note that similar devices with superconducting electrodes enable the controllable transfer of not only discrete single electrons, but also discrete single Cooper pairs between two or more superconducting Bose-Einstein condensates, and also an electrostatic control of supercurrents. However, due to the quantum coherence of the condensates, any quantitative discussion of these devices requires quantum mechanics.[72]

Problem 2.34. Use the charge image method to calculate the full surface charges induced in the plates of a very broad, externally-unbiased plane capacitor of thickness D by a point charge q separated from one of the electrodes by distance d. Suggest at least one alternative method to obtain the same result.

Solution: The system of image charges that satisfies the corresponding Poisson equation and the boundary conditions ($\phi(0) = \phi(D) = 0$) for this problem has already been discussed in Sec. 2.6 of the lecture notes – see Fig. 2.28c, reproduced below in an extended version. Here the red balls denote point charges $+q$ (including the real one), while the blue ones, charges $-q$.



As a result, the total charge induced in any electrode surface (say, the one located at $z = 0$) may be calculated as was discussed in Sec. 2.9:

$$Q = \int_{z=0} \sigma d^2 r = -\varepsilon_0 \int_{z=0} \frac{\partial \phi}{\partial z}\Big|_{z=0} d^2 r = -\varepsilon_0 \int_{z=0} \frac{\partial}{\partial z} \sum_j \phi_j \Big|_{z=0} d^2 r = -\varepsilon_0 \int_0^\infty \frac{\partial}{\partial z} \sum_{j,\pm} \frac{\pm q}{4\pi\varepsilon_0 \left[\rho^2 + z_j^2\right]^{1/2}}\Big|_{z=0} 2\pi\rho d\rho, \quad (*)$$

where ρ is the distance from the z-axis, and z_j is the position of an image belonging to the j^{th} pair of adjacent charges, centered at $2jD$. Per Eq. (2.194) (but also as evident from the figure above),

$$z_j = 2jD \pm d\,.$$

[71] For more discussion of these issues, see the literature cited in Sec. 2.9 of the lecture notes.
[72] See, e.g., QM Sec. 2.8.

The differentiation and integration in Eq. (*) may be swapped with the summation, making two first of these operations easy; however, the final analytical summation is not too pleasing.

A simpler way to calculate Q is to apply the Gauss law to the system of charges (including the actual charge q) shown in the figure above. Let us first consider the set of *only the "positive"*[73] *charges* ($+q$) inside a round cylinder with its symmetry axis coinciding with the z-axis, a very large radius $R >> D$, and the following special choice of position of the lids (both parallel to capacitor planes): one just to the right of the surface of our interest, i.e. at $z = +0$, while the other lid located exactly in the middle between the actual charge and the first "positive" charge image on the right of it, i.e. at

$$z = L_+ \equiv \frac{d + (2D + d)}{2} \equiv D + d \, .$$

Due to the condition $R >> D$, the electric field E_+ of the "positive" charges at the lateral (curved) wall of the Gaussian cylinder is virtually uniform, normal to the z-axis, and is the same as of a continuous charge line with uniform linear density $\lambda_+ = q/2D$. Hence, according to the (easy) solution of Problem 1.1, $E_+ = \lambda_+/2\pi\varepsilon_0$, so its flux through the wall is

$$\int_{\rho=R>>D} (E_+)_n d^2r = \frac{\lambda_+ L_+}{\varepsilon_0} \equiv \frac{q}{2D\varepsilon_0}(D + d) .$$

Generally, the electric flux through the cylinder's lids should be calculated more carefully because some parts of them are at distances of the order of $d \sim D$ from the nearest charges. However, with our choice of lid positions, the flux through the lid located at $z = L_+$ equals zero due to the symmetry, while, per Eq. (2.3), the flux through the lid with $z = +0$ is proportional to the surface charge Q_+ of the electrode, induced by the "positive" charges:

$$\int_{z=0} (E_+)_n d^2r = -\frac{Q_+}{\varepsilon_0} ,$$

where the minus sign is due to the fact that we are calculating the flux coming *out of* the considered cylinder. Now taking into account that inside this cylinder we have just one "positive" charge q, the Gauss law for it reads

$$\oint (E_+)_n d^2r \equiv \frac{q}{2D\varepsilon_0}(D + d) - \frac{Q_+}{\varepsilon_0} = \frac{q}{\varepsilon_0} ,$$

giving

$$Q_+ = -\frac{q}{2}\left(1 - \frac{d}{D}\right).$$

Now we may repeat this calculation for the system of "negative" charges ($-q$), with the replacement of L_+ with the exact middle between two such charges, for example

$$L_- \equiv \frac{-d + (2D - d)}{2} \equiv D - d ,$$

the replacement of λ_+ with $\lambda_- = -q/2D$, and taking into account that there are no "negative" charges inside this new cylinder, with $+0 \leq z \leq L_-$. The result is similar:

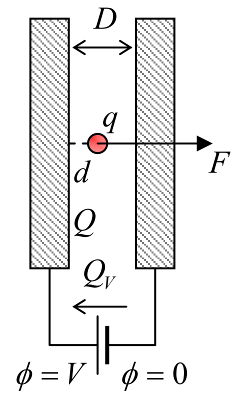
[73] I am taking this word in quotes because our calculation is correct for any sign of the actual point charge q.

$$Q_- = -\frac{q}{2}\left(1-\frac{d}{D}\right),$$

so the total surface charge $Q = Q_+ + Q_-$ of the surface is[74]

$$Q = -q\left(1-\frac{d}{D}\right). \qquad (**)$$

This result may be obtained much more easily by using the reciprocity theorem whose proof was the task of Problem 1.18. Indeed, let us take our problem's situation (with grounded capacitor electrodes, the point charge q between them, and the surface charge on the left electrode equal to Q) for the charge and potential distributions number 1, and those in the same capacitor and biased with voltage V as shown in the figure on the right but *without* the charge q, for distributions number 2. Then $\phi_1(\mathbf{r}) = 0$ at $z = 0$ and $z = D$, and $\rho_2(\mathbf{r}) = 0$ at $0 < z < D$, while, per Eq. (2.39), $\phi_2(\mathbf{r}) = V(1 - z/D)$, where z is the distance of the point $\mathbf{r}$ from the left electrode. So in the reciprocity theorem, we have to take



$$\int \rho_1(\mathbf{r})\phi_2(\mathbf{r})d^3r = QV + qV\left(1-\frac{d}{D}\right), \qquad \int \rho_2(\mathbf{r})\phi_1(\mathbf{r})d^3r = 0,$$

and the stated equality of these integrals immediately yields Eq. (**).

Finally, one more way to obtain the same result is by using the following simple reasoning. Let us assume that our plane capacitor, with the point charge inside, is connected to a battery fixing some voltage V independent of the charge position – see the figure above. Then the uniform external electric field $E = V/D$ creates a position-independent force of the magnitude $F = qE = qV/D$, which is applied to the charge and directed normally to the electrode surfaces. On a small displacement Δd, this force performs work $\Delta\mathscr{W} = F\Delta d = qV\Delta d/D$. On the other hand ("from the point of view of the battery"), the same work $\Delta\mathscr{W}$ has to be equal to $V\Delta Q_V$, where ΔQ_V is the charge that is transferred through the battery as a result of this displacement. But this charge has nowhere to go rather than to change the surface charge of the capacitor plate: $\Delta Q_V = \Delta Q$. Comparing these two expressions for $\Delta\mathscr{W}$, we get $\Delta Q = q(\Delta d/D)$, i.e. $Q = qd/D$ + const. Since the surface charge Q has to tend to $-q$ when the point charge approaches the surface ($d \to 0$) and hence becomes fully compensated by the closest surface charge, the constant in the last expression has to equal $-q$, so we return to Eq. (**).

An additional task for the reader: explain why the last two derivations of this result are not entirely independent.

Problem 2.35. Use the charge image method to calculate the potential energy of the electrostatic interaction between a point charge placed in the center of a spherical cavity that had been carved inside a grounded conductor, and the cavity's walls. Looking at the result, could it be obtained in a simpler way (or ways)?

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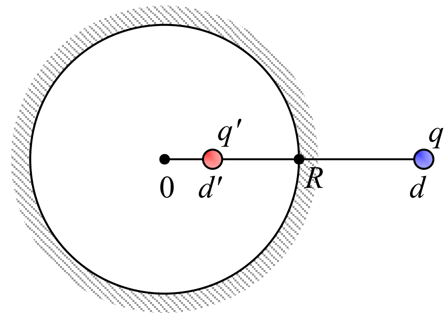
[74] With this result for Q on hand, the surface charge Q' of the counter-electrode (located at $z = D$) may be most simply calculated by applying the Gauss law to a similar cylinder, but with its lids inside the opposite capacitor's plates, so the electric field on them vanishes. This yields $Q + Q' + q = 0$, finally giving $Q' = -qd/D$.

Solution: First, let us calculate the force **F** exerted by the conductor on a point charge q' shifted by some distance $d' < R$ from the cavity's center – see the figure on the right. As it follows from the discussion of an example in Sec. 2.9 of the lecture notes (see Fig. 2.29a and its discussion), both the Poisson equation inside the cavity and the boundary conditions at the conductor's surface may be satisfied by adding, to the unperturbed Coulomb field of the real charge, that of an image charge q placed at distance $d > R$ from the center, with q and d related to q' and d' by Eqs. (2.198). Solving these equations for q and d (now functions of the given q' and d'), we get



$$d = \frac{R^2}{d'}, \qquad q = -q'\frac{d}{R} = -q'\frac{R}{d'}.$$

From here, the magnitude of the force **F** (directed toward the image charge, i.e. the point of the conducting surface, which is nearest to the real charge) is

$$F(d') = \frac{|qq'|}{4\pi\varepsilon_0}\frac{1}{(d-d')^2} = \frac{q'^2}{4\pi\varepsilon_0}\frac{R}{d'}\frac{1}{\left(R^2/d'-d'\right)^2} = \frac{q'^2}{4\pi\varepsilon_0}\frac{Rd'}{\left(R^2-d'^2\right)^2}.$$

As two sanity checks, the force vanishes at $d' \to 0$:

$$F(d') \to \frac{q'^2}{4\pi\varepsilon_0}\frac{d'}{R^3} \to 0, \qquad \text{at } \frac{d'}{R} \to 0,$$

in accord with the problem's symmetry, while at $d' \to R$, it tends to the expression

$$F_{\text{plane}}(R-\delta) = \frac{q'^2}{4\pi\varepsilon_0}\frac{1}{(2\delta)^2}, \quad \text{where } \delta \equiv R - d',$$

which we would get for the charge at the distance δ from a conducting plane.

Now the required interaction energy U may be defined as the difference between the energy of the (charge + sphere) system and the sum of energies of these two objects when they are far from each other. Hence, U may be calculated as the (minus) integral of the interaction force on a charge's path from infinity to the cavity center. Our results for the force **F** are only valid inside the sphere, but we may circumvent this problem by extending the integration from the center to some distance $\delta << R$ from the surface, and subtracting from the result the potential energy (2.191) provided by at that point by a conducting plane (reader, explain why):

$$U(0) = -\int_0^{R-\delta} F(x)dx - U_{\text{plane}}(R-\delta) = -\frac{q'^2}{4\pi\varepsilon_0}\left[\int_0^{R-\delta}\frac{R^2 x}{\left(R^2-x\right)^2}dx - \frac{1}{4\delta}\right] = -\frac{q'^2}{8\pi\varepsilon_0 R}. \qquad (*)$$

According to Eq. (2.192), the magnitude of this energy is twice larger than what a conducting plane would give at the distance R from the charge. The reason for such difference is clear: for a spherical cavity, all induced surface charges are at the same distance (R) from the initial charge, while only the closest charges on the conducting plane are at such distance, so the effect of their attraction is smaller.

A much simpler way to get Eq. (*) is to use Eq. (1.65) for the electric field energy, limiting the integral to the cavity interior, $r < R$ (since there is no field in the conductor). The divergence of the integral at $r \to 0$ disappears if U is understood as above, i.e. as the result of the difference between the charge's field in the cavity and that in the free space:

$$U \equiv \frac{\varepsilon_0}{2}\left(\int_{\text{cavity}} E^2_{\text{in cavity}}\, d^3r - \int_{\text{all space}} E^2_{\text{in free space}}\, d^3r\right) = \frac{\varepsilon_0}{2}\left(\int_{r<R}\left(E^2_{\text{in cavity}} - E^2_{\text{in free space}}\right) d^3r - \int_{r>R} E^2_{\text{in free space}}\, d^3r\right).$$

According to the Gauss law, if a charge is in the center, its field at $r < R$ is not affected by the conductor, so the first integral in the last expression vanishes, and we get

$$U = -\frac{\varepsilon_0}{2}\int_{r>R} E^2_{\text{in free space}}\, d^3r = -\frac{\varepsilon_0}{2}\int_R^\infty \left(\frac{q'}{4\pi\varepsilon_0 r^2}\right)^2 4\pi r^2 dr = -\frac{q'^2}{8\pi\varepsilon_0 R},$$

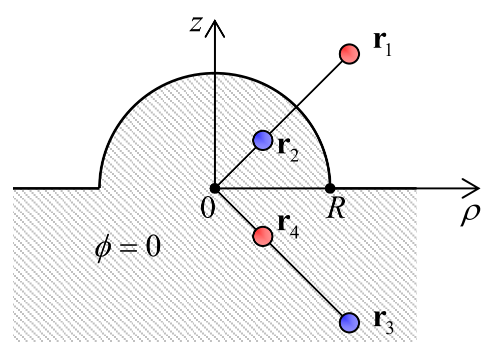
i.e. the same result as given by Eq. (*).

One more way to obtain the same result is to first calculate the small energy increment δU due to a small addition $\delta q << q$ to the charge, brought from infinity to the initial charge's location, i.e. the center of the sphere, by replacing the conducting sphere with a thin spherical shell with a uniformly distributed charge q. (Physically, such replacement of the conductor by the charge of its external surface is equivalent to the subtraction made at the previous approach.) Since, the due to the spherical symmetry, the shell's charge does not create an electric field inside it,

$$\delta U = -\int_0^\infty \delta F_{\text{shell}}\, dr = -\delta q\int_0^\infty E_{\text{shell}}\, dr = -\delta q\int_R^\infty E_{\text{shell}}\, dr = -\delta q\int_R^\infty \frac{q}{4\pi\varepsilon_0 r^2}\, dr = -\frac{q\,\delta q}{4\pi\varepsilon_0 R} \equiv \delta\left(-\frac{q^2}{8\pi\varepsilon_0 R}\right).$$

Integrating the result from $q = 0$ to $q = q'$, we again arrive at Eq. (*).

Problem 2.36. Use the method of charge images to find the Green's function of the system shown in the figure on the right, where the bulge on the (otherwise, plane) surface of a conductor has the shape of a semi-sphere of radius R.



Solution: Let a (real) charge q_1 be at point $\mathbf{r}_1$ in the free-space part of the system. Then, according to Eq. (2.198) of the lecture notes, all the boundary conditions may be satisfied using the three charge images shown in the figure on the right, with

$$q_2 = -q_1\frac{R}{r_1}, \quad \mathbf{r}_2 = \left(\frac{R}{r_1}\right)^2 \mathbf{r}_1; \qquad q_3 = -q_1, \quad \boldsymbol{\rho}_3 = \boldsymbol{\rho}_1, \quad z_3 = -z_1; \qquad q_4 = -q_3\frac{R}{r_3}, \quad \mathbf{r}_4 = \left(\frac{R}{r_3}\right)^2 \mathbf{r}_3;$$

where $\boldsymbol{\rho}_j$ are the horizontal components of the radius vectors $\mathbf{r}_j$ with the origin in the sphere's center. As a result, the Green's function may be represented simply as

$$G(\mathbf{r},\mathbf{r}_1) = \sum_{j=1}^{4}\frac{q_j/q_1}{\left|\mathbf{r}-\mathbf{r}_j\right|}.$$

However, when spelled out in the most suitable, cylindrical coordinates where $r_1 = (\rho_1^{\,2} + z_1^{\,2})^{1/2}$, this compact expression becomes somewhat bulky, because each denominator looks like

$$\left|\mathbf{r}-\mathbf{r}_j\right| = \left[\rho^2+\rho_j^2-2\rho\rho_j\cos\left(\varphi-\varphi_j\right)+\left(z-z_j\right)^2\right]^{1/2}.$$

Problem 2.37.* Use the spherical inversion expressed by Eq. (2.198) of the lecture notes to develop an iterative method for a more and more precise calculation of the mutual capacitance between two similar conducting spheres of radius R, with their centers separated by distance $d > 2R$.

Solution: There are several ways to develop the requested iterative process; the following one uses perhaps the most natural logic. At each step of the process, let us keep the total charge of the right sphere equal to $+Q$, and that of the left sphere equal to $-Q$, and keep the surface of each sphere equipotential (at potentials ϕ_L and ϕ_R, correspondingly), while calculating more and more exact values of these potentials, and hence that of the mutual capacitance

$$C \equiv \frac{Q}{V} = \frac{Q}{\phi_R - \phi_L}, \qquad (*)$$

by replacing the genuine distribution of the charge on the surface of each sphere with more and more extended systems of point image charges. At the 0th iteration, the distributed charges of sphere surfaces are replaced with single point charges $q_R^{(0)} = +Q$ and $q_L^{(0)} = -Q$, located at their centers. (This approximation correctly describes the field distribution at very large distances, i.e. it is asymptotically correct at $d/R \to \infty$.) In this approximation, the potentials of the sphere surfaces are

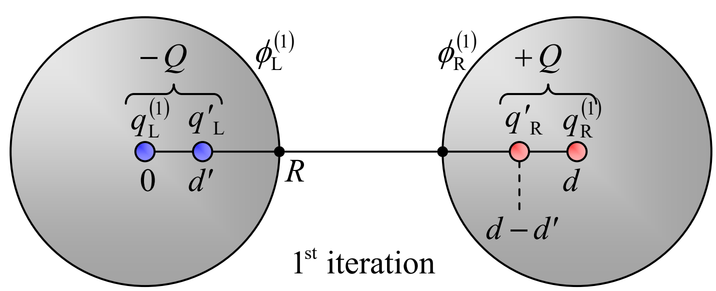
$$\phi_R^{(0)} = \frac{q_R^{(0)}}{4\pi\varepsilon_0 R} = +\frac{Q}{4\pi\varepsilon_0 R}, \qquad \phi_L^{(0)} = \frac{q_L^{(0)}}{4\pi\varepsilon_0 R} = -\frac{Q}{4\pi\varepsilon_0 R},$$

so their mutual capacitance,

$$C^{(0)} = \frac{Q}{\phi_R^{(0)} - \phi_L^{(0)}} = 2\pi\varepsilon_0 R,$$

does not depend on the distance d.

In order to make the 1st iteration, let us calculate the corrected value of ϕ_L using the fact proved in Sec. 2.9 of the lecture notes: if the field is induced by a point charge $q_R^{(0)}$ located at a distance $d > R$ from the sphere's center, then the potential of the sphere's surface equals zero if its genuine surface charge is replaced with a point charge $q'_L = -(R/d)q_R^{(0)} = -(R/d)Q$, located at the distance $d' = R^2/d$ from the center – see Eqs. (2.198) and the figure on the right. Now, following our general commitment, let us correct the central point charge q_L to such a value $q_L^{(1)}$ that the sum of all charges inside the left sphere is equal to $-Q$:



$$q_L^{(1)} + q'_L = -Q, \qquad \text{giving } q_L^{(1)} = -Q - q'_L = -Q + \frac{R}{d}Q = -Q\left(1-\frac{R}{d}\right).$$

This charge still keeps the surface potential of the left sphere constant, but changes its value to

$$\phi_{\rm L}^{(1)} = \frac{q_{\rm L}^{(1)}}{4\pi\varepsilon_0 R} = -\frac{Q}{4\pi\varepsilon_0 R}\left(1-\frac{R}{d}\right).$$

Generally, we would need to calculate a similar adjustment of the charge $q_{\rm R}$ and the potential $\phi_{\rm R}$, but due to the symmetry of our problem, the results are evident:

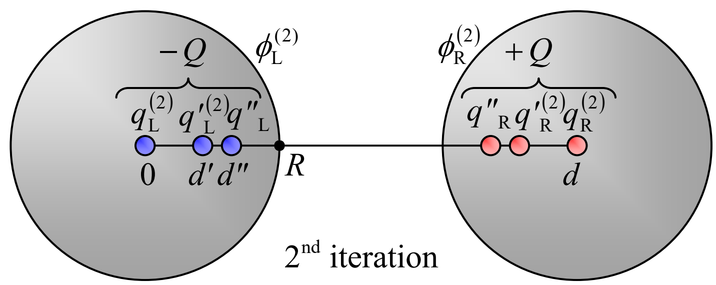
$$q_{\rm R}^{(1)} = -q_{\rm L}^{(1)} = Q\left(1-\frac{R}{d}\right), \qquad \phi_{\rm R}^{(1)} = -\phi_{\rm L}^{(1)} = \frac{Q}{4\pi\varepsilon_0 R}\left(1-\frac{R}{d}\right).$$

Now we may calculate the mutual capacitance in the 1st approximation:

$$C^{(1)} = \frac{Q}{\phi_{\rm R}^{(1)} - \phi_{\rm L}^{(1)}} = 2\pi\varepsilon_0 R \frac{1}{1-R/d}; \qquad (**)$$

this expression shows that C increases as the distance d between the spheres is reduced – as it should.

For the next, 2nd iteration, we have to use the same approach to correct the values of $\phi_{\rm L}$ and $\phi_{\rm R}$, by taking into account the corrected values of the fields of the two central charges, $q_{\rm L}^{(1)}$ and $q_{\rm R}^{(1)}$, as well as the existence of the image charges $q'_{\rm L}$ and $q'_{\rm R}$. According to Eq. (2.198), the latter step requires the introduction of new image charges, $q''_{\rm L}$ and $q''_{\rm R}$, located at different distances, $d'' = R^2/(d - d') = R^2/(d - R^2/d) \equiv R^2 d/(d^2 - R^2) > d'$, from the centers of the respective spheres – see the figure on the right.



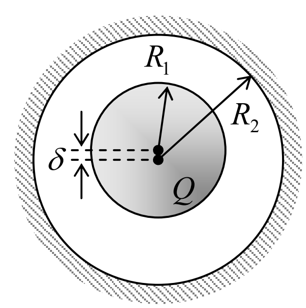
This process may be continued similarly, by adding, at each iteration, one more image charge inside each sphere, and then readjusting the prior charges and the surface potential. However, even the initial iteration (**) gives surprisingly accurate results if the ratio R/d is not too close to the convergence limit $(R/d)_{\max} = ½$; for example, even for d as small as $4R$ (i.e. with the gap between the spheres as small as the sphere's diameter), its error is close to 0.5%, and only at $d = 3R$, it increases to slightly more than 2%.

An additional task for the reader: by using the approach described at the end of the model solution of Problem 13 (but adjusted for the 3D geometry of our current problem), calculate an expression for C, which would be asymptotically correct in the limit $t \equiv d - 2R << R$.

And just for the reader's reference: by using the bispherical coordinates briefly mentioned in the solution of Problem 13, C may be calculated in the form of an explicit series:

$$C = 2\pi\varepsilon_0 \sum_{n=1}^{\infty} \frac{\sinh\left\{\ln\left[x + \left(x^2-1\right)^{1/2}\right]\right\}}{\sinh\left\{n\ln\left[x + \left(x^2-1\right)^{1/2}\right]\right\}}, \quad \text{where } x \equiv \frac{d}{2R} > 1.$$

Problem 2.38.* A conducting sphere of radius R_1, carrying an electric charge Q, is placed inside a spherical cavity of radius $R_2 > R_1$, carved inside another bulk conductor. Calculate the force exerted on the inner sphere if its center is displaced by a small distance $\delta << R_1, R_2 - R_1$ from that of the cavity – see the figure on the right.



Solution: As it follows from the solution of the previous problem, this problem could be solved (for an arbitrary $\delta < R_2 - R_1$) by building up a sequence of point image charges at the following distances from the *cavity's* center (here the term "positive" means a charge with the same sign as Q):

$$\begin{array}{ll} \text{positive charges} & \text{negative charges} \\ d = \delta \ \ (\text{charge } Q), & D = R_2^2 / \delta, \\ d' = \delta + R_1^2 / (D - d), & D' = R_2^2 / d', \\ d'' = \delta + R_1^2 / (D' - d), & D'' = R_2^2 / d'', \ldots, \end{array}$$

with the concurrent calculation/readjustment of their magnitudes. However, this solution would be in the form of an algorithm rather than an explicit series. Another possible analytical approach is using the bispherical coordinates, but for our current case of two unequal radii R_1 and R_2, it leads to calculations and results much more cumbersome than the last formula cited in the previous problem's solution.

However, for our assignment, i.e., for $\delta << R_1, R_2 - R_1$, a simple explicit result may be obtained – for example, in the following way. In the reference frame whose origin coincides with the center of the cavity, and the z-axis directed along the shift δ, the surface of the inner sphere is described by the following relation:

$$x'^2 + y'^2 + (z' - \delta)^2 = R_1^2,$$

where the prime signs distinguish a point on the sphere's surface from an arbitrary point in space. In the spherical coordinates, we may rewrite this relation as

$$(r' \sin\theta \cos\varphi)^2 + (r' \sin\theta \sin\varphi)^2 + (r' \cos\theta - \delta)^2 = R_1^2, \qquad \text{giving } \ r'^2 \sin^2\theta + (r' \cos\theta - \delta)^2 = R_1^2.$$

In the limit $\delta/R_1 \to 0$, the last expression reduces to the first-order approximation

$$r' = R_1 + \delta \cos\theta. \qquad (*)$$

Now let us calculate the potential distribution $\phi(r, \theta)$ between the spheres, taking into account its initial (radially-symmetric) part plus the first nonvanishing perturbation, which is, per Eq. (*), proportional to $\delta\cos\theta$. This means that in the general axially symmetric solution (2.172) of the Laplace equation we may drop all terms with $l > 1$:

$$\phi(r, \theta) = a_0 + \frac{b_0}{r} + \left(a_1 r + \frac{b_1}{r^2} \right) \cos\theta, \qquad \text{with } \ a_1, b_1 \propto \delta. \qquad (**)$$

Plugging this distribution into the appropriate boundary conditions,

$$\phi(r', \theta) \equiv \phi(R_1 + \delta\cos\theta, \theta) = \phi_0, \qquad \phi(R_2, \theta) = 0,$$

(where ϕ_0 is the so far unknown potential of the inner sphere, and the potential of the outer conductor is taken for zero), we get two equations,[75]

[75] Taking into account the variation, $\delta\cos\theta$, of radius r_1 in the terms with a_1 and b_1 would give us additional terms of the order of $(\delta\cos\theta)^2$ and higher, which may be ignored in our linear approximation.

$$a_0 + \frac{b_0}{R_1 + \delta\cos\theta} + \left(a_1 R_1 + \frac{b_1}{R_1^2}\right)\cos\theta \approx a_0 + \left(\frac{b_0}{R_1} - \frac{b_0\delta}{R_1^2}\cos\theta\right) + \left(a_1 R_1 + \frac{b_1}{R_1^2}\right)\cos\theta = \phi_0,$$

$$a_0 + \frac{b_0}{R_2} + \left(a_1 R_2 + \frac{b_1}{R_2^2}\right)\cos\theta = 0,$$

which should be satisfied at all θ. This requirement gives us a system of four equations for four unknown coefficients a and b:

$$a_0 + \frac{b_0}{R_1} = \phi_0, \qquad a_1 R_1 + \frac{b_1}{R_1^2} = \frac{b_0\delta}{R_1^2}; \qquad a_0 + \frac{b_0}{R_2} = 0, \qquad a_1 R_2 + \frac{b_1}{R_2^2} = 0. \qquad (***)$$

The equations for a_0 and b_0, and hence their solution,

$$a_0 = -\phi_0 \frac{R_1}{R_2 - R_1}, \qquad b_0 = \phi_0 \frac{R_1 R_2}{R_2 - R_1},$$

do not depend on a_1 and b_1. Before calculating a_1 and b_1, let us express the potential ϕ_0 of the inner sphere via its charge Q, by applying the Gauss law to a spherical surface of some radius r within the range $r_1 < r < R_2$:

$$\oint_S E_n d^2r \equiv -\oint_S \frac{\partial\phi}{\partial r} d^2r = \frac{Q}{\varepsilon_0}.$$

For the potential distribution given by Eq. (**), the surface integral equals

$$\oint_S \left[\frac{b_0}{r^2} - \left(a_1 - 2\frac{b_1}{r^3}\right)\cos\theta\right] d^2r = 2\pi r^2 \int_0^\pi \left[\frac{b_0}{r^2} - \left(a_1 - 2\frac{b_1}{r^3}\right)\cos\theta\right]\sin\theta d\theta = 4\pi b_0 = 4\pi\phi_0 \frac{R_2 - R_1}{R_1 R_2},$$

so the relation between ϕ_0 and Q is (in the linear approximation in δ only!) the same as at $\delta = 0$:[76]

$$\phi_0 = \frac{Q}{C}, \qquad \text{where } C = 4\pi\varepsilon_0 \frac{R_1 R_2}{R_2 - R_1} \equiv 4\pi\varepsilon_0 \left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

– cf. Eq. (2.56) of the lecture notes. Presently, we need this result only to express the coefficient b_0 via the given charge Q:

$$b_0 = \phi_0 \frac{R_1 R_2}{R_2 - R_1} = \frac{Q}{4\pi\varepsilon_0}.$$

Using this expression and solving the remaining two equations (***), we get

$$a_1 = -b_0 \frac{1}{R_2^3 - R_2^3}\delta = -\frac{Q}{4\pi\varepsilon_0}\frac{1}{R_2^3 - R_1^3}\delta, \qquad b_1 = b_0 \frac{R_2^3}{R_2^3 - R_2^3}\delta = \frac{Q}{4\pi\varepsilon_0}\frac{R_2^3}{R_2^3 - R_1^3}\delta,$$

so the spatial distribution of the radial electric field is

[76] This fact may be also proved by the following reasoning. The capacitance C between the spheres has to be a smooth function of δ, which may be represented as the Taylor series $C(\delta) = C(0) + (dC/d\delta)_{\delta=0}\delta + \ldots$. However, due to the evident mirror symmetry of the system with respect to the plane $\delta = 0$, the change of the sign of δ cannot result in a change of C, and hence $(dC/d\delta)_{\delta=0} = 0$.

$$E_r = -\frac{\partial\phi}{\partial r} = \frac{b_0}{r^2} + \left(-a_1 + 2\frac{b_1}{r^3}\right)\cos\theta = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r^2} + \frac{1}{R_2^3 - R_1^3}\left(1 + 2\frac{R_2^3}{r^3}\right)\delta\cos\theta\right].$$

Now let us use this field distribution to calculate E_r at the shifted surface of the inner sphere, i.e. at $r = r' = R_1 + \delta\cos\,\theta$. The small difference between r' and R_1 may be taken into account only in the largest, δ-independent term of the above expression for E_r, and only in the first, linear approximation:

$$\frac{1}{r'^2} = \frac{1}{(R_1 + \delta\cos\theta)^2} \approx \frac{1 - 2(\delta/R_1)\cos\theta}{R_1^2} \equiv \frac{1}{R_1^2} - \frac{2}{R_1^3}\delta\cos\theta\,.$$

(In all other terms, the difference between r' and R_1 may be ignored, because they are already proportional to δ.) As a result, we get

$$E_r\Big|_{r=r'} \approx \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{R_1^2} - \frac{2}{R_1^3}\delta\cos\theta + \frac{1}{R_2^3 - R_1^3}\left(1 + 2\frac{R_2^3}{R_1^3}\right)\delta\cos\theta\right] \equiv \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1^2} + \frac{3\delta\cos\theta}{R_2^3 - R_1^3}\right).$$

Per the solution of Problem 1, the normal force f_n per unit area of the inner sphere may be calculated as

$$\frac{dF_n}{dA} = \frac{\varepsilon_0}{2}E_r^2\Big|_{r=r'} = \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0}\right)^2\left(\frac{1}{R_1^2} + \frac{3\delta\cos\theta}{R_2^3 - R_1^3}\right)^2.$$

In the first, linear approximation in small $\delta\cos\theta$:

$$\frac{dF_n}{dA} = \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0 R_1^2}\right)^2\left(1 + \frac{6R_1^2\delta\cos\theta}{R_2^3 - R_1^3}\right).$$

Due to the axial symmetry of the problem, the net force may have only one (z-) component:

$$F = F_z = \oint_{r_1}\frac{dF_z}{dA}d^2r = \oint_{r_1}\frac{dF_n}{dA}\cos\theta\, d^2r = 2\pi R_1^2\int_0^\pi f_n\cos\theta\sin\theta d\theta.$$

At the integration, the net contribution of the unperturbed, radially-symmetric normal forces vanishes, while the first-perturbation term yields a very simple final result:

$$F = 2\pi R_1^2\,\frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0 R_1^2}\right)^2\frac{6R_1^2\delta}{R_2^3 - R_1^3}\int_0^\pi\cos^2\theta\sin\theta d\theta = 2\pi\frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0}\right)^2\frac{6\delta}{R_2^3 - R_1^3}\frac{2}{3} \equiv \frac{Q^2}{4\pi\varepsilon_0}\frac{\delta}{R_2^3 - R_1^3}.$$

As the simplest sanity check, at $R_1 \to 0$, when the inner sphere turns into a point charge, the result is reduced exactly to the corresponding limit of the byproduct of the solution of Problem 35:

$$F = \frac{Q^2}{4\pi\varepsilon_0}\frac{\delta}{R_2^3},$$

obtained there by using the charge image method. Another check may be obtained in the opposite limit, $R_1 \to R_2$, by calculating F using a combination of Eq. (2.28) with the approach demonstrated at the end of the model solution of Problem 13. (This additional simple exercise is highly recommended to the reader.)

Problem 2.39. Within the simple models of the electric field screening in conductors, discussed in Sec. 2.1 of the lecture notes, analyze the partial screening of the electric field of a point charge q by a planar conducting film of constant thickness $t << \lambda$, where λ is (depending on the charge carrier statistics) either the Debye or the Thomas-Fermi screening length – see, respectively, Eqs. (2.8) or (2.10). Assume that the distance d between the charge and the film is much larger than t.

Solution: Due to the condition $d >> t$, the gradient of the electrostatic potential inside the film is dominated by its component along the axis (say, z) normal to the film's plane, so we still may use the 1D equation (2.7):

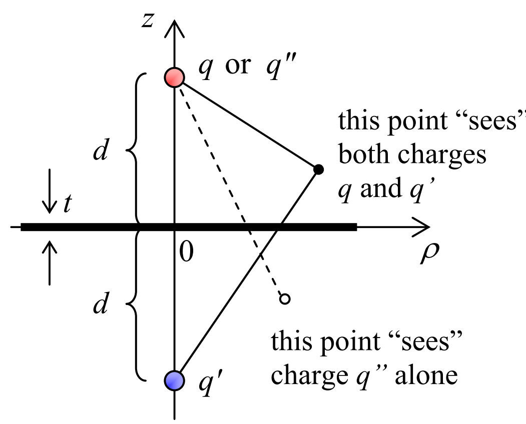
$$\frac{\partial^2 \phi}{\partial z^2} = \frac{1}{\lambda^2}\phi,$$

even if ϕ is relatively slowly (on distances of the order of d) changing in the plane of the film as well. Integrating both sides of this equation over the film's thickness t, we get

$$\frac{\partial \phi_+}{\partial z} - \frac{\partial \phi_-}{\partial z} = \frac{1}{\lambda^2}\int_{-t/2}^{+t/2} \phi dz, \qquad (*)$$

where the indices ± refer to the two surfaces of the film. This equation shows that if $t << \lambda$, the film can create only a small difference (of the order of $\phi\, t/\lambda^2$) of the derivatives, and hence just a negligible (of the order of $\phi\, t^2/\lambda^2$) relative variation of the potential inside it. As a result, Eq. (*) yields two boundary conditions:

$$\phi_+ = \phi_- \equiv \phi, \qquad \frac{\partial \phi_+}{\partial z} - \frac{\partial \phi_-}{\partial z} = \frac{t}{\lambda^2}\phi. \qquad (**)$$



Now, inspired by the charge-image analysis of the basic problem shown in Fig. 2.26 of the lecture notes, we may try to describe the field at $z > 0$ as a superposition of the fields by the real charge q and by its image q' located at the same distance d from the film but on its opposite side – see the figure on the right. However, now q' is not necessarily equal to $(-q)$; indeed, if $t \to 0$, all the film's effects, in particular the charge q', should vanish.

In addition, in contrast to the conducting half-space case, we should expect to find some field remnants on the other side of the film, at $z < 0$. This field cannot be contributed by the image charge q' because it would provide the potential's divergence at its location. Thus, in that half-space, we should try to use the real point source only, but with a re-normalized charge q'' rather than the genuine charge q – see the figure above. As a result, we may look for the potential's distribution in the form[77]

$$\phi(\rho, z) = \frac{1}{4\pi\varepsilon_0} \times \begin{cases} \dfrac{q}{\left[\rho^2 + (z-d)^2\right]^{1/2}} + \dfrac{q'}{\left[\rho^2 + (z+d)^2\right]^{1/2}}, & \text{for } z > 0, \\ \dfrac{q''}{\left[\rho^2 + (z-d)^2\right]^{1/2}}, & \text{for } z < 0. \end{cases}$$

[77] As will be discussed in Sec. 3.4 of the lecture notes, absolutely the same solution (though with different constants q' and q'') describes the electric field's penetration into a half-space filled with a linear dielectric.

(In the "coarse-grain" picture described by this formula, with the size scale given by $d >> t$, the film's thickness is negligible.)

Plugging this solution into the boundary conditions (**), we see that they are indeed satisfied (so this is indeed the unique solution of our boundary problem), provided that the effective charges q' and q'' obey the following two relations:

$$q + q' = q'', \qquad (q - q') - q'' = \frac{td}{\lambda^2} q''.$$

Solving this simple system of two linear equations, we get, in particular, the following expression for the q''/q ratio – which is, obviously, a relevant measure of the field's penetration behind the film:

$$\frac{q''}{q} = \frac{1}{1 + td/2\lambda^2}.$$

As the sanity check, q'' tends to q at $t \to 0$, and to zero at $t \to \infty$ – as it should. However, the most interesting feature of this result is that the ratio q''/q starts to drop as soon as t is increased to $\sim\lambda^2/d << \lambda$, i.e. the field behind the film may be well (though not exponentially well) screened even by a film *much thinner* than the screening length λ. Very unfortunately, the fact of such efficient screening of the electric field by thin conducting films (and of the similarly efficient shielding of the magnetic field by thin ferromagnetic and superconducting films[78]) is overlooked even in some very popular textbooks.

Problem 2.40. Prove the following expansion of the simplest Green's function (2.204) into a series over the Legendre polynomials:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r_>} \sum_{l=0}^{\infty} \left(\frac{r_<}{r_>} \right)^l \mathcal{P}_l(\cos\theta),$$

where $r_>$ is the largest of the two scalars $r \equiv |\mathbf{r}| \geq 0$ and $r' \equiv |\mathbf{r}'| \geq 0$, while $r_<$ is the smallest of them.

Solution: Let us direct the z-axis along the radius vector $\mathbf{r}'$. Then the Green's function (2.204), considered as a function of $\mathbf{r}$ alone, has to be axially symmetric and its Legendre-polynomial expansion is given by the general Eq. (2.172):

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \left(a_l r^l + \frac{b_l}{r^{l+1}} \right) \mathcal{P}_l(\cos\theta), \qquad \text{for } \mathbf{r}' = \mathbf{n}_z r', \qquad (*)$$

where in our case, the polar angle θ is also the angle between the radius vectors $\mathbf{r}$ and $\mathbf{r}'$. To find the coefficients a_l and b_l, let us consider the particular case when the point $\mathbf{r}$ is positioned on the z-axis as well: $\mathbf{r} = \mathbf{n}_z r$. In this case, Eq. (*) is reduced to

$$\frac{1}{|r - r'|} = \sum_{l=0}^{\infty} \left(a_l r^l + \frac{b_l}{r^{l+1}} \right). \qquad (**)$$

Let us first consider the case $r < r'$, and Taylor-expand the left-hand side of this relation in r at point $r = 0$:

[78] See, e.g., Secs. 5.6. 6.3, and 6.4 of the lecture notes.

$$\frac{1}{r'-r} = \sum_{l=0}^{\infty} \frac{1}{l!} \frac{\partial^l}{\partial r^l} \left(\frac{1}{r'-r} \right)_{r=0} r^l \,. \qquad (***)$$

Calculating the involved derivatives one by one,

$$\frac{\partial^0}{\partial r^0} \left(\frac{1}{r'-r} \right)_{r=0} = \frac{1}{(r'-r)_{r=0}} = \frac{1}{r'},$$

$$\frac{\partial^1}{\partial r^1} \left(\frac{1}{r'-r} \right)_{r=0} = \frac{1}{(r'-r)^2{}_{r=0}} = \frac{1}{r'^2},$$

$$\frac{\partial^2}{\partial r^2} \left(\frac{1}{r'-r} \right)_{r=0} = \frac{\partial}{\partial r} \frac{1}{(r'-r)^2}_{r=0} = \frac{1 \cdot 2}{(r'-r)^3}_{r=0} = \frac{1 \cdot 2}{r'^3}, \ldots,$$

we see that

$$\frac{\partial^l}{\partial r^l} \left(\frac{1}{r'-r} \right)_{r=0} = \frac{l!}{r'^{(l+1)}},$$

so the expansion (***) becomes

$$\frac{1}{r'-r} = \sum_{l=0}^{\infty} \frac{r^l}{r'^{(l+1)}} \equiv \frac{1}{r'} \sum_{l=0}^{\infty} \left(\frac{r}{r'} \right)^l, \qquad \text{for } r < r'.$$

For the opposite case $r' < r$, we may perform a similar expansion in r', getting

$$\frac{1}{r-r'} = \sum_{l=0}^{\infty} \frac{r'^l}{r'^{(l+1)}} \equiv \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{r'}{r} \right)^l, \qquad \text{for } r' < r.$$

Comparing these two expressions with Eq. (**), we see that

$$a_l = \begin{cases} 1/r'^{(l+1)}, & \text{for } r < r', \\ 0, & \text{for } r' < r, \end{cases} \qquad b_l = \begin{cases} 0, & \text{for } r < r', \\ r'^l, & \text{for } r' < r. \end{cases}$$

This equality, together with Eq. (*), proves the expansion given in the assignment.

Just for the reader's reference: for the general direction of the z-axis, this Green's function may be represented as an expansion over the spherical harmonics (mentioned in Sec. 2.8):

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{4\pi}{r_>} \sum_{l=0}^{\infty} \frac{1}{2l+1} \left(\frac{r_<}{r_>} \right)^l \sum_{m=-l}^{+l} \left[Y_l^m(\theta', \varphi') \right]^* Y_l^m(\theta, \varphi).$$

Problem 2.41. Use the expansion that was the subject of the previous problem to confirm the analysis, in Sec. 2.9 of the lecture notes, of the system shown in Fig. 2.29: a grounded conducting sphere of radius R and a point charge q located at distance $d > R$ from its center.

Solution: For the confirmation, let us calculate the potential of the sphere itself, induced by the suggested system of charges. In the spherical coordinates used in Fig. 2.29, the distance $r' = d$ of the point-charge source from the origin is larger than $r = R$ of any point on the sphere's surface. So, in the expansion derived in the previous problem's solution, we have to take $r_> = d$ and $r_< = R$, and it yields

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{d}\sum_{l=0}^{\infty}\left(\frac{R}{d}\right)^{l}\mathcal{P}_l(\cos\theta).$$

Hence, according to Eq. (1.37), the electrostatic potential induced by the point charge q (alone, without the effect of the sphere's charge) on the sphere's surface is

$$\phi_q=\frac{q}{4\pi\varepsilon_0 d}\sum_{l=0}^{\infty}\left(\frac{R}{d}\right)^{l}\mathcal{P}_l(\cos\theta).$$

On the other hand, using the same expansion for the image charge $q' = -(R/d)q$ located at distance $d' = R^2/d < R$ from the center, so we have to take $r_> = R$ and $r_< = d'$, getting

$$\phi_q{}'=\frac{q'}{4\pi\varepsilon_0 R}\sum_{l=0}^{\infty}\left(\frac{d'}{R}\right)^{l}\mathcal{P}_l(\cos\theta)=\frac{-(R/d)q}{4\pi\varepsilon_0 R}\sum_{l=0}^{\infty}\left[\frac{(R^2/d)d}{R}\right]^{l}\mathcal{P}_l(\cos\theta)\equiv-\frac{q}{4\pi\varepsilon_0 d}\sum_{l=0}^{\infty}\left[\frac{R}{d}\right]^{l}\mathcal{P}_l(\cos\theta).$$

We see that two contributions exactly cancel: $\phi \equiv \phi_q + \phi_{q'} = 0$ for any θ, thus satisfying the requirement to have $\phi = 0$ at any point of the grounded sphere.

This example shows that the Legendre-polynomial expansion of the Green's function, proved in the previous problem's solution, is in full compliance with the spherical inversion.

Problem 2.42. Suggest a convenient definition of the Green's function for 2D electrostatic problems, and calculate it for:

(i) the unlimited free space, and
(ii) the free space above a conducting plane.

Use the latter result to re-solve Problem 21.

Solution: As was discussed at the beginning of Sec. 2.10 of the lecture notes, the Green's function $G(\mathbf{r}, \mathbf{r}')$ for 3D electrostatic problems is defined as the partial solution (in the specific geometry of the problem) of the Poisson equation (1.41) for the electrostatic potential at point $\mathbf{r}$, provided that the right-hand side of the equation is proportional to the 3D delta function $\delta(\mathbf{r} - \mathbf{r}')$ – see Eq. (2.208):

$$\nabla^2 G(\mathbf{r},\mathbf{r}')=-4\pi\delta(\mathbf{r}-\mathbf{r}').$$

Now, if the field is two-dimensional (say, is a function of only the coordinates x and y), it is described by the same Poisson equation with $\partial/\partial z = 0$, so the charge distribution is also z-independent:

$$\nabla^2\phi=\nabla_\rho^2\phi\equiv\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi=-\frac{\rho(x,y)}{\varepsilon_0},$$

where $\boldsymbol{\rho} = \{x, y\}$ is the 2D radius vector. Hence the most natural definition of the 2D Green's function $G(\boldsymbol{\rho}, \boldsymbol{\rho}')$ is the solution of this equation with the similar, but 2D delta-functional right-hand side:[79]

$$\nabla_\rho^2 G(\boldsymbol{\rho},\boldsymbol{\rho}')=-4\pi\delta(\boldsymbol{\rho}-\boldsymbol{\rho}').$$

[79] As was noted in Sec. 2.10, numerical coefficients in such definitions is a matter of convention, not affecting the final solution of any particular problem.

Physically, this solution is the potential created by a thin straight line, normal to the $[x, y]$ plane, with the "unit" linear charge density λ (in the sense that $\lambda/4\pi\varepsilon_0 = 1$). This formulation allows us to readily calculate this Green's function for simple geometries.

(i) In the unlimited free space, we know the electric field induced by a thin line of charges with linear density λ – see Problem 1.1: $\mathbf{E} = \mathbf{n}_\rho E$, with

$$E = \frac{\lambda}{2\pi\varepsilon_0 \rho},$$

where ρ is the shortest distance from the field observation point to the line of charges, i.e. just the 2D distance between the observation point and the point (taken for the 2D origin) where the charged line, parallel to the z-axis, pierces the $[x, y]$ plane. From here, the electrostatic potential is

$$\phi = -\int E d\rho = -\frac{\lambda}{2\pi\varepsilon_0}\ln\rho + \text{const} \equiv \frac{\lambda}{4\pi\varepsilon_0}\times\left(-2\ln\rho + \text{const}\right).$$

Hence the 2D Green's function is

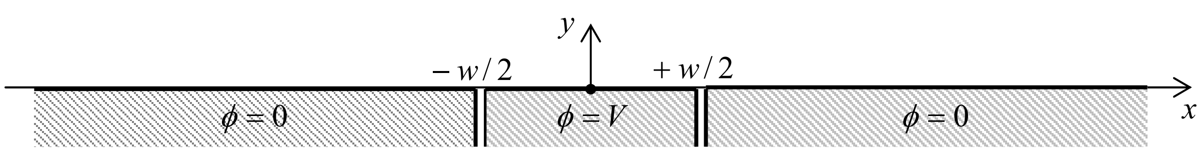
$$G(\boldsymbol{\rho}, \boldsymbol{\rho}') = -2\ln\left|\boldsymbol{\rho} - \boldsymbol{\rho}'\right| + \text{const} \equiv -\ln\left|\boldsymbol{\rho} - \boldsymbol{\rho}'\right|^2 + \text{const},$$

where $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ are the 2D radius vectors of the observation point and the charged line's trace ("cross-section"), respectively.[80]

(ii) For a charged line parallel to a grounded conducting plane $y = 0$ and crossing the $[x, y]$ plane at the point $\boldsymbol{\rho}' = \{x', y'\}$, both the Poisson equation and the boundary condition, $\phi(x,0) = 0$, may be satisfied by the introduction of an image line with the charge density $-\lambda$, crossing the $[x, y]$ plane at the point $\boldsymbol{\rho}'' = \{x', -y'\}$. Hence the Green's function of the system is

$$G(\boldsymbol{\rho}, \boldsymbol{\rho}') = -\ln\left|\boldsymbol{\rho} - \boldsymbol{\rho}'\right|^2 + \ln\left|\boldsymbol{\rho} - \boldsymbol{\rho}''\right|^2 \equiv -\ln\left[(x-x')^2 + (y-y')^2\right] + \ln\left[(x-x')^2 + (y+y')^2\right]. \qquad (*)$$

For the particular system that was the subject of Problem 21 (see the figure below), all we need from the Green's function is the normal component of its gradient at the surface.



A straightforward differentiation of Eq. (*) yields

[80] An additional simple exercise for the reader: use the same approach as in the model solution of Problem 41 to prove that this Green's function may be represented as the following series:

$$G(\boldsymbol{\rho}, \boldsymbol{\rho}') = -2\ln\rho_> + 2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\rho_<}{\rho_>}\right)^n \cos n\left(\varphi - \varphi'\right) + \text{const},$$

where $\rho_>$ is the largest, and $\rho_<$ is the smallest of ρ and ρ', while φ and φ' are the polar angles of the 2D vectors $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$. This expansion may be very useful for the solution of some 2D problems.

$$\left.\frac{\partial G}{\partial n'}\right|_S \equiv \left.\frac{\partial G}{\partial y'}\right|_{y'=0} = \frac{4y}{(x-x')^2+y^2}.$$

Plugging this expression into the 2D version of the main formula (2.210) of the Green's function theory (without the first term describing the spatially-distributed charge), we get

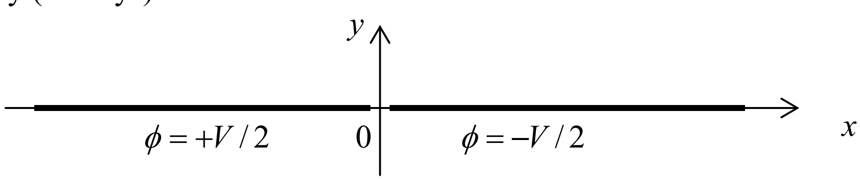
$$\phi(\boldsymbol{\rho}) = \frac{1}{4\pi}\sum_k \phi_k \oint_{l_k} \frac{\partial G(\boldsymbol{\rho},\boldsymbol{\rho}')}{\partial n'} dr' = \frac{V}{4\pi}\int_{-w/2}^{+w/2} \frac{4y}{(x-x')^2+y^2} dx'.$$

This integral may be readily worked out using the substitution $\xi \equiv (x-x')/y$, giving[81]

$$\phi = -\frac{V}{\pi}\int_{(x+w/2)/y}^{(x-w/2)/y} \frac{d\xi}{\xi^2+1} = \frac{V}{\pi}\left[\tan^{-1}\frac{x+w/2}{y} - \tan^{-1}\frac{x-w/2}{y}\right]. \qquad (**)$$

This formula gives exactly the same result as the integral (****) in the model solution of Problem 21. (See the plots in that solution.) Actually, Eq. (**) may be obtained by working out that integral, but the above Green's-function solution is much simpler. Note also alternatively the problem could be solved using the 3D Green's function given by Eq. (2.211) of the lecture notes, but this would cost us one more integration (along the z-axis).

Problem 2.43. A conducting plane is separated into two parts with a very narrow straight cut, and voltage V is applied between the resulting half-planes – see the figure below. Use the Green's function method to find the distribution of the electrostatic potential and the electric field everywhere in the space. Compare the result with Eq. (2.83) of the lecture notes. In hindsight, could the problem be solved in an even simpler way (or ways)?



Solution: Let us use the coordinates shown in the figure above, with the z-axis directed along the cut. For our 2D geometry ($\partial/\partial z = 0$), and in the absence of free charges, the basic formula (2.210) of the Green's function theory is reduced to

$$\phi(\rho) = \frac{1}{4\pi}\sum_k \phi_k \oint_{l_k} \frac{\partial G(\boldsymbol{\rho},\boldsymbol{\rho}')}{\partial n'} d\rho',$$

where, for our choice of coordinates, $\boldsymbol{\rho} = \{x, y\}$, while $G(\boldsymbol{\rho}, \boldsymbol{\rho}')$ is the 2D Green's function, and the integration should be extended along all boundaries of conductors' cross-sections, numbered with index k. For a semi-space limited by a conducting plane, the Green's function and its normal derivative have been calculated in the solution of the previous problem:

$$G(\boldsymbol{\rho},\boldsymbol{\rho}') = -2\ln|\boldsymbol{\rho}-\boldsymbol{\rho}'| + 2\ln|\boldsymbol{\rho}-\boldsymbol{\rho}''| = -\ln\left[(x-x')^2+(y-y')^2\right] + \ln\left[(x-x')^2+(y+y')^2\right],$$

[81] See, e.g., MA Eq. (6.3d).

$$\left.\frac{\partial G}{\partial y'}\right|_{y'=0} = \frac{4y}{(x-x')^2 + y^2},$$

so for our current problem, we get[82]

$$\phi = \frac{(+V/2)}{4\pi}\int_{-\infty}^{0}\frac{4y}{(x-x')^2+y^2}dx' + \frac{(-V/2)}{4\pi}\int_{0}^{+\infty}\frac{4y}{(x-x')^2+y^2}dx' = \frac{V}{\pi}\tan^{-1}\frac{x}{|y|}.$$

But this $\tan^{-1}$ is just $(\pi/2 - |\varphi|)$, where φ is the usual polar angle measured from the positive direction of the x-axis and defined on the interval $[-\pi, +\pi]$, so our result may be rewritten as

$$\phi = \frac{V}{\pi}\left(\frac{\pi}{2} - |\varphi|\right). \qquad (*)$$

Now we may readily find the electric field. According to MA Eq. (10.2) with $\partial/\partial\rho = \partial/\partial z = 0$, the field has only one (azimuthal) component

$$E_\varphi = -\frac{1}{\rho}\frac{\partial\phi}{\partial\varphi} = \frac{V}{\pi\rho}\operatorname{sgn}(\varphi), \qquad \text{where } \rho \equiv \left(x^2 + y^2\right)^{1/2}.$$

Just as one could expect, the field increases (indeed, diverges) as we approach the infinitely narrow gap that holds the non-zero voltage V.

Now looking back at Eq. (*), we see that this very simple result might be obtained in several other ways. First, it is the limiting case, for $t \to 0$, of Eq. (2.83) of the lecture notes. Second, it is a particular case (for $\beta = 0$) of Problem B formulated and analyzed in the model solution of Problem 14(i). Finally, the solution of the latter problem, and Eq. (*) as well, could be just guessed (and then verified) from the independence of the boundary conditions of ρ, and the simple form of the Laplace equation for any function $\phi(\varphi)$:

$$\frac{1}{\rho^2}\frac{d^2\phi}{d^2\varphi} = 0.$$

This equation tells us that ϕ should be a linear function of the angle in each of the two sectors where it is valid: $0 < \varphi < +\pi$, and $-\pi < \varphi < 0$.

Problem 2.44. Use the last result of Problem 42 and one of the conformal mappings discussed in Sec. 4 to find one more solution of Problem 18.

Solution: As was discussed in Sec. 2.4 of the lecture notes, the analytic complex function (2.81),

$$\mathscr{z} = \sin\frac{\pi \mathscr{w}}{2}, \qquad (*)$$

maps the interior of the gap $-w/2 \leq u \leq +w/2$, $0 \leq v < \infty$ (just with both coordinates normalized to $w/2$) on the complex plane $\mathscr{w} = u + iv$,[83] onto the upper half-plane, $y \geq 0$ on the complex plane $\mathscr{z} = x + iy$ – see

[82] See MA Eq. (6.3d).

[83] I hope that my usage of different fonts excludes any chance of confusion between the complex function $\mathscr{w}$ and the real parameter w.

Fig. 2.11 and its discussion. Hence it maps the boundary problem considered in Problem 18 onto that solved in Problem 42 (and, in a different form, in Problem 21). What remains is just to plug into Eq. (**) of that solution the function (*) spelled out for its real and imaginary parts, with the scale (w/2) for both coordinates:

$$x = \frac{w}{2}\sin\frac{\pi u}{w}\cosh\frac{\pi v}{w}, \qquad y = \frac{w}{2}\cos\frac{\pi u}{w}\sinh\frac{\pi v}{w},$$

and then make the replacements $u \to x$ and $v \to y$ – just in order to comply with the notation used in the model solution of Problem 18. The resulting formula,

$$\phi = \frac{V}{\pi}\left[\tan^{-1}\frac{\sin\frac{\pi x}{w}\cosh\frac{\pi y}{w}+1}{\cos\frac{\pi x}{w}\sinh\frac{\pi y}{w}} - \tan^{-1}\frac{\sin\frac{\pi x}{w}\cosh\frac{\pi y}{w}-1}{\cos\frac{\pi x}{w}\sinh\frac{\pi y}{w}}\right].$$

looks very different from the final formula of the solution of Problem 18,

$$\phi = \frac{4V}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos\frac{\pi(2n-1)x}{w}\exp\left\{-\frac{\pi(2n-1)y}{w}\right\},$$

but they give exactly the same results for every point $\{x, y\}$ within the free-space region $-w/2 \le x \le +w/2$, $0 \le y < \infty$.

Problem 2.45. Calculate the 2D Green's function for the free spaces:

(i) outside a round conducting cylinder, and

(ii) inside a round cylindrical hole in a conductor.

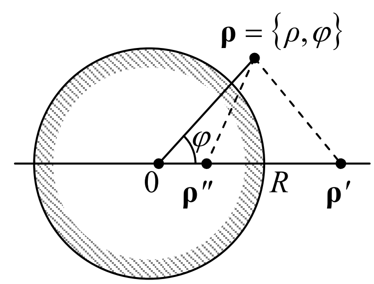
Solutions:

(i) Taking into account the solution of Task (i) of Problem 42,

$$G(\boldsymbol{\rho},\boldsymbol{\rho}')\Big|_{\text{unlimited space}} = -2\ln\left|\boldsymbol{\rho}-\boldsymbol{\rho}'\right| + \text{const}$$

(where $\boldsymbol{\rho} = \{\rho, \varphi\}$ is the 2D radius vector of a point), and inspired by the solution of a similar 3D problem (see Eqs. (2.197)-(2.198) of the lecture notes), let us try to look for the solution of our current problem in a similar form:

$$G(\boldsymbol{\rho},\boldsymbol{\rho}') = -2\ln\left|\boldsymbol{\rho}-\boldsymbol{\rho}'\right| + 2a\ln\left|\boldsymbol{\rho}-\boldsymbol{\rho}''\right| + b \equiv -\ln\left|\boldsymbol{\rho}-\boldsymbol{\rho}'\right|^2 + a\ln\left|\boldsymbol{\rho}-\boldsymbol{\rho}''\right|^2 + b, \qquad (*)$$



where $\boldsymbol{\rho}''$ is the charge image point whose distance from the cylinder's axis is related to that of the actual source point $\boldsymbol{\rho}'$ (see the figure on the right) by the inversion relation similar to the first of Eqs. (2.198):

$$\rho'' = \frac{R^2}{\rho'}.$$

The coefficients a (physically, the relative magnitude of the image charge) and b are still to be determined.

Per the solution of Problem 42, Eq. (*) satisfies the 2D Laplace equation (at $\boldsymbol{\rho} \neq \boldsymbol{\rho}'$, $\boldsymbol{\rho}''$) for any a and b, so these coefficients may be found (and thus the solution (*) proved) by requiring the Dirichlet boundary condition

$$G(\boldsymbol{\rho}, \boldsymbol{\rho}') = 0, \quad \text{for } \boldsymbol{\rho} \equiv \{R, \varphi\},$$

to be satisfied at any point at the boundary, i.e. for any polar angle φ. Counting φ from the direction from the center toward the points $\boldsymbol{\rho}'$ and $\boldsymbol{\rho}''$ (so $\boldsymbol{\rho}' = \{\rho', 0\}$ and $\boldsymbol{\rho}'' = \{\rho'', 0\}$), and applying basic trigonometry to the figure above, we may spell out this condition as

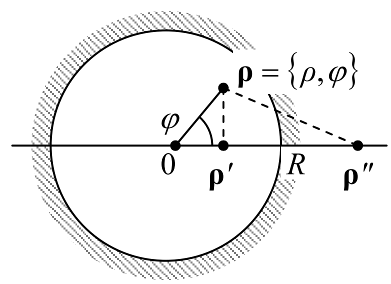
$$\begin{aligned}
&-\ln\left(\rho'^2 + R^2 - 2R'\rho'\cos\varphi\right) + a\ln\left(\rho'^2 + R^2 - 2R\rho'\cos\varphi\right) + b \\
&\equiv -\ln\left(\rho'^2 + R^2 - 2R\rho'\cos\varphi\right) + a\ln\left[\left(\frac{R^2}{\rho'}\right)^2 + R^2 - 2R\left(\frac{R^2}{\rho'}\right)\cos\varphi\right] + b \\
&\equiv -\ln\left(\rho'^2 + R^2 - 2R\rho'\cos\varphi\right) + a\ln\left[\frac{R^2}{\rho'^2}\left(R^2 + \rho'^2 - 2R\rho'\cos\varphi\right)\right] + b = 0.
\end{aligned}$$

It is evident that this condition is satisfied if $a = 1$ and $b = -\ln(R^2/\rho'^2) \equiv 2\ln(\rho'/R)$, so, finally, the Green's function is

$$G(\boldsymbol{\rho}, \boldsymbol{\rho}') = -\ln\left|\boldsymbol{\rho} - \boldsymbol{\rho}'\right|^2 + \ln\left|\boldsymbol{\rho} - \boldsymbol{\rho}''\right|^2 - \ln\frac{R^2}{\rho'^2} \equiv 2\ln\frac{\rho'}{R}\frac{\left|\boldsymbol{\rho} - \boldsymbol{\rho}''\right|}{\left|\boldsymbol{\rho} - \boldsymbol{\rho}'\right|}. \qquad (**)$$

In terms of electrostatics, the result $a = 1$ may be interpreted as the image charge (or rather its linear density) having, in contrast to the 3D cases, the same magnitude as the real linear charge density, though still the opposite sign. This equality quenches the logarithmic divergence of the Green's function (pertinent to the unlimited free space) at large distances.

(ii) Repeating the solution of Task (i) for this geometry, we see that Eq. (**) is valid for the "inner" problem as well, but now with $\rho' < R < \rho''$ (see the figure on the right). In particular, on the central line of the cylindrical hole (i.e. at $\boldsymbol{\rho} = 0$),



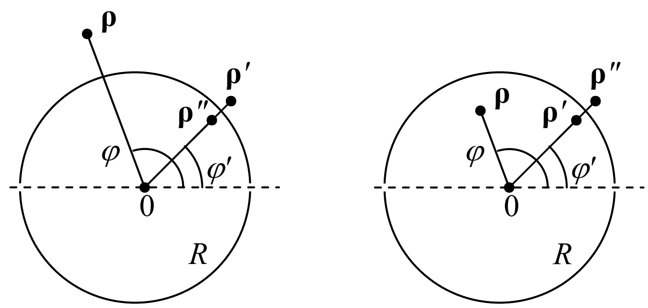
$$G(0, \boldsymbol{\rho}') = 2\ln\frac{\rho''}{R} = 2\ln\frac{R}{\rho'} > 0\,.$$

<u>Problem 2.46</u>. Solve Problem 17(i) using the Green's function method.

Solution: For our problem, the 2D form of Eq. (2.210), with no space charge and the half-pipe potentials specified in the problem's assignment, yields

$$\phi(\boldsymbol{\rho}) = \frac{1}{4\pi}\frac{V}{2}\operatorname{sgn}(\rho - R)\left(\int_0^{\pi} - \int_{-\pi}^{0}\right)\frac{\partial G(\boldsymbol{\rho}, \boldsymbol{\rho}')}{\partial \rho'}\Big|_{\rho'=R} R\,d\varphi', \qquad (*)$$

where $G(\boldsymbol{\rho}, \boldsymbol{\rho}')$ is the 2D Green's function of the thin, hollow, grounded pipe, and the sgn function is due to the opposite signs of the scalar product of the vector $\boldsymbol{\rho}'$ by the outer normal $\mathbf{n}$ to the surface for the external problem ($\rho > R$, $\boldsymbol{\rho}'\cdot\mathbf{n} > 0$, see the left panel of the figure below) and the internal problem ($\rho < R$, $\boldsymbol{\rho}'\cdot\mathbf{n} < 0$, shown on the right panel of that figure).



As was shown in the solution of the previous problem, the Green's function has the same form in both cases:

$$G(\boldsymbol{\rho},\boldsymbol{\rho}') = 2\ln\frac{\rho'}{R}\frac{|\boldsymbol{\rho}-\boldsymbol{\rho}''|}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|},$$

where the charge image point $\boldsymbol{\rho}''$ is located on the same radius as the source point $\boldsymbol{\rho}'$, at the distance $\rho'' = R^2/\rho'$ from the center. In the notation used in the figure above,

$$G(\boldsymbol{\rho},\boldsymbol{\rho}') = 2\ln\rho' + \ln\left[\rho^2 + \rho''^2 - 2\rho\rho''\cos(\varphi'-\varphi)\right] - \ln\left[\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi'-\varphi)\right] + \text{const}.$$

Taking into account that at $\rho' = R$, the derivative $d\rho''/d\rho' = -(R/\rho')^2$ equals (-1), this expression yields

$$\frac{\partial G(\boldsymbol{\rho},\boldsymbol{\rho}')}{\partial\rho'}\Big|_{\rho'=R} = \frac{2}{R}\left[1 - \frac{2R^2 - 2\rho R\cos(\varphi'-\varphi)}{\rho^2 + R^2 - 2\rho R\cos(\varphi'-\varphi)}\right] \equiv \frac{2}{R}\frac{\rho^2 - R^2}{\rho^2 + R^2 - 2\rho R\cos(\varphi'-\varphi)}.$$

Now using MA Eq. (6.3c),[84] with

$$a - b = \left(\rho^2 + R^2\right) + 2R\rho \equiv (\rho + R)^2, \qquad \left(a^2 - b^2\right)^{1/2} = \left[\left(\rho^2 + R^2\right)^2 - (2\rho R)^2\right]^{1/2} \equiv \left|\rho^2 - R^2\right|,$$

we get

$$\int \frac{\partial G(\boldsymbol{\rho},\boldsymbol{\rho}')}{\partial\rho'}\Big|_{\rho'=R} d\varphi' = \frac{4}{R}\operatorname{sgn}(\rho - R)\tan^{-1}\left[\frac{\rho+R}{|\rho-R|}\tan\frac{\varphi'-\varphi}{2}\right].$$

At the substitution of this expression into Eq. (*), the product of two front sgn functions yields +1, and we get

$$\begin{aligned}\phi(\boldsymbol{\rho}) &= \frac{V}{2\pi}\left\{\tan^{-1}\left[\frac{\rho+R}{|\rho-R|}\tan\frac{\varphi'-\varphi}{2}\right]_{\varphi'=0}^{\varphi'=\pi} - \tan^{-1}\left[\frac{\rho+R}{|\rho-R|}\tan\frac{\varphi'-\varphi}{2}\right]_{\varphi'=-\pi}^{\varphi'=0}\right\}\\ &= \frac{V}{2\pi}\left\{2\tan^{-1}\left[\frac{\rho+R}{|\rho-R|}\cot an\frac{\varphi}{2}\right] + 2\tan^{-1}\left[\frac{\rho+R}{|\rho-R|}\tan\frac{\varphi}{2}\right] - \pi\operatorname{sgn}(\sin\varphi)\right\},\end{aligned}$$

This is already the final answer, and its numerical plots coincide with those shown in the model solution of Problem 17(i). However, for the analytical comparison of the results, we may rewrite it as

---

[84] This formula is applicable here because in our case, $a^2 - b^2 = (\rho^2 - R^2)^2 \geq 0$, i.e. $a^2 \geq b^2$.

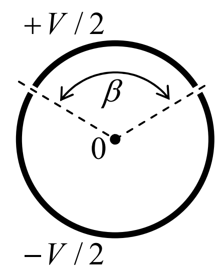
$$\phi(\boldsymbol{\rho}) = \frac{V}{\pi}\left(\alpha + \alpha' - \frac{\pi}{2}\operatorname{sgn}(\sin\varphi)\right), \qquad \text{with } \tan\alpha = \frac{\rho + R}{|\rho - R|}\cot an\frac{\varphi}{2}, \qquad \tan\alpha' = \frac{\rho + R}{|\rho - R|}\tan\frac{\varphi}{2},$$

so by using the well-known formula for the tangent of a sum, we get

$$\tan(\alpha + \alpha') \equiv \frac{\tan\alpha + \tan\alpha'}{1 - \tan\alpha\tan\alpha'} = \frac{\rho + R}{|\rho - R|}\frac{1}{1 - (\rho + R)^2/(\rho - R)^2}\left(\operatorname{cotan}\frac{\varphi}{2} + \tan\frac{\varphi}{2}\right)$$
$$= -\frac{|\rho^2 - R^2|}{4\rho R}\frac{\sin^2(\varphi/2) + \cos^2(\varphi/2)}{\sin(\varphi/2)\cos(\varphi/2)} \equiv -\frac{|\rho^2 - R^2|}{2\rho R\sin\varphi},$$

and our result may be rewritten in the form derived in the solution of Problem 17(i):

$$\phi(\boldsymbol{\rho}) = \frac{V}{\pi}\left\{\tan^{-1}\left[-\frac{|\rho^2 - R^2|}{2\rho R\sin\varphi}\right] + \frac{\pi}{2}\operatorname{sgn}(\sin\varphi)\right\} \equiv \frac{V}{\pi}\tan^{-1}\frac{2\rho R\sin\varphi}{|\rho^2 - R^2|}.$$



An advantage of the Green's function approach, as well as of the variable separation method, over the conformal mapping is that they may be more readily used in less symmetric situations – for example, in a similar problem with the pipe cut into two fragments as shown in the figure on the right, where β is an arbitrary angle rather than exactly π as in the problem solved above. (Carrying out such a calculation, using both methods, is a very useful additional exercise, highly recommended to the reader.)

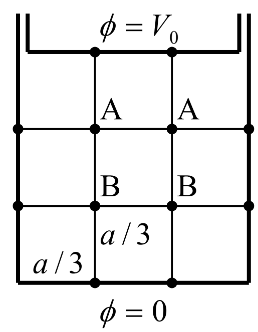
Problem 2.47. Solve the 2D boundary problem that was discussed in Sec. 2.11 of the lecture notes (Fig. 2.34) by using:

(i) the finite difference method with the finer square mesh $h = a/3$, and

(ii) the variable separation method.

Compare the results at the mesh points, and comment.

Solutions:



(i) Due to the problem's symmetry, there are only two types of internal nodes in this mesh: A and B (see the figure on the right), so we need only two finite difference equations, each describing a 5-point "cross" (see Fig. 2.33b) with its center in each of these points. For these two cases, Eq. (2.221) yields:

- points A: $\qquad 0 + \phi_A + V_0 + \phi_B - 4\phi_A = 0,$

- points B: $\qquad 0 + \phi_B + \phi_A + 0 - 4\phi_B = 0.$

Solving this simple system of two linear equations, we get

$$\phi_A = \frac{3}{8}V_0 = 0.375\,V_0, \qquad \phi_B = \frac{1}{8}V_0 = 0.125\,V_0.$$

As a simple sanity check, the potential in the cross-section's center, which is located in the middle of diagonal segments AB, may be estimated by the linear interpolation of these results, giving

$$\phi\big|_{\text{center}} \approx \frac{\phi_A + \phi_B}{2} = \frac{3/8 + 1/8}{2} V_0 \equiv \frac{1}{4} V_0,$$

i.e. the exact value – see Sec. 2.11 of the lecture notes.

(ii) In full analogy with the solution (2.95) of the similar 3D problem, obtained by the variable separation method in Sec. 2.5 of the lecture notes, the solution of this 2D problem is

$$\phi(x, y) = \sum_{n=1}^{\infty} c_n \sin\frac{\pi n x}{a} \sinh\frac{\pi n y}{a},$$

where the coefficients c_n have to be found from the boundary condition on the top lid ($y = a$):

$$V_0 = \sum_{n=0}^{\infty} c_n \sin\frac{\pi n x}{a} \sinh \pi n.$$

Performing the reciprocal Fourier transform (i.e., multiplying both parts of this equation by $\sin(\pi n' x/a)$, and integrating each of them over x from 0 to a), we get

$$c_n = \frac{1}{(a/2)\sinh \pi n} V_0 \int_0^a \sin\frac{\pi n x}{a} dx = \begin{cases} 4V_0/\pi n \sinh \pi n, & \text{for } n = 2m-1, \\ 0, & \text{for } n = 2m, \end{cases} \quad \text{with } m = 1, 2, 3,...$$

thus proving Eq. (2.224) of the lecture notes. In the a/3 mesh nodes, this (exact) formula yields the values

$$\phi_A = \frac{4}{\pi} V_0 \sum_{m=1}^{\infty} \frac{\sin[\pi(2m-1)/3] \sinh[2\pi(2m-1)/3]}{(2m-1)\sinh[\pi(2m-1)]} \approx 0.38072\, V_0,$$

$$\phi_B = \frac{4}{\pi} V_0 \sum_{m=1}^{\infty} \frac{\sin[2\pi(2m-1)/3] \sinh[\pi(2m-1)/3]}{(2m-1)\sinh[\pi(2m-1)]} \approx 0.11928\, V_0.$$

Note that though these series are formally infinite, virtually all their values are provided by just a few first terms, because the exponential character of the sinh functions at large values of their arguments ensures a very fast convergence of the sums.

Comparing the results provided by the two methods, we see that the finite difference method's error, even with this rather coarse mesh, is just a few percent.

Chapter 3. Dipoles and Dielectrics

Problem 3.1. Prove Eqs. (3.3)-(3.4) of the lecture notes, starting from Eqs. (1.38) and (3.2).

Solution: In Sec. 3.1 of the lecture notes, the first two terms of the multipole expansion (3.3) have already been derived from the general Eq. (1.38),

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r',$$

so we need only to calculate the third (quadrupole) term from the corresponding term on the right-hand side of Eq. (3.2):

$$\phi_\text{q}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{r}')\frac{1}{2!}\sum_{j,j'=1}^{3}\frac{\partial^2 f(R)}{\partial r_j \partial r_{j'}}\Big|_{\mathbf{r}'=0} r'_j r'_{j'} d^3r', \qquad (*)$$

for our case $f(R) = 1/R$, where $R = |\,\mathbf{R}\,|$ and $\mathbf{R} \equiv \mathbf{r} - \mathbf{r}'$. Spelling out the function f via the Cartesian coordinates $R_j = r_j - r_{j'}$ of the vector $\mathbf{R}$,

$$f(R) = \left[\sum_{j=1}^{3}\left(r_j - r'_j\right)^2\right]^{-1/2},$$

makes the first differentiation ($\partial/\partial r_j$) in Eq. (*) elementary:

$$\frac{\partial f(R)}{\partial r_j} = -\left[\sum_{j=1}^{3}\left(r_j - r'_j\right)^2\right]^{-3/2}\left(r_j - r'_j\right).$$

However, the result of the second differentiation ($\partial/\partial r_{j'}$) depends on whether the indices j and j' coincide, because if they do, then we have to differentiate r_j as well:

$$\frac{\partial^2 f(R)}{\partial r_j \partial r_{j'}} \equiv \frac{\partial}{\partial r_{j'}}\left(\frac{\partial f}{\partial r_j}\right) = \frac{3}{2}\left[\sum_{j=1}^{3}\left(r_j - r'_j\right)^2\right]^{-5/2} 2\left(r_j - r'_j\right)\left(r_{j'} - r'_{j'}\right) - \delta_{jj'}\left[\sum_{j=1}^{3}\left(r_j - r'_j\right)^2\right]^{-3/2}$$
$$\equiv \frac{3R_jR_{j'}}{R^5} - \frac{\delta_{jj'}}{R^3},$$

so the particular value participating in Eq. (*) is

$$\frac{\partial^2 f(R)}{\partial r_j \partial r_{j'}}\Big|_{\mathbf{r}'=0} = \frac{3r_jr_{j'}}{r^5} - \frac{\delta_{jj'}}{r^3} \equiv \frac{3r_jr_{j'} - \delta_{jj'}r^2}{r^5},$$

and may be moved out of the integral over the $\mathbf{r}'$ space. The resulting expression for the quadrupole potential,

$$\phi_\text{q}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{2r^5}\sum_{j,j'=1}^{3}\left(3r_jr_{j'} - \delta_{jj'}r^2\right)\int \rho(\mathbf{r}')\, r'_j r'_{j'} d^3r', \qquad (**)$$

though correct, is still not quite convenient for most applications: it is beneficial to move the account for the special character of the diagonal components of the sum into the integration over r'.

The standard way of doing that is shown in Eq. (3.4) of the lecture notes; plugging that expression for $\mathcal{Q}_{jj'}$ into the last term of Eq. (3.3), we may rewrite the latter as

$$\phi_{\rm q}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{2r^5}\sum_{j,j'=1}^{3} r_j r_{j'} \int \rho(\mathbf{r}')\left(3r'_j r'_{j'} - \delta_{jj'} r'^2\right) d^3r' ; \qquad (***)$$

let us prove that Eqs. (***) and the already proved Eq. (**) are equivalent. The off-diagonal terms under both sums are clearly similar (they differ only by the location of the factor 3), so the sums over off-diagonal elements are equal as well. Hence we need to analyze only the sums of the diagonal elements:

$$\text{in Eq. (**)}: \qquad \sum_{j=j'=1}^{3}\left(3r_j r_{j'} - \delta_{jj'} r^2\right)\int \rho(\mathbf{r}')\, r'_j r'_{j'} d^3r' = \sum_{j=1}^{3}\left(3r_j^2 - r^2\right)\int \rho(\mathbf{r}')\, r'^2_j d^3r'$$

$$= 3\sum_{j=1}^{3} r_j^2 \int \rho(\mathbf{r}')\, r'^2_j d^3r' - r^2 \int \rho(\mathbf{r}') r'^2 d^3r';$$

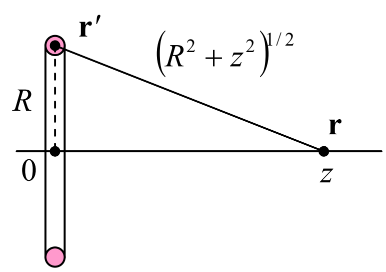
$$\text{in Eq. (***)}: \qquad \sum_{j=j'=1}^{3} r_j r_{j'} \int \rho(\mathbf{r}')\left(3r_j r_{j'} - \delta_{jj'} r^2\right) d^3r' = \sum_{j=1}^{3} r_j^2 \int \rho(\mathbf{r}')\left(3r'^2_j - r'^2\right) d^3r'$$

$$= 3\sum_{j=1}^{3} r_j^2 \int \rho(\mathbf{r}')\, r'^2_j d^3r' - r^2 \int \rho(\mathbf{r}') r'^2 d^3r' .$$

We see that the results are identical, thus completing the required proof.

<u>Problem 3.2</u>. A thin ring of radius R is charged with a constant linear density λ. Calculate the exact distribution of the electrostatic potential along the symmetry axis of the ring, and prove that at large distances, $r >> R$, the three leading terms of its multipole expansion are indeed correctly described by Eqs. (3.3)-(3.4) of the lecture notes.

Solution: Due to the axial symmetry of the system, the calculation of its electrostatic potential from Eq. (1.38), integrated over the cross-section of the ring, is elementary (see the figure on the right):



$$\phi = \frac{\lambda}{4\pi\varepsilon_0}\int_{\rm ring}\frac{dr'}{|r - r'|} = \frac{\lambda}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{Rd\varphi'}{\left(R^2 + z^2\right)^{1/2}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\frac{2\pi R}{\left(R^2 + z^2\right)^{1/2}} \equiv \frac{\lambda R}{2\varepsilon_0 z} f(\xi),$$

where

$$\xi \equiv \left(\frac{R}{z}\right)^2, \qquad f(\xi) \equiv (1+\xi)^{-1/2}.$$

Let us expand this function $f(\xi)$ into the Taylor series in the argument ξ at point $\xi = 0$:

$$f(\xi) = (1+\xi)^{-1/2}\Big|_{\xi=0} + \frac{d}{d\xi}(1+\xi)^{-1/2}\Big|_{\xi=0}\xi + \frac{1}{2}\frac{d^2}{d\xi^2}(1+\xi)^{-1/2}\Big|_{\xi=0}\xi^2 + ... = 1 - \frac{1}{2}\xi + \frac{3}{8}\xi^2 + ...$$

At large distances from the ring, where ξ is small, the function $f(\xi)$ is well approximated by these three leading terms, and we get

$$\phi \approx \frac{\lambda}{2\varepsilon_0}\left[\frac{R}{z} - \frac{1}{2}\left(\frac{R}{z}\right)^3 + \frac{3}{8}\left(\frac{R}{z}\right)^5\right], \qquad \text{at } z >> R. \tag{*}$$

This expression should be compared with the first three terms of the quadrupole expansion (3.3):

$$\phi(\mathbf{r}) \approx \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r}Q + \frac{1}{r^3}\sum_{j=1}^{3} r_j p_j + \frac{1}{2r^5}\sum_{j,j'=1}^{3} r_j r_{j'} \mathscr{Q}_{jj'}\right), \tag{**}$$

where

$$Q = \int \rho(\mathbf{r}')d^3r', \qquad p_j = \int \rho(\mathbf{r}')r'_j d^3r', \qquad \mathscr{Q}_{jj'} = \int \rho(\mathbf{r}')\left(3r'_j r'_{j'} - r'^2\delta_{jj'}\right)d^3r',$$

and $\rho(\mathbf{r}')$ is the electric charge density (per unit volume). In the standard polar coordinates, with the origin in the ring's center (see the figure above), the integration over the ring's length yields

$$Q = \lambda \int_0^{2\pi} R d\varphi' = 2\pi R\lambda;$$

$$p_x = \lambda \int_0^{2\pi} R\cos\varphi' \, Rd\varphi' = 0, \qquad p_y = \lambda \int_0^{2\pi} R\sin\varphi' \, Rd\varphi' = 0, \qquad p_z = \lambda \int_0^{2\pi} 0 \cdot Rd\varphi' = 0;$$

$$\mathscr{Q}_{xx} = \lambda \int_0^{2\pi} \left(3R^2\cos^2\varphi' - R^2\right)Rd\varphi' = \pi\lambda R^3, \qquad \mathscr{Q}_{yy} = \lambda \int_0^{2\pi} \left(3R^2\sin^2\varphi' - R^2\right)Rd\varphi' = \pi\lambda R^3,$$

$$\mathscr{Q}_{zz} = \lambda \int_0^{2\pi} \left(0 - R^2\right)Rd\varphi' = -2\pi\lambda R^3, \qquad \mathscr{Q}_{xy} = \mathscr{Q}_{yx} = \lambda \int_0^{2\pi} 3R^2 \sin\varphi'\cos\varphi' \, Rd\varphi' = 0,$$

$$\mathscr{Q}_{xz} = \mathscr{Q}_{zx} = \lambda \int_0^{2\pi} 3\cdot 0 \cdot R\cos\varphi' \, Rd\varphi' = 0, \qquad \mathscr{Q}_{yz} = \mathscr{Q}_{zy} = \lambda \int_0^{2\pi} 3\cdot 0 \cdot R\sin\varphi' \, Rd\varphi' = 0.$$

As a sanity check, the quadrupole moment matrix calculated above has zero trace (the sum of their diagonal elements) – as it should, for any system, by the very definition of the tensor $\mathscr{Q}_{jj'}$:

$$\mathrm{Tr}(\mathscr{Q}) \equiv \sum_{j=1}^{3} \mathscr{Q}_{jj} \equiv \sum_{j=1}^{3} \int \rho(\mathbf{r})\left(3r_j^2 - r^3\right)d^3r = \int \rho(\mathbf{r})\sum_{j=1}^{3}\left(3r_j^2 - r^2\right)d^3r = \int \rho(\mathbf{r})\left(3r^2 - 3r^2\right)d^3r = 0.$$

Now plugging these results into Eq. (**) written for z-axis, i.e. with $r_1 = r_2 = 0$, and $r = r_3 = z$,

$$\phi(z) \approx \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{z}Q + \frac{1}{z^2}p_z + \frac{1}{2z^3}\mathscr{Q}_{zz}\right),$$

we get

$$\phi(z) \approx \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{z}2\pi R\lambda + \frac{1}{z^2}\cdot 0 - \frac{1}{2z^3}2\pi\lambda R^3\right) \equiv \frac{1}{4\pi\varepsilon_0}\left(\frac{2\pi R\lambda}{z} - \frac{\pi\lambda R^3}{z^3}\right).$$

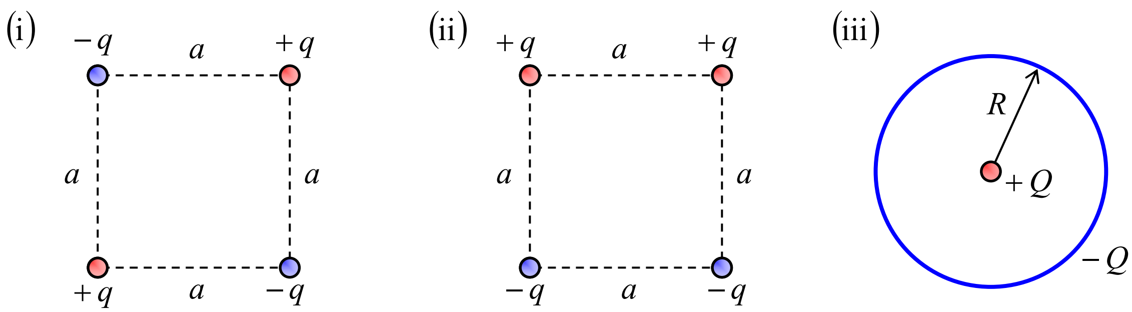
Comparing this expression with Eq. (*), we see that the quadrupole approximation exactly describes the three leading (including one vanishing) terms of that expansion. Evidently, for this simple geometry, the exact calculation is simpler, but in more complex cases, the multipole expansion may be the only way to carry out an (approximate) analytical calculation.

Problem 3.3. In suitable reference frames, calculate the dipole and quadrupole moments of the following systems (see the figures below):

(i) four point charges of the same magnitude but alternating signs, placed in the corners of a square;

(ii) a similar system but with a pair charge sign alternation; and

(iii) a point charge in the center of a thin ring carrying a similar but opposite charge uniformly distributed along its circumference.

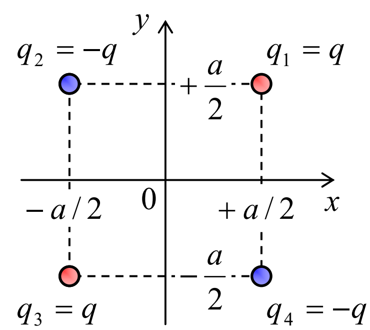


Solutions: For point charges, the definitions of the dipole and quadrupole moments of a system (see Eq. (3.4) of the lecture notes) take the following forms

$$p_j = \sum_k q_k \left(r_j\right)_k , \tag{*}$$

$$\mathscr{Q}_{jj'} = \sum_k q_k \left(3r_j r_{j'} - r^2 \delta_{jj'}\right)_k , \tag{**}$$

where the index k numbers the charges, the indices j, j' = 1, 2, 3 mark the Cartesian coordinates, and the prime sign of the radius vector components is dropped for brevity. Let us apply these formulas to systems (i) and (ii).



(i) For this system, in the reference frame and with the charge numbering shown in the figure on the right, Eq. (*) yields

$$p_x = q\frac{a}{2}(1+1-1-1) = 0, \qquad p_y = q\frac{a}{2}(1-1-1+1) = 0, \qquad p_z = 0,$$

where the order of terms in the parentheses follows the charge numbers and $z = 0$ for all of them. So, the dipole moment $\mathbf{p}$ completely vanishes. Similarly, Eq. (**) yields

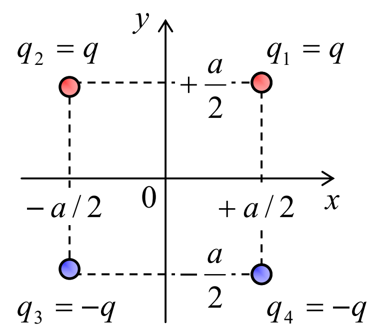
$$\mathscr{Q}_{xx} = q\left(\frac{a}{2}\right)^2 (1-1+1-1) = 0, \qquad \mathscr{Q}_{yy} = q\left(\frac{a}{2}\right)^2 (1-1+1-1) = 0,$$

$$\mathscr{Q}_{zz} = q\left(\frac{a}{2}\right)^2 (-2+2-2+2) = 0, \qquad \mathscr{Q}_{xy} = \mathscr{Q}_{yx} = q\left(\frac{a}{2}\right)^2 (3+3+3+3) = 3qa^2,$$

with all off-diagonal components involving the z-coordinate being equal to zero as well, so the quadrupole moment may be represented by the following off-diagonal matrix:

$$\mathscr{Q} = qa^2 \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(ii) For this system, in a similar reference frame (see the figure on the right):



$$p_x = q\frac{a}{2}(1-1+1-1) = 0, \qquad p_y = q\frac{a}{2}(1+1+1+1) = 2qa, \qquad p_z = 0,$$

so the system has a nonvanishing dipole moment $\mathbf{p} = 2qa\mathbf{n}_y$, and

$$\mathscr{Q}_{xx} = q\left(\frac{a}{2}\right)^2(1+1-1-1) = 0, \qquad \mathscr{Q}_{yy} = q\left(\frac{a}{2}\right)^2(1+1-1-1) = 0,$$

$$\mathscr{Q}_{zz} = q\left(\frac{a}{2}\right)^2(-2-2+2+2) = 0, \qquad \mathscr{Q}_{xy} = \mathscr{Q}_{yx} = q\left(\frac{a}{2}\right)^2(3-3-3+3) = 0,$$

with all off-diagonal components involving the z-coordinate being equal to zero as well. So in this case, it is the quadrupole moment that completely vanishes, while the dipole moment does not.

(iii) Generally speaking, for this system, we need to combine contributions from both the central point charge $+Q$ and the distributed charge of the ring, with the linear density $\lambda = -Q/2\pi R$. However, in a reference frame with its origin in the center of the ring, the former charge (now with $\mathbf{r} = 0$) yields no contribution to either p_j or $\mathscr{Q}_{jj'}$, and we may focus on the ring alone. Directing the axes so that $x = R\cos\varphi$, $y = R\sin\varphi$, and $z = 0$, and hence $r^2 = x^2 + y^2 = R^2$ for all points of the ring, we get $\rho d^3r = \lambda R d\varphi$, so

$$p_x = \lambda R \int_0^{2\pi} R\cos\varphi\, d\varphi = 0, \qquad p_x = \lambda R \int_0^{2\pi} R\sin\varphi\, d\varphi = 0, \qquad p_z = 0,$$

(i.e. the dipole moment $\mathbf{p}$ is completely vanishing), and

$$\mathscr{Q}_{xx} = \lambda R \int_0^{2\pi} \left(3x^2 - r^2\right) d\varphi = \lambda R \int_0^{2\pi} \left(3R^2\cos^2\varphi - R^2\right) d\varphi = \lambda R^3 \left(\frac{3}{2} - 1\right) 2\pi = -\frac{1}{2}QR^2,$$

$$\mathscr{Q}_{yy} = \lambda R \int_0^{2\pi} \left(3y^2 - r^2\right) d\varphi = \lambda R \int_0^{2\pi} \left(3R^2\sin^2\varphi - R^2\right) d\varphi = \lambda R^3 \left(\frac{3}{2} - 1\right) 2\pi = -\frac{1}{2}QR^2,$$

$$\mathscr{Q}_{zz} = \lambda R \int_0^{2\pi} \left(0 - r^2\right) d\varphi = -\lambda R \int_0^{2\pi} R^2 d\varphi = QR^2, \qquad \mathscr{Q}_{xy} = \mathscr{Q}_{yx} = \lambda R \int_0^{2\pi} 3xy\, d\varphi = \lambda R \int_0^{2\pi} R^2 \sin\varphi\cos\varphi\, d\varphi = 0,$$

with all other components involving the z-coordinate being equal to zero as well. So, the quadrupole tensor has the following diagonal matrix:

$$\mathscr{Q} = qR^2 \begin{pmatrix} -1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As a sanity check, the sums of the diagonal components of all the quadrupole tensors calculated above equal 0, as they should – see the solution of the previous problem.

Problem 3.4. Calculate the dipole and quadrupole moments of a thin spherical shell of radius R, carrying an electric charge with the areal density $\sigma = \sigma_0\cos\theta$. Discuss the relation between the results and the solution of Problem 2.28.

Solution: As was discussed in Sec. 1.1 of the lecture notes, we may describe the areal charge density σ of a thin spherical shell by taking the bulk density $\rho(\mathbf{r})$ equal to $\sigma\delta(r - R)$, so after their integration over r' from R–0 to R+0, the two last Eqs. (3.4) become

$$p_j \equiv \int \sigma(\mathbf{r}')r_j' d^2r', \qquad \mathcal{Q}_{jj'} \equiv \int \sigma(\mathbf{r}')\left(3r_j'r_{j'}' - r'^2\delta_{jj'}\right)d^2r',$$

where both integrals are over the surface of the shell. With the natural choice of the coordinate origin in the center of the sphere, these integrals become

$$\int(...)d^2r' = R^2\int_{4\pi}(...)d\Omega' = R^2\int_0^{2\pi}d\varphi'\int_0^{\pi}(...)\sin\theta' d\theta',$$

so for the given charge density $\sigma = \sigma_0\cos\theta$, we get

$$\begin{aligned} p_{\begin{Bmatrix} x \\ y \end{Bmatrix}} &= \sigma_0 R^3\int_0^{2\pi}\begin{Bmatrix}\cos\varphi' \\ \sin\varphi'\end{Bmatrix}d\varphi'\int_0^{\pi}\cos\theta'\sin^2\theta' d\theta' = 0, \\ p_z &= \sigma_0 R^3\int_0^{2\pi}d\varphi'\int_0^{\pi}\cos^2\theta'\sin\theta' d\theta' = \frac{4\pi}{3}\sigma_0 R^3, \end{aligned} \qquad (*)$$

$$\begin{aligned} \mathcal{Q}_{\begin{Bmatrix} xx \\ yy \end{Bmatrix}} &\equiv \sigma_0 R^4\int_0^{2\pi}d\varphi'\int_0^{\pi}\left(3\begin{Bmatrix}\cos^2\varphi' \\ \sin^2\varphi'\end{Bmatrix}\sin^2\theta' - 1\right)\cos\theta'\sin\theta' d\theta' = 0, \\ \mathcal{Q}_{zz} &\equiv \sigma_0 R^4\int_0^{2\pi}d\varphi'\int_0^{\pi}\left(3\begin{Bmatrix}\cos^2\varphi' \\ \sin^2\varphi'\end{Bmatrix}\cos^2\theta' - 1\right)\cos\theta'\sin\theta' d\theta' = 0, \end{aligned} \qquad (**)$$

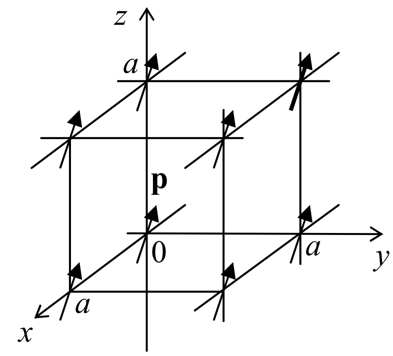
and all off-diagonal elements equal zero as well, because they are proportional to the azimuthal integrals of either $\sin\varphi$ (for $\mathcal{Q}_{xz}$ and $\mathcal{Q}_{zx}$), or $\cos\varphi$ (for $\mathcal{Q}_{yz}$ and $\mathcal{Q}_{zy}$), or $\sin\varphi\cos\varphi$ (for $\mathcal{Q}_{xy}$ and $\mathcal{Q}_{yx}$).

These results are in full accord with the result of Problem 2.28 for the exact field of the sphere outside it:

$$\phi = \frac{\sigma_0 R}{3\varepsilon_0 r^2}\cos\theta, \qquad \text{for } r > R.$$

Indeed, a comparison of the last formula with Eq. (3.7) of the lecture notes shows that this is the field of a dipole with moment $\mathbf{p} = \mathbf{n}_z p_z$, where p_z is given by the second of Eqs. (*). For such a purely dipole field, all higher multipole moments starting from the quadrupole one, have to vanish – just as Eq. (**) exemplifies.

Problem 3.5. For a regular cubic lattice of similarly oriented identical dipoles, calculate the electric field it creates at the location of each dipole.



Solution: In a Cartesian coordinate system with the origin at the location of one of the dipoles, and the axes directed along the lattice sides (see the figure on the right), the dipole coordinates are

$$x_{jkl} = aj, \quad y_{jkl} = ak, \quad z_{jkl} = al,$$

where j, k, and l are the integers numbering the lattice nodes, with $j = k = l = 0$ at the origin. Now we may use the last form of Eq. (3.13) and the linear superposition principle to calculate one of the Cartesian components (say, along the x-axis) of the field created at the location of the dipole at the origin by all other dipoles of the lattice:

$$E_x(0) = \frac{1}{4\pi\varepsilon_0 a^3} \sum_{j,k,l=-\infty}^{+\infty} \frac{3j(jp_x + kp_y + lp_z) - p_x(j^2 + k^2 + l^2)}{(j^2 + k^2 + l^2)^{5/2}},$$

with the point $j = k = l = 0$ excluded. The sums of all cross-terms, proportional to the products jk and jl, vanish due to the system's symmetry with respect to the inversion of any coordinate, so we get

$$E_x(0) = \frac{1}{4\pi\varepsilon_0 a^3} \sum_{j,k,l=-\infty}^{+\infty} \frac{\left[3j^2 - \left(j^2 + k^2 + l^2\right)\right]}{\left(j^2 + k^2 + l^2\right)^{5/2}} p_x .$$

Since all the partial sums participating in this expression are equal,

$$\sum_{j,k,l=-\infty}^{+\infty} \frac{j^2}{(j^2 + k^2 + l^2)^{5/2}} = \sum_{j,k,l=-\infty}^{+\infty} \frac{k^2}{(j^2 + k^2 + l^2)^{5/2}} = \sum_{j,k,l=-\infty}^{+\infty} \frac{l^2}{(j^2 + k^2 + l^2)^{5/2}},$$

we get $E_x(0) = 0$. But, due to the system symmetry with respect to the coordinate swap, the same result is valid for all other components of the dipole field. Hence $\mathbf{E}(0) = 0$, so (due to the similarity of all the dipoles of the system), at the location of each dipole, the field of all other dipoles vanishes.

Historically, this counter-intuitive result, first derived by H. Lorentz, served as the basis of his theoretical explanation of the initially-empirical Clausius-Mossotti formula (3.52) – see Problem 15 below.

Problem 3.6. Without carrying out an exact calculation, can you predict the spatial dependence of the interaction between various electric multipoles, including point charges (in this context, frequently called electric *monopoles*), dipoles, and quadrupoles? Based on these predictions, what is the functional dependence of the interaction between *homonuclear* diatomic molecules such as H_2, N_2, O_2, etc., on the distance between them when the distance is much larger than the molecular size?

Solution: According to Eq. (1.35) of the lecture notes, the electrostatic potential ϕ_m of a monopole is proportional to $1/r$. Since, according to Eq. (1.31), the potential energy U of a monopole in an external potential is proportional to the potential itself, the potential energy of monopole-monopole interaction is inversely proportional to the distance between them:

$$U_{\text{m-m}} \propto \frac{1}{r}.$$

This conclusion is of course elementary, but the above reasoning establishes the framework that may be readily extended to higher multipoles. For example, according to Eq. (3.7), the potential ϕ_d of a dipole scales as $1/r^2$. Hence its interaction with a monopole

$$U_{\text{d-m}} \propto \frac{1}{r^2}.$$

This conclusion is confirmed by Eqs. (3.15) with $\mathbf{E}_{\text{ext}} = \mathbf{E}_m = -\nabla\phi_m \propto 1/r^2$.

The same Eq. (3.15) may be used to calculate the dipole-dipole interaction, now taking $\mathbf{E}_{\text{ext}} = \mathbf{E}_d = -\nabla\phi_d \propto 1/r^3$, so

$$U_{\text{d-d}} \propto \frac{1}{r^3}$$

– the result quantified by Eq. (3.16).

The general trend is perhaps already clear to the reader, but let us follow it one step further. According to the multipole expansion whose proof was the subject of Problem 1, the quadrupole potential ϕ_q decreases with distance as $1/r^3$, so its interaction with a monopole scales as

$$U_{\text{q-m}} \propto \frac{1}{r^3}.$$

Again using Eq. (3.15), we may conclude that its interaction with a dipole is proportional to $\mathbf{E}_q = -\nabla\phi_q \propto 1/r^4$, i.e.

$$U_{\text{q-d}} \propto \frac{1}{r^4}.$$

Now we may formulate the general rule, which covers all the above cases: the interaction between two multipoles as a function of the distance r between them acquires one more factor $1/r$ each time when the order of any of the interacting multipoles is increased by 1. On this basis, we may safely predict that the interaction between two quadrupoles drops with the distance as

$$U_{\text{q-q}} \propto \frac{1}{r^5}.$$

The formal proof of this statement and the exact expression for $U_{\text{q-q}}$ may be obtained by the corresponding extension of Eq. (3.14) – the additional exercise highly recommended to the reader.

Due to their symmetry, homonuclear diatomic molecules do not have a dipole moment but may have a nonvanishing quadrupole moment. Indeed, a crude but fair model of their charge distribution is given by a nonuniform but symmetric linear density $\lambda(z) = \lambda(-z)$, where z is the molecule's symmetry axis, with the origin in its center. For such a density, the dipole moment vanishes, because its only significant component,

$$p_z = \int \lambda(z) z dz,$$

is proportional to an integral of an odd function in symmetric limits, i.e. is equal to zero. Using the above results, and those of the previous problem, one may conclude that the long-range intermolecular interaction is proportional to $1/r^5$.

Note, however, that molecules' quadrupole moments are typically so small that this interaction is rather insignificant. An even more important factor diminishing the importance of such interaction is that, according to Eq. (3.3), the quadrupole potential has a zero angular average, so if the mutual orientation of the interacting molecules is random (as it is, for example, in the gas phase), the quadrupole interaction is averaged out. This is why another long-range interaction (called the *London dispersion force*), with the potential energy proportional to $1/r^6$, is usually much more important. As was mentioned in Sec. 3.1 of the lecture notes, this effect results from the interaction of two mutually induced (rather than spontaneous) dipole moments, due to their quantum fluctuations.[85] Due to this mutual correlation of the induced dipole moments, this effect is not averaged out at the random orientation of the molecules and always has the same sign, corresponding to their mutual attraction.

Problem 3.7. Two similar electric dipoles, of a fixed magnitude p, located at a fixed distance r from each other, are free to change their directions. What stable equilibrium position(s) they may take as a result of their electrostatic interaction?

Solution: Directing the z-axis along the line connecting the dipoles, we may use the last form of Eq. (3.16) of the lecture notes for the potential energy of the dipole-dipole interaction:

$$U_{\text{int}} = \frac{1}{4\pi\varepsilon_0}\frac{p_{1x}\cdot p_{2x} + p_{1y}\cdot p_{2y} - 2p_{1z}\cdot p_{2z}}{r^3}.$$

Plugging into this relation the expressions for Cartesian components of both dipole moments via the polar and azimuthal angles of their orientation,

$$p_{jx} = p\sin\theta_j\cos\varphi_j, \qquad p_{jy} = p\sin\theta_j\sin\varphi_j, \qquad p_{jz} = p\cos\theta_j, \qquad \text{where } j = 1, 2,$$

we may rewrite the interaction potential as $U_{\text{int}} = Cf$, where $C \equiv (1/4\pi\varepsilon_0)p^2/r^3$ is just a constant, while f is a function of the angles θ_j and φ_j:

$$\begin{aligned} f &= \sin\theta_1\cos\varphi_1\sin\theta_2\cos\varphi_2 + \sin\theta_1\sin\varphi_1\sin\theta_2\sin\varphi_2 - 2\cos\theta_1\cos\theta_2 \\ &\equiv \sin\theta_1\sin\theta_2\cos\varphi - 2\cos\theta_1\cos\theta_2, \end{aligned}$$

where $\varphi \equiv \varphi_1 - \varphi_2$ is the mutual shift of the azimuthal angles of the dipole moment vectors. (The last form of the function f reflects the obvious fact that the interaction is invariant with respect to any simultaneous rotation of the vectors about the line connecting them.)

The stable equilibrium orientations of the dipoles correspond to the minima of this function of three arguments. In order to find them, let us use the system's symmetry. Since the dipoles are similar, and the dependence of each **p** on its θ is monotonic, the equilibrium angles θ_1 and θ_2 should be either equal or π-complementary, providing us with the following two options:

$$\begin{aligned} &\text{Option A}: \theta_1 = \theta_2 \equiv \theta, && \text{giving } f = f_0(\theta,\varphi) \equiv \sin^2\theta\cos\varphi - 2\cos^2\theta \equiv \cos\varphi - (2+\cos\varphi)\cos^2\theta, \\ &\text{Option B}: \theta_1 = \pi - \theta_2 \equiv \theta, && \text{giving } f = -f_0(\theta,\varphi). \end{aligned}$$

On the relevant segment $0 \le \theta \le \pi$, the function $f_0(\theta, \varphi)$ has two equal minima, $f_0(0, \varphi) = f_0(\pi, \varphi) = -2$, and one maximum, $f_0(\pi/2, \varphi) = \cos\varphi$. Since $|\cos\varphi|$ is always less than 2, this means that the function f has two minima, at $\theta = 0$ and $\theta = 0$ (when the azimuthal angle φ is inconsequential). So, the

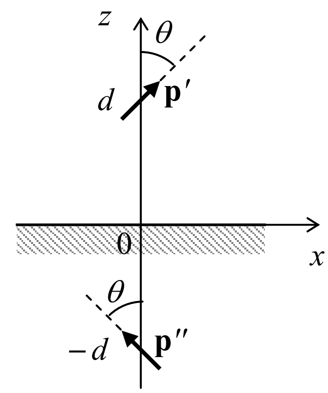
[85] A quantitative discussion of this interaction may be found, e.g., in QM Chapters 3, 5, and 6.

system has two similar stable equilibrium states, in which both dipoles are aligned with each other ($\mathbf{p}_1 = \mathbf{p}_2$), and are together oriented in either direction of the line connecting them.

Problem 3.8. An electric dipole is located above a grounded infinite conducting plane. Calculate:

(i) the distribution of the induced charge in the conductor,
(ii) the dipole-to-plane interaction energy, and
(iii) the force and the torque exerted on the dipole.

Solutions:



(i) The problem may be solved by the introduction of a dipole image, at the same distance d below the plane, and with the same dipole moment magnitude p as the original dipole, but reflected in the vertical plane normal to that containing the dipole moment vector – see the figure on the right. The simplest way to come up with this fact is to represent the dipole in the approximate form of two point charges, ($+q$) and ($-q$), slightly displaced along the direction of the dipole moment vector, and then construct the dipole image from the mirror images of these point charges in the conducting plane. However, so far this is just a guess, not a proof; let us prove this fact.

The net field of these two dipoles evidently satisfies the proper Poisson equation in the upper half-space, so the only thing we have to prove is that it also satisfies the boundary condition ($\phi = 0$) on the plane's surface. Let us generalize Eq. (3.7) of the lecture notes to the system of two dipoles (calling them $\mathbf{p}'$ and $\mathbf{p}''$, see the figure above), with the following Cartesian components:

$$p_x' = -p_x'' = p\sin\theta, \quad p_y' = p_y' = 0, \quad p_z' = p_z'' = p\cos\theta, \qquad (*)$$

and located, respectively, at points $\mathbf{r}'$ and $\mathbf{r}''$ with the following Cartesian coordinates:

$$x' = x'' = 0, \quad y' = y'' = 0, \quad z' = -z'' = d.$$

(Here x is the coordinate within the vertical plane that contains the vectors $\mathbf{p}'$ and $\mathbf{p}''$, i.e. in the plane of the above drawing.) In these coordinates, the generalization yields

$$\phi = \frac{1}{4\pi\varepsilon_0}\left[\frac{(\mathbf{r}-\mathbf{r}')\cdot\mathbf{p}'}{|\mathbf{r}-\mathbf{r}'|^3} + \frac{(\mathbf{r}-\mathbf{r}'')\cdot\mathbf{p}''}{|\mathbf{r}-\mathbf{r}''|^3}\right] = \frac{p}{4\pi\varepsilon_0}\left\{\frac{(z-d)\cos\theta + x\sin\theta}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} + \frac{(z+d)\cos\theta - x\sin\theta}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right\}.$$

This expression shows that the potential indeed vanishes everywhere on the surface ($z = 0$), thus proving our guess.

Now the induced surface charge density may be calculated from Eq. (2.3) of the lecture notes as

$$\sigma = -\varepsilon_0 \frac{\partial\phi}{\partial z}\Big|_{z=0},$$

giving

$$\sigma = \frac{p}{2\pi}\frac{(2d^2 - x^2 - y^2)\cos\theta - 3dx\sin\theta}{(x^2+y^2+d^2)^{5/2}}.$$

(ii) The potential energy of interaction between the actual dipole and its image (i.e. the conducting plane) may be calculated using Eqs. (3.16), with the additional factor ½ because the image dipole is induced by the actual one – see the discussion following that formula in the lecture notes. Taking $r = 2d$, and using Eqs. (*), we get

$$U_{\text{int}} = -\frac{1}{8\pi\varepsilon_0}\frac{p^2}{(2d)^3}\left(1+\cos^2\theta\right). \qquad (**)$$

Note that for any angle θ, the interaction energy is negative, with its magnitude increasing at $d \to 0$, i.e. an electric dipole is always attracted to a conductor.

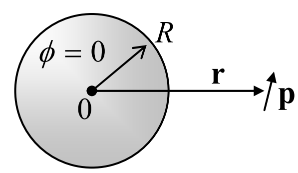
(iii) Now we can use Eq. (**) to calculate the force and the torque acting of the dipole. As should be clear from the symmetry of that expression (namely, its independence on the horizontal position of the dipole), the force has only one nonvanishing component,

$$F_z = -\frac{\partial U_{\text{int}}}{\partial d} = -\frac{1}{4\pi\varepsilon_0}\frac{3p^2}{16d^4}\left(1+\cos^2\theta\right) < 0,$$

so the force is directed toward the plane. The torque vector also has only one Cartesian component, normal to the plane of the drawing:

$$\tau_y = -\frac{\partial U_{\text{int}}}{\partial \theta} = -\frac{1}{4\pi\varepsilon_0}\frac{p^2}{16d^3}\sin 2\theta .$$

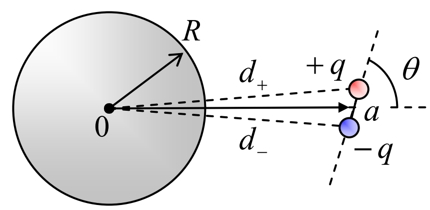
(Alternatively, this result may be obtained from Eq. (3.17) of the lecture notes.) The torque disappears at $\theta = 0$, π, and $\pm\pi/2$, i.e. in all positions in which the dipole moment is aligned with the field created by its image. Of these configurations, only the former two, $\theta = 0$ (the dipole directed up), and $\theta = \pi$ (directed down), are stable with respect to dipole rotation, because they correspond to the minima of the interaction energy (**).[86]



Problem 3.9. Calculate the net charge Q induced in a grounded conducting sphere of radius R by a dipole $\mathbf{p}$ located at point $\mathbf{r}$ outside the sphere – see the figure on the right.

Solution: The simplest way to solve this problem is to recall that according to Eq. (3.9) of the lecture notes, the dipole $\mathbf{p}$ may be represented as a limiting case of a couple of equal but opposite charges $\pm q$ displaced by the vector $\mathbf{a} = \mathbf{p}/q$, at $a \to 0$ but $q \to \infty$, so that $qa = p =$ const. For our problem's geometry, in the first approximation in the small parameter $a/r << 1$, the distances of the components of such a pair from the sphere's center are

$$d_\pm = r \pm \frac{a}{2}\cos\theta,$$



where θ is the angle between the vectors $\mathbf{r}$ and $\mathbf{p}$ – see the figure on the right. Now applying Eq. (2.199) to each charge of the pair, we get

[86] In principle, such bistable systems of spontaneous molecular dipoles may be used as binary memory cells, potentially scalable to atomic size,. However, to be practicable, such cells would need atomic-scale devices for their state change and readout.

$$Q = -\frac{R}{d_+}q + \frac{R}{d_-}q = Rq\left[-\frac{1}{r+(a/2)\cos\theta} + \frac{1}{r-(a/2)\cos\theta}\right].$$

The limit of this expression at $a/r \to 0$,

$$Q \to \frac{Rqa}{r^2}\cos\theta \equiv \frac{Rp}{r^2}\cos\theta \equiv \frac{R\,\mathbf{r}\cdot\mathbf{p}}{r^3},$$

gives the solution of our problem.

<u>Problem 3.10</u>. Use two different approaches to calculate the energy of interaction between a grounded conductor and an electric dipole **p** placed in the center of a spherical cavity of radius R, carved in the conductor.

Solution: Let us use the spherical coordinates $\{r, \theta, \varphi\}$, with the origin in the center of the sphere, and the z-axis directed along the dipole moment vector **p**.

<u>Approach 1</u>. As we know from the solution of Problem 2.28, if a thin spherical layer of radius R is charged with areal density

$$\sigma = \sigma_0 \cos\theta, \qquad (*)$$

it creates, outside it, a purely dipole field corresponding to the dipole moment $\mathbf{p}_\sigma = \mathbf{n}_z(4\pi/3)\sigma_0 R^3$. Hence, if we select σ_0 to satisfy the condition $\mathbf{p} + \mathbf{p}_\sigma = 0$, i.e. take

$$\sigma_0 = -\frac{3}{4\pi R^3}p, \qquad \text{so } \sigma = -\frac{3}{4\pi R^3}p\cos\theta, \qquad (**)$$

the net field of these two dipoles vanishes at $r \geq R$, thus satisfying the boundary condition $\phi(R) = 0$.

Now let us use one more result of the solution of Problem 2.28, namely that the internal electric field induced by the distribution (*) is uniform and equal to

$$\mathbf{E}_{\text{in}} = -\frac{\sigma_0}{3\varepsilon_0}\mathbf{n}_z.$$

With the σ_0 given by the first of Eqs. (**), this field of dipole-induced surface charges is

$$\mathbf{E}_{\text{in}} = \frac{\mathbf{p}}{4\pi\varepsilon_0 R^3}.$$

So, according to Eq. (3.15b), the dipole-cavity interaction energy is

$$U = -\frac{p^2}{8\pi\varepsilon_0 R^3}.$$

<u>Approach 2</u>. The model solution of Problem 2.35 had the following by-product: the force **F** of attraction of a point charge q, displaced from the center of a spherical cavity by distance $z \ll R$, is directed to the nearest point of the surface, and is equal to

$$F(z) = \frac{q^2}{4\pi\varepsilon_0 R^3}z. \qquad (***)$$

Let us represent our dipole **p** as a result of a gradual separation of two point charges $+q$ and $-q$, shifting each of them from the center of the cavity, along the z-axis, to the final positions $\pm a/2$, where $a = p/q \ll R$ – see Eq. (3.9) of the lecture notes. Physically, the force (***) is exerted by the surface charge induced by this point charge. In our case, there are two such distributions (induced by each of the two charges), and each of them is exerted on both charges, so the full work of the forces at the dipole formation is

$$\mathscr{W} = 4\int_0^{a/2} F(z)dz = 4\frac{q^2}{4\pi\varepsilon_0 R^3}\int_0^{p/2q} zdz = 4\frac{q^2}{4\pi\varepsilon_0 R^3}\frac{1}{2}\left(\frac{p}{2q}\right)^2 = \frac{p^2}{8\pi\varepsilon_0 R^3}.$$

(Note that this calculation does not take into account the forces of the direct interaction of the dipole charges, because they are responsible for the built-in internal energy of the dipole itself, and in particular, do not depend on R.)

Since at the beginning of the build-up, the dipole did not exist, the potential energy of the interaction at the end of the build-up interaction is just minus the calculated work, giving the same result as in Approach 1.

<u>Problem 3.11</u>. A plane separating two halves of otherwise free space is densely and uniformly (with a constant areal density n) filled with electric dipoles, with similar moments **p** oriented normally to the plane.

(i) Use two different approaches to calculate the electrostatic potential at distances $d \gg 1/n^{1/2}$ on both sides of the plane.
(ii) Give a physical interpretation of your result.
(iii) Use the result to calculate the potential distribution created in space by a spherical surface of radius R, densely and uniformly filled with radially oriented dipoles.

Solutions:

(i) At the stated condition $d \gg 1/n^{1/2}$, the dipole set may be treated as a 2D continuum, with the polarization

$$\mathbf{P} = n\mathbf{p}\delta(z) = \mathbf{n}_z np\delta(z), \qquad (*)$$

where the z-axis is normal to the plane, whose position is taken for $z = 0$. This approximation allows us to proceed using either of the following approaches.

<u>Approach 1</u>. Plugging Eq. (*) into Eq. (3.27) of the lecture notes, we get

$$\phi(\mathbf{r}) = \frac{np}{4\pi\varepsilon_0}\int \mathbf{n}_z \cdot \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\delta(z)d^3r' = \frac{np}{4\pi\varepsilon_0}\int \frac{z}{|\mathbf{r}-\mathbf{r}'|^3}d^2r' = \frac{np}{4\pi\varepsilon_0}\int \frac{z}{\left(\rho^2+z^2\right)^{3/2}}d^2r',$$

where ρ is the in-plane component of the vector ($\mathbf{r}' - \mathbf{r}$), and the integration is over the whole plane. This integral may be readily worked out in polar coordinates:

$$\int \frac{z}{\left(\rho'^2+z^2\right)^{3/2}}d^2r' = 2\pi\int_0^\infty \frac{z}{\left(\rho'^2+z^2\right)^{3/2}}\rho' d\rho' = 2\pi\frac{\operatorname{sgn}(z)}{2}\int_0^\infty \frac{d\xi}{(\xi+1)^{3/2}} = 2\pi\frac{\operatorname{sgn}(z)}{2}2 \equiv 2\pi\operatorname{sgn}(z),$$

finally giving

$$\phi(\mathbf{r}) = \frac{np}{2\varepsilon_0}\,\mathrm{sgn}(z). \qquad (**)$$

As usual, the potential is only defined up to an additive constant, but such an addition would not affect the finite leap of the potential across the dipole layer:

$$\Delta\phi \equiv \phi(z+0) - \phi(z-0) = \frac{np}{\varepsilon_0}. \qquad (***)$$

Approach 2. Due to the symmetry of this problem with respect to any translation parallel to the plane, the electrostatic potential may depend only on the z-coordinate, so the Poisson equation for the macroscopic electrostatic potential takes the form

$$\frac{d^2\phi}{dz^2} = -\frac{\rho_{\mathrm{ef}}}{\varepsilon_0},$$

where ρ_{ef} is given by Eq. (3.30) of the lecture notes, which in our 1D case is reduced to

$$\rho_{\mathrm{ef}} = -\frac{dP_z}{dz}.$$

Integrating the Poisson equation over z once, and using Eq. (*), we get

$$\frac{d\phi}{dz} = -\frac{1}{\varepsilon_0}\int \rho(z)dz = \frac{P(z)}{\varepsilon_0} + c_1 = \frac{np}{\varepsilon_0}\delta(z) + c_1,$$

where the physical sense of the constant c_1 is the (minus) electric field outside of the interface. Per the Gauss law, the dipole layer cannot create such a field because its net charge density σ equals zero, so in the absence of other field sources, $c_1 = 0$. Now integrating the last equation again over an infinitesimal interval across the plane (so the contribution of the integral of c_1 is negligible), we return to Eq. (**).

(ii) The physical sense of the potential's leap (***) at $z = 0$ becomes clear from the simple cartoon in which each dipole $\mathbf{p}$ is represented by two equal and opposite charges $\pm q$ separated by a small distance vector $\mathbf{a} = \mathbf{p}/q$ – see Eq. (3.9) of the lecture notes. Because of that, the dipole layer we are discussing may be faithfully represented by two slightly separated charge layers, with equal and opposite areal densities $\pm\sigma = \pm nq$. (Such a system is frequently called the *double electric layer*.) As we know from the numerous discussions of such a system in Charters 1 and 2, the electric field induced by the charges between these layers equals $E_z = -nq/\varepsilon_0$, so the potential difference it creates is $\Delta\phi = -E_z a = nqa/\varepsilon_0 \equiv np/\varepsilon_0$, thus returning us to Eq. (***).

(iii) According to the Gauss law, since the sphere carries no net charge, the electric field not only inside it but also outside it has to be zero, so the electrostatic potential should be constant in each of these regions:

$$\phi(\mathbf{r}) = \begin{cases} \phi_{\mathrm{in}} = \mathrm{const}, & \text{for } r < R, \\ \phi_{\mathrm{out}} = \mathrm{const}, & \text{for } r > R, \end{cases}$$

with the origin of $\mathbf{r}$ in the center of the sphere. Very close to any smooth surface, it may be well approximated with a plane, so the boundary condition (***) is valid for two points very close to the sphere's surface, giving

$$\phi_{\mathrm{out}} - \phi_{\mathrm{in}} = \frac{np}{\varepsilon_0}.$$

With the usual convention $\phi(\infty) = 0$, this result gives

$$\phi(\mathbf{r}) = \begin{cases} -np/\varepsilon_0, & \text{for } r < R, \\ 0, & \text{for } r > R. \end{cases} \qquad (****)$$

All the above results, based on the approximation of the dipole system with a 2D continuum, are valid if the average distance $n^{-1/2}$ between the nearest dipoles is much smaller than any other linear scale of the problem. For the case of the sphere, this role is played by the distance $d = |r - R|$ of the observation point from the sphere's surface and its radius R, so Eq. (****) is valid if

$$nR^2, n(r-R)^2 >> 1.$$

Actually, these conditions of continuity are applicable to any smooth interface, provided that R is understood as the smallest of its two curvature radii.

Problem 3.12. Prove Eq. (24) of the lecture notes.

Hint: You may like to use the basic Eq. (1.9) to spell out the left-hand side of Eq. (24), change the order of integration over **r** and **r**', and then contemplate the physical sense of the inner integral.

Solution: The first two steps suggested in the *Hint* yield

$$\int_V \mathbf{E}(\mathbf{r})d^3r = \frac{1}{4\pi\varepsilon_0}\int_V d^3r \int d^3r' \rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \equiv -\int d^3r' \left[\frac{1}{4\pi\varepsilon_0}\int_V d^3r \rho(\mathbf{r}')\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3}\right]. \qquad (*)$$

A comparison of the contents of the square brackets with the same Eq. (1.9) shows that this is just the electric field $\mathbf{E}_c(\mathbf{r}')$ that would be created by a distributed charge with a *constant* (for the purposes of the **r**-space integration) density $\rho(\mathbf{r}')$ in the considered volume V. By the condition of Eq. (24), this volume is a sphere of any radius R large enough to contain all the charges, so the **r**'-space integration in Eq. (*) may be limited to the region with $r' < R$. For the observation points satisfying this condition, the field $\mathbf{E}_c(\mathbf{r}')$ was calculated in Sec. 1.2, and we may use for it the vector form of Eq. (1.22) with the appropriate change of notation ($\mathbf{r} \to \mathbf{r}'$):

$$\mathbf{E}_c(\mathbf{r}') = \frac{\rho(\mathbf{r}')}{3\varepsilon_0}\mathbf{r}', \qquad \text{for } r' < R.$$

As a result, Eq. (*) is reduced to

$$\int_V \mathbf{E}(\mathbf{r})d^3r = -\int_{r'<R} \mathbf{E}_c(r')d^3r' = -\frac{1}{3\varepsilon_0}\int_{r'<R} \rho(\mathbf{r}')\mathbf{r}'d^3r'.$$

But according to Eq. (3.6), the last integral is just the total electric dipole moment **p** of our system, so the last equality coincides with Eq. (24).

Problem 3.13. A sphere of radius R is made of a material with a uniform spontaneous polarization $\mathbf{P}_0$. Calculate the electric field everywhere in space – both inside and outside of the sphere, and compare the result for the internal field with Eq. (3.24) of the lecture notes.

Solution: According to the general Eq. (3.31), the effective charges with the density $\rho_{ef} \equiv -\nabla\cdot\mathbf{P}$ create the electric field exactly as the stand-alone charges – which are absent in this problem. Since the polarization **P** is uniform both inside the sphere ($\mathbf{P} = \mathbf{P}_0$) and outside it ($\mathbf{P} = 0$), ρ_{ef} is different from zero only on the sphere's surface, forming the effective surface charge density σ_{ef} that may be found by the integration of Eq. (3.31) through an infinitesimal interval of the outer normal **n** to the surface:

$$\sigma_{\text{ef}} = \int_{n=-0}^{n=+0} \rho_{\text{ef}} dn = -\int_{n=-0}^{n=+0} \nabla \cdot \mathbf{P}\, dn = P_0 \cos\theta,$$

where θ is the angle between the vector $\mathbf{P}_0$ and the local normal to the surface. With the z-axis directed along the vector $\mathbf{P}_0$, θ becomes the usual polar angle of the spherical coordinates with the origin in the sphere's center. But this means that the spatial distribution of the electrostatic potential ϕ and the electric field $\mathbf{E} = -\nabla\phi$ in this problem are exactly the same as were calculated in the solution of Problem 2.28, with the replacement $\sigma_0 \to P_0$:

$$\phi_{\text{in}} = \frac{P_0 r}{3\varepsilon_0}\cos\theta, \qquad \phi_{\text{out}} = \frac{P_0 R^3}{3\varepsilon_0 r^2}\cos\theta. \qquad (*)$$

Comparing the second of these expressions with Eq. (3.7) of the lecture notes, we see that the electric field outside of the sphere coincides with that of an electric dipole with the moment

$$\mathbf{p} = \frac{4\pi}{3} R^3 \mathbf{P}_0 = V\mathbf{P}_0, \qquad (**)$$

equal to the sum of all elementary dipoles within the sphere – the result that could be conjectured even before the formal solution of the problem. On the other hand, the first of Eqs. (*) describes a uniform electric field inside the sphere, with

$$\mathbf{E}_{\text{in}} = -\frac{\mathbf{P}_0}{3\varepsilon_0}, \qquad (***)$$

and $\mathbf{D}_{\text{in}} \equiv \varepsilon_0\mathbf{E}_{\text{in}} + \mathbf{P}_0 = (2/3)\mathbf{P}_0$. (Note that the vectors **E** and **D** have *opposite* directions, once again demonstrating the radical difference between these two macroscopic fields.)

Note also another interesting way to derive Eq. (***). In Sec. 3.4 of the lecture notes, we have obtained the following result for the internal field and polarization of a sphere made of a *linear* dielectric (with $\mathbf{P} \propto \mathbf{E}$) , placed into a uniform external field $\mathbf{E}_0$ – see Eqs. (3.65):

$$\mathbf{E}_{\text{in}} = \frac{3}{\kappa+2}\mathbf{E}_0, \qquad \mathbf{P} = 3\varepsilon_0\frac{\kappa-1}{\kappa+2}\mathbf{E}_0. \qquad (****)$$

The field $\mathbf{E}_{\text{in}}$ may be considered a sum of the external field $\mathbf{E}_0$ and the field $\mathbf{E}_{\text{self}}$ created by the polarization of the sphere itself. For the latter field, Eqs. (****) yield

$$\mathbf{E}_{\text{self}} \equiv \mathbf{E}_{\text{in}} - \mathbf{E}_0 = \frac{3}{\kappa+2}\mathbf{E}_0 - \mathbf{E}_0 \equiv -\frac{\kappa-1}{\kappa+2}\mathbf{E}_0 \equiv -\frac{\mathbf{P}}{3\varepsilon_0},$$

i.e. the equality similar to Eq. (***). This is natural because the field induced by a polarization of some volume does not depend on whether this polarization is induced or spontaneous.

One more note: by using Eq. (**), Eq. (***) may be recast as

$$\mathbf{E}_{\text{in}} = -\frac{\mathbf{p}}{3\varepsilon_0}\frac{1}{V}, \qquad \text{so } \int_V \mathbf{E}_{\text{in}} d^3r = -\frac{\mathbf{p}}{3\varepsilon_0},$$

in agreement with the general Eq. (3.24) whose proof was the subject of the previous problem.

Problem 3.14. Calculate the electric field at the center of a cube made of a material with the uniform spontaneous polarization $\mathbf{P}_0$ of arbitrary orientation.

Solution: As was discussed in the previous problem, the effect of a uniform polarization of a volume is equivalent to its surface's charging with the areal charge density

$$\sigma_{\text{ef}} = \mathbf{P} \cdot \mathbf{n}, \qquad (*)$$

where $\mathbf{n}$ is the unit vector outer-normal to the surface. Let us orient the coordinate axes along the cube's sides and consider, for example, the two faces normal to the x-axis. For them, Eq. (*) yields constant charge densities $(\sigma_{\text{ef}})_x = \pm P_x$, where $P_x \equiv \mathbf{P}\cdot\mathbf{n}_x$ is the x-component of the vector $\mathbf{P}_0$. Obviously, these two face charges give equal contributions to the electric field in the center of the cube, directed normally to those faces, i.e. along the x-axis. The calculation of the magnitude of the field induced by one cube's face carrying a constant charge density σ was one of the tasks of Problem 1.7; the result was $|E| = \sigma/6\varepsilon_0$. Replacing σ with σ_{ef}, doubling the result (to reflect the equal contributions of both charged faces), and taking into account the field's direction (from the positive charge to the negative one), we see that the x-normal faces yield

$$\mathbf{E}_x = -\frac{P_x}{3\varepsilon_0}\mathbf{n}_x,$$

with no contributions to other field components because of the system's symmetry. Now repeating this analysis for the other two face pairs, we may summarize our result as

$$\mathbf{E} = \mathbf{E}_x + \mathbf{E}_y + \mathbf{E}_z = -\frac{1}{3\varepsilon_0}\left(P_x\mathbf{n}_x + P_y\mathbf{n}_y + P_z\mathbf{n}_z\right) \equiv -\frac{\mathbf{P}_0}{3\varepsilon_0}, \qquad (**)$$

for any polarization's orientation and independently of the cube's size.

Note that this is exactly the same result as was obtained in the solution of the previous problem for the spontaneously polarized sphere. Also, by multiplying both sides of Eq. (**) by the volume V of the cube, and taking into account that its total dipole moment of the cube is $\mathbf{p} = \mathbf{P}_0 V$, we may rewrite our result in the form of Eq. (3.24) of the lecture notes (whose proof was the task of Problem 13):

$$\int_V \mathbf{E}_{\text{in}}\, d^3r = -\frac{\mathbf{p}}{3\varepsilon_0},$$

showing that the validity of this relation extends beyond the spherical geometry.

Problem 3.15. Derive the Clausius-Mossotti formula (see Eq. (3.52) of the lecture notes) by combining Eq. (3.24) with the result of the solution of Problem 5.

Solution: Due to the linear superposition principle, the actual (microscopic) field $\mathbf{E}_{\text{m}}(\mathbf{r})$ inside a dipole medium may be represented as the sum,

$$\mathbf{E}_{\text{m}}(\mathbf{r}) = \mathbf{E}_{\text{ext}} + \mathbf{E}_{\text{self}}(\mathbf{r}), \qquad (*)$$

where $\mathbf{E}_{\text{ext}}$ is the field of the distant stand-alone charges (say, those located outside of the considered sample)[87] and the field $\mathbf{E}_{\text{self}}$ of the medium itself. On the atomic/molecular scale, the former of these

[87] Per the discussion following Eq. (3.33) of the lecture notes, $\mathbf{E}_{\text{ext}}$ is just the "would-be" field $\mathbf{D}/\varepsilon_0$.

fields may be considered uniform, while the latter of them may vary from point to point very substantially.

First, let us average Eq. (*) over a macroscopic sphere of volume V so large that, near its center, the field of the outer molecules is negligible, applying Eq. (3.24) to the last term, $\langle \mathbf{E}_{\text{self}}(\mathbf{r})\rangle$:[88]

$$\frac{1}{V}\int_V \mathbf{E}_{\text{m}}(r)d^3r \equiv \mathbf{E} = \mathbf{E}_{\text{ext}} - \frac{\mathbf{P}}{3\varepsilon_0}.$$

Here $\mathbf{E}$ is the usual average (macroscopic) electric field, while $\mathbf{P}$ is the average polarization of the medium. In an isotropic linear dielectric, they are related by Eq. (3.45), $\mathbf{P} = (\kappa - 1)\varepsilon_0\mathbf{E}$, so the last displayed relation may be rewritten as

$$\frac{\mathbf{P}}{\varepsilon_0(\kappa-1)} = \mathbf{E}_{\text{ext}} - \frac{\mathbf{P}}{3\varepsilon_0}. \qquad (**)$$

On the other hand, we may apply the same Eq. (*) to the center of one of the molecules, taking it for $\mathbf{r} = 0$:

$$\mathbf{E}_{\text{m}}(0) = \mathbf{E}_{\text{ext}} + \mathbf{E}_{\text{self}}(0).$$

Modeling molecules as point dipoles, we may associate $\mathbf{E}_{\text{m}}(0)$ with the field participating in Eqs. (3.48)-(3.49), so this equality becomes

$$\frac{\mathbf{P}}{\alpha n} = \mathbf{E}_{\text{ext}} + \mathbf{E}_{\text{self}}(0). \qquad (***)$$

Now let us apply, to the last term, the result of Problem 5 (obtained for a simple but representative model of the dipole lattice): $\mathbf{E}_{\text{self}}(0) = 0$. With this, the combination of Eqs. (**) and (***) immediately yields the Clausius-Mossotti relation

$$\kappa - 1 = \frac{\alpha n/\varepsilon_0}{1 - \alpha n/3\varepsilon_0}. \qquad (****)$$

Note that this derivation substantially depends on the variation of the field $\mathbf{E}_{\text{self}}(\mathbf{r})$ from point to point – in particular, on the relation $\mathbf{E}_{\text{self}}(0) \neq \langle \mathbf{E}_{\text{self}}(\mathbf{r})\rangle$. Quantitatively, it is based on modeling molecules with point dipoles, explaining why Eq. (****) does not work well for many condensed materials in that the effective size of the constituent molecules is close to the average distance between them.

Problem 3.16. Stand-alone charge Q is distributed, in some way, within the volume of a body made of a uniform linear dielectric with a dielectric constant κ. Calculate the total polarization charge Q_{ef} residing on the surface of the body, provided that it is surrounded by free space.

Solution: Let us apply the *macroscopic* Gauss law (see Eq. (3.34) of the lecture notes) to two closed surfaces – one (S_{in}) immediately inside the body's surface and one (S_{out}) immediately outside it, so the stand-alone charge inside them is the same (Q):

$$\oint_{S_{\text{in}}} D_n d^2r = Q, \qquad \oint_{S_{\text{out}}} D_n d^2r = Q. \qquad (*)$$

[88] As the solution of the previous problem illustrates, this relation is not limited to the spherical geometry, which, due to its isotropy, represents the average of $\mathbf{E}_{\text{self}}$ pretty fairly.

The inner surface is inside the dielectric, so at all its points, $\mathbf{D} = \varepsilon\mathbf{E}$, i.e. $D_n = \varepsilon E_n$ (where $\varepsilon \equiv \kappa\varepsilon_0$ is the dielectric's permittivity), while at the outer surface, $\mathbf{D} = \varepsilon_0\mathbf{E}$, i.e. $D_n = \varepsilon_0 E_n$. Plugging these relations into Eqs. (*), we get

$$\oint_{S_{\text{in}}} E_n d^2r = \frac{Q}{\varepsilon}, \qquad \oint_{S_{\text{out}}} E_n d^2r = \frac{Q}{\varepsilon_0}. \qquad (**)$$

But according to the "microscopic" Gauss law (see Eq. (1.16) of the lecture notes), applied to the very slim volume between these two surfaces,

$$\oint_{S_{\text{out}}} E_n d^2r - \oint_{S_{\text{in}}} E_n d^2r = \frac{Q_{\text{ef}}}{\varepsilon_0},$$

because this volume, by the problem's conditions, does not contain any stand-alone surface charges but may have a surface polarization charge Q_{ef}. Plugging in the integrals from Eqs. (**), we get

$$\frac{Q}{\varepsilon_0} - \frac{Q}{\varepsilon} = \frac{Q_{\text{ef}}}{\varepsilon_0}, \qquad \text{giving } Q_{\text{ef}} = Q\left(1 - \frac{\varepsilon_0}{\varepsilon}\right) \equiv Q\left(1 - \frac{1}{\kappa}\right).$$

Note that this result does not describe the part of the polarization ("effective") charge with the density $\rho_{\text{ef}} = -\nabla\cdot\mathbf{P}$ that may be distributed in the volume of the body. (This density vanishes only if the polarization inside the body is uniform: $\mathbf{P}$ = const.)

Problem 3.17. In two separate experiments, a thin plane sheet of a linear dielectric with κ = const is placed into a uniform external electric field $\mathbf{E}_0$, in two different ways:

(i) with the sheet's surfaces parallel to the electric field, and
(ii) with its surfaces normal to the field.

For each case, find the electric field $\mathbf{E}$, the electric displacement $\mathbf{D}$, and the polarization $\mathbf{P}$ inside the dielectric, sufficiently far from the sheet's edges.

Solution: The same reasoning that was used in Sec. 3.4 of the lecture notes to analyze free-space slits in dielectrics, based on the continuity of E_τ and D_n at the "major" (larger) surfaces of the sheet, gives the following results:

(i) for the sheet parallel to the electric field:[89]

$$\mathbf{E} = \mathbf{E}_0, \qquad \text{and hence } \mathbf{D} \equiv \varepsilon\mathbf{E} = \kappa\varepsilon_0\mathbf{E}_0, \quad \mathbf{P} = (\kappa - 1)\varepsilon_0\mathbf{E}_0;$$

(ii) for the sheet normal to the field:

$$\mathbf{D} = \mathbf{D}_0 \equiv \varepsilon_0\mathbf{E}_0, \qquad \text{and hence } \mathbf{E} \equiv \frac{\mathbf{D}}{\kappa\varepsilon_0} = \frac{\mathbf{E}_0}{\kappa}, \quad \mathbf{P} = \frac{\kappa - 1}{\kappa}\varepsilon_0\mathbf{E}_0.$$

We see that in the latter case, both fields, $\mathbf{D}$ and $\mathbf{E}$, are κ times weaker than in the former one.

[89] Note that this result was also derived in Sec. 3.5 from the condition of the minimum of the Gibbs potential energy (3.78).

Problem 3.18. A fixed dipole **p** is placed in the center of a spherical cavity of radius R, carved inside a uniform linear dielectric. Calculate the electric field distribution everywhere in the system.

Hint: You may start with the assumption that the field at $r > R$ has a distribution typical for a dipole. However, be ready for surprises.

Solution: Following the *Hint*, let us look for the electric potential's distribution in the form

$$\phi = \frac{1}{4\pi\varepsilon_0} \times \begin{cases} \dfrac{p'\cos\theta}{r^2}, & \text{for } r \geq R, \\ \dfrac{p\cos\theta}{r^2} + \zeta(r,\theta), & \text{for } r \leq R, \end{cases}$$

where $\zeta(r,\theta)$ is an axially symmetric function that satisfies the Laplace equation (added to accommodate the promised surprise). With this assumption, the boundary conditions at $r = R$ take the form

$$\begin{aligned} &\phi = \text{const}: \quad \frac{p'\cos\theta}{R^2} = \frac{p\cos\theta}{R^2} + \zeta(R,\theta), \\ &\kappa\frac{\partial\phi}{\partial n} \equiv \kappa\frac{\partial\phi}{\partial r} = \text{const}: \quad -\kappa\frac{2p'\cos\theta}{R^3} = -\frac{2p\cos\theta}{R^3} + \frac{\partial\zeta}{\partial r}(R,\theta). \end{aligned} \qquad (*)$$

Representing the function $\zeta(r,\theta)$ as a series over the Legendre polynomials (see, e.g., Eq. (2.172) of the lecture notes), we see, from the boundary conditions, that only the first term of that expansion, $a_1 r\cos\theta$, may have a nonvanishing magnitude. For the coefficients p' and a_1, Eqs. (*) turn into a system of two linear equations:

$$\frac{p'}{R^2} = \frac{p}{R^2} + a_1 R, \qquad -\kappa\frac{2p'}{R^3} = -\frac{2p}{R^3} + a_1,$$

with the solution

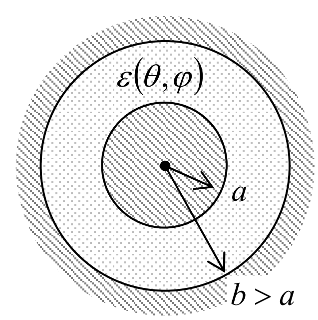
$$p' = \frac{3p}{2\kappa+1}, \quad a_1 = -\frac{2(\kappa-1)p}{(2\kappa+1)R^3}.$$

Hence the electric field outside the spherical cavity indeed corresponds to a single dipole, but with a different dipole moment $\mathbf{p}' = 3\mathbf{p}/(2\kappa + 1)$, while that inside the sphere is the sum of the original dipole field and (here is the promised surprise) an additional *uniform* field

$$\mathbf{E}_{\text{in}} = \frac{1}{4\pi\varepsilon_0}\frac{2(\kappa-1)}{2\kappa+1}\frac{\mathbf{p}}{R^3}.$$

As a sanity check, in the limit $\kappa \to \infty$, this field approaches the κ-independent value that was calculated in the model solution of Problem 10. On the other hand, in the limit $\kappa \to 1$ (say, of no dielectric at all), the uniform field vanishes.

Problem 3.19. A spherical capacitor (see the figure on the right) is filled with a linear dielectric whose permittivity ε depends on the spherical angles θ and φ, but not on the distance r from the system's center. Derive an explicit expression for its capacitance C.



Solution: Let us prove that the boundary problem for the field distribution inside the capacitor is satisfied by the following radially-directed fields:

$$\mathbf{E} = \mathbf{n}_r E(r), \qquad \mathbf{D} = \mathbf{n}_r \varepsilon(\theta,\varphi) E(r).$$

Since the curl of such a vector **E** equals zero,[90] the first macroscopic Maxwell equation for **E**, given by Eq. (3.36) of the lecture notes, is satisfied. On the other hand, the macroscopic Maxwell equation (3.32) for **D** is satisfied if $\nabla\cdot\mathbf{D} = 0$. For the selected form of **D**, this equation is reduced to[91]

$$\frac{d}{dr}\left[r^2 E(r)\right] = 0.$$

Its straightforward integration yields the same result,

$$E(r) = \frac{c_1}{r^2},$$

as for the capacitor without the dielectric (i.e. with $\varepsilon = \varepsilon_0 = \text{const}$), which was analyzed in Sec. 2.3(iii) of the lecture notes, so the boundary conditions for the corresponding electrostatic potential,

$$\phi(r) = \phi(a) - \int_a^r E(r)dr = \phi(a) + \frac{c_1}{r} + c_2,$$

may be satisfied by the same choice of the constants $c_{1,2}$ as in that case.

Hence, our proof has been accomplished, and now we may use Eq. (2.53) to write the expression for the electric field on the surface of any conductor – say, the inner one:

$$E(a) = \frac{V}{a^2}\left(\frac{1}{a} - \frac{1}{b}\right)^{-1}.$$

Only at this stage, the difference between the current and dielectric-free cases cuts in, because the areal density of stand-alone charges on the surface of the conductor has to be calculated by using Eq. (3.54) rather than Eq. (2.3):

$$\sigma_a = D(a) = \varepsilon(\theta,\varphi)E(a) = \varepsilon(\theta,\varphi)\frac{V}{a^2}\left(\frac{1}{a} - \frac{1}{b}\right)^{-1},$$

so the capacitance is

$$C \equiv \frac{Q_a}{V} = \frac{1}{V}\int_{r=a}\sigma_a d^2r = \frac{a^2}{V}\int_{r=a}\sigma_a d\Omega = \left(\frac{1}{a} - \frac{1}{b}\right)^{-1}\int_{4\pi}\varepsilon(\theta,\varphi)d\Omega \equiv \left(\frac{1}{a} - \frac{1}{b}\right)^{-1}\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\varphi\ \varepsilon(\theta,\varphi). \quad (*)$$

For the particular case $\varepsilon = \text{const}$, this double integral is equal to $4\pi\varepsilon$, and we immediately get an evident generalization of the (easy) solution of Problem 2.8(i):

$$C = 4\pi\varepsilon\left(\frac{1}{a} - \frac{1}{b}\right)^{-1} \equiv 4\pi\varepsilon\frac{ab}{b-a}. \quad (**)$$

Evidently, Eq. (*) may be interpreted as just a result of the connection, in parallel, of many elementary spherical sectors $d\Omega = \sin\theta d\theta d\varphi$, each with a capacitance dC described similarly to Eq. (**):

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[90] See, e.g., MA Eq. (10.11) with $f_\theta = f_\varphi = 0$ and $\partial/\partial\theta = \partial/\partial\varphi = 0$.

[91] See, e.g., MA Eq. (10.10) with $f_\theta = f_\varphi = 0$.

$$dC = \varepsilon(\theta,\varphi)\frac{ab}{b-a}d\Omega..$$

Problem 3.20. A spherical capacitor similar to that considered in the previous problem is now filled with a linear dielectric whose permittivity depends only on the distance from the center. Obtain an explicit expression for its capacitance, and spell it out for the particular case $\varepsilon(r) = \varepsilon(a)(r/a)^n$.

Solution: As was discussed at the end of Sec. 3.4 of the lecture notes, in the case of spatial-dependent permittivity $\varepsilon(\mathbf{r})$, we may find the distribution of the electrostatic potential (defined by Eq. (1.33), $\mathbf{E} = -\nabla\phi$) by solving the Maxwell equation (3.32) for $\mathbf{D}$ together with the relation (3.46) valid for linear dielectrics:

$$\nabla\cdot\mathbf{D} \equiv \nabla\cdot[\varepsilon(\mathbf{r})\mathbf{E}] \equiv -\nabla\cdot[\varepsilon(\mathbf{r})\nabla\phi] = \rho. \qquad (*)$$

In our spherically-symmetric case, ϕ may depend only on the distance r from the capacitor's center (which we will take for the spherical coordinates' origin), so in the region $a < r < b$ where $\rho = 0$,[92] this equation is reduced to[93]

$$-\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\varepsilon(r)\frac{\partial\phi}{\partial r}\right] = 0.$$

Its first integral is, obviously,

$$\frac{\partial\phi}{\partial r} = \frac{c}{r^2\varepsilon(r)},$$

where c is an integration constant. This constant may be expressed via the inner conductor's charge Q by applying the macroscopic Gauss law (3.34) to a sphere slightly larger than it:

$$4\pi a^2 D(a+0) \equiv -4\pi a^2\varepsilon(a)\frac{\partial\phi}{\partial r}\Big|_{r=a+0} \equiv -4\pi a^2\varepsilon(a)\frac{c}{a^2\varepsilon(a)} \equiv -4\pi c = Q,$$

so our first integral becomes[94]

$$\frac{\partial\phi}{\partial r} = -\frac{Q}{4\pi r^2\varepsilon(r)}.$$

Now we may integrate this equation again to get a formal solution for the potential's distribution in the dielectric layer,

$$\phi(r) = \phi(a) - \frac{Q}{4\pi}\int_a^r\frac{dr}{r^2\varepsilon(r)} \qquad \text{for } a \le r \le b,$$

and in particular, for the voltage between the electrodes and hence their mutual capacitance:

[92] As a (hopefully, unnecessary) reminder, in Eq (*), ρ is the density of stand-alone charges not bound into the dielectric layer's dipoles.

[93] See, e.g., MA Eqs. (10.8) and (10.10), with $\partial/\partial\theta = \partial/\partial\varphi = 0$.

[94] Admittedly, this formula could be obtained immediately via the application of the same Gauss law to a sphere of radius r. However, since this problem is my first (and the only) example of using Eq. (*) with a continuous function $\varepsilon(\mathbf{r})$, I wanted to demonstrate a more general approach to its solution, which is applicable to less symmetric geometries as well.

$$V \equiv \phi(a) - \phi(b) = \frac{Q}{4\pi}\int_a^b \frac{dr}{r^2\varepsilon(r)}, \qquad \text{so } C \equiv \frac{Q}{V} = 4\pi \bigg/ \int_a^b \frac{dr}{r^2\varepsilon(r)} . \qquad (**)$$

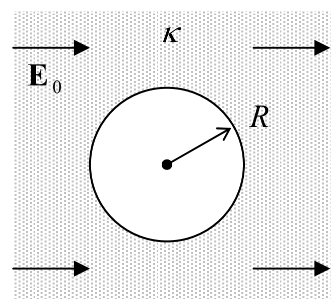
This expression may be interpreted as a result of the connection in series of many elementary quasi-plane capacitors of thickness dr and area $A(r) = 4\pi r^2$, with their reciprocal capacitances obeying Eq. (3.55): $d(C^{-1}) = dr/A(r)\varepsilon(r)$ – see the discussion of Eq. (3.58) in the lecture notes.

For the particular case given in the assignment, the integration is elementary, giving

$$C = 4\pi\varepsilon(a)a \times \begin{cases} (n+1)\big/\left[1-(a/b)^{n+1}\right], & \text{for } n \neq -1, \\ 1/\ln(b/a), & \text{for } n = -1. \end{cases}$$

As a sanity check, in the particular case of a uniform dielectric case $\varepsilon(r) = \varepsilon =$ const, i.e. $n = 0$, this result coincides with the formula obtained (for the same case) in the solution of the previous problem:

$$C = 4\pi\varepsilon \bigg/ \left(\frac{1}{a} - \frac{1}{b}\right).$$



Problem 3.21. A uniform electric field $\mathbf{E}_0$ has been created (by distant external sources) inside a uniform linear dielectric. Find the electric field's change created by carving out a cavity in the shape of a round cylinder of radius R, with its axis normal to the external field – see the figure on the right.

Solution: Introducing the usual polar coordinates, we can use the general solution (2.112) of the Laplace equation. Based on our experience with using it for the problem shown in Fig. 2.15 of the lecture notes, we may immediately look for the solution of this equation in the following form:

$$\phi\Big|_{\rho \leq R} = a_1 \rho \cos\varphi, \qquad \phi\Big|_{\rho \geq R} = \left(-E_0\rho + \frac{b_1}{\rho}\right)\cos\varphi,$$

where the coefficient a_1 has the physical sense of the uniform field inside the cavity (with the minus sign). Using the boundary conditions of continuity of ϕ and $\varepsilon\partial\phi/\partial n$ (and hence of $\kappa\partial\phi/\partial\rho$) on the cavity surface ($\rho = R$), we get the following system of two linear equations,

$$a_1 R = -E_0 R + \frac{b_1}{R}, \qquad a_1 = \kappa\left(-E_0 - \frac{b_1}{R^2}\right),$$

for two unknown coefficients, a_1 and b_1. Solving the system, we get:

$$a_1 = -\frac{2\kappa}{\kappa+1}E_0, \quad b_1 = -\frac{\kappa-1}{\kappa+1}E_0 R^2 .$$

As a result, the electrostatic potential's distribution may be represented as

$$\phi\Big|_{\rho < R} = -\frac{2\kappa}{\kappa+1}E_0\rho\cos\varphi,$$

$$\phi\big|_{\rho>R} = -E_0\left(\rho + \frac{\kappa-1}{\kappa+1}\frac{R^2}{\rho}\right)\cos\varphi \equiv -E_0 x\left(1+\frac{\kappa-1}{\kappa+1}\frac{R^2}{x^2+y^2}\right),$$

where x is the Cartesian coordinate along the initial field.

As a (necessary for us all :-) sanity check, at $\kappa = 1$ (uniform space with no dielectric), the potential distribution is the same at both $\rho > R$ and $\rho < R$:

$$\phi = \phi_0 = -E_0\rho\cos\theta,$$

and corresponds to the uniform field $\mathbf{E}_0 = E_0\mathbf{n}_x$.

In the general case, the electric field, $\mathbf{E} = -\nabla\phi$, is:

$$\mathbf{E}\big|_{\rho<R} = \frac{2\kappa}{\kappa+1}E_0\mathbf{n}_x, \qquad \mathbf{E}\big|_{\rho>R} = E_0\mathbf{n}_x + \frac{\kappa-1}{\kappa+1}E_0\left[\mathbf{n}_x\frac{R^2(y^2-x^2)}{\left(x^2+y^2\right)^2} - \mathbf{n}_y\frac{2R^2xy}{\left(x^2+y^2\right)^2}\right].$$

From here, the electric field's change from its original value $\mathbf{E}_0$ is:

$$\Delta\mathbf{E}\big|_{\rho<R} = \left(\frac{2\kappa}{\kappa+1}-1\right)E_0\mathbf{n}_x \equiv \frac{\kappa-1}{\kappa+1}E_0\mathbf{n}_x, \qquad \Delta\mathbf{E}\big|_{\rho>R} = \frac{\kappa-1}{\kappa+1}E_0\left[\mathbf{n}_x\frac{R^2(y^2-x^2)}{\left(x^2+y^2\right)^2} - \mathbf{n}_y\frac{2R^2xy}{\left(x^2+y^2\right)^2}\right].$$

It is curious that in the limit $\kappa \to \infty$, the internal electric field *increases* by exactly $E_0\mathbf{n}_x$, i.e. by 100% of its initial value, while **D** *drops* dramatically (by a factor of κ) as a result of the cavity carving. The reader is invited to interpret this fact in the light of the thin-gap "experiments" discussed in Sec. 3.4 of the lecture notes – see Fig. 3.9.

Problem 3.22. Similar small spherical particles, made of a linear dielectric, are dispersed in free space with a low concentration $n << 1/R^3$, where R is the particle's radius. Calculate the average dielectric constant of such a medium. Compare the result with the apparent but wrong answer

$$\bar{\kappa} - 1 = (\kappa-1)nV, \qquad \text{(WRONG!)}$$

(where κ is the dielectric constant of the particle's material and $V = (4\pi/3)R^3$ is its volume), and explain the origin of the difference.

Solution: Due to the given condition $nR^3 << 1$, we may neglect the mutual electrostatic interaction between the particle dipole moments **p** induced by an external electric field $\mathbf{E}_0$, so for each **p**, we may use Eq. (3.64) of the lecture notes:

$$\mathbf{p} = 4\pi\varepsilon_0 R^3\frac{\kappa-1}{\kappa+2}\mathbf{E}_0.$$

Now we may calculate the average polarization of the medium by just adding all moments in a unit volume:

$$\bar{\mathbf{P}} = n\mathbf{p} = 4\pi nR^3\frac{\kappa-1}{\kappa+2}\varepsilon_0\mathbf{E}_0.$$

By applying Eq. (3.37) to the average values of **P** and κ, we get

$$\overline{\kappa} = 1 + 4\pi nR^3 \frac{\kappa - 1}{\kappa + 2} \equiv 1 + (\kappa - 1)nV \frac{3}{\kappa + 2}. \qquad (*)$$

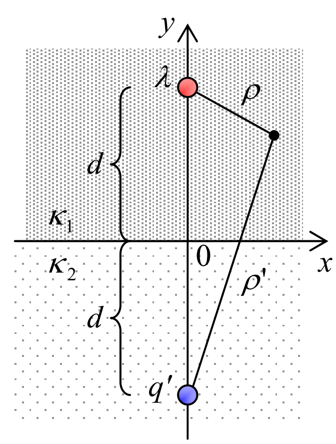
The formula shows that the apparent result cited in the assignment is valid only asymptotically when the last fraction on its right-hand side of Eq. (*) approaches 1, i.e. in the limit $\kappa \to 1$ when the particle material's polarization is very weak. The reason for this difference is clear from the solution of the single sphere polarization problem in Sec. 3.4, and in particular from Fig. 3.11b and Eq. (3.65): due to the non-uniform distribution of the electric field in the vicinity of the sphere (caused by its polarization), the field inside the sphere (albeit uniform) differs, by exactly the factor $3/(\kappa + 2)$, from the external field far from the sphere. The comparison shows that generally, the average dielectric constant depends not only on the concentration, volume, and material of the dispersed particles but also on their shape and orientation in space.

Note also in the limit $\kappa \to \infty$, the correct result (*) approaches Eq. (3.51) for a diluted medium of metallic spheres, which was derived in the lecture notes in a similar way.

<u>Problem 3.23</u>. A straight thin filament, uniformly charged with linear density λ, is positioned parallel to the plane separating two uniform linear dielectrics, at a distance d from it. Calculate the electric potential's distribution everywhere in the system.

Solution: Using the macroscopic Gauss law (3.34) for a straight filament with the linear charge λ in a *uniform* medium with dielectric constant κ, for its radial-oriented fields we readily get

$$D = \frac{\lambda}{2\pi\rho}, \quad \text{so that } E = \frac{D}{\varepsilon} = \frac{\lambda}{2\pi\varepsilon_0\kappa\rho},$$



where ρ is the distance between the observation point and the closest point of the filament. (In the coordinate system shown in the figure on the right, $\rho^2 = x^2 + (y - d)^2$.) Hence its electrostatic potential is

$$\phi = -\frac{\lambda}{2\pi\varepsilon_0\kappa}\ln\rho + \text{const} \equiv \frac{\lambda}{4\pi\varepsilon_0\kappa}\ln\frac{1}{x^2 + (y-d)^2} + \text{const}. \qquad (*)$$

Now repeating the argumentation used in Sec. 3.4 of the lecture notes to solve the problem shown in Fig. 3.12, but replacing the point charges q, q', and q'' with linear charge densities λ, λ', and λ'' (see the figure), and the basic Coulomb law $\phi = q/4\pi\varepsilon_0 r$ with Eq. (*), we get the following 2D analog of Eq. (3.66):

$$\phi = \frac{1}{4\pi\varepsilon_0} \times \begin{cases} \dfrac{1}{\kappa_1}\left[\lambda \ln\dfrac{1}{x^2 + (y-d)^2} + \lambda' \ln\dfrac{q'}{x^2 + (y+d)^2}\right] + \text{const}, & \text{for } y \geq 0, \\ \dfrac{\lambda''}{\kappa_2}\ln\dfrac{1}{x^2 + (y-d)^2} + \text{const}, & \text{for } y \leq 0, \end{cases} \qquad (**)$$

where the charge densities λ' and λ'' still need to be found from the boundary conditions at the interface ($y = 0$). Using Eqs. (**) to calculate the corresponding tangential and normal components of the fields on both sides of the interface,

$$E_\tau = E_x = \frac{1}{2\pi\varepsilon_0}\frac{x}{x^2+d^2}\times\begin{cases}(\lambda+\lambda')/\kappa_1, & \text{for } y=+0,\\ \lambda''/\kappa_2, & \text{for } y=-0,\end{cases}$$

$$E_n = E_y = -\frac{1}{2\pi\varepsilon_0}\frac{d}{x^2+d^2}\times\begin{cases}(\lambda-\lambda')/\kappa_1, & \text{for } y=+0,\\ \lambda''/\kappa_2, & \text{for } y=-0,\end{cases}$$

and requiring them to satisfy Eqs. (3.37) and (3.56) for all x, we get the following simple system of linear equations:

$$\frac{\lambda+\lambda'}{\kappa_1} = \frac{\lambda''}{\kappa_2}, \qquad \lambda-\lambda' = \lambda'',$$

which gives

$$\lambda' = \frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\lambda, \qquad \lambda'' = \frac{2\kappa_2}{\kappa_1+\kappa_2}\lambda. \qquad (***)$$

Formulas (**) and (***) give the full solution of our problem. Note that despite its different dimensionality, it yields essentially the same results for the image charges as for the system discussed in the lecture notes. Indeed, in the particular case considered in Sec. 3.4: $\kappa_1 = 1$ and $\kappa_2 \equiv \kappa$, Eqs. (***) give exactly the same results for the ratios λ'/λ and λ''/λ as Eq. (3.68) gives for the ratios q'/q and q''/q. Of the new results given by the more general Eqs. (***), note an interesting dependence of the image charge's sign on the difference $\kappa_1 - \kappa_2$: if this difference is positive, the density λ' of the charge observed from the points with $y > 0$ has the *same* sign as the real one.

Let me add one more (hopefully, unnecessary) note: there is no need to be concerned about the divergence of Eqs. (**) at large distances from the filament because they still yield finite values of the electric field $\mathbf{E} = -\nabla\phi$, and hence of the field energy density $u \propto E^2$. Indeed, such divergence of the electrostatic potential with distance is typical for many applications of this concept (starting from the uniform electric field's description) and is an artifact of the assumption of the infinite spatial extent of the system under analysis – see, e.g., the discussion of Eqs. (2.49)-(2.50) in the lecture notes.

Problem 3.24. A point charge q is located at a distance $d > R$ from the center of a sphere of radius R, made of a uniform linear dielectric with permittivity ε.

(i) Calculate the electrostatic potential's distribution in all the space, for an arbitrary ratio d/R.

(ii) For large d/R, use two different approaches to calculate the interaction force and the energy of interaction between the sphere and the charge, in the first nonzero approximation in $R/d \ll 1$.

Hint: Task (i) cannot be carried out using the method of charge images, so you may like to use the expansion of the function $1/|\mathbf{r}-\mathbf{r}'|$ in the series over the Legendre polynomials, whose proof was the subject of Problem 2.40.

Solution: Aligning the z-axis, with its origin in the sphere's center, along the direction toward the point charge, let us look for the potential in the following natural form:

$$\phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\times\begin{cases}\sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta), & \text{for } r \le R,\\ \dfrac{1}{|\mathbf{r}-\mathbf{r}'|} + \sum_{l=0}^{\infty}\dfrac{b_l}{r^{l+1}}P_l(\cos\theta), & \text{for } R \le r.\end{cases} \qquad (*)$$

Here the regular parts of the field follow the corresponding (non-diverging) parts of the general solution (2.172) of the Laplace equation for our axially-symmetric case, while the singular term, with $\mathbf{r}' = \mathbf{n}_z d$, takes care of the point charge's field in its vicinity. Following the *Hint*, we may represent this term as a similar expansion:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l \mathcal{P}_l(\cos\theta),$$

where $r_>$ is the largest of the two scalars r and $r' \equiv d$, and $r_<$ is the smallest of them. Plugging it into Eq. (*), we get the following expressions for the potential and its normal derivatives on the sphere's surface:

$$\phi\Big|_{r=R} = \frac{q}{4\pi\varepsilon_0}\times\begin{cases}\displaystyle\sum_{l=0}^{\infty} a_l R^l P_l(\cos\theta), & \text{for } r = R-0,\\ \displaystyle\sum_{l=0}^{\infty}\left(\frac{R^l}{d^{l+1}}+\frac{b_l}{R^{l+1}}\right)P_l(\cos\theta), & \text{for } r = R+0,\end{cases}$$

$$\frac{\partial\phi}{\partial r}\Big|_{r=R} = \frac{q}{4\pi\varepsilon_0}\times\begin{cases}\displaystyle\sum_{l=0}^{\infty} a_l l\, R^{l-1} P_l(\cos\theta), & \text{for } r = R-0,\\ \displaystyle\sum_{l=0}^{\infty}\left(\frac{l\,R^{l-1}}{d^{l+1}}-\frac{(l+1)b_l}{R^{l+2}}\right)P_l(\cos\theta), & \text{for } r = R+0.\end{cases}$$

According to the boundary conditions (3.37) and (3.56), in our case we should have

$$\phi\Big|_{r=R-0} = \phi\Big|_{r=R+0}, \qquad \kappa\frac{\partial\phi}{\partial r}\Big|_{r=R-0} = \frac{\partial\phi}{\partial r}\Big|_{r=R+0},$$

where $\kappa \equiv \varepsilon/\varepsilon_0$. Due to the orthogonality of the Legendre polynomials, these equalities should hold for each l of the series. This requirement gives us the following pair of linear equations for each pair of the coefficients a_l and b_l:

$$a_l R^l = \frac{R^l}{d^{l+1}}+\frac{b_l}{R^{l+1}}, \qquad \kappa a_l l R^{l-1} = \frac{lR^{l-1}}{d^{l+1}}-\frac{(l+1)b_l}{R^{l+2}},$$

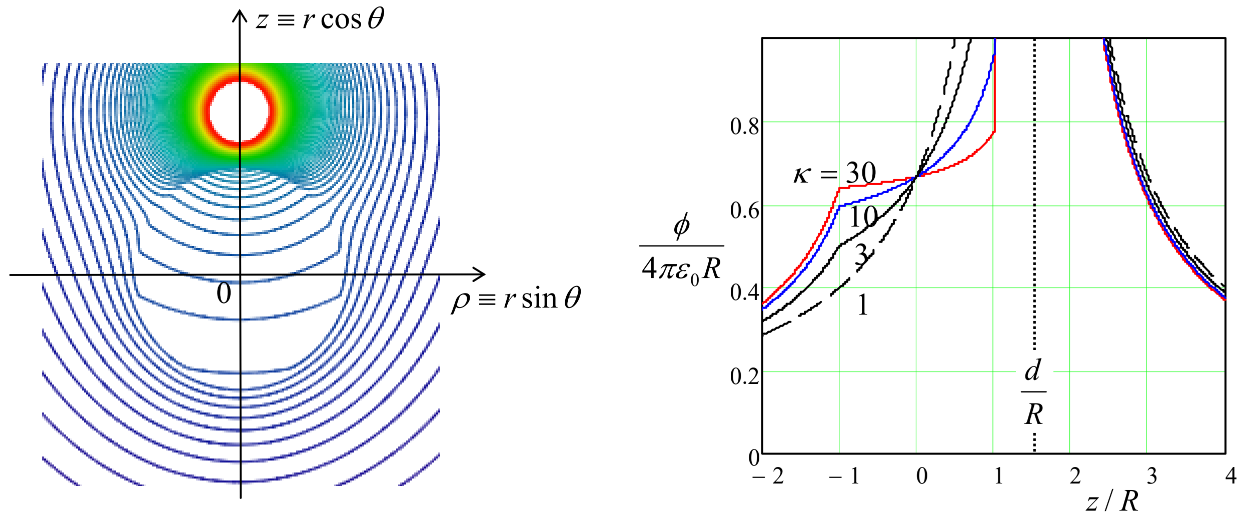
which may be readily solved, giving

$$a_l = \frac{2l+1}{\kappa l+(l+1)}\frac{1}{d^{l+1}}, \qquad b_l = -\frac{l(\kappa-1)}{\kappa l+(l+1)}\frac{R^{2l+1}}{d^{l+1}}. \qquad (**)$$

With these results plugged in, Eqs. (*) give the full solution of Task (i).

The left figure below shows the resulting general pattern of the equipotential planes (more exactly, their intersection with a plane passing through the sphere's center and the point charge) for a particular case $d/R = 1.5$ and $\kappa = 10$. This pattern clearly shows the partial expulsion of the electric field from the high-κ dielectric, which becomes full at $\kappa \to \infty$, i.e. for a conductor – see Fig. 2.29 of the lecture notes.

The right figure below shows a more quantitative description of this trend: plots of the potential's distribution along the symmetry axis of the system for a few values of κ (and for the same charge position $d = 1.5R$ as in the left figure). Note that the potential at the center of the sphere does not depend on κ; indeed, Eqs. (*) and (**) show that $\phi(0) = (q/4\pi\varepsilon_0)a_0 = q/4\pi\varepsilon_0 d$, i.e. exactly the potential the charge would produce at that point in the absence of the sphere.



(ii) With the solution of Task (i) on hand, we may readily calculate all global characteristics of the charge-to-sphere interaction. In particular, according to the basic Eq. (1.60), the sphere-to-charge interaction energy U may be calculated as

$$U = \frac{q}{2}\phi_s\bigg|_{\substack{r=d\\ \theta=0}},$$

where ϕ_s is the potential induced by the sphere at the point charge's position. This potential is given by the second line of Eq. (*) less the singular term describing the field of the charge itself. Since according to Eq. (**), $b_0 = 0$,[95] at $r \to \infty$, this potential is dominated by the dipole term with $l = 1$:

$$\phi_s\bigg|_{\substack{r=d\\ \theta=0}} \to \frac{q}{4\pi\varepsilon_0}\frac{b_1}{d^2} = -\frac{q}{4\pi\varepsilon_0}\frac{\kappa-1}{\kappa+2}\frac{R^3}{d^4},$$

so we get

$$U = -\frac{1}{4\pi\varepsilon_0}\frac{1}{2}\frac{\kappa-1}{\kappa+2}\frac{q^2R^3}{d^4}, \qquad \text{for } R \ll d. \qquad (***)$$

Since for all realistic materials, $\kappa > 1$, this formula describes the mutual attraction of the sphere and the charge – for any sign of q. The attraction force may be calculated as

$$F = -\frac{\partial U}{\partial d} = \frac{1}{4\pi\varepsilon_0}2\frac{\kappa-1}{\kappa+2}\frac{q^2R^3}{d^5}, \qquad \text{for } R \ll d. \qquad (****)$$

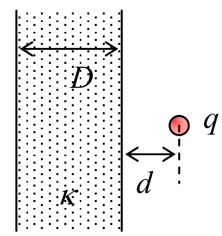
The last formula may be also obtained without any appeal to the exact solution. Indeed, at $R \ll d$, the point charge's field $E = q/4\pi\varepsilon_0 d^2$ at the sphere's location is nearly uniform on the scale of its radius $R \ll d$. From the problem solved in Sec. 3.4 of the lecture notes, we know that such a uniform field induces, in a dielectric sphere, a dipole moment of the magnitude $p = 4\pi\varepsilon_0 ER^3(\kappa - 1)/(\kappa + 2)$, directed along the initial electric field – see Eq. (3.64). Hence we can use the general formula for the

[95] This is natural because the corresponding term with $l = 0$, proportional to $1/r$, describes the far field of the sphere's net electric charge – which it does not have.

radial component of the dipole field, $E_r = 2p\cos\theta/4\pi\varepsilon_0 d^3$ (see the first form of Eq. (3.13) of the lecture notes), with $\theta = 0$, to calculate the interaction force:

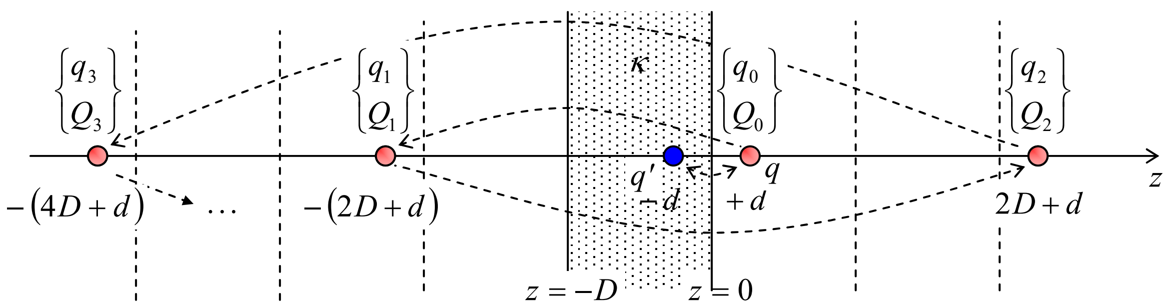
$$F = qE_r = q\frac{2}{4\pi\varepsilon_0 d^3}p = q\frac{2}{4\pi\varepsilon_0 d^3}4\pi\varepsilon_0 R^3\frac{\kappa-1}{\kappa+2}E = q\frac{2}{4\pi\varepsilon_0 d^3}4\pi\varepsilon_0 R^3\frac{\kappa-1}{\kappa+2}\frac{q}{4\pi\varepsilon_0 d^2}.$$

This is the same result as in Eq. (****). Now we may readily obtain Eq. (***) by either integrating the calculated force over d from ∞ to r, or just by using Eq. (3.15b) of the lecture notes for the interaction energy of a dipole with the field that has induced it.



Problem 3.25. Calculate the spatial distribution of the electrostatic potential induced by a point charge q located at distance d from a very wide parallel plate, of thickness D, made of a linear dielectric – see the figure on the right.

Solution: Let us direct the z-axis normally to the plate so that the radius vector of the charge is $\mathbf{r} = \{0, 0, d\}$, as shown in the figure below, and examine what system of charge images may describe the field everywhere in the system, while satisfying the boundary conditions (3.35) and (3.37) on both surfaces of the plate.



From Sec. 3.4 of the lecture notes (see Fig. 3.12 and its discussion) we know that the conditions on the plate's right surface ($z = 0$) may be satisfied by using, besides the actual charge q, just two of its images: the charge

$$q' = -\frac{\kappa-1}{\kappa+1}q \qquad (*)$$

located at $z = -d$ and describing the additional field in the free space (at $z \geq 0$), and the charge

$$Q_0 = \frac{2}{\kappa+1}q \qquad (**)$$

(in Sec. 3.4, called q'') co-located with the actual charge ($z = +d$) but describing the field inside the dielectric (at $z \leq 0$).

Now let us try to satisfy the boundary conditions on the opposite surface of the plate, located at $z = -D$, by introducing an image charge Q_1 located at $z = -(2D + d)$, i.e. at the same distance $(D + d)$ from this surface as the charge Q_0 (see the figure above) and describing the additional field inside the dielectric. The most important point here (and in this solution as a whole) is that the value (**) of the charge Q_0 cannot be adjusted, because the charge Q_1 (as well as all other image charges we will introduce later) gives a different spatial dependence of the potential inside the dielectric, so their net potential is

$$\phi_1(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{Q_0}{\left[\rho^2 + (z-d)^2\right]^{1/2}} + \frac{Q_1}{\left[\rho^2 + (z+2D+d)^2\right]^{1/2}}\right\}, \qquad \text{for } -D \leq z \leq 0, \qquad (***)$$

(where $\rho \equiv (x^2 + y^2)^{1/2}$ is the distance of the field observation point $\mathbf{r}$ from the axis z), so any adjustment of Q_0 would irreparably violate the boundary conditions at $z = 0$.

However, to balance the field (***) in the dielectric, and hence on the right of the border $z = -D$, with that in the free space left of the border (i.e. at $z < -D$), we need a new contribution that would have, at the border, the same dependence on ρ. Such contribution may be provided by another charge image located at the same distance ($D + d$) from that border as Q_0 and Q_1. This charge cannot be co-located with Q_1, because that would violate the Laplace equation at that point, $z = -(2D + d)$, which does not have a real charge. Hence, this new image charge (let us call it q_0) has to be co-located with the charge Q_0, i.e. sit at the same point $z = d$ (see the figure above), creating the contribution

$$\phi_1(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q_0}{\left[\rho^2 + (z-d)^2\right]^{1/2}}, \qquad \text{for } z \leq -D, \qquad (****)$$

to the potential.[96] Now applying the boundary conditions (3.35) and (3.37) at $z = -D$ to Eqs. (***) and (****), we get a system of equations somewhat similar to Eqs. (3.67),

$$Q_0 + Q_1 = q_0, \qquad \kappa(Q_0 - Q_1) = q_0,$$

which allows us to express the new image charges via the already determined Q_0:

$$Q_1 = \frac{\kappa - 1}{\kappa + 1}Q_0, \qquad q_0 = \frac{2\kappa}{\kappa + 1}Q_0.$$

(As an intermediate sanity check, at $\kappa = 1$, i.e. with no dielectric plate at all, $Q_1 = 0$, and $q_0 = Q_0 = q$ – as it should be because in this case, the field everywhere is just that of the real point charge q.)

Now we should revisit the surface $z = 0$, and compensate the field contribution by the charge Q_1 in the dielectric (see Eq. (***) above) by an absolutely similar introduction of two new charge images located at the same distance, $2D + d$, from that boundary: charge Q_2 at $z = 2D + d$ and describing a new contribution to the field in the dielectric, and charge q_1 co-located with Q_1, which describes a new contribution (besides those from q and q') to the field in the free space outside the dielectric. The boundary conditions relating Q_1, Q_2, and q_1 are exactly the same as the above relations for Q_0, Q_1, and q_0, and lead to similar expressions:

$$Q_2 = \frac{\kappa - 1}{\kappa + 1}Q_1 = \left(\frac{\kappa - 1}{\kappa + 1}\right)^2 Q_0, \qquad q_1 = \frac{2\kappa}{\kappa + 1}Q_1 = \frac{2\kappa}{\kappa + 1}\frac{\kappa - 1}{\kappa + 1}Q_0.$$

Similarly returning to the surface $z = -D$, then to surface $z = 0$, etc. (see arrows in the figure above, showing the sequence of new charge image introduction), we get an infinite system of image charges

[96] Here and later, I will use lower-case letters q to denote any charge image describing a contribution to the field in a free space region (necessarily on the plate's side opposite to that charge's location), while capital-case letters Q are reserved for the charges giving contributions into the field inside the plate.

$$Q_n = \left(\frac{\kappa - 1}{\kappa + 1}\right)^n Q_0, \quad \text{and} \quad q_n = \frac{2\kappa}{\kappa + 1} Q_n = \frac{2\kappa}{\kappa + 1}\left(\frac{\kappa - 1}{\kappa + 1}\right)^n Q_0, \qquad \text{for } n \geq 0,$$

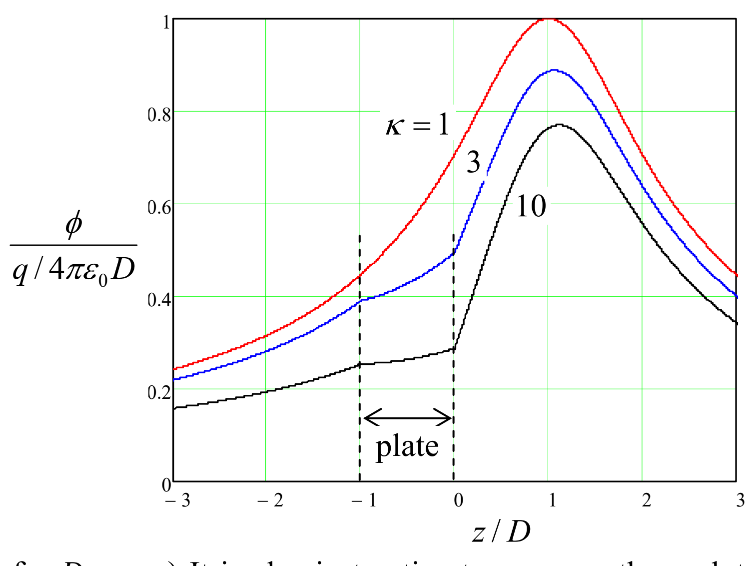
located at $z = nD + d = 2mD + d$ for even $n = 2m$, and at $z = -(n + 1)D - d = -2mD - d$ for odd $n = 2m - 1$ (with $m = 0, 1, 2,\ldots$). From this series, using Eq. (**) for Q_0, and taking into account the contributions from the actual charge q and the "irregular" image charge q' given by Eq. (*),[97] we may readily construct the potential distribution in the system:

$$\phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\frac{4\kappa}{(\kappa+1)^2}\sum_{m=0}^{\infty}\left(\frac{\kappa-1}{\kappa+1}\right)^{2m}\frac{1}{\left[\rho^2 + (z - 2mD - d)^2\right]^{1/2}}, \qquad \text{for } z < -D,$$

$$\phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\frac{2}{\kappa+1}\sum_{m=0}^{\infty}\left(\frac{\kappa-1}{\kappa+1}\right)^{2m-1}\times\left\{\begin{aligned}&\frac{\kappa-1}{\kappa+1}\frac{1}{\left[\rho^2 + (z - 2mD - d)^2\right]^{1/2}}\\ &+\frac{(1-\delta_{m0})}{\left[\rho^2 + (z + 2mD + d)^2\right]^{1/2}}\end{aligned}\right\}, \quad \text{for } -D \leq z < 0,$$

$$\phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\times\left\{\begin{aligned}&\frac{1}{\left[\rho^2 + (z-d)^2\right]^{1/2}} - \frac{\kappa-1}{\kappa+1}\frac{1}{\left[\rho^2 + (z+d)^2\right]^{1/2}}\\ &+\frac{4\kappa}{(\kappa+1)^2}\sum_{m=1}^{\infty}\left(\frac{\kappa-1}{\kappa+1}\right)^{2m-1}\frac{1}{\left[\rho^2 + (z + 2mD + d)^2\right]^{1/2}}\end{aligned}\right\}, \quad \text{for } 0 \leq z.$$

The figure on the right shows plots of this potential as a function of z, for the particular values $\rho = d = D$, and three representative values of the dielectric constant κ. (The necessary sanity check is that at both borders, the potential ϕ is continuous, while its derivative $\partial\phi/\partial z$ jumps by a factor of κ.) The plots show that the increase of κ suppresses the electric field not only behind the dielectric plate but also inside and in front of it, i.e. in the vicinity of the actual point charge. (According to Eqs. (3.66) and (3.68) of the lecture notes, the last effect is even stronger for a dielectric half-space, which may be viewed as the particular case of our current problem for $D \to \infty$.) It is also instructive to compare these plots with the right figure in the solution of the previous problem, which is a spherical analog of this one.



[97] This charge (the only one with the sign opposite to that of the actual charge q) drops out of the regular sequence $\{Q_n, q_n\}$ because this is the only charge that may be located inside the dielectric plate, and hence cannot have a co-located partner (say, Q'), which would give a diverging contribution into the field inside the plate.

As an additional exercise, I can recommend the reader to solve this problem also by the variable separation method, using the following (generally useful) expansion:

$$\frac{1}{\left(\rho^2+z^2\right)^{1/2}}=\int_0^\infty J_0(k\rho)e^{-k|z|}dk\,,$$

where $J_0(\xi)$ is the Bessel function of the first kind – see Sec. 2.7 of the lecture notes.

Problem 3.26. Discuss the physical nature of Eq. (3.76) of the lecture notes. Apply your conclusions to a material with a fixed (field-independent) polarization $\mathbf{P}_0(\mathbf{r})$, and calculate the electric field's energy of a uniformly polarized sphere (see Problem 13 above).

Solution: In a dipole media with a field-independent number n of elementary dipole moments $\mathbf{p}$ per unit volume, $\delta\mathbf{P} = n\delta\mathbf{p}$, per the general Eq. (3.33), the electric energy density's variation, given by Eq. (3.76), equals

$$\delta u = \mathbf{E}\cdot\delta\left(\varepsilon_0\mathbf{E}+\mathbf{P}\right)\equiv\varepsilon_0\mathbf{E}\cdot\delta\mathbf{E}+\mathbf{E}\cdot\delta\mathbf{P}=\delta u_0+n\mathbf{E}\cdot\delta\mathbf{p},$$

where $u_0 \equiv (\varepsilon_0/2)E^2$ is the energy density of the electric field as such – see Eq. (1.65) of the lecture notes. But the product $\mathbf{E}\cdot\delta\mathbf{p}$ is just the elementary energy transfer from the electric field to the internal degrees of freedom of the elementary dipole $\mathbf{p}$, resisting its polarization.[98] (For example, for the simplest model of the dipole: a point charge q deviating from its equilibrium position by distance $\mathbf{r}$, with $\mathbf{p} = q\mathbf{r}$, this product is $\mathbf{E}\cdot\delta\mathbf{p} = \mathbf{E}\cdot q\delta\mathbf{r} = \mathbf{F}\cdot\delta\mathbf{r}$, where $\mathbf{F} = q\mathbf{E}$ is the force exerted by the field.) Hence, the deviation of the electric field energy U from the free-space value (1.65) is due to the additional energy, possibly of non-electric origin, stored inside elementary dipoles – for example, polarized atoms or molecules.

For a medium with a field-independent polarization $\mathbf{P}_0(\mathbf{r})$ (for example, an electret or a saturated ferroelectric), the internal energy of the elementary dipoles does not depend on the electric field ($\delta\mathbf{P} = 0$), so it is reasonable to associate with the field only the energy density u_0. Hence, the full electric field energy may be calculated as an integral,

$$U = U_0 = \frac{\varepsilon_0}{2}\int E^2 d^3r\,, \qquad (*)$$

over the whole space where the electric field is significant – both inside and outside the volume filled with the polarized medium. However, in some cases it may be technically easier to calculate the energy using the equivalent Eq. (1.60), with the charge density replaced by the effective charge (3.30):

$$U=\frac{1}{2}\int\rho_{\text{ef}}(\mathbf{r})\phi(\mathbf{r})d^3r=-\frac{1}{2}\int\nabla\cdot\mathbf{P}_0(\mathbf{r})\phi(\mathbf{r})d^3r\,, \qquad (**)$$

because this integration may be limited to the polarized medium – moreover, if $\mathbf{P}_0$ is spatially independent inside it, then just to its surface.

For example, for Problem 13 (the uniformly polarized sphere), where the effective charge resides on the sphere's surface S, with the areal density $\sigma_{\text{ef}} = P_0\cos\theta$, Eq. (**) expression yields

[98] This expression is in agreement with Eq. (3.15a), despite its apparently different sign. For example, if the dipole moment *increases* by $\delta\mathbf{p}$ in a fixed external field $\mathbf{E}_{\text{ext}}$, then the energy of its interaction with the field described by Eq. (3.15a) *decreases* by $\delta\mathbf{p}\cdot\mathbf{E}_{\text{ext}}$, and hence its internal energy *increases* by that amount.

$$U = \frac{1}{2}\int_S \sigma_{\text{ef}}\phi\big|_{r=R}\, d^2r = \frac{R^2 P_0}{2}\int_0^{2\pi} d\varphi \int_0^{\pi} \cos\theta\, \phi\big|_{r=R} \sin\theta d\theta .$$

Plugging in the potential's distribution over the surface of the sphere, calculated in the solution of that problem,

$$\phi\big|_{r=R} = \frac{P_0 R}{3\varepsilon_0}\cos\theta ,$$

we get

$$U = \frac{R^2 P_0}{2}\frac{P_0 R}{3\varepsilon_0}\int_0^{\pi}\cos^2\theta \sin\theta d\theta = \frac{P_0^2 R^3}{6\varepsilon_0}\int_0^{2\pi} d\varphi \int_0^{\pi}\cos^2\theta\sin\theta d\theta = \frac{P_0^2 R^3}{6\varepsilon_0}2\pi\frac{2}{3} = \frac{2\pi P_0^2 R^3}{9\varepsilon_0} .$$

A straightforward integration by using Eq. (*) and the results obtained in the model solution of Problem 13 (a good additional exercise for the reader), confirms this result, with one-third of the energy U localized inside the sphere and two-thirds, outside it.

Problem 3.27. Use Eqs. (3.73) and (3.81) of the lecture notes to calculate the force of attraction of a plane capacitor's plates, for two cases:

(i) the capacitor is charged to voltage V, and then disconnected from the battery,[99] and
(ii) the capacitor remains connected to the battery.

Solution: The attraction force may be calculated as

$$F = \frac{\partial U}{\partial d} ,$$

where d is the distance between the plates, and U is the appropriate potential energy. For case (i), when the capacitor is disconnected from the battery, the only relevant energy is that of the capacitor itself, so we have to use Eq. (3.73). In this particular case, with the uniform field $E = V/d$ inside the volume Ad between the plates, and a negligible field outside it, this formula gives

$$U = \frac{\varepsilon}{2}E^2 Ad = \frac{\varepsilon}{2}\left(\frac{V}{d}\right)^2 Ad \equiv \frac{\varepsilon A V^2}{2d} ,$$

where A is the capacitor's area.

Before the partial differentiation over d, however, we have to take into account that as d is varied, so is the voltage V. In order to spell out this dependence, we may use the fact that since the capacitor is disconnected from the battery, its electric charge Q is not affected by d, so using the well-known expression for the capacitance C (see Eq. (3.55) of the lecture notes), we get

[99] "Battery" is a common if misleading term for what is usually either a single *electric accumulator* or a single *galvanic element*. (The last term stems from the name of Luigi Galvani, a pioneer of electric current studies. Another term derived from his name is the *galvanic connection*, meaning a direct connection of two conductors, enabling a dc current flow – see the next chapter.) The term "battery" had to be, in all fairness, reserved for the connection of several electric accumulators or galvanic elements in series – as was pioneered in 1800 by L. Galvani's friend Alexander Volta.

$$V = \frac{Q}{C} = \frac{Qd}{\varepsilon A}, \qquad \text{and hence } U = \frac{\varepsilon A}{2d}\left(\frac{Qd}{\varepsilon A}\right)^2 \equiv \frac{Q^2}{2\varepsilon A}d\,.$$

(Actually, for the particular case of no dielectric filling, i.e. $\varepsilon = \varepsilon_0$, this expression in the form

$$\frac{U}{A} = \frac{\sigma^2}{2\varepsilon_0}d\,,$$

where $\sigma = Q/A$ is the areal density of the plate's charge, was already obtained in the solution of Problem 1.15.) Now the differentiation over d yields

$$F = \frac{\partial}{\partial d}\left(\frac{Q^2}{2\varepsilon A}d\right) = \frac{Q^2}{2\varepsilon A} \equiv \frac{(CV)^2}{2\varepsilon A} \equiv \frac{\varepsilon A V^2}{2d^2}. \qquad (*)$$

This result (which coincides with the solution of Problem 2.1, with the natural replacement $\varepsilon_0 \to \varepsilon$) may be verified by a direct calculation of the force per unit area, $F/A = \sigma E/2$, with the factor of ½ resulting from the fact that the force exerted on one plate is due to the electric field $E/2$, produced by the other plate alone.

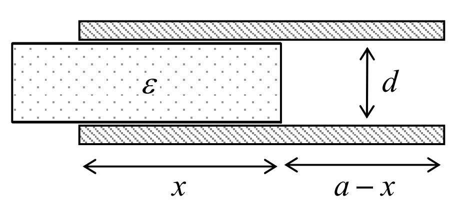
However, in case (ii), when the voltage V is fixed by the battery, the use of the "ordinary" potential energy (3.73) would give a wrong result:

$$F = \frac{\partial}{\partial d}\left(\frac{\varepsilon A V^2}{2d}\right) = -\frac{\varepsilon A V^2}{2d^2}, \qquad \text{(WRONG!)} \qquad (**)$$

which differs from the correct result (*) by the sign. The reason for this error is that if the capacitor stays connected with the battery during the virtual change of d, we have to take into account the total energy of the (capacitor + battery) system, or at least of its part depending on d. This is exactly the definition of the Gibbs potential energy U_G, given (for the electrostatic field's case) by Eq. (3.81). Since according to that expression (besides an inconsequential constant), $U_G = -U$, there is no need to do the calculations again, and we may just revert the sign in the wrong Eq. (**), turning it to the correct Eq. (*).

So, the attraction force between the plates, at the same voltage between them, does not depend on whether the capacitor is still connected to the battery or not.

Let me hope that this simple example illustrates the difference between the potential energies U and U_G very clearly.



Problem 3.28. A slab made of a linear dielectric is partly inserted into a plane capacitor – see the figure on the right. Assuming the simplest (cylindrical) geometry of the system, calculate the force exerted by the field on the slab, for the same two cases as in the previous problem.

Solution: As it follows from the analysis of a similar system in Sec. 3.4 of the lecture notes (see Fig. 10a), the insertion of the stub, with its lateral surface normal to the capacitor plates, does not perturb the electric field uniformity inside the capacitor. Also, since the capacitor plates are equipotential, the electric field in both parts of its volume is the same: $E = V/d$, where V is the voltage across it.

Now we are ready to proceed to the requested calculations. In the first case, when the capacitor is disconnected from the battery, the relevant energy U is given by Eq. (3.73), and we can (neglecting the fringe field effects) calculate it as the sum of energies of the two parts of its interior:

$$U = U_x + U_{a-x} = \left[x\frac{\varepsilon E^2}{2} + (a-x)\frac{\varepsilon_0 E^2}{2} \right] bd , \qquad (*)$$

where a is the complete width of the capacitor (see the figure above) and b is its other planar size (in the direction formal to the plane of its drawing). If we calculated the force F_x from this expression as $-\partial U/\partial x$, we would get a wrong (negative) result because if the capacitor is disconnected from the battery, E also depends on x. Hence we should first calculate what is indeed constant at the variation of x – the full electric charge Q of the capacitor, which is also the sum of the contributions from its two parts:

$$Q = Q_x + Q_{a-x} = b\left[x\sigma_x + (a-x)\sigma_{a-x}\right] = b\left[xD_x + (a-x)D_{a-x}\right] = b\left[x\varepsilon E + (a-x)\varepsilon_0 E\right], \qquad (**)$$

so

$$E = \frac{Q}{b\left[\varepsilon x + \varepsilon_0 (a-x)\right]}.$$

Plugging the last relation into Eq. (*), we get

$$U = \frac{Q^2}{2}\frac{d}{b\left[\varepsilon x + \varepsilon_0 (a-x)\right]}.$$

(As a sanity check, each term in the square brackets, after its multiplication by b/d, is the capacitance of the corresponding part of the system, so this expression for U describes the energy (2.15) of the parallel connection of these partial capacitors, as was discussed in Sec. 3.4.) Now we are ready to calculate the requested force:

$$F_x = -\frac{\partial U}{\partial x} = \frac{Q^2}{2}\frac{(\varepsilon - \varepsilon_0)d}{b\left[\varepsilon x + \varepsilon_0 (a-x)\right]^2};$$

plugging in Q from Eq. (**), we get

$$F_x = -\frac{\partial U}{\partial x} = \frac{(\varepsilon - \varepsilon_0)E^2}{2} bd . \qquad (***)$$

Since for any realistic dielectric, $\varepsilon > \varepsilon_0$, this result shows that $F_x > 0$, i.e. the force pulls the slab into the capacitor – see the figure above. Note also the product bd is just the area of the interface between the slab and the free space inside the capacitor, so we may rewrite our result in the form

$$\frac{F_x}{A_{\text{interface}}} = \frac{(\varepsilon - \varepsilon_0)E^2}{2},$$

hinting that it may be applicable not only to this particular geometry. (As will be discussed in Chapter 9, this turns out to be true.)

Now proceeding to the case when the applied voltage V and hence the applied field $E = V/d$ are fixed, here the appropriate potential energy is the Gibbs energy U_G. For our particular case when, per Eq. (3.45), the polarization of the dielectric material of the slab does not depend on x but is proportional to E: $\mathbf{P} = (\varepsilon - \varepsilon_0)\mathbf{E}$, it is beneficial to derive this energy from Eq. (3.86):

$$U_G = -\frac{1}{2}\int_{[0,x]} PEd^3r + \text{const} = -\frac{(\varepsilon - \varepsilon_0)E^2 xbd}{2},$$

so an elementary differentiation over x yields the same Eq. (**).

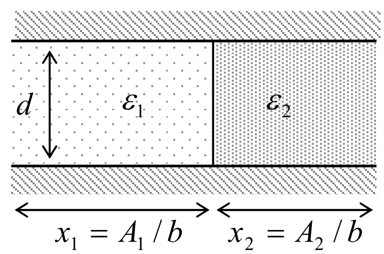
Two major conclusions may be drawn from this solution. The first one is (in hindsight :-) almost trivial, especially after the solution of the previous problem: the force exerted on the slab does not depend on whether the capacitor is still connected to the battery or not if the electric field in it is still the same. Another fact to remember is that for some calculations, the Gibbs energy U_G is much more convenient than the "usual" electrostatic energy U.

Problem 3.29. For each of the two capacitors shown in Fig. 3.10 of the lecture notes, calculate the electric force exerted on the interface between two different dielectrics, in terms of the fields in the system.

Solution: As the discussions in Sec. 3.5 of the lecture notes and the two previous problems show, the force does not depend on which electric field parameter of the capacitor is considered fixed: its full charge Q or the voltage V across it, so let us use the first assumption. In this case, the relevant potential energy of the system, whose gradient (with the minus sign) is the force, is given by Eq. (3.73) of the lecture notes. Reviewing the arguments that were used in Sec. 2.2 to derive Eq. (2.15),

$$U = \frac{Q^2}{2C},$$

we see that it remains valid for capacitors with linear dielectrics,[100] so we can use the results for C obtained in Sec. 3.4.



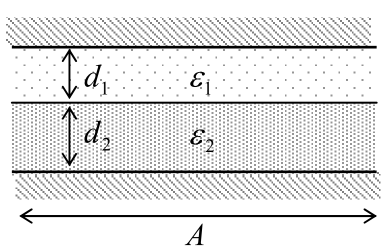
For the capacitor with the dielectric interface normal to the capacitor's plates (see the figure on the right), whose capacitance is given by Eq. (3.57), we get

$$U = \frac{Q^2}{2C} = \frac{Q^2 d}{2(\varepsilon_1 A_1 + \varepsilon_2 A_2)} = \frac{Q^2 d}{2b(\varepsilon_1 x_1 + \varepsilon_2 x_2)},$$

where b is the linear size of the capacitor in the direction normal to the plane of the drawing. Now differentiating this expression over $x \equiv x_1 = \text{const} - x_2$, we obtain

$$F_x = -\frac{\partial U}{\partial x} = \frac{\partial U}{\partial x_2} - \frac{\partial U}{\partial x_1} = \frac{Q^2(\varepsilon_1 - \varepsilon_2)d}{2b(\varepsilon_1 x_1 + \varepsilon_2 x_2)^2} = \frac{(\varepsilon_1 - \varepsilon_2)E^2}{2} bd, \qquad \text{i.e. } \frac{F_x}{A_\text{interface}} \equiv \frac{F_x}{bd} = \frac{(\varepsilon_1 - \varepsilon_2)E^2}{2},$$

where $E = V/d = Q/Cd$ is the electric field – common for both parts of the capacitor. Note that the result obtained in the previous problem is just the particular case of this one, for $\varepsilon_1 = \varepsilon_0\kappa$ and $\varepsilon_2 = \varepsilon_0$.



In contrast, as was already discussed in Sec. 3.4, in the capacitor with the dielectric boundary parallel to the electrodes (see the figure on the right), the field common for both dielectrics is $D = \sigma = Q/A$, so it is

[100] This is an important qualification – see the discussion in Sec. 3.5.

more natural to express the final result via this field. Using Eq. (3.59) of the lecture notes to write

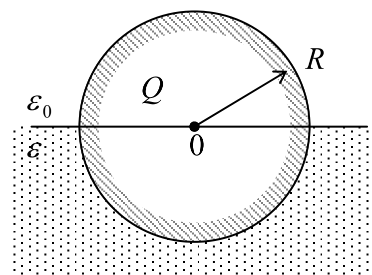
$$U = \frac{Q^2}{2C} = \frac{Q^2}{2A}\left(\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}\right),$$

and taking $y \equiv d_2 = \text{const} - d_1$, we get

$$F_y = -\frac{\partial U}{\partial y} = \frac{\partial U}{\partial d_1} - \frac{\partial U}{\partial d_2} = \frac{Q^2}{2A}\left(\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2}\right) = \frac{D^2}{2}\left(\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2}\right)A, \qquad \text{i.e. } \frac{F_y}{A} = \frac{D^2}{2}\left(\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2}\right).$$

In the limit $\varepsilon_2 \to \infty$, corresponding to the second material turning into a conductor, i.e. to a plane capacitor of thickness $d \equiv d_1$, this result is reduced to that of Problem 27, with $\varepsilon = \varepsilon_1$.

Note also for both considered systems, the force exerted on the interface is directed toward the dielectric with the lowest permittivity, i.e. the electric field always "tries" to pull in the material with the higher dielectric constant, pushing out that with the lower ε.



Problem 3.30. One half of a conducting sphere of radius R, carrying electric charge Q, is submerged into a half-space filled with a linear dielectric with permittivity ε – see the figure on the right. Calculate the electric force exerted on the sphere by the dielectric.

Solution: Repeating the arguments given in the model solution of Problem 19 for a similar (and actually, more general) system, we see that the electric field outside the sphere is purely radial:

$$E(r) = \frac{c}{r^2}, \qquad \text{for } R \le r,$$

with the same constant c in the free space and in the dielectric. This constant may be related to the charge Q by applying the macroscopic Gauss law (3.34) to a sphere of radius $r \ge R$:

$$\oint_S D_r d^2r \equiv 2\pi r^2 \varepsilon_0 E(r) + 2\pi r^2 \varepsilon E(r) \equiv 2\pi(\varepsilon_0 + \varepsilon)c = Q, \quad \text{so } c = \frac{Q}{2\pi(\varepsilon_0+\varepsilon)}, \text{ i.e. } E(r) = \frac{Q}{2\pi(\varepsilon_0+\varepsilon)r^2}.$$

According to the solution of Problem 27 (see also Problem 2.1), this field, normal to the conductor's surface, exerts on it the following elementary radial force directed outward:

$$\frac{dF_r}{dA} = \frac{E^2(R)}{2} \times \begin{cases} \varepsilon_0, & \text{in the free-space part,} \\ \varepsilon, & \text{in the submerged part.} \end{cases}$$

At the integration over the sphere's surface, only one Cartesian component (normal to the dielectric's surface) of each dF_r gives a nonvanishing contribution to the net force exerted on the sphere:

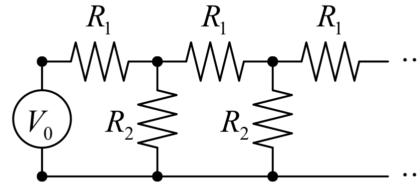
$$F_z = \oint_S \frac{dF_z}{dA} d^2r = \oint_S \frac{dF_r}{dA}\cos\theta d^2r = 2\pi R^2 \int_0^\pi \frac{dF_r}{dA}\cos\theta \sin\theta d\theta$$

$$= 2\pi R^2 \frac{E^2(R)}{2}\left(\varepsilon_0 \int_0^{\pi/2} \cos\theta \sin\theta d\theta + \varepsilon \int_{\pi/2}^{\pi} \cos\theta \sin\theta d\theta\right) = 2\pi R^2 \frac{E^2(R)}{2}\left(\frac{\varepsilon_0}{2} - \frac{\varepsilon}{2}\right) \equiv -\frac{1}{8\pi}\frac{Q^2}{R^2}\frac{\varepsilon - \varepsilon_0}{(\varepsilon+\varepsilon_0)^2}.$$

So, for all realistic dielectric media, with $\varepsilon > \varepsilon_0$, the field pulls the sphere toward the dielectric half-space (regardless of the sign of the sphere's charge). This is exactly what we could expect from the solution of the three previous problems.

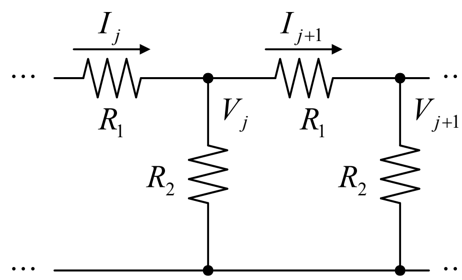
Curiously, in our current case, the force as a function of $\varepsilon \geq 0$ has a maximum at $\varepsilon = 3\varepsilon_0$, i.e. at $\kappa = 3$. The reason for this behavior is that at $\kappa \to \infty$, the effective capacitance of the lower (dielectric-embedded) part of the sphere increases, leading (at fixed charge Q) to the drop of its potential, i.e. to the decrease of the electric field everywhere in the system, and hence of the attraction force it exerts.

Chapter 4. DC Current



Problem 4.1. DC voltage V_0 is applied to the open end of a semi-infinite chain of lumped Ohmic resistors, shown in the figure on the right. Calculate the voltage across the j^{th} link of the chain.

Solution: This problem is a close analog of Problem 2.4 for a similar chain of lumped capacitances. As in its solution, since all links of the chain are similar, it is sufficient to analyze just a couple of them – say, with numbers j and $(j + 1)$, where j is the "distance" from the open end. Denoting the currents through the "series" resistors R_1 and the voltages across the "parallel" resistors R_2 as shown in the figure on the right, the 1st Kirchhoff law (4.7a) for the left of the two displayed circuit nodes gives



$$I_j - I_{j+1} = \frac{V_j}{R_2}, \qquad (*)$$

while the 2nd Kirchhoff law (4.7b) for the displayed loop reads

$$V_j - V_{j+1} = R_1 I_{j+1}. \qquad (**)$$

These relations are absolutely similar to Eqs. (*) of the model solution of Problem 2.4 (with the natural replacements $Q \to I$ and $C \to 1/R$), so we may use the same arguments to look for the solution of the infinite system of equations (*) and (**), written for $j = 1, 2, \ldots$, in a similar form:

$$V_j \propto I_j \propto e^{-\lambda j}, \qquad \text{so that } \frac{V_{j+1}}{V_j} = \frac{I_{j+1}}{I_j} = \alpha \equiv e^{-\lambda} < 1.$$

Indeed, using the last of these relations to eliminate V_{j+1} and I_{j+1} from Eqs. (*) and (**), we get a system of two linear homogenous equations for the two variables V_j and I_j:

$$(1-\alpha) I_j = \frac{V_j}{R_2}, \qquad (1-\alpha) V_j = R_1 \alpha I_j .$$

These equations are compatible (and hence the assumed solution is valid for all j) if the determinant of the system equals zero:

$$\begin{vmatrix} 1-\alpha & -1/R_2 \\ -\alpha R_1 & 1-\alpha \end{vmatrix} = 0, \qquad \text{i.e. if } \alpha^2 - \left(2 + \frac{R_1}{R_2}\right)\alpha + 1 = 0 .$$

The needed root $\alpha < 1$ of this quadratic equation is

$$\alpha = 1 + \frac{R_1}{2R_2} - \left(\frac{R_1}{R_2} + \frac{R_1^2}{4R_2^2}\right)^{1/2}. \qquad (***)$$

Since our definition of V_j (see the figure above) complies, for $j = 0$, with the boundary value V_0, the requested voltage across the j^{th} link of the chain is

$$V_j = V_0 \alpha^j \equiv V_0 e^{-\lambda j},$$

where α is given by Eq. (***).

In the limit $R_1/R_2 \to \infty$, this relation yields

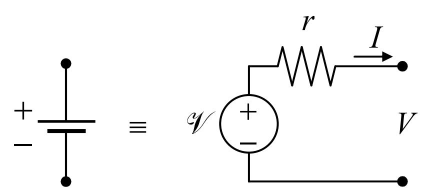
$$\alpha \to \frac{R_1}{2R_2} >> 1.$$

This result means that the applied voltage does not propagate noticeably beyond the first resistance R_1 of the chain, because the next, relatively small resistance R_2 effectively "grounds" the voltage across it – see the figure above. In the opposite limit, $R_1/R_2 \to 0$, Eq. (***) yields

$$\alpha \to 1 - \left(\frac{R_1}{R_2}\right)^{1/2} \to 1, \qquad \text{so that } \lambda \equiv \ln\frac{1}{\alpha} \to \left(\frac{R_1}{R_2}\right)^{1/2} << 1.$$

According to the definition of the parameter λ, this strong inequality means that the applied voltage penetrates deep into the chain, decaying at the "distance" of the order of $N \equiv 1/\lambda \approx (R_2/R_1)^{1/2} >> 1$.

Problem 4.2. It is well known that properties of many dc current sources (e.g., batteries) may be reasonably well represented as a connection in series of a *perfect voltage source* and an Ohmic *internal resistance*. Discuss the option, and possible advantages, of using a different equivalent circuit that would include a *perfect current source*.



Solution: The well-known representation mentioned in the assignment is shown in the figure on the right. Here the circle denotes a *perfect voltage source* – an imaginary two-terminal circuit element that sustains a fixed voltage $\mathscr{V}$,[101] regardless of what is connected to it. (Naturally, it is impossible to implement such an ideal element separately from the battery's internal resistance r; indeed, by short-circuiting its terminals, we would get infinite current at the non-zero voltage $\mathscr{V}$, i.e. an infinite power.) Let us apply the 2nd Kirchhoff law (4.7b) to the loop formed by this equivalent circuit and the (unspecified) external circuit to write the relation between the voltage V at the battery's terminals and the current I flowing through the battery:[102]

$$V = \mathscr{V} - rI. \qquad (*)$$

This is just an analytical representation of the equivalent circuit shown above (and hence of the battery's properties), and is certainly well known to the readers from their undergraduate studies.

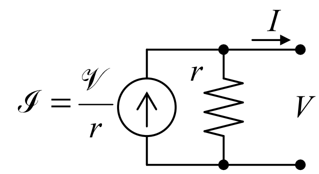
The fact which attracts much less attention is that the following equivalent form of Eq. (*),

$$I = \mathscr{I} - \frac{V}{r}, \qquad \text{with } \mathscr{I} \equiv \frac{\mathscr{V}}{r}, \qquad (**)$$

may be graphically represented with another equivalent circuit – see the figure below. Here the circle with an arrow is a common notation of a *perfect current source* defined as an imaginary two-terminal

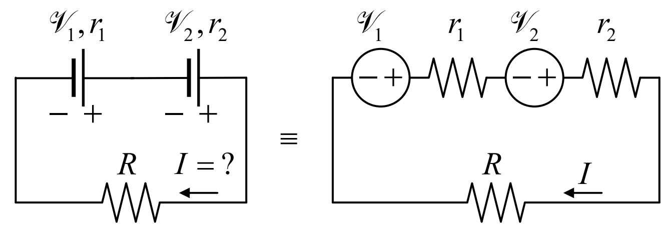
[101] This is exactly the formal definition of the electromotive force $\mathscr{V}$. (Note that in the circuit theory, it is frequently denoted as either E or $\mathscr{E}$.)

[102] This relation, in the limit $R_i \to 0$, explains why the perfect voltage source is sometimes called the "battery with zero internal resistance".



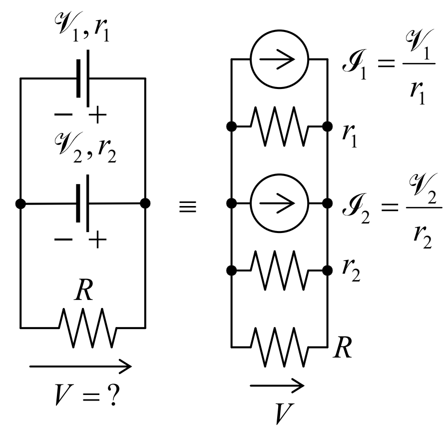
circuit element that sustains a certain current $\mathscr{I}$ (in our case, equal to $\mathscr{V}/r$), regardless of the circuit it is connected to. Evidently, such an ideal element is also impossible to implement separately, without the parallel connection of a finite resistance r, because it would push the same current $\mathscr{I}$ even through empty space, creating infinite voltage and hence supplying infinite power.)

Due to the mathematical equivalence of Eqs. (*) and (**), the two equivalent circuits shown above always give the same final result, but each of them may be more or less convenient, depending on the problem. For example, if we need to calculate the current in the circuit shown on the first panel of the figure on the right, it is more convenient to use the first equivalent circuit of each battery, with its perfect voltage source and internal resistance in series. Indeed, as the second panel of that figure shows, its application reduces the full circuit to a connection, in series, of the net e.m.f. ($\mathscr{V}_1 + \mathscr{V}_2$) to the total resistance ($r_1 + r_2 + R$), immediately giving the final result:



$$I = \frac{\mathscr{V}_1 + \mathscr{V}_2}{r_1 + r_2 + R}.$$

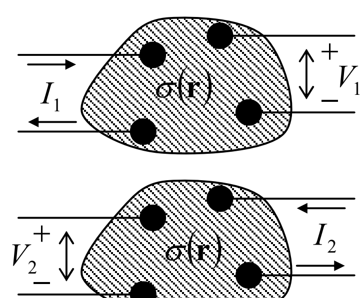
However, using this equivalent circuit for the solution of the (infamous :-) problem shown in the lower figure on the right is much less convenient, and requires using the heavy artillery of the Kirchhoff laws (4.7). On the other hand, using the perfect-current-source alternative immediately reduces the full circuit to a simple connection in parallel of an ideal source of the total current ($\mathscr{I}_1 + \mathscr{I}_2$) and the total Ohmic conductance ($1/r_1 + 1/r_2 + 1/R$) of the three resistors, so the requested voltage across the circuit is calculated very similarly to the current in the first problem:



$$V = \frac{\mathscr{I}_1 + \mathscr{I}_2}{1/r_1 + 1/r_2 + 1/R} \equiv \frac{\mathscr{V}_1/r_1 + \mathscr{V}_2/r_2}{1/r_1 + 1/r_2 + 1/R}.$$

(The current through the "load" R is of course just V/R.)

Problem 4.3. Prove the following *Rayleigh-Lorentz-Carson reciprocity relation*: the results of the two separate experiments shown schematically in the figure on the right, with an arbitrary Ohmic conductor with four electrodes/terminals, are related as $I_1V_2 = I_2V_1$.



Hint: Try to apply the same approach as was used to prove Green's reciprocity relation of electrostatics in Problem 1.18, but with proper modifications.

Solution: Let us consider the integral almost similar to that as in the solution of Problem 1.18:

$$J \equiv \int_V \sigma(\mathbf{r})\mathbf{E}_1(\mathbf{r}) \cdot \mathbf{E}_2(\mathbf{r}) d^3r,$$

where $\mathscr{V}$ is the whole volume of the conductor, $\sigma(\mathbf{r})$ is its Ohmic conductivity, and $\mathbf{E}_1(\mathbf{r})$ and $\mathbf{E}_2(\mathbf{r})$ are the electric field distributions in the two experiments under consideration. First, let us apply the Ohm law (4.8) to $\mathbf{E}_1(\mathbf{r})$ and Eq. (1.33) to $\mathbf{E}_2(\mathbf{r})$; then the integral may be rewritten as

$$J = \int_{\mathscr{V}} \mathbf{j}_1 \cdot \mathbf{E}_2 d^3r = -\int_{\mathscr{V}} \mathbf{j}_1 \cdot \nabla\phi_2 d^3r,$$

where $\mathbf{j}_1(\mathbf{r})$ is the current distribution in the first experiment and $\phi_2(\mathbf{r})$ is the potential's distribution in the second experiment. Using the same integration by parts as in the solution of Problem 1.18, we get

$$J = \int_{\mathscr{V}} \phi_2 (\nabla \cdot \mathbf{j}_1) d^3r - \int_{\mathscr{V}} \nabla \cdot (\phi_2 \mathbf{j}_1) d^3r.$$

Since we are considering dc (i.e. time-independent) conductivity, the first term vanishes due to the continuity relation – see Eq. (4.6). Now using the divergence theorem to transform the second term to a surface integral (again, just as this was done in the solution of Problem 1.18), we get

$$J = -\int_S (\phi_2 \mathbf{j}_1)_n d^2r,$$

where S is the full surface of the conductor, and $\mathbf{n}$ is the outer normal to it. Nonvanishing contributions to this integral are given only by the interfaces $S_L^{\pm}$ of two current-carrying electrodes in the first experiment (shown in the figure above as two left bold points). On each of these interfaces, ϕ_2 is constant because, in the second experiment, they carry no current. Marking these potential's values by upper indices $\pm$, we get

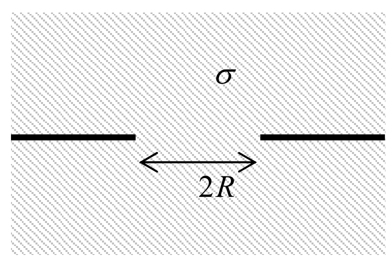
$$J = -\phi_2^+ \int_{S_L^+} (\mathbf{j}_1)_n d^2r - \phi_2^- \int_{S_L^-} (\mathbf{j}_1)_n d^2r.$$

But these two integrals are nothing else than the full currents flowing out of the conductor, equal to $(-I_1)$ for the interface $^+$, and $(+I_1)$ for the electrode $^-$, so

$$J = \phi_2^+ I_1 - \phi_2^- I_1 . \equiv I_1 V_2 .$$

Now repeating the same calculation with the indices 1 and 2 swapped, and requiring the two results to be equal, we get the reciprocity relation $I_1V_2 = I_2V_1$.[103]

Note that this relation is conditioned by the Ohmic (linear) conductivity of the sample.



Problem 4.4. Calculate the resistance between two large uniform Ohmic conductors separated by a very thin, plane, insulating partition with a circular hole of radius R in it – see the figure on the right.

Hint: You may like to use the same oblate spheroidal coordinates that were discussed in Sec. 2.4 of the lecture notes.

Solution: Selecting the parameter R of the degenerate ellipsoidal coordinates defined by Eqs. (2.59) of the lecture notes,

[103] This simple proof was suggested by J. Carson only in 1930 (mimicking an earlier work on elasticity theory by Lord Rayleigh), well after it had been derived from a more general reciprocity relation proved in 1896 by H. Lorentz, by using the full set of Maxwell equations – see Chapters 6 and 7.

$$x = R\cosh\alpha\sin\beta\cos\varphi,$$
$$y = R\cosh\alpha\sin\beta\sin\varphi,$$
$$z = R\sinh\alpha\cos\beta,$$

to be equal to the hole's radius, and placing the coordinate origin into its center, we see that the partition (outside the hole) corresponds to $\beta = \pi/2 =$ const, while for the conductor points slightly above the partition, $z \approx R\,(\pi/2 - \beta)\sinh\alpha$, so

$$\left.\frac{\partial\phi}{\partial n}\right|_{\substack{z=0\\ r>R}} = \left.\frac{\partial\phi}{\partial z}\right|_{\substack{z=0\\ r>R}} = -\frac{1}{R\sinh\alpha}\frac{\partial\phi}{\partial\beta}.$$

This means that if the potential is a function of α alone, the boundary condition of having no current flow through the partition is automatically satisfied. (Due to the mirror symmetry of the problem with respect to the partition plane, the same conclusion is valid for the lower semi-space as well.) From the disk capacitance problem's solution in Sec. 2.4 of the lecture notes, we already know that the Laplace equation may be also satisfied with the potential distribution (2.64):

$$\phi = c_1\tan^{-1}(\sinh\alpha) + c_2. \qquad (*)$$

For the sake of notation simplicity, let us accept the (anti)symmetric boundary conditions at infinity:

$$\phi \to -\frac{V}{2}\operatorname{sgn}(z) = -\frac{V}{2}\operatorname{sgn}(\alpha), \quad \text{at } |z| \to \infty, \ \text{ i.e. at } |\alpha| \to \infty,$$

where V is the voltage applied between the distant points of the conductors; in this case, the solution (*) takes the simple form

$$\phi = -\frac{V}{\pi}\tan^{-1}(\sinh\alpha).$$

Note that according to this formula, the contribution of distant parts of the conductor (with $r >> R$) to the total voltage drop is negligible; this is why the exact shape of the external electrodes is not important, provided that they are not too small and are not too close to the hole.

What remains is to calculate the total current

$$I = \int_S j_n d^2r = -\sigma\int_S \frac{\partial\phi}{\partial n}d^2r,$$

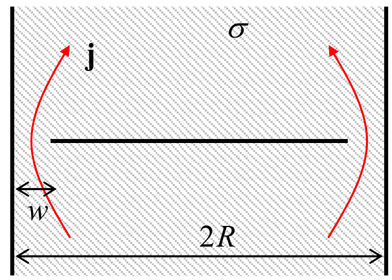
where S is an arbitrary surface crossed by the whole current flow. (Due to the current continuity, the result does not depend on the choice of the surface.) The easiest choice is evidently the plane $z = 0$ – more exactly, its fraction inside the hole, $r < R$. Near the plane, the coordinate relation for z is reduced to $z \approx R\alpha\cos\beta$, while our solution, to just $\phi \approx -(V/\pi)\alpha$, so $-\partial\phi/\partial z = V/\pi R\cos\beta$. Since at this plane, the distance of a point from the center is $\rho \equiv (x^2 + y^2)^{1/2} = R\sin\beta$, so $R\cos\beta = (R^2 - \beta^2)^{1/2}$, we get

$$I = -\sigma\int_{\substack{z=0\\ \rho<R}}\frac{\partial\phi}{\partial z}d^2\rho = -2\pi\sigma\int_0^R\frac{\partial\phi}{\partial z}\rho d\rho = 2\pi\sigma\frac{V}{\pi}\int_0^R\frac{\rho d\rho}{R\cos\beta} = \sigma V\int_{\rho=0}^{\rho=R}\frac{d(\rho^2)}{\left(R^2-\rho^2\right)^{1/2}} = 2\sigma VR,$$

so the hole's resistance V/I is just $1/2\sigma R$.

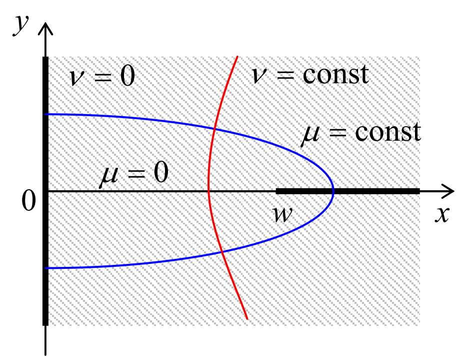
This *Maxwell* (or "Maxwell-Rayleigh") *formula* for resistance is frequently memorized as the ratio of the conductor's resistivity $1/\sigma$ to the hole's diameter $2R$, without any numerical coefficients.

Problem 4.5. A very narrow plane crack inside a round conducting wire of radius R does not reach its surface by a small distance w – see the figure on the right. Assuming that the Ohmic conductivity σ of the wire's material is otherwise constant, calculate the electric resistance of the obstacle in the first approximation in small $w/R << 1$.



Hint: You may like to use the same elliptic coordinates as were employed at the solution of Problem 2.12.

Solution: Due to the condition $w/R << 1$, the local distribution of the electric potential ϕ created by the current flowing around the crack's edge may be considered a function of just two Cartesian coordinates x and y shown in the figure on the right. In this approximation, the function should satisfy the 2D Laplace equation,



$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0,$$

and the following boundary conditions:

$$\frac{\partial\phi}{\partial x}=0, \quad \text{for } x=0,$$

(due to the absence of the current flow $j_x = -\sigma\partial\phi/\partial x$ through the wire's surface);

$$\frac{\partial\phi}{\partial y}=0, \quad \text{for } y=0 \text{ and } w\leq x,$$

(due to the absence of the current flow $j_y = -\sigma\partial\phi/\partial y$ through the crack). Also, due to the evident mirror symmetry of our problem with respect to the crack's plane, we may take $\phi(x,-y) = -\phi(x,y)$, giving

$$\phi=0, \qquad \text{for } y=0 \text{ and } 0\leq x\leq w.$$

Now let us compare these conditions with the definition of the elliptic coordinates (which were already encountered in Sec. 2.4 of the lecture notes) with their scale parameter equal to w:[104]

$$x+iy=w\cosh(\mu+i\nu), \quad \text{i.e.} \begin{cases} x=w\cosh\mu\cos\nu, \\ y=w\sinh\mu\sin\nu, \end{cases} \quad \text{with } -\infty<\mu<+\infty,\ -\frac{\pi}{2}\leq\nu\leq+\frac{\pi}{2}.$$

(In these coordinates, the lines of constant μ are ellipses, while those of constant ν are hyperbolas, with the focus at the point $\{x = w, y = 0\}$ – see the figure above.) The comparison shows that all above boundary conditions are satisfied by ϕ being a function of just one variable, μ.[105] Since $\boldsymbol{w} = w\cosh \boldsymbol{z}$ is

[104] For our current purposes, this variable range, which allows μ to change sign, is more convenient than the similarly legitimate (and more common) choice $0 \leq \mu \leq \infty$, $0 \leq \nu \leq \pi$.

[105] This immediately means that the hyperbolas ν = const, orthogonal to the equipotential lines μ = const at each point (again, see the figure above), coincide with the current flow lines.

an analytic function, the Laplace equation in the curvilinear orthogonal coordinates μ and ν has the same simple form as in the Cartesian coordinates x and y – see the discussion in Sec. 2.4. Plugging $\phi = \phi(\mu)$ into this equation, we immediately see that it this function has to be linear, so with our problem's symmetry, we may take simply

$$\phi = C\mu. \qquad (*)$$

For relatively large distances from the origin, i.e. at $|\mu| >> 1$, the 2D radius

$$\rho \equiv \left(x^2 + y^2\right)^{1/2} = w\left(\sinh^2 \mu + \cos^2 \nu\right)^{1/2}$$

tends to $w\,|\sinh\mu| \approx (w/2)e^{|\mu|}$, so $|\mu| \to \ln(2\rho/w)$ and $\mu \to \mathrm{sgn}(y)\ln(2\rho/w)$, i.e.

$$\phi \to C\,\mathrm{sgn}(y)\ln\frac{2\rho}{w}, \qquad \mathbf{j} = -\sigma\nabla\phi \to -\sigma\frac{C}{\rho}\mathrm{sgn}(y)\mathbf{n}_\rho, \qquad \text{for } \frac{\rho}{w} >> 1.$$

Since this formula may be used only if ρ is still much smaller than the wire's radius R (to keep our 2D Cartesian approximation legitimate), this radial current density, multiplied by the arc $(\pi/2)\rho$ of its spread on the $[x, y]$ plane, has to be equal to the linear density J of the current flowing along the wall. This requirement gives[106]

$$C = -\frac{2J}{\pi\sigma}, \qquad \text{i.e. } \phi \to -\frac{2J}{\pi\sigma}\mathrm{sgn}(y)\ln\frac{2\rho}{w}, \qquad \text{for } \frac{\rho}{w} >> 1.$$

Hence the voltage drop across the crack region is

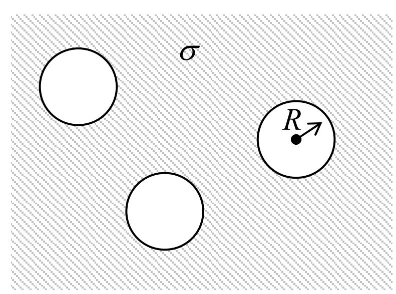
$$V \equiv \phi(\mu_{\min}) - \phi(\mu_{\max}) = \frac{4J}{\pi\sigma}\ln\frac{2\rho_{\max}}{w},$$

where $\rho_{\max} >> w$ is the cutoff distance of our 2D analysis, which is of the order of the wire's radius R. Taking $\rho_{\max} = cR$, where $c \sim 1$, for the crack's contribution to the wire's resistance, we get

$$\frac{V}{I} = \frac{V}{2\pi RJ} = \frac{2}{\pi^2 R\sigma}\ln\frac{2cR}{w} \equiv \frac{2}{\pi^2 R\sigma}\left(\ln\frac{2R}{w} + \ln c\right).$$

(Since $R >> w$, the choice of the constant $c \sim 1$ has a very weak effect on the result.)

Perhaps the most striking feature of this result is its very weak (logarithmic) dependence on the width w; this weakness is due to the effectively 2D geometry of this problem – to be compared with the essentially 3D geometry of the previous problem.



Problem 4.6. Calculate the effective (average) conductivity σ_{ef} of a medium with many empty spherical cavities of radius R, carved at random positions in a uniform Ohmic conductor (see the figure on the right), in the limit of low density $n << R^{-3}$ of the cavities.

Hint: You may like to use the analogy with an electric-dipole medium – see, e.g. Sec. 3.2 of the lecture notes.

[106] An alternative way to obtain this result is to calculate J from the same Eq. (*) by integrating $j_y = -\sigma\partial\phi/\partial y$ over the interval $0 \le x \le w$ at $y \to 0$ (i.e. $\mu \to 0$), where $x \to w\cos\nu$ and $y \to w\mu\sin\nu$.

Solution: As was calculated in Sec. 4.3 of the lecture notes, a single spherical cavity cut in a conductor with a uniform current flow of density $\mathbf{j}_0$ creates outside it an additional dipole electric field described by the second term in Eq. (4.26)

$$\phi_{\rm d} = -\frac{j_0}{\sigma}\frac{R^3}{2r^2}\cos\theta .$$

According to Eq. (3.7), this field corresponds to the following electric dipole moment:

$$\mathbf{p} = -4\pi\varepsilon_0 \frac{R^3}{2\sigma}\mathbf{j}_0 .$$

If the concentration of the cavities is low ($nR^3 << 1$), their dipole fields do not affect each other and hence add up independently to form the following average electric polarization:

$$\mathbf{P} = n\mathbf{p} = -4\pi\varepsilon_0 \frac{nR^3}{2\sigma}\mathbf{j}_0 .$$

Our analysis of electrically polarized media in Sec. 3.2 has shown that the polarization (divided by ε_0) is effectively subtracted from the field created by external charges. In particular, Eq. (3.33) may be rewritten as

$$\mathbf{E} = \mathbf{E}_0 - \frac{\mathbf{P}}{\varepsilon_0}, \qquad \text{with } \mathbf{E}_0 \equiv \frac{\mathbf{D}}{\varepsilon_0}, \tag{*}$$

where $\mathbf{E}$ is the macroscopic electric field, and the vector $\mathbf{D}$ is defined by the external charge density ρ via Eq. (3.32). Since the Sec. 3.2 arguments concerning the effect of polarization did not use any assumptions about the dipole media's nature, Eq. (*) may be used for conductors as well. In this case, $\mathbf{E}_0$ is the field created by an external e.m.f., which would be equal to $\mathbf{E}$ at $\mathbf{P} = 0$, i.e. $\mathbf{E}_0 = \mathbf{j}_0/\sigma$. This means that in our case, the average field is

$$\mathbf{E} = \mathbf{E}_0 - \frac{\mathbf{P}}{\varepsilon_0} = \frac{\mathbf{j}_0}{\sigma} + 4\pi\frac{nR^3}{2\sigma}\mathbf{j}_0 \equiv \frac{\mathbf{j}_0}{\sigma_{\rm ef}},$$

where $\sigma_{\rm ef}$ is the effective conductance we were looking for:

$$\sigma_{\rm ef} = \frac{\sigma}{1+2\pi nR^3} \approx \sigma(1-2\pi nR^3) \leq \sigma .$$

The last (approximate) equality is valid because our analysis is limited to the small cavity concentrations, $nR^3 << 1$, when the correction to σ is relatively small. Of course, the very fact of the conductivity reduction, i.e. the negative sign of the difference $\Delta\sigma \equiv \sigma_{\rm ef} - \sigma$ due to the cavities and the proportionality of the small ratio $\Delta\sigma/\sigma$ to the dimensionless product nR^3 could be predicted from handwaving arguments, so the main result of our calculation is the particular (and rather large) numerical coefficient 2π before the product.

Problem 4.7. In two separate experiments, a narrow gap, possibly of irregular width, between two close, perfectly conducting electrodes is filled with some material: in the first case, a uniform linear dielectric with an electric permittivity ε, and in the second case, a uniform conducting material with an Ohmic conductivity σ. Neglecting the fringe effects, calculate the relation between the mutual capacitance C between the electrodes (in the first case) and the dc resistance R between them (in the second case).

Solution: According to Eqs. (3.53) and (4.14), the electrostatic potential distribution $\phi(\mathbf{r})$ inside both materials satisfies the same (Laplace) differential equation. The boundary conditions on the electrode surfaces ($\phi(\mathbf{r}_1) = \phi_1$ = const, and $\phi(\mathbf{r}_2) = \phi_2$ = const) are also the same. (If fringe effects are negligible, the boundary conditions on the lateral surfaces of the materials are not important.) Thus the potential's distribution $\phi(\mathbf{r})$ in both cases is the same, and defines, in the first case, the electrode surface charge density (3.54):

$$\sigma_s = -\varepsilon \frac{\partial \phi}{\partial n},$$

and in the second case, the current density (4.8) on the surfaces:

$$j_n = -\sigma \frac{\partial \phi}{\partial n}.$$

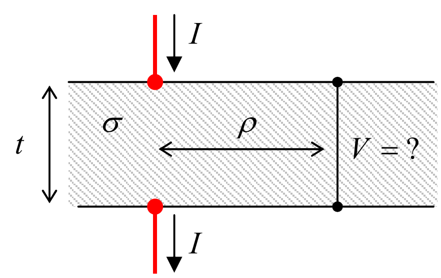
As a result, the ratio σ_s/j_n, and hence the ratio of the full charge Q to the full dc current I, and hence the ratio of the mutual capacitance $C \equiv Q/V$ to the dc conductance $G \equiv 1/R \equiv I/V$, all are independent of the exact potential distribution (and hence of the electrode shape):

$$\frac{C}{G} = \frac{Q}{I} = \frac{\sigma_s}{j_n} = \frac{\varepsilon}{\sigma}.$$

According to this formula, the product RC ($\equiv C/G$) depends only on one material parameter ratio:

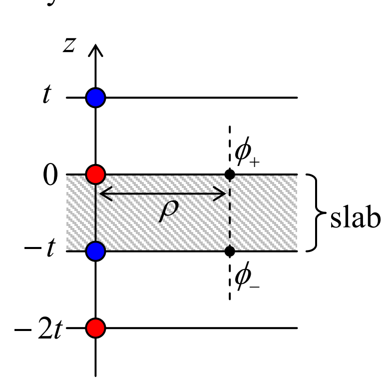
$$RC = \frac{\varepsilon}{\sigma} \equiv \frac{\varepsilon_0 \kappa}{\sigma}.$$

Note that both these effects (of the surface charge accumulation and Ohmic conductivity) may coexist in the same material, and in this case, the RC product is equal to its relaxation time constant τ_r – see Eq. (4.10) and its discussion.



Problem 4.8. Calculate the voltage V across a uniform, wide resistive slab of thickness t, at distance ρ from the points of the injection/pickup of the dc current I passed across the slab – see the figure on the right.

Solution: Following the discussion in Sec. 4.3 of the lecture notes, both the Laplace equation for the electric potential ϕ inside the conductor (related to the dc current density as $\mathbf{j} = \sigma\mathbf{E} = -\sigma\nabla\phi$) and the boundary condition at its surface outside of the injection/pickup points ($j_n = -\sigma\partial\phi/\partial n = 0$) may be satisfied by complementing the actual slab with an infinite stack of imaginary slabs, with a periodic pattern of mirror images of the injection/pickup points – see the figure on the right, where the points' colors code the current sign.



With the coordinate frame selected as shown in this figure, the points of current injection (shown red) are located at $z_k = 2kt$, while points of current pickup (shown blue) are at $z_k' = (2k - 1)t$, where k are integers. Since the current's magnitude at each injection/ejection point is $2I$ (as it follows, e.g., from Fig. 4.8 of the lecture notes, with $d \to 0$), it creates, in the slab stack, the following spherically symmetric distributions of the current density and the electric field:

$$\mathbf{j}_k(\mathbf{r}) = \frac{(-1)^k 2I}{4\pi|\mathbf{r}-\mathbf{r}_k|^3}(\mathbf{r}-\mathbf{r}_k), \qquad \mathbf{E}_k(\mathbf{r}) = \frac{\mathbf{j}_k}{\sigma} = \frac{(-1)^k I}{2\pi\sigma|\mathbf{r}-\mathbf{r}_k|^3}(\mathbf{r}-\mathbf{r}_k),$$

and hence the following spherically symmetric distribution of the potential:

$$\phi_k(z) = \frac{(-1)^k I}{2\pi\sigma|\mathbf{r}-\mathbf{r}_k|} = \frac{(-1)^k I}{2\pi\sigma\left[(kt-z)^2+\rho^2\right]^{1/2}}.$$

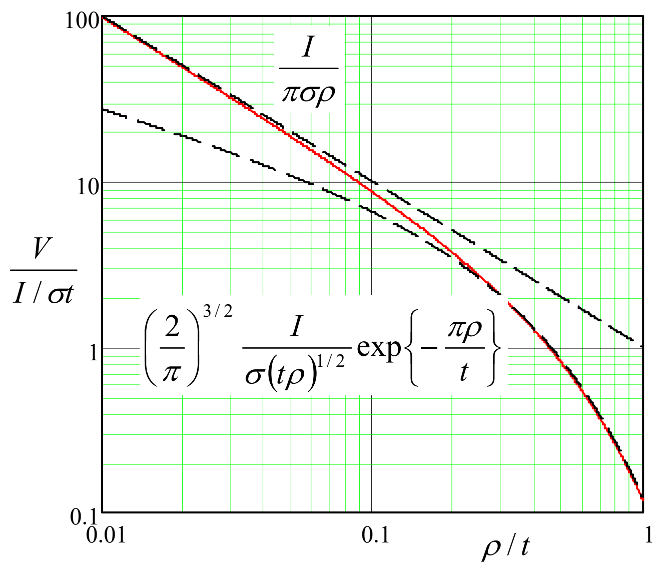
Summing up these contributions for all k, for the top point of voltage pick-up ($z = 0$), we get

$$\phi_+ = \frac{I}{2\pi\sigma}\sum_{k=-\infty}^{+\infty}\frac{(-1)^k}{\left(k^2t^2+\rho^2\right)^{1/2}}.$$

From the last figure above, it is evident that in this symmetric situation, the potential at the lower pick-up point is equal and opposite, $\phi_- = -\phi_+$, so the voltage V is

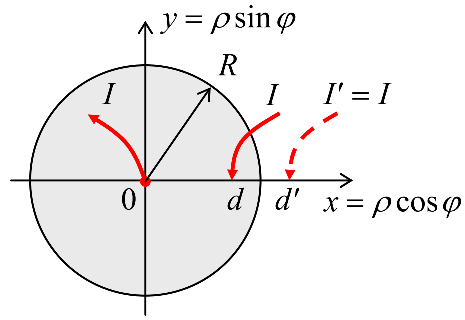
$$V \equiv \phi_+ - \phi_- = 2\phi_+ = \frac{I}{\pi\sigma}\sum_{k=-\infty}^{+\infty}\frac{(-1)^k}{\left(k^2t^2+\rho^2\right)^{1/2}}.$$

This result is shown (on the most appropriate log-log scale) by the solid red line in the figure on the right. At close distances, $\rho/t \to 0$, the sum is dominated by the term with $k = 0$, so the voltage does not depend on t and increases as $I/\pi\sigma\rho$, reflecting the potential's divergence near the current's injection and pickup points. (This divergence would disappear at an account for a non-zero size of the electrodes.)



In the opposite limit $\rho/t >> 1$, the voltage decreases with distance as $\rho^{-1/2}\exp\{-\pi\rho/t\}$. Note that this law may be readily conjectured from even the first step of the variable separation approach to this problem – which is left for the reader's additional exercise. Indeed, since ϕ has to be a $2t$-periodic function of z, its leading spatial harmonic is proportional to $\exp\{\pm ik_1z\}$ with $k_1 = \pi/t$. Hence, in order to satisfy the Laplace equation, its dependence on $\rho >> t$ (when this z-harmonic dominates) is proportional to the Bessel function $K_0(k_1\rho) \to \rho^{-1/2}\exp\{-k_1\rho\}$, with the same k_1.

<u>Problem 4.9</u>. Calculate the distribution of the dc current's density in a thin, round, uniform resistive disk, if the current is inserted into some point at the disk's rim, and picked up in the center.



Solution: The image method enables us to solve analytically even a more general problem when the current I is inserted into an arbitrary point of the disk, at distance d from the center – see the figure on the right. Indeed, following the discussion at the end of Sec. 4.3 of the lecture notes, let us look for the solution of the problem in the form of the current in an unlimited thin layer made of the same material as the disk, with an additional current I' inserted into a point at a certain distance d' from the center, on the same radius. According to Eq. (4.14) of the lecture notes, the distribution of the electric potential in the disk (i.e. at $\rho < R$), outside of the current injection and pickup points, has to satisfy the same 2D Laplace equation as in 2D problems of electrostatics – for example, Problem 2.46. Guided by the model solution of that problem, we may guess that the proper location of the additional current's injection is in the inversion point with $d' = R^2/d$ and that the image current I' exactly equals the actual injection current I.

Taking into account that, according to the solution of Problem 2.45, the Green's function of the 2D Poisson equation for the free space is proportional to ln $|\boldsymbol{\rho} - \boldsymbol{\rho}'|$, this solution has the form

$$\phi(\boldsymbol{\rho}) = C_0 \ln \rho + C_d \ln\left(\rho^2 + d^2 - 2\rho d \cos\varphi\right)^{1/2} + C_{d'} \ln\left(\rho^2 + d'^2 - 2\rho d' \cos\varphi\right)^{1/2}, \qquad (*)$$

where the argument of each log function is the distance of the observation point, with the polar coordinates $\{\rho, \varphi\}$, from one of the current injection/pickup points – see the figure above. The constants C should be selected so that the total current flowing out of each contact equals the corresponding insertion/pickup current. For example, the first term, describing the current pickup point from the center of the disk, gives the linear current density

$$\mathbf{J} = t\mathbf{j} = t\sigma\mathbf{E} = -t\sigma\nabla\phi = -t\sigma\frac{\partial\phi}{\partial\rho}\mathbf{n}_\rho = -\frac{t\sigma C_0}{\rho}\mathbf{n}_\rho,$$

where t is the disk's thickness, so the above requirement,

$$\oint_{\text{around } \rho=0} J_n dr \equiv 2\pi\rho J_\rho = -I,$$

yields

$$C_0 = \frac{I}{2\pi\sigma t}. \qquad (**)$$

The constants in two other terms of Eq. (*) are evidently similar by magnitude but opposite by sign: $C_d = C_{d'} = -C_0$, because they describe the *insertion* of similar currents I.

With these constants, Eq. (*) satisfies the 2D Laplace equation for our problem and the boundary conditions at the current injection/pickup points, and the last step is to prove that it also satisfies the boundary condition at the disk's rim:

$$J_n\Big|_{\rho=R} \equiv -t\sigma\frac{\partial\phi}{\partial\rho}\Big|_{\rho=R} = 0.$$

The proof is straightforward; indeed, according to Eq. (*), in an arbitrary point of the disk,

$$\frac{1}{C_0}\frac{\partial\phi}{\partial\rho} = \frac{\partial}{\partial\rho}\left[\ln\rho - \frac{1}{2}\ln\left(\rho^2 + d^2 - 2\rho d\cos\varphi\right) - \frac{1}{2}\ln\left(\rho^2 + d'^2 - 2\rho d'\cos\varphi\right)^{1/2}\right]$$
$$= \frac{1}{\rho} - \frac{\rho - d\cos\varphi}{\rho^2 + d^2 - 2\rho d\cos\varphi} - \frac{\rho - d'\cos\varphi}{\rho^2 + d'^2 - 2\rho d'\cos\varphi}.$$

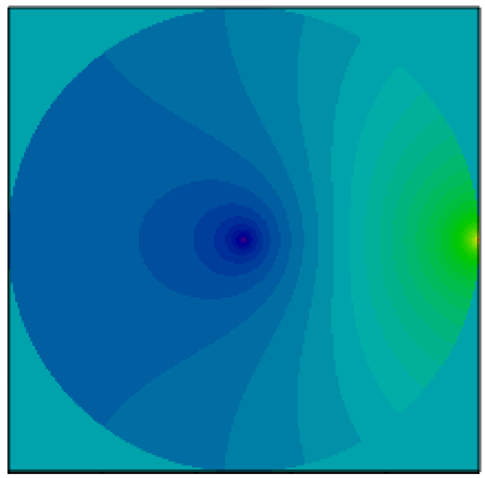
Now taking $\rho = R$, and using the inversion rule $d' = R^2/d$, we get

$$\frac{1}{C_0}\frac{\partial\phi}{\partial\rho}\Big|_{\rho=R}=\frac{1}{R}-\frac{R-d\cos\varphi}{R^2+d^2-2Rd\cos\varphi}-\frac{R-(R^2/d)\cos\varphi}{R^2+(R^2/d)^2-2R(R^2/d)\cos\varphi}$$
$$\equiv\frac{1}{R}\left[1-\frac{R^2-Rd\cos\varphi}{R^2+d^2-2Rd\cos\varphi}-\frac{d^2-Rd\cos\varphi}{d^2+R^2-2Rd\cos\varphi}\right]=0.$$

So, Eq. (*) with proper constants (again, $C_d = C_{d'} = -C_0 = -I/2\pi t\sigma$) is indeed the correct electric potential's distribution. Now we may simplify it for our case of current injection at the disk's rim, $d = d' = R$, getting a very simple expression:

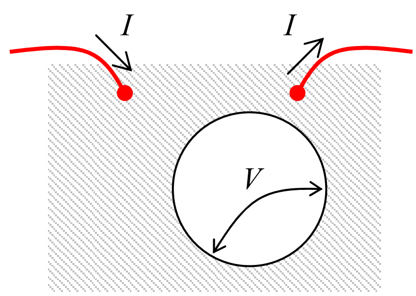
$$\phi(\mathbf{\rho})=\frac{I}{2\pi t\sigma}\left[\ln\rho-\ln\left(\rho^2+R^2-2\rho R\cos\varphi\right)\right]\equiv\frac{I}{2\pi t\sigma}\ln\frac{\rho}{\rho^2+R^2-2\rho R\cos\varphi}.\qquad(***)$$

From here, the current density components are



$$J_\rho=-t\sigma\frac{\partial\phi}{\partial\rho}=\frac{I}{2\pi}\frac{\rho^2-R^2}{\rho\left(\rho^2+R^2-2\rho R\cos\varphi\right)},$$
$$J_\varphi=-t\sigma\frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}=\frac{I}{2\pi}\frac{2R\sin\varphi}{\rho^2+R^2-2\rho R\cos\varphi}.$$

The figure on the right shows the color-coded contour plot of the distribution (**), with the boundaries between the different colors showing the equipotential surfaces, so the current density vectors are normal to them. The plot shows that the current flows from the injection point on the rim mostly straight to the pickup point in the center.



Problem 4.10. DC current is passed between two point electrodes connected to a wide, thin, uniform resistive sheet – see the figure on the right. Use the solution of the previous problem to prove, without much new calculation, that cutting a round hole in the sheet (outside of the current injection/extraction points) doubles the voltage between any two points on its border.

Solution: As was discussed in the model solution of the previous problem, the electric potential due to the point injection of current I into a uniform thin sheet (with its extraction at infinity) equals $C\ln\rho_+$, where ρ_+ is the distance of the observation point from the injection point, and $C \propto I$. The extraction of the same current, at a different point, may be described by a similar expression with the equal but opposite current and the distance ρ_- from the extraction point. So, the full potential distribution in the sheet, in the absence of the hole, is[107]

$$\phi_{\text{without the hole}}=C\left(\ln\rho_+-\ln\rho_-\right).\qquad(*)$$

[107] At this point, we do not need the assumption of the current pickup/injection at infinity anymore, because at large distances, the contributions of the real injection and extraction of the same current cancel each other.

From the same solution, the boundary condition $j_n = 0$ at a circular border between the sheet and an insulator (like that inside the hole we are discussing) is satisfied by addition, to any initial potential of the type $C\ln\rho$, of the potential $C\ln\rho'$ where ρ' is the distance from the field source point inverted in a circle, plus a constant. (This constant depends only on the positions of the source and the circle, but not on that of the potential's measurement point, so it does not contribute to the voltage between any two points.) So, the total potential, in our case, is

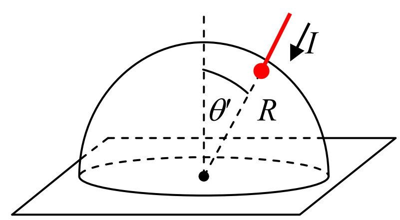
$$\phi_{\text{with the hole}} = C\left[\left(\ln\rho_+ + \ln\rho_+'\right) - \left(\ln\rho_- + \ln\rho_-'\right)\right] + \text{const}\,. \qquad (**)$$

Next, as was discussed in Sec. 4.3 of the lecture notes, the distribution of ϕ in such a uniform resistive sheet, outside of the current injection/extraction points, is governed by the same 2D Laplace equation as in cylindrical electrostatic systems. But from the (easy!) calculation in the solution of Problem 2.45, we know that adding, to the potential $C\ln\rho$, the image potential $-C\ln\rho'$ turns the net potential $C(\ln\rho - \ln\rho')$ into a constant (also not contributing to the voltage), at any point of the circle's border. Hence, on that border (only!) $C\ln\rho' = C\ln\rho + \text{const}$. Plugging this equality, written for both field source points $\rho_\pm$, into Eq. (**), and comparing the result with Eq. (*), we get

$$\phi_{\text{with the hole}} = 2\phi_{\text{without the hole}} + \text{const}', \qquad (***)$$

where the new constant is also the same for all points on the circle's border. Hence, the hole's cut doubles the potential difference (i.e. the voltage) between any two points of its border.

Finally, note that this equality remains valid for *any* areal distribution of the injected and extracted dc currents, provided that they are all outside of the hole's interior. Indeed, any such distribution may be represented as a sum of point pairs of elementary source/drain currents $\pm dI$ flowing into elementary areas $d^2\rho$. Our result (***) is valid for each such pair, and hence (due to the linear superposition principle, applicable here due to the Ohm law's linearity) for the whole distribution as well.



Problem 4.11. The rim of a hemispherical thin shell, of radius R and thickness $t \ll R$, made of a uniform Ohmic conductor, is connected to a plane ground electrode. Calculate the distribution of the electrostatic potential created in the shell by a dc current I injected into it through a small-size electrode located at a polar angle $\theta' < \pi/2$ from the symmetry axis – see the figure on the right.

Hint: You may like to use the variable substitution $\rho \equiv \tan(\theta/2)$ to map the hemisphere onto a unit circle.

Solution: Due to the small thickness of the shell, the distribution of the electrostatic potential in it (outside of the current injection point) obeys the 2D Laplace equation[108]

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{\partial^2\phi}{\partial\varphi^2} = 0, \qquad \text{for } 0 \le \theta \le \frac{\pi}{2}.$$

With the substitution suggested in the *Hint*, it becomes

[108] See, e.g., MA Eq. (10.9) with $\partial/\partial r = 0$.

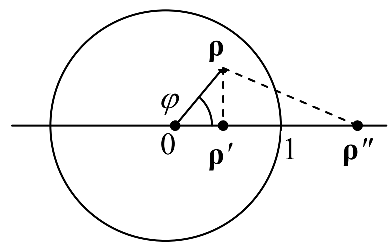
$$\rho \frac{\partial}{\partial \rho}\left(\rho \frac{\partial \phi}{\partial \rho}\right)+\frac{\partial^2 \phi}{\partial \varphi^2}=0. \qquad (*)$$

But this is exactly the Laplace equation in the polar coordinates.[109] Hence, on the planar circular area of a unit radius: $0 \leq \rho \equiv \tan(\theta/2) \leq 1$, onto which the suggested substitution maps the initial hemisphere with $0 \leq \theta \leq \pi/2$, the scalar potential ϕ obeys the Laplace equation as well.

In the initial geometry, the ground electrode attached to the hemisphere's rim keeps it at a constant potential, which we may take for zero. Then, since this rim is mapped onto the circle's rim, with $\rho = 1$, Eq. (*) should be solved with the boundary condition $\phi|_{\rho=1} = 0$. But the Green's function of this boundary problem, describing a delta-functional source of the electric field, was calculated in Problem 2.45:

$$G(\boldsymbol{\rho}, \boldsymbol{\rho}') = 2 \ln\left(\rho' \frac{|\boldsymbol{\rho} - \boldsymbol{\rho}''|}{|\boldsymbol{\rho} - \boldsymbol{\rho}'|}\right), \qquad (**)$$

where $\boldsymbol{\rho}$ is the 2D radius vector of the observation point, $\boldsymbol{\rho}' \equiv \tan(\theta'/2)$ is such vector of the source point, while the vector $\boldsymbol{\rho}''$ with the length $\rho'' = 1/\rho' \geq 1$ is the result of the source point's inversion in the circle with a unit radius – see the figure on the right, in which the source point's azimuthal angle is taken for zero. Hence, by using the well-known geometric expression for the length of the triangle's side opposite to angle φ, and then returning to the original variable θ, we may write the solution of our problem as



$$\phi = c \ln\left(\rho'^2 \frac{|\boldsymbol{\rho} - \boldsymbol{\rho}''|^2}{|\boldsymbol{\rho} - \boldsymbol{\rho}'|^2}\right) = c \ln\left[\rho'^2 \frac{\rho^2 + (1/\rho')^2 - 2(\rho/\rho')\cos\varphi}{\rho^2 + \rho'^2 - 2\rho\rho'\cos\varphi}\right]$$
$$\equiv c \ln\left[\frac{\rho^2\rho'^2 + 1 - 2\rho\rho'\cos\varphi}{\rho^2 + \rho'^2 - 2\rho\rho'\cos\varphi}\right] \equiv c \ln\left[\frac{\tan^2(\theta/2)\tan^2(\theta'/2) + 1 - 2\tan(\theta/2)\tan(\theta'/2)\cos\varphi}{\tan^2(\theta/2) + \tan^2(\theta'/2) - 2\tan(\theta/2)\tan(\theta'/2)\cos\varphi}\right]. \qquad (***)$$

What remains is to express the constant c via the injected current I. This may be done, for example,[110] by noticing that the potential's distribution would not change if the current is injected not into one point of the hemisphere (this is physically impossible anyway), but into an infinitesimal area that is mapped onto a similarly small circle with the center at the point $\boldsymbol{\rho}'$.[111] Hence we may calculate c by comparing the limit of the first form of Eq. (***) at $\boldsymbol{\rho} \to \boldsymbol{\rho}'$,

$$\phi \to -c \ln\left(|\boldsymbol{\rho} - \boldsymbol{\rho}'|^2\right) + c \ln\left(\rho'^2 |\boldsymbol{\rho}' - \boldsymbol{\rho}''|^2\right) \equiv -2c \ln|\boldsymbol{\rho} - \boldsymbol{\rho}'| + \text{const},$$

with Eqs. (*)-(**) of the solution of Problem 9: near the point of injection of current I into a thin film,

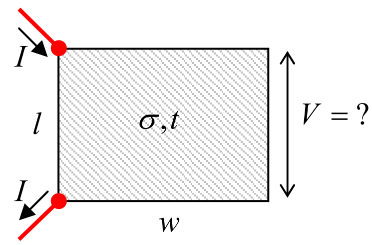
[109] See, e.g., MA Eq. (10.3) with $\partial/\partial z = 0$.

[110] Another way to get the same result (recommended to the reader as an additional exercise and as a sanity check) is to calculate the linear density of the current flowing through the hemisphere's rim, $J = jt = \sigma t E_\theta = -(\sigma t/R)(\partial\phi/\partial\theta)$ at $\theta = \pi/2$, from Eq. (***), integrate the result over the length $2\pi R$ of the rim, and then require the integral to equal I.

[111] The fact that the corresponding area on the initial hemisphere is an ellipse rather than a circle and is proportional to R^2 is irrelevant here, due to the logarithmic character of the potential's distribution.

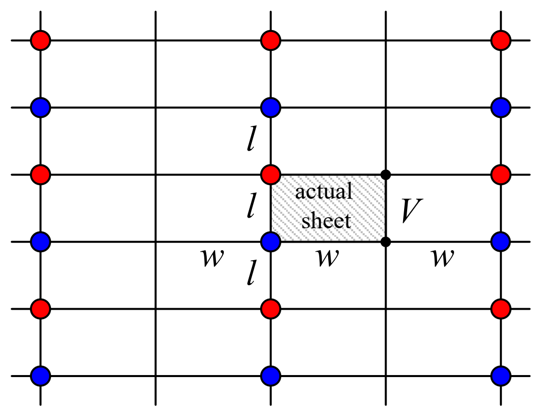
$$\phi \to -\frac{I}{2\pi\sigma t}\ln\left(\left|\boldsymbol{\rho}-\boldsymbol{\rho}'\right|\right)+\text{const},$$

immediately giving $c = I/4\pi\sigma t$.



Problem 4.12. A rectangle of area $l \times w$ is cut from a uniform resistive sheet of thickness $t \ll l, w$. Use two different approaches to calculate the voltage V between its two adjacent corners, induced by the dc current I that is passed between the two other corners – see the figure on the right.

Hint: Besides the charge/current image method, you may like to consider using the variable separation method, with due respect to the current injection/extraction points.



Solution: Extending the solution of Problem 9, we see that the real situation may be replaced with an infinite resistive sheet with a rectangular mesh of current injection/extraction points, shown in the figure on the right. Here each red dot means an injection of current $4I$, while a blue point, an extraction of a similar current. (The factor 4 is due to the merger of 4 equal currents, due to the same-sign reflection in both sides of the square – cf. Fig. 4.8 of the lecture notes.) Indeed, as it is clear from the figure, the net current, which is the vector sum of the currents radially spreading from each injection point, does not cross any border of the actual sheet (shown by hatching), thus satisfying the boundary conditions. In our 2D case, each of these currents creates the following current density:

$$\mathbf{j}_k(\boldsymbol{\rho}) = \frac{I_k}{2\pi t}\frac{(\boldsymbol{\rho}-\boldsymbol{\rho}_k)}{\left|\boldsymbol{\rho}-\boldsymbol{\rho}_k\right|^2} = \pm\frac{4I}{2\pi t}\frac{(\boldsymbol{\rho}-\boldsymbol{\rho}_k)}{\left|\boldsymbol{\rho}-\boldsymbol{\rho}_k\right|^2},$$

and hence the following electric field:

$$\mathbf{E}_k(\boldsymbol{\rho}) = \frac{\mathbf{j}_k(\boldsymbol{\rho})}{\sigma} = \pm\frac{2I}{\pi t\sigma}\frac{(\boldsymbol{\rho}-\boldsymbol{\rho}_k)}{\left|\boldsymbol{\rho}-\boldsymbol{\rho}_k\right|^2},$$

which may be represented as the (minus) gradient of the following electrostatic potential:

$$\phi_k(\boldsymbol{\rho}) = \mp\frac{2I}{\pi t\sigma}\ln\left|\boldsymbol{\rho}-\boldsymbol{\rho}_k\right| + \text{const} \equiv \mp\frac{I}{\pi t\sigma}\ln\left[\left|\boldsymbol{\rho}-\boldsymbol{\rho}_k\right|^2\right] + \text{const}.$$

As usual, the constant in the potential is unimportant; indeed, it drops out from the voltage V, i.e. the potential difference between the two points we are interested in, at the corners of the sheet – see the figure above again. As the figure shows, for these points,

$$\left|\boldsymbol{\rho}-\boldsymbol{\rho}_k\right|^2 = \left[w^2(\Delta k)^2 + l^2(\Delta k')^2\right],$$

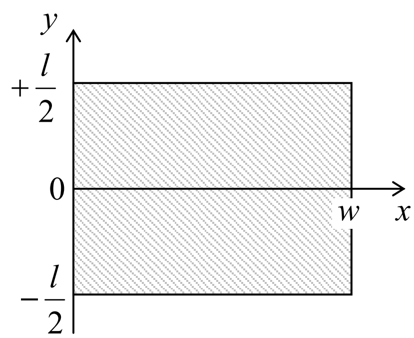
where k and k' are the integers listing, respectively, the "horizontal" and "vertical" distances, in the mesh cell units, between the corner of interest and the current injection/extraction points. Now using the linear superposition principle to sum up the contributions of all the images, we get

$$V = \frac{2I}{\pi t \sigma} \sum_{k,k'=-\infty}^{+\infty} \ln \frac{(2k+1)^2 r^2 + (2k'+1)^2}{(2k+1)^2 r^2 + (2k')^2}, \qquad (*)$$

where the ratio $r \equiv w/l$ is the "form factor" of our rectangle.

Unfortunately, for large values of r, the double series in Eq. (*) is very inconvenient for practical calculations, because it converges very slowly, requiring a careful balance of the sum truncation limits at even very large numbers of k and k'. (Such poor convergence of the sums over multiple charge/current images is typical for 2D geometries.)

In order to improve the situation, we may use an alternative approach, for example, the variable separation method. (As we know from Sec. 2.5, for our current rectangular geometry it does not require any special functions.) The only new[112] issue we should be careful about is the description of the currents' injection and extraction at the corners of our rectangular sheet. This may be done, for example, by artificially moving these points by an infinitesimal distance along one of the borders – the mathematical trick that obviously does not change the physical current distribution. Which of the borders is preferable, depends on our choice of the basic functions.



For example, in the coordinate frame shown in the figure on the right, let us first use naturally periodic trigonometric functions along the x-axis, by taking the partial solution of the 2D Laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \qquad \text{for } 0 \le x \le w, \quad -\frac{l}{2} \le y \le +\frac{l}{2},$$

in the following form – cf., e.g., Eq. (2.95) of the lecture notes:[113]

$$\phi_n = \left(c_n \cos k_n x + s_n \sin k_n x\right) \sinh k_n y, \qquad \text{for } k_n \ne 0.$$

In this case, it is beneficial to shift the current injection/extraction points slightly along the "horizontal" (x-) sides of the rectangle, keeping the boundary conditions on its "vertical" (y-) sides purely homogeneous:

$$j_x\Big|_{x=0,w} = 0, \quad \text{and hence } \frac{\partial \phi}{\partial x}\Big|_{x=0,w} = 0$$

These two conditions immediately give $s_n = 0$, $k_n = \pi n/w$, so the full solution takes the form

$$\phi = c_0 y + \sum_{n=1}^{\infty} c_n \cos\frac{\pi n x}{w} \sinh\frac{\pi n y}{w}, \qquad \text{giving } \frac{\partial \phi}{\partial y}\Big|_{y=l/2} = c_0 + \sum_{n=1}^{\infty} c_n \frac{\pi n}{w} \cos\frac{\pi n x}{w} \cosh\frac{\pi n l}{2w},$$

where the first terms describe an x-independent partial solution of our Laplace equation. Now we may represent the injection of current I into the point $\{x = \varepsilon, y = +l/2\}$,[114] with $0 < \varepsilon << l$, by the requirement

[112] Indeed, in Chapters 2 and 3 the variable separation method was discussed for problems described by the Laplace equation, while our current problem is formally described by the Poisson equation, if with a delta-functional right-hand side.

[113] Another y-function satisfying the Laplace equation, $\cosh k_n y$, obviously does not satisfy our problem's symmetry – or rather asymmetry: $\phi(x,-y) = -\phi(x, y)$.

[114] The extraction of the current from the mirror point at the border $y = -l/2$ is already taken care of by the assumed y-asymmetry of our solution.

$$j_y\Big|_{y=l/2} = -\frac{I}{t}\delta(x-\varepsilon), \qquad \text{and hence } \frac{\partial\phi}{\partial y}\Big|_{y=l/2} = \frac{I}{t\sigma}\delta(x-\varepsilon).$$

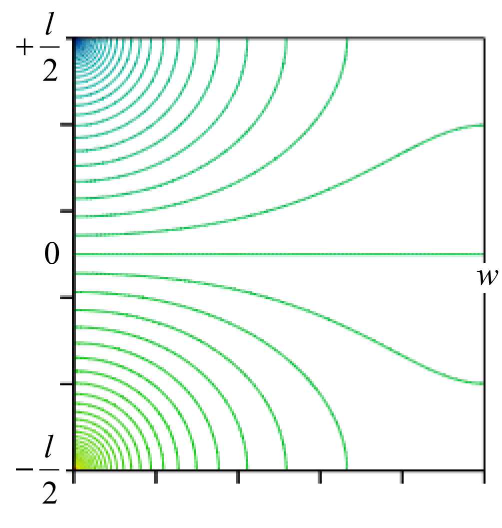
Now employing the orthogonality of the functions $\cos(\pi nx/w)$ in the usual way, in the limit $\varepsilon \to 0$, we get

$$c_0 = \frac{I}{t\sigma w}; \qquad c_n = \frac{I}{t\sigma w}\frac{2}{\pi n\cosh(\pi nl/2w)}, \quad \text{for } n \neq 0,$$

so the final solution is

$$\phi = \frac{I}{t\sigma}\left[\frac{y}{w} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos\frac{\pi nx}{w}\frac{\sinh(\pi ny/w)}{\cosh(\pi nl/2w)}\right]. \qquad (**)$$

This form of the result is very convenient for small and moderate values of the ratio $r \equiv w/l$; for example, the figure on the right shows the equipotential lines for the case of the square (r = 1). The lines are naturally concentric near the left-side corners, i.e. the current injection and extraction points. Note also that the lines are much denser on the side connecting these corners, while only a small part of them reaches the opposite side of the rectangle. This means that the voltage between the corners of that side (which is the main subject of this problem) should be rather small – much smaller than the value $V_0 = (I/t\sigma)(l/w)$ given by the first term of Eq. (**). (That value is reached only in the limit $r \equiv w/l \to 0$ when the current is virtually uniformly distributed across the shorter side of the rectangle.) This conclusion is quantified by Eq. (**), which yields



$$V \equiv \phi\Big|_{\substack{x=0\\ y=+l/2}} - \phi\Big|_{\substack{x=0\\ y=-l/2}} = \frac{I}{t\sigma}\left[\frac{1}{r} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\tanh\frac{\pi n}{2r}\right]. \qquad (***)$$

The single sum in this result converges better than the double sum in Eq. (*),[115] in particular giving the proper limit $V \to V_0 \equiv (I/t\sigma)/r$ at $r \to 0$, but still rather poorly for large values of r. Moreover, even getting an asymptotic expression for V in the limit $r \to \infty$ from Eq. (***) is not straightforward.

This fact justifies using one more approach to this problem: also the variable separation, but with exponential/hyperbolic rather than trigonometric functions along the x-axis:[116]

$$\phi = \sum_{n=1}^{\infty} c_n \cosh[k_n(x-w)]\sin k_n y,$$

where the shift of the x-argument of the cosh function by w immediately enforces the proper boundary condition at the current-free side of the rectangle:

$$j_x\Big|_{x=w} = 0, \quad \text{and hence } \frac{\partial\phi}{\partial x}\Big|_{x=w} = 0.$$

[115] For the particular case $r = 1$, illustrated by the last figure above, Eq. (***) yields $V \approx 0.221(I/t\sigma)$.

[116] Note that in this case, adding a special partial solution for $k_n = 0$ is unnecessary, because the term proportional to y may be described by its extension over the functions $\sin k_n y$ with $n \neq 0$.

At this approach, we better make the boundary conditions on the x-sides homogeneous as well:

$$j_y\Big|_{y=\pm l/2} = 0, \quad \text{and hence } \frac{\partial \phi}{\partial y}\Big|_{y=\pm l/2} = 0,$$

immediately getting the following eigenvalue spectrum: $\cos(k_n l/2) = 0$, i.e. $k_n = \pi(2n-1)l$, so

$$\phi = \sum_{n=1}^{\infty} c_n \cosh\frac{\pi(2n-1)(x-w)}{l}\sin\frac{\pi(2n-1)y}{l},$$

$$\frac{\partial \phi}{\partial x}\Big|_{x=0} = \frac{\pi}{l}\sum_{n=1}^{\infty} c_n(2n-1)\sinh\frac{\pi(2n-1)w}{l}\sin\frac{\pi(2n-1)y}{l}.$$

The price to pay for such simplicity is the necessary shift of the current injection/extraction points from the corners to close points $y_+ = l/2 - \varepsilon$ and $y_- = -l/2 + \varepsilon$, with $\varepsilon << l$, so the boundary condition becomes

$$j_x\Big|_{x=0} = \frac{I}{t}\delta\left(y - \frac{l}{2} + \varepsilon\right), \qquad \text{i.e. } \frac{\partial \phi}{\partial x}\Big|_{x=0} = -\frac{I}{t\sigma}\delta\left(y - \frac{l}{2} + \varepsilon\right), \qquad \text{for } y \geq 0.$$

(Again, the boundary condition for $y \leq 0$ has already been taken care of by the properly asymmetric form of the potential as a function of y.) Now repeating the standard procedure of finding the coefficients c_n, in the limit $\varepsilon \to 0$, we get the following final solution:

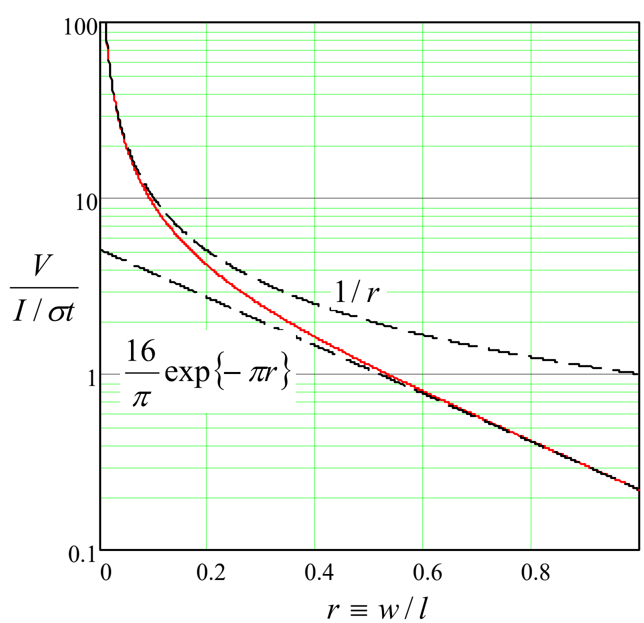
$$\phi = \frac{I}{t\sigma}\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\frac{\cosh[\pi(2n-1)(x-w)/l]}{\sinh[\pi(2n-1)w/l]}\sin\frac{\pi(2n-1)y}{l},$$

giving the following expression for the sought-for voltage:

$$V \equiv \phi\Big|_{\substack{x=0\\ y=+l/2}} - \phi\Big|_{\substack{x=0\\ y=-l/2}} = \frac{I}{t\sigma}\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)\sinh[\pi(2n-1)r]}.$$

Though these expressions much differ from Eqs. (**) and (***), they give exactly the same results, in addition converging much faster for *larger* values of r. Moreover, the last formula makes it elementary to obtain the following asymptotic expression for the limit $r \to \infty$, when the first term dominates the sum:

$$V \to \frac{I}{t\sigma}\frac{16}{\pi}\exp\{-\pi r\}.$$



The figure on the right summarizes our results for the function $V(r)$, showing its plot (red line) and two asymptotes (dashed lines). It is interesting that the crossover between these two limits takes place at a relatively small ratio $r \sim 0.3$.

Finally, note that the calculated voltage is always directly proportional to the so-called *sheet resistance* $R_\square \equiv 1/t\sigma$. This notion is very popular in experimental and engineering practice because it characterizes the average lateral conductive properties of a thin sheet or a thin film, even if it is nonuniform at the distances of the order of t. The usual notation of this constant (alternatively called the "resistance per square") is due to the fact that according to Eq. (4.21), $R_\square$ is the resistance of a square of any size, cut of a thin resistive sheet, provided that the current is uniformly distributed in it – say, flows between two electrodes contacting opposite sides of the square.

Problem 4.13.* The simplest reasonable model of a vacuum diode consists of two parallel planar metallic electrodes of area A, separated by a gap of thickness $d << A^{1/2}$: a "cathode" that emits electrons into the gap, and an "anode" that absorbs the electrons arriving from the gap at its surface. Calculate the dc I-V curve of the diode, i.e. the relation between the average current I flowing between the electrodes and the dc voltage V applied between them, using the following simplifying assumptions:

(i) due to the effect of the negative space charge of the emitted electrons, the current I is much lower than the emission ability of the cathode,
(ii) the initial velocity of the emitted electrons is negligible, and
(iii) the direct Coulomb interaction of electrons (besides the space charge effect) is negligible.

Solution: Due to the system's symmetry, the average current is uniformly distributed over the electrode area, and directed along the axis (say, x) normal to their surfaces, so we may integrate the relation $\mathbf{j} = \rho\mathbf{v}$, where ρ is the space charge density, over the area to calculate the total current:[117]

$$I = Aj_x = -A\rho(x)v_x(x). \qquad (*)$$

Due to the same symmetry, the electrostatic potential ϕ also depends only on the same coordinate: $\phi = \phi(x)$. Since the direct electron-electron interaction is negligible, the total energy of each electron, consisting of its electrostatic energy $q\phi(x) = -e\phi(x)$ and the kinetic energy $m_e v^2/2$, is conserved during its motion from the cathode to the anode, i.e. does not depend on x. Since the initial velocity of electrons is also negligible, $\boldsymbol{v} = \mathbf{n}_x v_x$ everywhere in the gap, and taking the cathode's electrostatic potential for zero, we may write this energy conservation law as

$$\frac{m_e v_x^2(x)}{2} - e\phi(x) = 0. \qquad (**)$$

Now we may use Eqs. (*) and (**) to eliminate v_x, and hence express the local charge density as a function of the local value of the electrostatic potential: $\rho(x) = -I/Av_x(x) = -I/A[2e/m_e\phi(x)]^{1/2}$. Plugging this expression into the Poisson equation (1.41), with $\nabla^2 = d^2/dx^2$ for our 1D geometry, we get the following equation for the distribution of the electrostatic potential along the x-axis:

$$\frac{d^2\phi}{dx^2} = \frac{C}{\phi^{1/2}}, \qquad \text{with } C \equiv \frac{I}{\varepsilon_0 A}\left(\frac{m_e}{2e}\right)^{1/2}.$$

For the first integration of the equation, we may use the very common[118] trick of multiplying and dividing the second derivative by $d\phi$:

[117] The minus sign in this expression reflects the usual convention to take the anode on the right of the cathode, so the electrons, with their negative charges $q = -e$, move from left to right ($v_x > 0$), and their current flows to the left. At this notation, $I \geq 0$, while $\rho(x)$ is negative.
[118] For example, this trick is used in classical mechanics for the simplest derivation of the work-energy principle – see, e.g., CM Eq. (1.20).

$$\frac{d^2\phi}{dx^2} \equiv \frac{d}{dx}\left(\frac{d\phi}{dx}\right) = \frac{d}{d\phi}\left(\frac{d\phi}{dx}\right)\frac{d\phi}{dx} = \frac{1}{2}\frac{d}{d\phi}\left[\left(\frac{d\phi}{dx}\right)^2\right],$$

so the equation takes the form

$$d\left[\left(\frac{d\phi}{dx}\right)^2\right] = 2C\frac{d\phi}{\phi^{1/2}}.$$

Now we may integrate both parts of this relation, getting

$$\left(\frac{d\phi}{dx}\right)^2 = 2C\int\frac{d\phi}{\phi^{1/2}} = 4C\phi^{1/2} + \text{const}. \qquad (***)$$

Now comes the (only :-) nontrivial point of the solution. Per assumption (i), if the electric field $E = -d\phi/dx$ on the cathode's surface was substantially negative (with the force $\mathbf{F} = -e\mathbf{E}$, acting on each electron, directed towards the anode), it would extract a much larger current than what we are calculating. On the other hand, if the field was substantially positive, it would suppress the emission altogether, chasing the emitted electrons back to the cathode. Hence E has to be at the very emission threshold, and a good approximation may be done by taking $E = 0$ on the cathode's surface. In this approximation, the integration constant in Eq. (***) equals zero. Extracting the square root of both parts of the resulting relation (with the physically justified positive sign),

$$\frac{d\phi}{dx} = 2C^{1/2}\phi^{1/4}, \qquad \text{i.e.}\ \frac{d\phi}{\phi^{1/4}} = 2C^{1/2}dx,$$

we may readily integrate its last form, taking into account the second boundary condition, on the anode's surface: $\phi(d) = V$, where the cathode surface's position is taken for $x = 0$. The integration yields[119]

$$\int_0^V \frac{d\phi}{\phi^{1/4}} \equiv \frac{4}{3}V^{3/4} = 2C^{1/2}\int_0^d dx \equiv 2C^{1/2}d.$$

After squaring both sides of this relation and using the definition of the constant C:

$$\frac{16}{9}V^{3/2} = 4Cd^2 \equiv 4\frac{I}{\varepsilon_0 A}\left(\frac{m_\mathrm{e}}{2e}\right)^{1/2} d^2,$$

we obtain the requested dc I-V curve:

$$I = \frac{4}{9}\varepsilon_0\left(\frac{2e}{m_\mathrm{e}}\right)^{1/2}\frac{A}{d^2}V^{3/2}.$$

This is the famous *Child-Langmuir equation* (frequently called the "$V^{3/2}$ law"),[120] one of the theoretical foundations of all vacuum electronics. Please note its substantial difference from Ohm's law $I = V/R$; physically, this difference is due to the effect of the negative space charge of the electrons, which suppresses the electric field on the cathode's surface to $E \approx 0$ – and hence the current to the values

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[119] The same integration, but with an arbitrary upper limit (on the interval $0 \le x \le d$) yields the following spatial distribution of the electrostatic potential: $\phi \propto x^{4/3}$, so the space charge distribution is $|\rho| \propto \phi^{-1/2} \propto x^{-2/3}$.

[120] It was first derived in 1911 by C. Child for ionic currents, and in 1913 by I. Langmuir for electrons.

much lower than the cathode's emission ability. The $V^{3/2}$ law is very well obeyed by many vacuum electronic devices unless the applied voltage is so large that it dissolves the electron "cloud".

Note that very similar space-charge effects may take place in semiconductor structures, but because of the frequent scattering of electrons by impurities, the charge carriers (electrons and holes) do not move ballistically (as was assumed in the above calculation) but rather drift through the material, dissipating their energy at frequent collisions. Because of that, the quantitative result of the (conceptually, similar) theory is different – see the next problem.

Problem 4.14.* Calculate the space-charge-limited current in a system with the same geometry as in the previous problem, and using the same assumptions besides that now the emitted charge carriers do not fly ballistically, but drift in accordance with the Ohm law, with the conductivity given by Eq. (4.13) of the lecture notes: $\sigma = q^2\mu n$, with a constant mobility μ.[121]

Hint: In order to get a realistic result, assume that the medium in which the charge carriers move has a certain dielectric constant κ unrelated to the carriers.

Solution: Let us combine the Ohm law for the current, with the conductivity σ taken in the suggested form,

$$I = Aj = A\sigma(x)E(x) = Aq\mu\rho(x)E(x), \tag{*}$$

where $\rho(x) = qn(x)$ is the space charge density, with the 1D version of the Maxwell equation (3.53):

$$\frac{dE}{dx} = \frac{\rho(x)}{\kappa\varepsilon_0}. \tag{**}$$

For that, we may, for example, express $\rho(x)$ from Eq. (*) and plug it into Eq. (**), getting the following differential equation for the electric field's distribution:

$$E\frac{dE}{dx} = \frac{I}{Aq\mu\kappa\varepsilon_0}, \qquad \text{i.e. } EdE = \frac{I}{Aq\mu\kappa\varepsilon_0}dx..$$

Integrating both sides of the last equation, we get

$$\frac{E^2}{2} = \frac{I}{Aq\mu\kappa\varepsilon_0}x + \text{const}.$$

Now by using exactly the same argumentation as in the previous problem to conclude that the proper boundary condition at the emission surface ($x = 0$) is $E = 0$, we see that the integration constant equals zero, and taking the appropriate (negative) sign before the square root,[122] get

$$\frac{d\phi}{dx} = -E = \left(\frac{2I}{Aq\mu\kappa\varepsilon_0}x\right)^{1/2}. \tag{***}$$

Integrating this equation with the boundary conditions $\phi = 0$ at $x = 0$ and $\phi = V$ at $x = d$, we obtain

[121] As was mentioned in Sec. 4.2 of the lecture notes, the approximation of a constant (in particular, field- and charge-density-independent) mobility is most suitable for semiconductors.

[122] In our system with, say, positive anode voltage V, the electric field should be directed from the anode to the cathode, i.e. (for our coordinate choice) should be negative.

$$V = \left(\frac{2I}{Aq\mu\kappa\varepsilon_0}\right)^{1/2} \int_0^d x^{1/2} dx = \left(\frac{2I}{Aq\mu\kappa\varepsilon_0}\right)^{1/2} \frac{2}{3} d^{3/2}.$$

This is the final result, which is usually represented in the reciprocal form,

$$I = \frac{9}{8}\varepsilon_0 \kappa\mu q \frac{A}{d^3} V^2,$$

and called the *Mott-Gurney law* (first derived in 1948). Note that it gives a current dependence on the applied voltage, $I \propto V^2$, which significantly differs from both the integral form of the Ohm law, $I \propto V$ (valid in semiconductors only at a negligible space charge accumulation) and the Child-Langmuir formula, $I \propto V^{3/2}$ (valid only at the ballistic motion of charge carriers).

As an interesting by-product, we may readily use Eq. (**) and then Eq. (***) with $x = d$ to calculate the full space charge (per unit area):

$$\frac{Q}{A} \equiv \int_0^d \rho(x) dx = \kappa\varepsilon_0 \int_0^d \frac{dE}{dx} dx = \kappa\varepsilon_0 E(d) = -\kappa\varepsilon_0 \left(\frac{2Id}{Aq\mu\kappa\varepsilon_0}\right)^{1/2}.$$

Now plugging in our final result for the current I, we get

$$\frac{Q}{A} \equiv -\kappa\varepsilon_0 \frac{3}{2}\frac{V}{d} \equiv -\frac{3}{2} C_0 V,$$

where $C_0 = \kappa\varepsilon_0/d$ is the capacitance (also per unit area) of the same device when it is empty of charge carriers – see Eq. (3.55). This increase of the effective capacitance (by 50%) is natural, because the field lines starting on the surface charges of the anode, which in an empty device would continue all the way (d) to the cathode, now end up on the closer spatially-distributed charges.

Problem 4.15. Prove that the distribution of dc currents in a uniform Ohmic conductor with a given voltage distribution along its surface corresponds to the minimum of the total energy dissipation rate ("Joule heat").

Solution: According to Eq. (4.40) of the lecture notes, the total Joule heat power (i.e. the electric energy dissipation rate) in a uniform Ohmic conductor may be expressed as

$$\mathscr{P} = \sigma \int_V (\nabla\phi)^2 d^3r. \qquad (*)$$

According to Eq. (4.14), the distribution of the scalar potential ϕ in a uniform Ohmic conductor (with σ = const) satisfies the Laplace equation, just as in free space. Hence, its distribution is the same as in a similar free-space volume V with the same boundary conditions $\phi|_S$. But as we know from Chapter 1 (see the model solution of Problem 1.17), the latter distribution corresponds to the minimum of the electrostatic energy

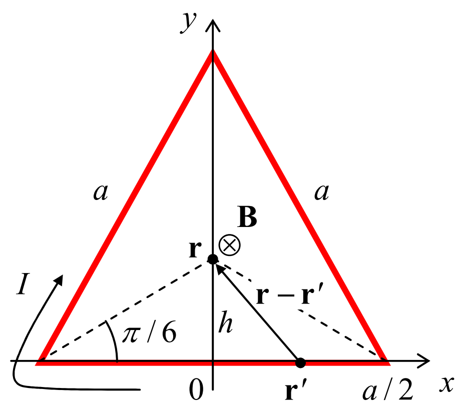
$$U = \frac{\varepsilon_0}{2}\int_V E^2 d^3r = \frac{\varepsilon_0}{2}\int_V (\nabla\phi)^2 d^3r,$$

i.e. to the minimum of the same integral as in Eq. (*).

Chapter 5. Magnetism

Problem 5.1. DC current I flows around a thin wire loop bent into the form of a plane equilateral triangle with side a. Calculate the magnetic field in the center of the loop.

Solution: According to Eq. (5.18) of the lecture notes, for our planar system, the magnetic field $d\mathbf{B}$ of each wire element $d\mathbf{r}'$, at an observation point $\mathbf{r}$ in the same plane as the current, is directed normally to the plane, so all dB add up as scalars. Due to the system's symmetry, the total field is six times that induced by one-half of each side. So, introducing the Cartesian coordinates as shown in the figure on the right, with $\mathbf{r} = \{0, h, 0\}$ and $\mathbf{r}' = \{x', 0, 0\}$, we may use Eq. (5.18) to write



$$B = 6\frac{\mu_0 I}{4\pi}\int_0^{a/2}\frac{h\,dx'}{\left(x'^2 + h^2\right)^{3/2}} = 6\frac{\mu_0 I}{4\pi h}\int_0^{a/2h}\frac{d\xi}{\left(\xi^2+1\right)^{3/2}},$$

where $\xi \equiv x'/h$, while $h = (a/2)\tan(\pi/6) = a/2\sqrt{3}$ is the height of the triangle – see the figure above. Working out the last integral,[1] we finally get

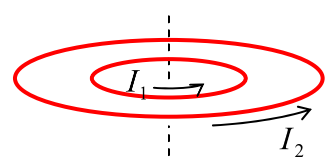
$$B = 6\frac{\mu_0 I}{4\pi h}\frac{a/2h}{\left[(a/2h)^2+1\right]^{1/2}} = 6\frac{\mu_0 I}{4\pi h}\frac{\sqrt{3}}{\left[\left(\sqrt{3}\right)^2+1\right]^{1/2}} \equiv \frac{3\sqrt{3}}{4\pi}\frac{\mu_0 I}{h} = \frac{9}{2\pi}\frac{\mu_0 I}{a}.$$

As a sanity check, it is useful to rewrite this result in the form

$$B = \frac{\mu_0 I}{2R_{\text{ef}}}, \qquad \text{with } R_{\text{ef}} = \frac{2\pi}{3\sqrt{3}}h \approx 1.21h,$$

where per Eq. (5.24) of the lecture notes, R_{ef} is the effective radius of the wire loop. Looking at the figure above again, we see that our result, $R_{\text{ef}} \approx 1.2h$, makes sense.

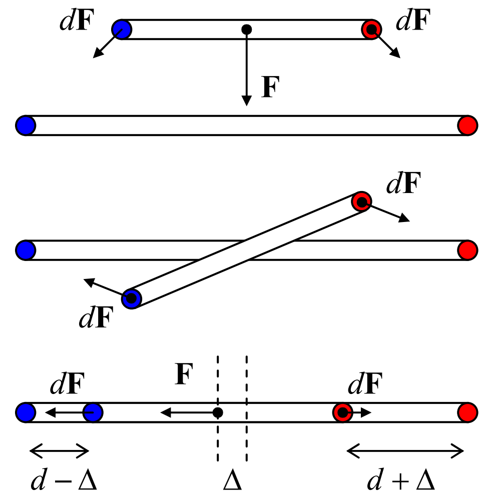
Problem 5.2. A circular wire loop, carrying a fixed dc current, is placed inside a similar but larger loop, carrying a fixed current in the same direction – see the figure on the right. Use semi-quantitative arguments to analyze the mechanical stability of the coaxial and coplanar position of the inner loop with respect to its possible angular, axial, and lateral displacements.



Solution: The stability may be discussed using a semi-quantitative analysis of magnetic forces. Indeed, since according to the basic Eq. (5.1) of the lecture notes, the currents flowing in the same directions attract each other, and this force decreases with the distance d between them, the interaction of the closest parts of two loops plays the most important role.

[1] See, e.g., MA Eq. (6.2b).

Hence, a small *axial* displacement of the inner ring from the outer ring's plane results in a rise of a net force **F** dragging it back – see the top panel of the figure on the right, where colors code the current direction – in or out of the plane of the drawing.



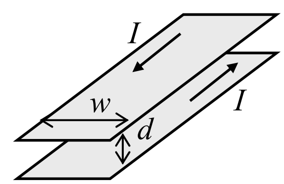
Very similarly, an *angular* displacement of the inner loop causes the elementary forces $d\mathbf{F}$ exerted on it to create a mechanical torque pulling it back to the coplanar position – see the middle panel of the figure.

The analysis of the *lateral* displacement is a bit more subtle. According to Eq. (5.7) of the lecture notes, the attraction of two currents (per unit length) scales approximately[2] as $1/d$, where d is the distance between them. At a lateral displacement of the loop by a small distance $\Delta << d$ (see the bottom panel of the figure on the right), the nearest distances d for one half of the ring decrease by $\sim\Delta$, and for the other half, increase by the same amount. However, the function $f(d) \equiv 1/d$ grows faster at the argument's decrease than drops at its increase:

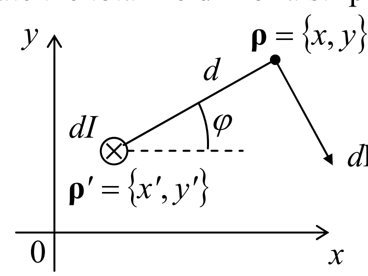
$$f(d\pm\Delta) \equiv \frac{1}{d\pm\Delta} = \frac{1}{d\left(1\pm\Delta/d\right)} \approx \frac{1}{d}\left(1\mp\frac{\Delta}{d}+\frac{\Delta^2}{d^2}\right),$$

so we may expect the net force to be directed toward the nearest part of the outer ring. Hence, the coaxial and coplanar position of the inner loop is *stable* with respect to its axial and angular displacements,[3] but is *unstable* with respect to its lateral displacements.

Problem 5.3. Two planar, parallel, long, thin conducting strips of width w, separated by distance d, carry equal but oppositely directed currents I – see the figure on the right. Calculate the magnetic field in the plane located in the middle between the strips, assuming that the flowing currents are uniformly distributed across the strip widths.



Solution: Due to the linear superposition principle, we may calculate the total field **B** of a strip as a vector sum of elementary fields $d\mathbf{B}$ induced by elementary currents $dI = (I/w)dw$ flowing in each elementary segment $dw << w, d$ of the strip's widths, considering it as a thin wire. The magnetic field of such a wire was calculated in Sec. 5.1 of the lecture notes (see Eq. (5.20) and its derivation), and we need just to rewrite it in a form more convenient for our current purposes. Directing the z-axis along the current (see the figure on the right), we get



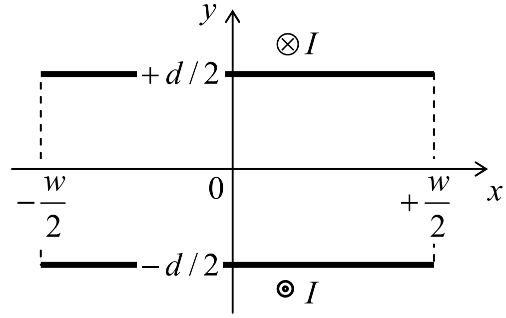
[2] As a reminder, Eq. (5.7) is strictly valid only for infinite straight wires, but as we know, for example from Eq. (5.24), the results for other similar geometries scale almost similarly.

[3] Eq. (5.53) of the lecture notes allows easy confirmation of these two conclusions using potential energy arguments, but I will defer this task until the conclusion of the final discussion of the magnetic field energy (or rather energies) in the next chapter.

$$dB_x = dB\sin\varphi = \frac{\mu_0}{2\pi d}dI\frac{\Delta y}{d} = \frac{\mu_0}{2\pi}\frac{y-y'}{(x-x')^2+(y-y')^2}dI,$$

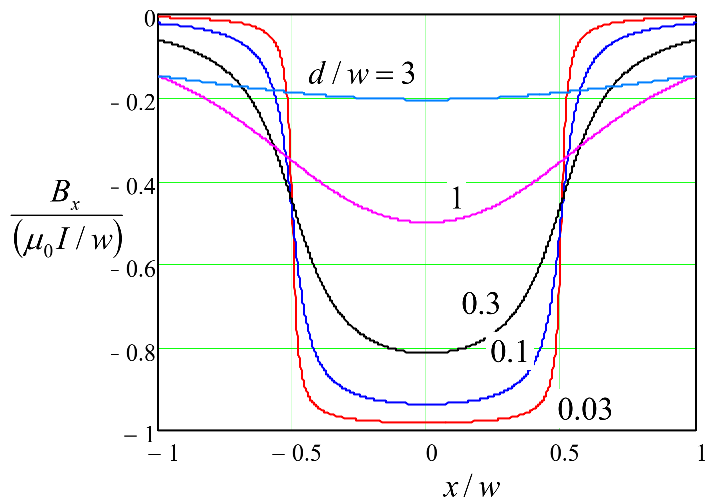
$$dB_y = -dB\cos\varphi = -\frac{\mu_0}{2\pi d}dI\frac{\Delta x}{d} = -\frac{\mu_0}{2\pi}\frac{x-x'}{(x-x')^2+(y-y')^2}dI,$$

where $\boldsymbol{\rho} = \{x, y\}$ is the 2D radius vector of the field observation point, while $\boldsymbol{\rho}' = \{x', y'\}$ is that of the source (wire segment).



Now we should integrate this expression over both current-carrying strips. Selecting the reference frame in the natural way shown in the figure on the right, so the mid-plane in question corresponds to $y = 0$, we see that due to the problem's symmetry, the integral of dB_y vanishes, while the "horizontal" component of the field is twice that of a single strip:

$$B_x = -2\frac{\mu_0}{2\pi}\int_{x'=-w/2}^{x'=w/2}\frac{d/2}{(x-x')^2+(d/2)^2}dI$$
$$= -\frac{\mu_0}{\pi}\frac{I}{w}\frac{d}{2}\int_{-w/2}^{+w/2}\frac{dx'}{(x-x')^2+(d/2)^2}$$
$$= -\frac{\mu_0 I}{\pi w}\left(\tan^{-1}\frac{x-w/2}{d/2} - \tan^{-1}\frac{x+w/2}{d/2}\right).$$



The figure on the right shows the plots of B_x as a function of x, for several values of the d/w ratio. If $d >> w$, the field is relatively low, distributed along axis x over a broad interval $\Delta x \sim d >> w$, and may be well approximated by the sum of fields of two thin wires.

On the other hand, as the distance between the strips is reduced, the field becomes localized within the gap between them. (At $d/w \to 0$, the so-called "fringe fields" at $|x| > w/2$ become negligible.) Moreover, the field becomes uniform and tends to an x- and d-independent value:

$$B_x = -\frac{\mu_0 I}{w}.$$

A simple explanation of this important result and its discussion are the subjects of the next problem.

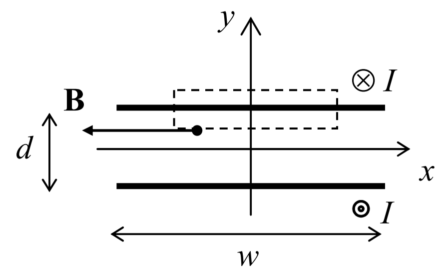
<u>Problem 5.4</u>. For the system studied in the previous problem, but now only in the limit $d << w$, calculate:

(i) the distribution of the magnetic field in space,
(ii) the vector potential of the field,
(iii) the magnetic force (per unit length) exerted on each strip, and

(iv) the magnetic energy and self-inductance of the loop formed by the strips (per unit length).

Solutions:

(i) As was shown in the previous problem, in the limit $d/w \to 0$, the magnetic field is localized in the gap between the strips and is uniform. Applying the Ampère law, given by Eq. (5.37) of the lecture notes, to the contour shown with the dashed line in the figure on the right (which shows the cross-section of the system), we get the same result as was obtained in the previous problem's solution,



$$B = -B_x = \mu_0 \frac{I}{w}, \qquad (*)$$

in a much simpler way, and for arbitrary y between $-d/2$ and $+d/2$.

(ii) According to Eq. (5.28) of the lecture notes, in our system, the vector potential has to be directed along the z-axis, i.e. along the current, and be independent of z. From the structure of that formula, we may also argue that at $d << w$ and well inside the gap, $\mathbf{A}$ should not depend on x either (because the strip edges are not "visible" from those points). Hence we may look for the vector potential in the form

$$\mathbf{A} = A(y)\mathbf{n}_z .$$

Calculating the curl of such a vector,

$$\nabla \times \mathbf{A} = \mathbf{n}_x \frac{\partial A}{\partial y},$$

and requiring it to be equal to the vector $\mathbf{B} = -B\mathbf{n}_x$ described by Eq. (*), we get

$$\mathbf{A} = -\mu_0 \frac{I}{w} y \mathbf{n}_z + \text{const.}$$

This result could be also obtained by the direct integration of Eq. (5.28) along x' and z'. (Alternatively, we can integrate Eq. (5.51) written for each current component $dI = Idx'/w$, along the x'-axis.)

Note, however, that a uniform magnetic field, like our $\mathbf{B} = -B\mathbf{n}_x$, cannot "tell" one transverse coordinate (say, y) from the other one (z), and remains the same for different distributions of the vector potential, for example[4]

$$\mathbf{A} = \mu_0 \frac{I}{w} z \mathbf{n}_y + \text{const},$$

or a linear combination of these two functions. This is one more manifestation of the gauge invariance of the magnetic field with respect to any transformation $\mathbf{A} \to \mathbf{A} + \nabla\chi$ – see Eq. (5.46).

(iii) As it follows from Eq. (5.1), the total magnetic force exerted on each strip is directed along the y-axis and corresponds to the strips' repulsion. To calculate its magnitude, it would be wrong to plug Eq. (*) into Eq. (5.15), because an elementary current (like a point electric charge) does not exert force on itself.[5] As evident from the system's symmetry (and as may be readily checked) using the Ampère law, the field created by a single strip is twice lower:

[4] Note that this distribution, of course, does *not* satisfy Eq. (5.28).

[5] Of course, different components Jdx of the current in a strip do exert nonvanishing (x-directed) forces on each other, but their sum over the strip equals zero.

$$B_1 = \mu_0 \frac{I}{2w} \ ,$$

so, according to Eq. (5.15), the force magnitude (per unit length)

$$\frac{F}{l} = \frac{1}{l} \int_{\text{strip}} j B_1 d^3 r = I B_1 = \mu_0 \frac{I^2}{2w}.$$

This expression may be also represented as an integral, over the strip's area, of the positive pressure of the magnetic field:

$$\mathcal{P} \equiv \frac{F}{lw} = \frac{\mu_0 I^2}{2w^2} = \frac{B^2}{2\mu_0} = u\,,$$

where $u = U/V_{\text{gap}} = U/lwd$ is the magnetic energy's density – see Eq. (5.57b).

(iv) Since the full magnetic field is uniform inside the gap (with the cross-section area dw) and vanishes outside of it, the magnetic energy per unit length is just[6]

$$\frac{U}{l} = \frac{1}{l} \int_{\text{gap}} \frac{B^2}{2\mu_0} d^3 r = \frac{B^2}{2\mu_0} dw = \mu_0 \frac{d}{w} \frac{I^2}{2} = u \frac{V_{\text{gap}}}{l}.$$

From this result and Eq. (5.72), we immediately get the following expression for the loop's self-inductance (also per unit length):

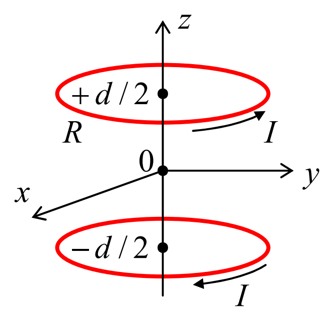
$$\frac{L}{l} = \mu_0 \frac{d}{w}.$$

The same result may be also calculated as the ratio $(\Phi/l)/I$, where Φ is the magnetic flux in the gap between the strips:

$$\frac{\Phi}{l} = \frac{1}{l} \int_{\text{gap}} B_n d^2 r = Bt = \mu_0 \frac{I}{w} t\,.$$

This simple formula, which shows a clear way toward the reduction of current loop inductances (bring the counterpart conductors as close to each other as possible!), is very important for applications in electronics because self-inductances may play a negative role in high-performance integrated circuits, reducing their maximum speed/bandwidth.

<ins>Problem 5.5</ins>. Calculate the magnetic field distribution near the center of the system of two similar, plane, round, coaxial wire coils, carrying equal but oppositely directed currents – see the figure on the right.



[6] This result poses an additional question for the reader: how can the strip *repulsion* (i.e. their "desire" to *increase* d, and hence the volume V_{gap} and the product $U = uV_{\text{gap}}$) be compatible with the general trend for any system to *decrease* its potential energy? As a reminder, for the *electrostatic* force exerted on a conductor, this paradox does not exist because the force direction corresponds to *negative* pressure – see, e.g., the solution of Problem 2.1. Thinking about this issue may be a good preparation for the general discussion, in Sec. 6.2 of the lecture notes, of the relation between the two magnetic energies given by Eqs. (5.53) and (5.54).

Solution: In order to find the field $\mathbf{B}(x, y, z)$ exactly on the system's axis, we may combine two versions of Eq. (5.23) of the lecture notes, applied to each of the loops:

$$\mathbf{B}(0,0,z) = \frac{\mu_0 I}{2}\left\{-\frac{R^2}{\left[R^2+(d/2+z)^2\right]^{3/2}}+\frac{R^2}{\left[R^2+(d/2-z)^2\right]^{3/2}}\right\}\mathbf{n}_z,$$

where z is the axis directed along the system's symmetry axis, with the origin in its center – see the figure above. Per this formula, at the center of the system ($z = 0$), the field vanishes, while for small ($|z| \ll R, d$) but nonvanishing deviations from that point, it may be found from the linear term of the Taylor expansion of its right-hand side:

$$\mathbf{B}(0,0,z) \approx \frac{\mu_0 I}{2}\frac{\partial}{\partial z}\left\{\frac{R^2}{\left[R^2+(d/2+z)^2\right]^{3/2}}-\frac{R^2}{\left[R^2+(d/2-z)^2\right]^{3/2}}\right\}_{z=0} z\mathbf{n}_z = \frac{3\mu_0 IdR^2}{2\left[R^2+(d/2)^2\right]^{5/2}} z\mathbf{n}_z.$$

Obviously, this expression may be rewritten as

$$\left.\frac{\partial B_z}{\partial z}\right|_{\mathbf{r}=0} = \frac{3\mu_0 IdR^2}{2\left[R^2+(d/2)^2\right]^{5/2}}.$$

Now, in the Cartesian coordinates, the general Eq. (5.29) reads

$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=0.$$

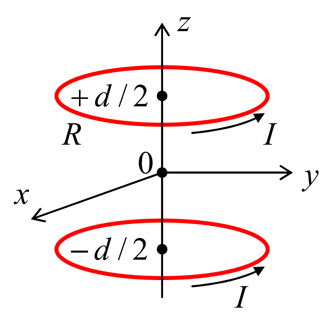
Due to the axial symmetry of our system, we may argue that at $z = 0$, the first two partial derivatives have to be equal to each other. Hence, the last two relations yield

$$\left.\frac{\partial B_x}{\partial x}\right|_{\mathbf{r}=0} = \left.\frac{\partial B_y}{\partial y}\right|_{\mathbf{r}=0} = -\frac{1}{2}\left.\frac{\partial B_z}{\partial z}\right|_{\mathbf{r}=0} = -\frac{3\mu_0 IdR^2}{4\left[R^2+(d/2)^2\right]^{5/2}},$$

so in the linear approximation in small x, y, and z we may write

$$\mathbf{B}(x,y,z) \approx \frac{3\mu_0 IdR^2}{4\left[R^2+(d/2)^2\right]^{5/2}}\left(-x\mathbf{n}_x - y\mathbf{n}_y + 2z\mathbf{n}_z\right), \qquad \text{for } r \ll d, R.$$

Such a *quadrupole magnetic field* pattern may provide trapping of a particle with zero net electric charge but a non-zero total spin (and hence a nonvanishing magnetic moment), within a certain range of its initial velocity. A brief discussion of magnetic traps, and references to additional literature on the subject, may be found in Sec. 9.7 of the lecture notes.



Problem 5.6. The two-coil system considered in the previous problem now carries equal and *similarly* directed currents – see the figure on the right. Calculate what should be the ratio d/R for the second derivative $\partial^2 B_z/\partial z^2$ to equal zero at $z = 0$.[7]

[7] This *Helmholtz coils* system, producing a highly uniform magnetic field near its center, is broadly used in physical experiment.

Solution: Just as in the model solution of the previous problem, by using Eq. (5.23) for the field of each loop, we get

$$B_z = \frac{\mu_0 I}{2}\sum_{\pm}\frac{R^2}{\left[R^2 + (d/2 \pm z)^2\right]^{3/2}} \equiv \frac{\mu_0 I R^2}{2}\sum_{\pm}\left[R^2 + \left(\frac{d}{2} \pm z\right)^2\right]^{-3/2}.$$

Note that at an arbitrary ratio d/R, the first derivative

$$\frac{\partial B_z}{\partial z} = \frac{\mu_0 I R^2}{2}\sum_{\pm}\left[\mp 3\left(\frac{d}{2} \pm z\right)\right]\left[R^2 + \left(\frac{d}{2} \pm z\right)^2\right]^{-5/2},$$

vanishes in the center of the system (at $z = 0$) – just as one might expect from the system's symmetry. Now calculating the second derivative, we get

$$\frac{\partial^2 B_z}{\partial z^2}\Big|_{z=0} = \frac{\mu_0 I R^2}{2}\sum_{\pm}\left\{\left[R^2 + \left(\frac{d}{2} \pm z\right)^2\right] - 5\left(\frac{d}{2} \pm z\right)^2\right\}\left[R^2 + \left(\frac{d}{2} \pm z\right)^2\right]^{-7/2}\Big|_{z=0}$$

$$= \mu_0 I R^2\left[R^2 - 4\left(\frac{d}{2}\right)^2\right]\left[R^2 + \left(\frac{d}{2}\right)^2\right]^{-7/2}$$

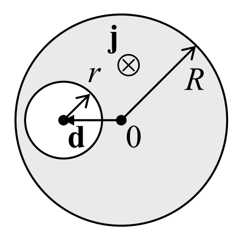
This expression evidently turns to zero at

$$R^2 - 4\left(\frac{d}{2}\right)^2 = 0, \qquad \text{i.e. at } R = d.$$

This is exactly the relation used for the Helmholtz coils' design. Note that due to the system's symmetry, the third derivative, $\partial^3 B_z/\partial z^3$, vanishes at $z = 0$ as well (this one for any d/R), so the deviation of the field from its value in the center,

$$B_z\Big|_{\substack{z=0\\ d=R}} = \mu_0 I \frac{R^2}{\left[R^2 + (d/2)^2\right]^{3/2}}\Big|_{d=R} = \left(\frac{4}{5}\right)^{3/2}\frac{\mu_0 I}{R},$$

grows very slowly, as $(z/R)^4$, with the deviation from that point.



Problem 5.7. DC current of a constant density j flows along a round cylindrical wire of radius R, with a round cylindrical cavity of radius r cut in it. The cavity's axis is parallel to that of the wire but offset from it by a distance $d < R - r$ (see the figure on the right). Calculate the magnetic field inside the cavity.

Solution: We may formally represent this current distribution as an algebraic sum of the following two components:

– a current of the constant density j inside an uncut wire of the given radius R, and
– a current of the density ($-j$) inside the cavity of the radius r.

Applying the Ampère law to the first (axially-symmetric) component, just as this was done in Sec. 5.2 of the lecture notes, we get the result expressed by Eqs. (5.38). For this field inside the wire, with $j(\rho)$ = const, the first of those formulas yields.

$$B_1(\boldsymbol{\rho}) = \frac{\mu_0 j}{2}|\boldsymbol{\rho}|, \qquad \text{for } |\boldsymbol{\rho}| \le R,$$

where $\boldsymbol{\rho}$ is the 2D radius vector of the field measurement point, referred to the center of the wire's cross-section. With the account of the azimuthal direction of the field, this result may be represented in the following vector form:

$$\mathbf{B}_1(\boldsymbol{\rho}) = \frac{\mu_0 j}{2}\mathbf{n}_j \times \boldsymbol{\rho}, \qquad \text{for } |\boldsymbol{\rho}| \le R,$$

where $\mathbf{n}_j$ is the unit vector directed as the current is – in the figure above, into the plane of the drawing.

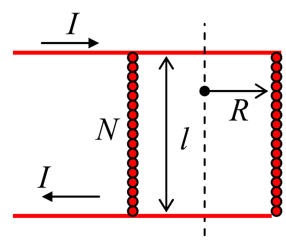
Now we may carry out an absolutely similar calculation for the second component of the current, which is also axially symmetric but with the symmetry axis offset from the origin by a vector **d** (see the figure above), so the Ampère contour C has to be centered to that point. The result is

$$\mathbf{B}_2(\boldsymbol{\rho}) = -\frac{\mu_0 j}{2}\mathbf{n}_j \times (\boldsymbol{\rho} - \mathbf{d}), \qquad \text{for } |\boldsymbol{\rho} - \mathbf{d}| \le r,$$

so the net field inside the cavity is

$$\mathbf{B}(\boldsymbol{\rho}) \equiv \mathbf{B}_1(\boldsymbol{\rho}) + \mathbf{B}_2(\boldsymbol{\rho}) = \frac{\mu_0 j}{2}\mathbf{n}_j \times \mathbf{d}.$$

Rather counter-intuitively, the field is uniform! It is directed normally to the center displacement vector **d**, and its magnitude is equal to that of the component field $\mathbf{B}_1$ at the center of the cavity.



Problem 5.8. Calculate the magnetic field's distribution along the axis of a straight solenoid (see Fig. 5.6a of the lecture notes, partly reproduced on the right) of a finite length l, and a round cross-section of radius R. Assume that the solenoid has many ($N >> 1, l/R$) wire turns, uniformly distributed along its length.

Solution: Because of the high wire turn density $n \equiv N/l >> 1/R$, and hence a virtually circular shape of each turn, the solenoid's field may be represented as a sum of fields of N circular current loops. The field induced by a single loop was calculated in Sec. 5.1 of the lecture notes – see Eq. (5.23). Generalizing that relation to an arbitrary position (z') of the loop, we get

$$B_1 = \frac{\mu_0 I}{2}\frac{R^2}{\left[R^2 + (z - z')^2\right]^{3/2}}.$$

Now we can use the linear superposition principle to find the field induced by the solenoid as a whole:

$$B_N = \frac{\mu_0 I}{2}\sum_{k=-N/2}^{N/2}\frac{R^2}{\left[R^2 + (z - z_k)^2\right]^{3/2}} \equiv \frac{\mu_0 I}{2}\sum_{k=-N/2}^{+N/2}\frac{R^2}{\left[R^2 + (z - kl/N)^2\right]^{3/2}},$$

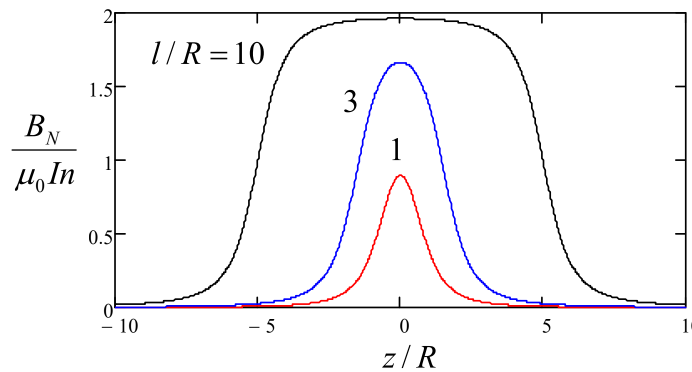
where the index k numbers wire turns, starting from the middle of the solenoid. For $N >> 1$, this sum is well approximated by the corresponding integral

$$B_N = \frac{\mu_0 I}{2}\int_{-N/2}^{+N/2}\frac{R^2}{\left[R^2 + (z - kl/N)^2\right]^{3/2}}dk = \frac{\mu_0 I}{2}n\int_{-l/2}^{+l/2}\frac{R^2}{\left[R^2 + (z - z')^2\right]^{3/2}}dz'.$$

This is a table integral,[8] giving finally:

$$B_N = \frac{\mu_0 In}{2} \frac{z-z'}{\left[R^2+(z-z')^2\right]^{1/2}}\Bigg|_{z'=+l/2}^{z'=-l/2} = \frac{\mu_0 In}{2}\left\{\frac{z+l/2}{\left[R^2+(z+l/2)^2\right]^{1/2}} - \frac{z-l/2}{\left[R^2+(z-l/2)^2\right]^{1/2}}\right\}.$$

The figure on the right shows typical field profiles given by this formula. For points well inside a long solenoid ($l >> |z|$, R), the expression in the figure brackets tends to 2, so our result approaches the simple Eq. (5.40): $B_N = \mu_0 In$. Note, however, that a close approach to this value requires rather large values of the l/R ratio, i.e. very long solenoids. (In practice, this handicap may be remedied by using either the toroidal solenoids sketched in Fig. 5.6 of the lecture notes, or the straight solenoid's filling with a high-μ magnetic material – see Sec. 5.6.)



On the other hand, well outside of the solenoid, i.e. for $|z| >> R$, l, our result yields

$$B_N \approx \frac{\mu_0 InR^2 l}{2|z|^3} \equiv \frac{\mu_0}{4\pi}\frac{2m_N}{|z|^3}, \qquad \text{where } m_N \equiv \pi R^2 IN = Nm_1,$$

i.e. the dipole field (5.99), with the dipole moment m_N, which is N times larger than that ($m_1 = AI$) of a single current loop – see Eq. (5.97).

<u>Problem 5.9</u>. A thin round disk of radius R, carrying an electric charge of a constant areal density σ, rotates about its axis with a constant angular velocity ω. Calculate:

(i) the magnetic field on the disk's axis,
(ii) the magnetic moment of the disk,

and relate these results.

Solutions:

(i) The rotating disk may be fairly represented as a set of narrow elementary rings of radius ρ and width $d\rho << \rho$, each carrying a ring current of the magnitude $dI = Jd\rho = \sigma\omega\rho d\rho$ and thus creating an elementary field dB described by Eq. (5.23) of the lecture notes (with the notation replacements $B \to dB$, $I \to dI$, $R \to \rho$):

$$dB = \frac{\mu_0 dI}{2}\frac{\rho^2}{\left(\rho^2+z^2\right)^{3/2}} = \frac{\mu_0\sigma\omega}{2}\frac{\rho^3 d\rho}{\left(\rho^2+z^2\right)^{3/2}},$$

where z is the distance of the field observation point from the ring's plane. Since the fields of all the elementary rings have the same direction (along the disk's axis), we may sum up their contributions to the total field as scalars:

[8] See, e.g. MA Eq. (6.2b).

$$B = \int_{\rho=0}^{\rho=R} dB = \frac{\mu_0 \sigma \omega}{2} \int_0^R \frac{\rho^3 d\rho}{\left(\rho^2 + z^2\right)^{3/2}} = \frac{\mu_0 \sigma \omega}{2} \frac{|z|}{2} \int_{\rho=0}^{\rho=R} \frac{\xi d\xi}{\left(\xi + 1\right)^{3/2}},$$

where $\xi \equiv \rho^2/z^2$. The last integral may be readily worked out as

$$\int \frac{\xi d\xi}{(\xi+1)^{3/2}} \equiv \int \frac{(\xi+1) d\xi}{(\xi+1)^{3/2}} - \int \frac{d\xi}{(\xi+1)^{3/2}} \equiv \int \frac{d\xi}{(\xi+1)^{1/2}} - \int \frac{d\xi}{(\xi+1)^{3/2}} = 2(\xi+1)^{1/2} + \frac{2}{(\xi+1)^{1/2}} \equiv \frac{2(\xi+2)}{(\xi+1)^{1/2}},$$

and we finally get

$$B = \frac{\mu_0 \sigma \omega}{2} \frac{|z|}{2} \left[\frac{2\left(\rho^2/z^2 + 2\right)}{\left(\rho^2/z^2 + 1\right)^{1/2}} \right]_{\rho=0}^{\rho=R} = \mu_0 \sigma \omega \left[\frac{R^2 + 2z^2}{2\left(R^2 + z^2\right)^{1/2}} - |z| \right]. \qquad (*)$$

(ii) Very similarly, the magnitude of the magnetic moment **m** of the system (evidently, also directed along the symmetry axis z) may be calculated by summing up the elementary ring contributions given by Eq. (5.97):

$$dm = AdI = \pi\rho^2 dI = \pi\sigma\omega\rho^3 d\rho,$$

and getting

$$m = \int_{\rho=0}^{\rho=R} dm = \pi\sigma\omega \int_0^R \rho^3 d\rho = \frac{\pi\sigma\omega R^4}{4}. \qquad (**)$$

In order to relate the above results, let us find the limit of Eq. (*) at large distances, $|z| >> R$:

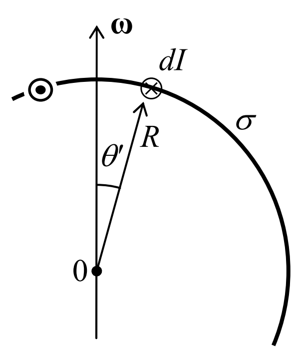
$$B = \mu_0 \sigma \omega |z| \left[\frac{1 + R^2/2z^2}{\left(1 + R^2/z^2\right)^{1/2}} - 1 \right] \rightarrow \mu_0 \sigma \omega |z| \left[\left(1 + \frac{R^2}{2z^2}\right)\left(1 - \frac{R^2}{2z^2} + \frac{3R^4}{8z^4}\right) - 1 \right] \rightarrow \mu_0 \sigma \omega \frac{R^4}{8|z|^3}. \ (***)$$

On the other hand, for the observation point on the magnetic dipole's axis ($\mathbf{r} = |z|\mathbf{m}/m$), the general expression for the magnetic dipole's field, following from Eq. (5.99) of the lecture notes, is

$$B = \frac{\mu_0}{4\pi} \frac{2m}{|z|^3}.$$

With the m given by Eq. (**), this expression coincides with that Eq. (***), i.e. our two results do match.

<u>Problem 5.10</u>. A thin spherical shell of radius R, with charge Q uniformly distributed over its surface, is rotated about its diameter with a constant angular velocity ω. Calculate the distribution of the magnetic field everywhere in space.



Solution: As viewed from the lab reference frame, the rotation creates ring-like surface currents with the linear density $J = \sigma v$, where $\sigma = Q/4\pi R^2$ is the (constant) surface charge density, $v = \omega R \sin\theta'$ is the linear velocity, and θ' is the polar angle of the elementary current's location, measured from the rotation axis – see the figure on the right, showing a part of the shell's cross-section. As was discussed in Sec. 5.4 of the lecture notes, at large distances $r >> R$, each elementary ring current

$$dI = JRd\theta' = \frac{Q}{4\pi R^2}\omega R\sin\theta' Rd\theta' \equiv \frac{Q\omega}{4\pi}\sin\theta' d\theta',$$

creates a magnetic field with the same dipole distribution (5.99), with the elementary dipole moment (5.97):

$$dm = A'dI = \pi\left(R\sin\theta'\right)^2\frac{Q\omega}{4\pi}\sin\theta' d\theta' \equiv \frac{1}{4}Q\omega R^2\sin^3\theta' d\theta'.$$

Due to the similarity of these distributions, the long-distance magnetic field of the whole shell is also similar, with the magnetic dipole moment m that may be calculated by summing dm of all elementary currents, i.e. integrating dm over the segment $0 \leq \theta' \leq \pi$ – see the figure above:[9]

$$m = \int dm = \frac{1}{4}Q\omega R^2\int_0^\pi \sin^3\theta' d\theta' = \frac{1}{4}Q\omega R^2\frac{4}{3} \equiv \frac{1}{3}Q\omega R^2.$$

After having solved so many problems on uniform spheres and thin spherical shells in this course, the reader may readily guess that the dipole field distribution (5.99),

$$\mathbf{H} = \frac{1}{4\pi}\frac{3\mathbf{r}(\mathbf{r}\cdot\mathbf{m}) - \mathbf{m}r^2}{r^5} \equiv \frac{m}{4\pi}\frac{2\mathbf{n}_r\cos\theta + \mathbf{n}_\theta\sin\theta}{r^3}, \qquad (*)$$

is valid not only at large distances, but for any $r > R$, and that the field inside the shell is uniform:

$$\mathbf{H} = \frac{\boldsymbol{\omega}}{\omega}H_0 = H_0\left(\mathbf{n}_r\cos\theta - \mathbf{n}_\theta\sin\theta\right), \qquad \text{for } r < R, \qquad (**)$$

where H_0 is some constant.

Indeed, as we already know, the fields (*) and (**) do satisfy the stationary Maxwell equations (5.36) in the current-free regions. Hence, due to the uniqueness of the solutions of linear boundary problems, it is sufficient to prove that, at a certain choice of the scalar constant H_0, these fields also satisfy the boundary conditions at the shell, i.e. at $r = R$. These conditions,

$$H_\theta\big|_{r=R+0} - H_\theta\big|_{r=R-0} = J, \qquad H_r\big|_{r=R+0} - H_r\big|_{r=R-0} = 0,$$

follow from, respectively, Eqs. (5.116) and (5.119) of the lecture notes. For our fields (*) and (**), with the dipole moment $m = Q\omega R^2/3$ and the linear current density $J = (Q/4\pi R^2)\omega R\sin\theta$ calculated above, the conditions take the following form:

$$\left(\frac{1}{4\pi}\frac{1}{3}Q\omega R^2\frac{\sin\theta}{R^3}\right) - \left(-H_0\sin\theta\right) = \frac{Q}{4\pi R^2}\omega R\sin\theta, \qquad \left(\frac{1}{4\pi}\frac{1}{3}Q\omega R^2\frac{2\cos\theta}{R^3}\right) - \left(H_0\cos\theta\right) = 0.$$

One can readily see that if

$$H_0 = \frac{Q\omega}{6\pi R},$$

both boundary conditions are satisfied for any θ, so we have indeed found the (unique) solution of the problem.

[9] The integral over $d\theta'$ may be readily worked out using the usual variable replacement $\xi \equiv \cos\theta'$, so that $\sin^3\theta' d\theta' = -\sin^2\theta' d(\cos\theta') = -(1 - \xi^2)d\xi$.

Problem 5.11. A sphere of radius R, made of an insulating material with a uniform electric charge density ρ, rotates about its diameter with a constant angular velocity ω. Calculate the magnetic field distribution inside the sphere and outside it.

Solution: We may represent the sphere as a set of thin spherical shells, each with radius $R' \leq R$, thickness dR', and charge $dQ = \rho dV = \rho(4\pi R'^2 dR')$. As was discussed in the model solution of the previous problem, such a shell creates outside it a purely dipole magnetic field, with the spatial distribution

$$d\mathbf{H} = \frac{1}{4\pi}\frac{3\mathbf{r}(\mathbf{r}\cdot d\mathbf{m}) - r^2 d\mathbf{m}}{r^5},$$

where $d\mathbf{m}$ is the shell's magnetic dipole moment directed along the axis of rotation, with the magnitude

$$dm = \frac{1}{3}dQ\omega R'^2 = \frac{4\pi}{3}\rho\omega R'^4 dR',$$

and inside it, a uniform field,

$$d\mathbf{H} = dH_0(\mathbf{n}_r \cos\theta - \mathbf{n}_\theta \sin\theta),$$

of magnitude

$$dH_0 = \frac{dQ\omega}{6\pi R'} = \frac{2}{3}\rho\omega R' dR'.$$

As the result of the addition of these elementary fields, the total field *outside* the sphere (for $r \geq R$) has the same dipole distribution, with the total dipole moment $\mathbf{m}$ whose magnitude m that may be calculated by the scalar addition of the elementary moments dm:

$$m = \int dm = \frac{4\pi}{3}\rho\omega\int_0^R R'^4 dR' = \frac{4\pi}{15}\rho\omega R^5. \qquad (*)$$

In order to calculate the field *inside* the rotating sphere, i.e. at distances $r < R$ from its center, we should sum up the dipole field induced by all thin shells with $R' < r$, giving the total dipole moment

$$m = \frac{4\pi}{15}\rho\omega r^5,$$

and the uniform magnetic fields created by all thin shells with $R' \geq r$, whose magnitude is the scalar sum of all shell contributions:

$$H_0 = \int dH_0 = \frac{2}{3}\rho\omega\int_r^R R' dR' = \frac{1}{3}\rho\omega\left(R^2 - r^2\right).$$

Using an alternative form of the dipole field representation,[10]

$$\mathbf{H} = \frac{1}{4\pi}m\frac{2\mathbf{n}_r \cos\theta + \mathbf{n}_\theta \sin\theta}{r^3}, \qquad (**)$$

these two contributions may be readily summed up, giving

[10] See, e.g., the first form of Eq. (3.13) of the lecture notes.

$$\mathbf{H} = \frac{1}{4\pi}\frac{4\pi}{15}\rho\omega r^5 \frac{2\mathbf{n}_r \cos\theta + \mathbf{n}_\theta \sin\theta}{r^3} + \frac{1}{3}\rho\omega\left(R^2 - r^2\right)\left(\mathbf{n}_r \cos\theta - \mathbf{n}_\theta \sin\theta\right)$$
$$= \frac{\rho\omega r}{15}\left[\mathbf{n}_r\left(5R^2 - 3r^2\right)\cos\theta + \mathbf{n}_\theta\left(6r^2 - 5R^2\right)\sin\theta\right], \quad \text{for } r \leq R.$$

As useful sanity checks: on the sphere's surface (i.e. at $r = R$), the result gives a purely dipole field:

$$\mathbf{H}\Big|_{r=R} = \frac{\rho\omega R^3}{15}\left(\mathbf{n}_r 2\cos\theta + \mathbf{n}_\theta \sin\theta\right),$$

which coincides with the field following from Eqs. (*) and (**) for $r = R$. On the other hand, in the sphere's center ($r = 0$), the field is naturally directed along the rotation axis:

$$\mathbf{H}\Big|_{r=0} = \frac{\rho\omega R^3}{3}\left(\mathbf{n}_r \cos\theta - \mathbf{n}_\theta \sin\theta\right) = \frac{\rho R^3}{3}\boldsymbol{\omega}.$$

Problem 5.12. A conducting sphere with no total electric charge is rotated about its diameter with a constant angular velocity ω, in a uniform constant external magnetic field $\mathbf{B}$ directed along the rotation axis. Assuming that the sphere's contribution to the magnetic field is negligibly small, calculate the stationary distribution of the electric charge density inside the sphere and on its surface, and the electrostatic potential both inside and outside the sphere. Quantify the above assumption.

Solution: By definition, a conductor is a medium with some charge carriers (e.g., electrons in metals) free to move under the effect of the Lorentz force:

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)$$

– see, e.g., Eq. (5.10) of the lecture notes. In a stationary state with no currents, this force has to vanish, so the charges have to self-distribute in a way to make their electric field $\mathbf{E}$ balance the magnetic field's effect:

$$\mathbf{E} = -\mathbf{v}\times\mathbf{B}.$$

where the charge's velocity $\mathbf{v}$ in the lab reference frame is due to the sphere's rotation as a whole:[11] $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$, so

$$\mathbf{E} = -\left(\boldsymbol{\omega}\times\mathbf{r}\right)\times\mathbf{B}.$$

Here $\mathbf{r}$ is the observation point's radius vector with the origin at any static point; for what follows, it is convenient for us to place this origin in the sphere's center.

In our simple case, the directions of the vectors $\boldsymbol{\omega}$ and $\mathbf{B}$ coincide, so by taking this direction for the z-axis, decomposing the radius vector as $\mathbf{r} = \mathbf{n}_z z + \mathbf{n}_\rho\rho$ (where $\mathbf{n}_\rho \perp \mathbf{n}_z$, so $\mathbf{n}_z \cdot \mathbf{n}_\rho = 0$), and using the *bac minus cab* rule,[12] we get

$$\mathbf{E} = -\left[\mathbf{n}_z\omega\times\left(\mathbf{n}_z z + \mathbf{n}_\rho\rho\right)\right]\times\mathbf{n}_z B = -\left(\mathbf{n}_z\times\mathbf{n}_\rho\right)\times\mathbf{n}_z\omega\rho B = -\mathbf{n}_\rho\omega B\rho.$$

[11] If there is any doubt about this expression for $\mathbf{v}$, please consult CM Eq. (4.9).
[12] See, e.g., MA Eq. (7.5).

Hence, the field **E** is normal to the rotation axis and proportional to the observation point's distance ρ from it. Now we may use the Maxwell equation (1.27) to calculate the corresponding distribution of the electric charge density ρ:[13]

$$\rho = \varepsilon_0 \nabla \cdot \mathbf{E} = -\varepsilon_0 \omega B \nabla \cdot (\mathbf{n}_\rho \rho) = -2\varepsilon_0 \omega B .$$

Hence the volumic density of the induced electric charge inside the sphere is constant.

The distribution of the electric potential inside the sphere may be readily calculated directly from **E**, by using Eq. (1.33):[14]

$$\nabla \phi = -\mathbf{E} = \mathbf{n}_\rho \omega B \rho .$$

Since our **E** is axially symmetric and does not have any z-component, we may take $\phi(\mathbf{r}) = \phi(\rho)$, so according to MA Eq. (10.2), $\nabla \phi = \mathbf{n}_\rho \, d\phi/d\rho$, and this equation becomes merely

$$\frac{d\phi}{d\rho} = \omega B \rho, \qquad \text{i.e. } d\phi = \omega B \rho d\rho ,$$

which is very easy to integrate:

$$\phi = \phi_0 + \frac{\omega B \rho^2}{2} \equiv \phi_0 + \frac{\omega B r^2}{2} \sin^2 \theta, \qquad \text{for } r \leq R . \qquad (*)$$

Here the integration constant ϕ_0 has the sense of the potential's value at the rotation axis (and in particular, in the sphere's center), and θ is the usual polar angle. In particular, on the sphere's surface ($r = R$) the potential is

$$\phi = \phi_0 + \frac{\omega B R^2}{2} \sin^2 \theta \equiv \phi_0 + \frac{\omega B R^2}{2}\left(1 - \cos^2 \theta\right), \qquad \text{for } r = R . \qquad (**)$$

At $r > R$, where $\rho = 0$, the potential satisfies the Laplace equation, which has to be solved with the axially symmetric boundary condition (**). Hence its solution is also axially symmetric and has to obey Eq. (2.172). After dropping all the terms diverging at $r \to \infty$ (which clearly should vanish in our case) and also the term proportional to b_0/r (which describes the Coulomb field $\mathbf{E} \propto 1/r^2$ of the net charge of the sphere, in our case equal to zero), and taking the potential's value a_0 at infinity for zero (just for simplicity), it reads

$$\phi(r, \theta) = \sum_{l=1}^{\infty} \frac{b_l}{r^{l+1}} \mathcal{P}_l(\cos\theta), \qquad \text{for } r \geq R, \qquad (***)$$

where $\mathcal{P}_l(\xi)$ are the Legendre polynomials. In order to find the remaining coefficients b_l, it is instrumental to express Eq. (**) as a linear combination of these polynomials as well. This is easy: from the third line of Eq. (2.170), we have

$$1 - \xi^2 = \frac{2}{3}\left[1 - \mathcal{P}_2(\xi)\right],$$

so by using the first line of Eq. (2.170) as well, Eq. (**) may be rewritten as

[13] The last step uses MA Eq. (10.4) with $f_\rho = \rho$, and $f_\varphi = f_z = 0$.

[14] Note that in this case, Eq. (1.46) of the lecture notes, also obtained for a constant charge density inside a sphere, is inapplicable because its derivation used the additional assumption of the spherical symmetry of the function $\varphi(\mathbf{r})$, which is clearly invalid in this case since the field **E** does not obey this symmetry.

$$\phi = \phi_0 + \frac{\omega B R^2}{3}\left[1 - \mathcal{P}_2(\cos\theta)\right] \equiv \left(\phi_0 + \frac{\omega B R^2}{3}\right)\mathcal{P}_0(\cos\theta) - \frac{\omega B R^2}{3}\mathcal{P}_2(\cos\theta).$$

Now requiring Eq. (***) to match this expression at $r = R$, we get

$$\phi_0 + \frac{\omega B R^2}{3} = 0, \qquad \frac{b_1}{R^2} = 0, \qquad \frac{b_2}{R^3} = -\frac{\omega B R^2}{3}, \qquad \text{and } b_{l>2} = 0,$$

so the external field of the sphere is a pure electric-quadrupole one:

$$\phi = -\frac{\omega B R^5}{3r^3}\mathcal{P}_2(\cos\theta) \equiv -\frac{\omega B R^5}{6r^3}\left(3\cos^2\theta - 1\right), \qquad \text{for } r \geq R. \qquad (****)$$

What remains is to calculate the density σ of the surface charge of the sphere. For that, we may use the evident generalization of Eq. (2.3) for the case when the electric field has nonvanishing normal components on both sides of the surface:

$$\sigma = -\varepsilon_0\left(\frac{\partial\phi}{\partial r}\Big|_{r=R+0} - \frac{\partial\phi}{\partial r}\Big|_{r=R-0}\right).$$

With our Eqs. (*) and (****), this formula yields

$$\sigma = -\varepsilon_0 \omega B R\left\{\frac{1}{2}\left(3\cos^2\theta - 1\right) - \left(1 - \cos^2\varphi\right)\right\} \equiv \frac{\varepsilon_0 \omega B R}{2}\left(3 - 5\cos^2\theta\right).$$

As a sanity check, the surface integral of this density,

$$Q_\text{s} \equiv \oint_{r=R} \sigma d^2 r = 2\pi R^2 \int_0^\pi \sigma \sin\theta d\theta = 2\pi R^2 \frac{\varepsilon_0 \omega B R}{2} 2\int_0^1 \left(3 - 5\xi^2\right) d\xi = \frac{8\pi}{3}\varepsilon_0 \omega B R^3,$$

is equal and opposite to the volume integral of the bulk density of charge,

$$Q_\text{b} \equiv \int_{r<R} \rho d^3 r = -2\varepsilon_0 \omega B \int_{r<R} \rho d^3 r = -\frac{8\pi}{3}\varepsilon_0 \omega B R^3,$$

as it has to be for the sphere with zero total charge.

Finally, from the solution of the two previous problems, we know that the rotation of bulk and surface charges Q_s and Q_b of a sphere induces their own (generally, differently distributed) magnetic field $H_\text{ind} \sim Q\omega/R$. In our current case, this field's magnitude is of the order of $\varepsilon_0\omega^2 B R^2$, so

$$\frac{B_\text{ind}}{B} \sim \mu_0 \frac{H_\text{ind}}{B} \sim \varepsilon_0 \mu_0 (\omega R)^2 = \frac{v_\text{max}^2}{c^2},$$

where $v_\text{max} \equiv \omega R$ is the largest linear velocity of the sphere's points. Hence the used assumption $B_\text{ind} \ll B$ is valid for any non-relativistic rotation.

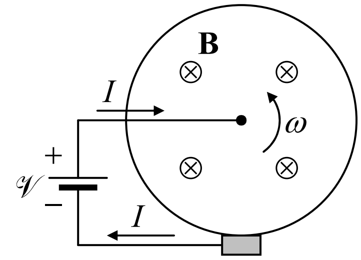
<u>Problem 5.13</u>.* The simplest version of the famous *homopolar* (or "unipolar") *motor*[15] is a thin round conducting disk, placed into a uniform magnetic field normal to its plane, with dc current passed between the disk's center and a sliding electrode ("brush") on its rim – see the figure below.

[15] This device was invented by Michael Faraday in 1821 (i.e. well before his much-celebrated work on electromagnetic induction) and is frequently called the *Faraday disk*.

(i) Express the torque rotating the disk via its radius R, the magnetic field $\mathbf{B}$, and the current I.

(ii) If the disk is allowed to rotate about its axis, and the motor is driven by a battery with e.m.f. $\mathscr{V}$, calculate its stationary angular velocity ω, neglecting friction and the electric circuit's resistance.

(iii) Now assuming that the current' path (battery + wires + contacts + disk itself) has a non-zero resistance $\mathscr{R}$, derive and solve the equation for the time evolution of ω, and analyze the solution.



Solutions:

(i) Applying Eq. (5.8) of the lecture notes to an elementary area of the disk, we get the elementary Lorenz force

$$d\mathbf{F} = \mathbf{J}(\boldsymbol{\rho}) \times \mathbf{B} d^2\rho \, ,$$

where $\mathbf{J}$ is the *linear* density (measured, e.g., in A/m) of the current in the disk, and $\boldsymbol{\rho}$ is the 2D radius vector in the disk's plane. Relative to the disk's center (i.e. the rotation axis), this elementary force creates the following elementary rotating torque:[16]

$$d\boldsymbol{\tau} = \boldsymbol{\rho} \times d\mathbf{F} = \boldsymbol{\rho} \times [\mathbf{J}(\boldsymbol{\rho}) \times \mathbf{B}] d^2\rho \, .$$

Since the vector $d\mathbf{F}$ is perpendicular to the vector $\mathbf{B}$, i.e. lies in the disk's plane, and so is vector $\boldsymbol{\rho}$, all elementary torque vectors $d\boldsymbol{\tau}$ are directed along the disk's symmetry axis, and hence ad up as scalars. Moreover, the radial component of $d\mathbf{F}$, proportional to the angular component $J_\varphi \mathbf{n}_\varphi$ of the current, does not contribute to the net torque $\boldsymbol{\tau}$, so its magnitude is[17]

$$\tau = \int_S \rho J_\rho(\boldsymbol{\rho}) B d^2\rho \, , \qquad (*)$$

where J_ρ is the radial component of the current density, and S is the disk's area.

Superficially, it may look like that in order to complete the calculation of the torque, we need to know in detail the distribution of the current density $\mathbf{J}$ (or at least of its radial component J_ρ) over the disk's area – for example, the one calculated in the solution of Problem 4.9 in the approximation of point injection and pickup of the current. However, this is not so. Indeed, spelling out Eq. (*) in the form

$$\tau = B \int_0^R \rho d\rho \int_0^{2\pi} \rho d\varphi \; J_\rho(\boldsymbol{\rho}),$$

we may notice that since $\rho d\varphi$ is an elementary arc of a circle of the radius ρ, the internal integral is just the total current I flowing out from the disk's center. Due to the charge conservation (see the discussion in Sec. 4.1 of the lecture notes), this current cannot depend on ρ, and we may complete the integration as follows:

$$\tau = BI \int_0^R \rho d\rho = BI \frac{R^2}{2} \cdot \qquad (**)$$

[16] See, e.g., CM Eq. (1.34).

[17] Formally, this result may be obtained by using the well-known "bac minus cab rule" – see MA Eq. (7.5):

$$d\boldsymbol{\tau} / d^2\rho = \boldsymbol{\rho} \times (\mathbf{J} \times \mathbf{B}) = \mathbf{J}(\boldsymbol{\rho} \cdot \mathbf{B}) - \mathbf{B}(\boldsymbol{\rho} \cdot \mathbf{J}) = 0 - \mathbf{B} J_\rho \, .$$

The background reason for this surprising simplicity will be discussed below.

(ii) The simplest (though somewhat formal) way to fulfill this task is to apply the energy-work principle to an elementary time interval dt:

$$d\mathscr{W} \equiv \mathscr{V}dQ \equiv \mathscr{V}Idt = dT_{\text{rotation}} \equiv d\left(\frac{\mathscr{I}\omega^2}{2}\right),$$

where $\mathscr{I}$ is the disk's moment of inertia for rotation about its axis.[18] Dividing both sides of this equation by dt, and differentiating its right-hand side, we get

$$\mathscr{V}I = \mathscr{I}\omega\dot{\omega}\,.$$

But according to the basic equation of rotation dynamics,[19] the product $\mathscr{I}\dot{\omega}$ (i.e. the time derivative of the disk's angular momentum) should be equal to the applied torque τ, so using Eq. (**), we get

$$\mathscr{V}I = \omega\tau \equiv \omega BI\frac{R^2}{2}, \qquad \text{giving } \omega = \omega_0 \equiv \frac{2\mathscr{V}}{BR^2}. \qquad (***)$$

At first glance, this result is counter-intuitive. Since we have neglected the circuit's resistance, why could not the non-zero e.m.f. lead to an infinite growth of the current I, and hence of the torque, and hence an infinite speed-up of the disk's rotation? Some light on the underlying physics may be shed by the following simple model. Let the disk's conductivity be due to certain classical particles with charge q each. Then the disk's rotation with angular velocity ω results in the particles' motion (in the lab reference frame, i.e. relative to the magnetic field's source) with the velocity $v = \rho\omega$, and hence to the radially-directed Lorentz force (5.10) with the magnitude $F = qvB = q\omega\rho B$. The ratio $F/q = \omega\rho B$ may be viewed as the magnitude of some additional, rotation-induced electric field $\mathbf{E}_{\text{ind}}$; it is easy to check that for any sign of q, this field is directed against the current I. As a result, the total additional e.m.f.,

$$\mathscr{V}_{\text{ind}} = -\int_0^R \mathbf{E}_{\text{ind}} \cdot d\boldsymbol{\rho} = \int_0^R \omega\rho B d\rho = \omega B\int_0^R \rho d\rho = \omega B\frac{R^2}{2}, \qquad (****)$$

induced by the rotation between the disk's axis and the brush contact, has the polarity opposite to the battery's e.m.f. $\mathscr{V}$. So, the physics of Eq. (***) is that at $\omega = \omega_0$, $\mathscr{V}_{\text{ind}}$ exactly compensates $\mathscr{V}$.[20]

(iii) The last discussion allows for a ready generalization of the result to the case of nonvanishing resistance $\mathscr{R}$ of the current-supplying circuit. Indeed, by writing the 2nd Kirchhoff law (4.7b) with the account of the rotation-induced e.m.f.,

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[18] See, e.g., CM Secs. 4.2-4.3.

[19] See, e.g., CM Eqs. (4.38).

[20] This result is actually more general than the explored simple model and is one of the manifestations of Faraday's law of electromagnetic induction – to be discussed in Sec. 6.1 of the lecture notes. Indeed, it is easy to check that the product $\omega R^2/2$, which participates in Eq. (****), is just dA/dt, where A is the area swept by the circle's radius, so Eq. (****) may be rewritten as $\mathscr{V}_{\text{ind}} = d(BA)/dt$. Since, for a uniform magnetic field $\mathbf{B}$ normal to the disk's plane and directed against axis z, BA is just minus magnetic flux Φ through area A, Eq. (***) agrees with the induction law in its best-known form – see Eq. (6.1). (Note, however, that the exact specification of what exactly A is in this case, and hence the derivation of Eq. (****) from the Faraday law, require some care.) This discussion shows that the Faraday law is not something entirely new, and is at least consistent with the laws of magnetostatics. (For more discussion of this fact, see Sec. 6.2 of the lecture notes.)

$$\mathscr{V} - \mathscr{V}_{\text{ind}} = I\mathscr{R},$$

using the I expressed from Eq. (***) with $\tau = \mathscr{I}\dot{\omega}$, and $\mathscr{V}_{\text{ind}}$ from Eq. (****), we get

$$\mathscr{V} - \frac{\omega B R^2}{2} = \mathscr{I}\dot{\omega}\frac{2}{BR^2}\mathscr{R}.$$

This linear, first-order differential equation for ω may be rewritten in the standard form

$$\left(1 + t_0 \frac{d}{dt}\right)\omega = \omega_0,$$

where t_0 is the following time constant:

$$t_0 \equiv \frac{4\mathscr{I}\mathscr{R}}{B^2 R^4},$$

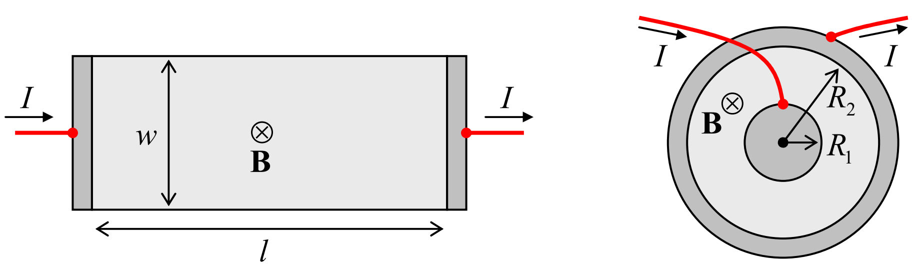
and ω_0 is the stationary value of ω, given by Eq. (***). This equation may be readily solved, giving

$$\omega(t) = \omega(0)e^{-t/t_0} + \omega_0\left(1 - e^{-t/t_0}\right).$$

This formula describes an exponential transient process, with the time constant t_0, from the initial rotation velocity $\omega(0)$ to its stationary value (***). This time constant is proportional to the circuit's resistance $\mathscr{R}$, making it clear why in Task (ii) of the problem (where $\mathscr{R}$ and hence t_0 were neglected) we received value (**) as a constant, without any apparent transient process.

Finally note that the same device, if driven by an external torque and with the battery $\mathscr{V}$ replaced with an electric load, may serve as a *homopolar generator* of dc e.m.f. (***).[21] Since it has only one current loop, for most practical purposes it is less efficient than the usual multi-coil generators but has certain technical advantages for producing short, very large pulses of current (in some cases, of several mega-amperes!) by using the gradually accumulated kinetic energy of massive flywheels, for such special purposes as driving railguns – see, e.g., Problem 6.7 below.

Problem 5.14. The reader is hopefully familiar with the classical Hall effect in the usual rectangular *Hall bar* geometry – see the left panel of the figure below. However, the effect takes a different form in the so-called *Corbino disk* – see the right panel of the figure below. (Dark shading shows the electrodes, with no appreciable resistance.) Analyze the effect in both geometries, assuming that in both cases, the conductors are thin and planar, have a constant Ohmic conductivity σ and a charge carrier density n, and that the applied magnetic field **B** is uniform and normal to the conductors' planes.



[21] It was also invented by M. Faraday and is sometimes called the "Faraday wheel".

Solution: In the Hall bar geometry, the current I is uniformly distributed across the sample, with the density $j = I/wt$, where t is the bar's thickness. As was discussed in Sec. 4.2, in an Ohmic conductor, this density equals $qn\langle v\rangle$, where $\langle v\rangle$ is the average "drift" velocity of the charge carriers (on top of their random thermal and/or quantum motion), so $\langle v\rangle = I/wtqn$. This velocity gives rise to an average magnetic component of the Lorentz force (5.10), with magnitude $\langle F\rangle = q\langle v\rangle B$, and directed normally to the current and the magnetic field, i.e. across the Hall bar. As the current is turned on, this force causes accumulation of the charges at the lateral sides of the bar, which stops when their electric field $\mathbf{E}$ becomes large enough to compensate for the magnetic force: $\mathbf{E} = -\langle \mathbf{v}\rangle\times\mathbf{B}$ at each point of the bar. So, at a stationary motion of the charge carriers along the current's direction, the electric field has to be uniform, hence providing a transverse voltage of magnitude $V = R_\mathrm{H}I$, where R_H is the so-called *Hall resistance*

$$R_\mathrm{H} \equiv \frac{V}{I} = \frac{wE}{I} = \frac{w\langle v\rangle B}{I} = \frac{B}{qnt} \ .$$

Note the following important features of this well-known result: first, R_H does not depend on w and l, i.e. on the Hall bar's size, and, second, in contrast to the usual Ohmic resistance (see Eqs. (4.9) and (4.13) of the lecture notes) it *is* sensitive to the sign of q. These features make the measurement of R_H the standard technique for finding out the sgn(q) and the charge carrier density n in semiconductors.

Now addressing the Corbino disk: due to the axial symmetry of its geometry, the radial component j_ρ of the current density has to be uniformly distributed over the angle, so at distance ρ from the center,

$$j_\rho = \frac{I}{2\pi t\rho}, \qquad \text{and } \langle v_\rho\rangle = \frac{j_\rho}{qn} = \frac{I}{2\pi tqn\rho},$$

causing an azimuthally-directed average magnetic force of the following magnitude:

$$\langle F_\varphi\rangle = q\langle v_\rho\rangle B = \frac{IB}{2\pi tn\rho}.$$

This field is equivalent, in its action on the carriers, to an azimuthal electric field

$$\langle E_\varphi\rangle = \frac{\langle F_\varphi\rangle}{q} = \frac{IB}{2\pi tqn\rho}.$$

However, in this axially-symmetric geometry, the resulting azimuthal motion of the charge carriers does not lead to their accumulation, and hence continues even in the stationary state, giving an azimuthal (circular) current with the following density:

$$j_\varphi = \sigma\langle E_\varphi\rangle = \frac{\sigma IB}{2\pi tqn\rho}.$$

Calculating the full circular current,

$$I_\varphi = t\int_{R_1}^{R_2} j_\varphi d\rho = I\,\frac{\sigma B}{2\pi qn}\ln\frac{R_2}{R_1},$$

we see that the I_φ/I ratio is independent of the disk's thickness t, and, according to Eq. (4.13), also of the charge carrier density:

$$\frac{I_\varphi}{I} = \frac{\omega_\mathrm{c}\tau}{2\pi}\ln\frac{R_2}{R_1},$$

where $\tau = \sigma m/q^2 n$ is the effective scattering time participating in the Drude formula (4.13), and $\omega_\mathrm{c} \equiv qB/m$ is the *cyclotron frequency* of the same charge carriers if they move ballistically in the same magnetic field. (For more about ω_c, see Sec. 9.6 of the lecture notes.)

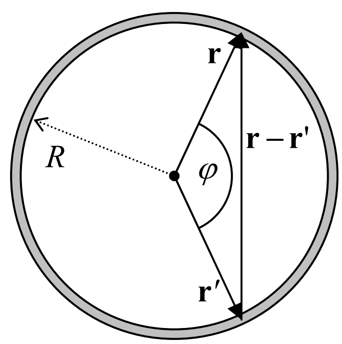
For typical conductors in practicable magnetic fields, $I_\varphi/I << 1$, and the magnetic field produced by the circular currents (which was ignored in the above solution) is much smaller than B.

Problem 5.15. A wire with a round cross-section of radius a has been bent into a round loop of radius $R >> a$. Prove the formula for its self-inductance, which was mentioned at the end of Sec. 5.3 of the lecture notes:

$$L = \mu_0 R \ln\frac{cR}{a}, \quad \text{with } c \sim 1.$$

Solution: The general expression (5.54) for the magnetic energy U, and Eq. (5.72) for the self-inductance of a loop, $L = U/(I^2/2)$, allow us to separate L into a short-distance contribution L_s from the region with $|\,\mathbf{r} - \mathbf{r}'| \sim a$, and a long-distance contribution L_l from the region outside of that range. For the latter, dominating contribution, we may use the Neumann formula (5.60). For the self-inductance of a loop, it reduces to

$$L_\mathrm{l} = \frac{\mu_0}{4\pi}\oint_l\oint_l \frac{d\mathbf{r}\cdot d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}, \qquad \text{for } |\mathbf{r}-\mathbf{r}'| >> a, \qquad (*)$$



where both integrals (in the space of the vectors $\mathbf{r}$ and $\mathbf{r}'$) are over the same loop's length. For our geometry (see the figure on the right), $|\mathbf{r} - \mathbf{r}'| = 2R\sin(\varphi/2)$ and $d\mathbf{r}\cdot d\mathbf{r}' = Rd\varphi dr'\cos\varphi$, where φ is the angle between the radius vectors $\mathbf{r}$ and $\mathbf{r}'$, with the origin at the loop's center. Due to the axial symmetry of this geometry, the integration over one of the variables (say, $\mathbf{r}'$) may be performed while keeping that angle φ fixed and gives

$$L_\mathrm{l} = \frac{\mu_0 R}{4}\int_{-\pi}^{+\pi}\frac{\cos\varphi\, d\varphi}{\sin(\varphi/2)}, \qquad \text{with } |\varphi| >> \frac{a}{R}.$$

Due to the last condition, we have to restrict the last integration to the angles outside a certain interval near the point $\varphi = 0$ – say, a symmetric interval $-\varepsilon/2 \le \varphi \le +\varepsilon/2$, where ε must be within the limits $a/R << \varepsilon << 1$. In this case, the integral is symmetric in φ, and may be readily calculated

$$L_\mathrm{l} \equiv \frac{\mu_0 R}{4}2\int_{\varepsilon/2}^{\pi}\frac{\cos\varphi d\varphi}{\sin(\varphi/2)} = \mu_0 R\int_{\varphi=\varepsilon/2}^{\varphi=\pi}\frac{\left[1-2\cos^2(\varphi/2)\right]\sin(\varphi/2)d(\varphi/2)}{1-\cos^2(\varphi/2)} = \mu_0 R\int_{\xi=0}^{\xi=\cos(\varepsilon/4)}\frac{\left(2\xi^2-1\right)d\xi}{1-\xi^2}$$

$$\equiv \mu_0 R\int_{\xi=0}^{\xi=\cos(\varepsilon/4)}\left[\frac{1}{2(1+\xi)}+\frac{1}{2(1-\xi)}-2\right]d\xi = \left[\frac{1}{2}\ln\frac{1+\xi}{1-\xi}-2\xi\right]_{\xi=0}^{\xi=\cos(\varepsilon/4)}$$

$$\equiv \mu_0 R\left[\frac{1}{2}\ln\frac{1+\cos(\varepsilon/4)}{1-\cos(\varepsilon/4)}-2\cos\frac{\varepsilon}{4}\right] \equiv \mu_0 R\left[\ln\left(\tan\frac{8}{\varepsilon}\right)-2\cos\frac{\varepsilon}{4}\right].$$

If we take $\varepsilon = Ca/R$, where C is a constant: $1 << C << R/a$, the above result reduces to

$$L_1 \approx \mu_0 R\left(\ln\frac{8R}{Ca} - 2\right).$$

Since, at $R >> a$ the magnetic field inside the wire and its close vicinity cannot be affected by its bending, the short-range contribution L_s to the full inductance of the loop may be fairly estimated using Eq. (5.80) of the lecture notes, by replacing b with a certain $C'a$, where $1 << C' << R/a$, i.e. $C' \sim C$, and integrating the result along the loop:

$$L_s \approx \mu_0 R\left(\ln C' + \frac{1}{4}\right).$$

We may see that this contribution is much smaller than L_l, so the approximate character of its evaluation does not affect the (very reasonable) accuracy of the final result:

$$L \equiv L_1 + L_s \approx \mu_0 R\left(\ln\frac{8C'R}{Ca} - 2 + \frac{1}{4}\right). \qquad (**)$$

(Here I have kept the term ¼ separate, because, as was already mentioned in Sec. 5.3 of the lecture notes, it is due to the magnetic field inside the wire and hence disappears in the important case when a high frequency of the field prevents its penetration into the wire's bulk.) This expression is equivalent to the formula given in the assignment, with

$$c = \frac{8C'}{C}\exp\left\{-2 + \frac{1}{4}\right\} \sim 1,$$

so our task has been accomplished. (A more careful evaluation of L_s shows[22] that $C' = C$, so in Eq. (**), these factors cancel, giving $c = 8\exp\{-7/4\} \approx 1.39$ for stationary or low-frequency fields, and $c = 8\exp\{-2\} \approx 1.08$ in the high-frequency limit.)

Please note an important byproduct of our solution: since at $a \to 0$, the short-range contribution does not depend on the long-range loop's geometry, a reasonably good approximation of the self-inductance of a thin-wire loop of an *arbitrary shape* is formally given by its long-distance contribution (*) alone, but with a more mild cut-off: $|\mathbf{r} - \mathbf{r}'| > a$.

Problem 5.16. Prove that:

(i) the self-inductance L of a current loop cannot be negative, and

(ii) each mutual inductance coefficient $L_{kk'}$, defined by Eq. (5.60) of the lecture notes, cannot be larger than $(L_{kk}L_{k'k'})^{1/2}$.

Solutions:

(i) Since, according to Eq. (5.57), the magnetic energy U of any system cannot be negative, Eq. (5.72) for a single current loop,

$$U = \frac{L}{2}I^2,$$

cannot give negative results for any current I, which is only possible if $L \geq 0$.

22 See, for example, W. R. Smythe, *Static and Dynamic Electricity*, 3rd ed., McGraw-Hill, 1968.

(ii) Let us apply a similar argument to the more general Eq. (5.59). It also cannot give a negative result for any combination of currents, in particular when only two of them, I_k and $I_{k'}$ with $k' \neq k$,[23] are different from 0, and hence this relation (taking into account Eq. (5.61), $L_{kk'} = L_{k'k}$) is reduced to a form similar to Eq. (5.62):

$$U = \frac{L_{kk}}{2} I_k^2 + L_{kk'} I_k I_{k'} + \frac{L_{k'k'}}{2} I_{k'}^2 . \qquad (*)$$

This means, in particular, that if we fix one of the currents (say, $I_{k'}$), U as a function of the other current (I_k) cannot be negative even at its minimum, where

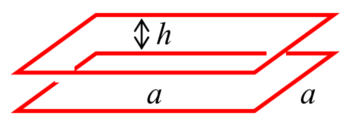
$$\frac{\partial U}{\partial I_k} \equiv L_{kk} I_k + L_{kk'} I_{k'} = 0 .$$

Plugging the resulting value, $I_k = -L_{kk'} I_{k'} / L_{kk}$, back into Eq. (*), we get

$$U_{\min} = \frac{L_{kk}}{2} \left(\frac{L_{kk'} I_{k'}}{L_{kk}} \right)^2 - L_{kk'} \frac{L_{kk'} I_{k'}}{L_{kk}} I_{k'} + \frac{L_{k'k'}}{2} I_{k'}^2 \equiv \left(1 - \frac{L_{kk'}^2}{L_{kk} L_{k'k'}} \right) \frac{L_{k'k'}}{2} I_{k'}^2 .$$

Now the requirement $U_{\min} \geq 0$ immediately gives the required proof of the inequality $L_{kk'} \leq (L_{kk} L_{k'k'})^{1/2}$.

Problem 5.17. Calculate the mutual inductance of two similar thin-wire square-shaped loops offset by distance h in the direction normal to their planes – see the figure on the right.



Solution: This problem may be most simply solved using the Neumann formula – see Eq. (5.60) of the lecture notes:

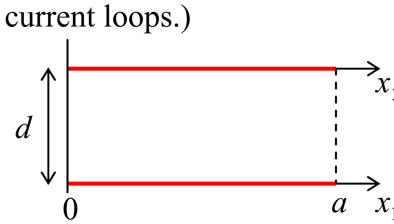
$$M \equiv L_{12} = L_{21} = \frac{\mu_0}{4\pi} \oint_{l_1} \oint_{l_2} \frac{d\mathbf{r}_1 \cdot d\mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} , \qquad (*)$$

where the indices 1 and 2 number the wire loops (in any order), and both contour integrals should follow the same (but arbitrary) direction. Due to the evident symmetry of the system, this integral falls into a sum of four equal components M_s, each corresponding to the integration along only one side of one of the squares. Moreover, since the scalar product $d\mathbf{r}_1 \cdot d\mathbf{r}_2$ vanishes for the sides normal to each other, and equals $\pm dr_1 dr_2$ for the parallel sides, M_s may be represented as

$$M_s = \frac{\mu_0}{4\pi} \left[f(a,h) - f\left(a, \sqrt{h^2 + a^2}\right) \right],$$

where $f(a, d)$ is the same double integral as in Eq. (*), but limited to just two parallel sides separated by distance d. (One may loosely think about $(\mu_0/4\pi) f(a, d)$ as the mutual inductance of these two wire segments, even though this notion is strictly defined only for two closed current loops.)

In order to calculate the function $f(a, d)$, it is convenient to use the Cartesian coordinates shown in the figure on the right; in them,



[23] If $k = k'$, the correctness of (ii) is obvious because $L_{kk'} \equiv L_{kk}$ cannot be larger than $(L_{kk} L_{k'k'})^{1/2} \equiv L_{kk}$, i.e. itself.

$$f(a,d) = \int_0^a dx_2 \int_0^a dx_1 \frac{1}{\left[(x_1 - x_2)^2 + d^2\right]^{1/2}}.$$

The inner integral is easy: by taking $(x_1 - x_2)/d \equiv \xi$, and then $\xi \equiv \sinh\alpha$, so $d\xi/d\alpha = \cosh\alpha$ and $(\xi^2 + 1)^{1/2} = \cosh\alpha$ cancel, we get

$$f(a,d) = \int_0^a dx_2 \int_{-x_2/d}^{(a-x_2)/d} \frac{d\xi}{\left(\xi^2+1\right)^{1/2}} = \int_0^a dx_2 \int_{\xi=-x_2/d}^{\xi=(a-x_2)/d} d\alpha = \int_0^a dx_2 \left[\sinh^{-1}\xi\right]_{-x_2/d}^{(a-x_2)/d}$$

$$\equiv \int_0^a \left[\sinh^{-1}\frac{a-x_2}{d} + \sinh^{-1}\frac{x_2}{d}\right] dx_2 .$$

This expression falls into a sum of two similar integrals, which may be readily worked out by parts:

$$\int \sinh^{-1}\xi d\xi = \xi \sinh^{-1}\xi - \int \xi d\left(\sinh^{-1}\xi\right) = \xi\sinh^{-1}\xi - \int \frac{\xi d\xi}{\left(\xi^2+1\right)^{1/2}} = \xi\sinh^{-1}\xi - \left(\xi^2+1\right)^{1/2}.$$

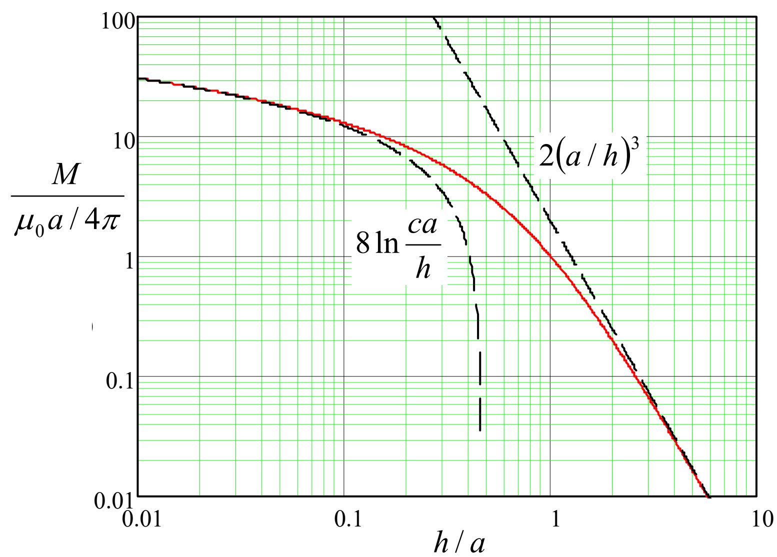
The result is

$$f(a,d) = 2\left[a\sinh^{-1}\frac{a}{d} + d - \left(d^2 + a^2\right)^{1/2}\right].$$

The red line in the figure on the right shows the final result for the mutual inductance

$$M = 4M_\text{s} = \frac{\mu_0}{\pi}\left[f(a,h) - f\left(a, \sqrt{h^2+a^2}\right)\right],$$

as a function of the h/a ratio. As the distance h between the loops (with a fixed size a) is increased, the mutual inductance coefficient decreases. At $h/a << 1$, this trend is only logarithmic because it is governed by the field very close to the wires:



$$M \to \frac{2\mu_0 a}{\pi}\left(\ln\frac{2a}{h} - 2 + \sqrt{2} - \sinh^{-1}1\right) \equiv \frac{2\mu_0 a}{\pi}\ln\frac{ca}{h}, \quad \text{with } c \approx 0.461, \quad \text{at } h/a \to 0$$

– see the lower dashed line in the figure above. (The weak divergence at $h \to 0$ disappears for wires of any non-zero thickness.)

On the other hand, at large distances between the loops, the mutual inductance decreases fast:

$$M \to \frac{\mu_0 a^4}{2\pi h^3}, \quad \text{at } h/a \to \infty \qquad (**)$$

– see the upper dashed line. This behavior may be readily understood: at a large distance $h >> a$ from a loop carrying current I, its field approaches that of a magnetic dipole $\mathbf{m} = I\mathbf{A}$, in our current case with area $A = a^2$ – see Eq. (5.97) of the lecture notes. According to Eq. (5.99), on its symmetry axis, the dipole field is directed along the axis and has the magnitude

$$B = \frac{\mu_0}{4\pi}\frac{2m}{h^3} = \frac{\mu_0}{4\pi}\frac{2Ia^2}{h^3}.$$

Piercing the counterpart loop, of the same area $A = a^2$ and with the plane normal to its lines, the field creates the magnetic flux

$$\Phi = Ba^2 = \frac{\mu_0}{4\pi}\frac{2Ia^4}{h^3},$$

so the ratio Φ/I, i.e. the mutual inductance of the loops (see, e.g., Eq. (5.67) and Fig. 5.7) is indeed given by Eq. (**).

Note that this approach (the decomposition of M into side-pair components) may be used for the expression of the mutual inductance of other systems of rectangular loops, including those offset sidewise within their common plane, via the same function $f(a, d)$.

Problem 5.18.* Estimate the values of magnetic susceptibility due to

(i) orbital diamagnetism, and

(ii) spin paramagnetism,

for a medium with negligible interactions between the induced molecular dipoles. Compare the results.

Hints: For Task (i), you may use the classical model described by Eq. (5.114) of the lecture notes – see Fig. 5.13. For Task (ii), assume the ordering of spontaneous magnetic dipoles $\mathbf{m}_0$, with a fixed magnitude m_0 of the order of the Bohr magneton μ_B, similar to the one sketched for electric dipoles in Fig. 3.7a.

Solutions:

(i) According to Eq. (5.114), the angular velocity of the torque-induced precession of the vector $\mathbf{L}$ (in this case called the *Larmor frequency*),

$$\Omega_L = \frac{2\pi}{\mathcal{T}_L} = \frac{2\pi}{2\pi L \sin\theta/(dL/dt)} = \frac{qB}{2m_e} \equiv \frac{\mu_B}{\hbar}B,$$

does not depend on the angle θ between these two vectors.[24] This motion creates the average current

$$I = qf_L = q\frac{\Omega_L}{2\pi} = \frac{q^2}{4\pi m_e}B = \frac{e^2}{4\pi m_e}B$$

(note its independence of the sign of q) and hence the induced magnetic moment may be estimated as[25]

$$\Delta m = I\pi\langle\rho^2\rangle = \frac{e^2 B}{4m_e}\langle\rho^2\rangle = \frac{e^2 B}{4m_e}\left(\langle x^2\rangle + \langle y^2\rangle\right),$$

[24] As a useful intermediate sanity check, $\hbar\Omega_L = \mu_B B$ is exactly the result given by quantum mechanics for the atomic energy level splitting by a magnetic field, due to the electron's orbital motion – see, e.g., QM Sec. 6.4.

[25] In this model, the basic dipole moments $\mathbf{m}$ are assumed to be disordered, so their net contribution to the magnetization vanishes.

where ρ is the distance of the electron from the precession axis, taken for axis z. Now let us add such contributions from Z electrons of an atom, assuming that the initial orientations of their rotation planes are random, and hence the averages of all Cartesian coordinate squares are equal:

$$\langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle = \frac{\langle r^2\rangle}{3}, \quad \text{so that } \langle x^2\rangle + \langle y^2\rangle = \frac{2}{3}\langle r^2\rangle.$$

With this assumption, for the orbital molecular susceptibility we get the following *Langevin formula*:

$$|\chi_\mathrm{m}| \equiv n\frac{\Delta m}{H} \approx n\frac{\Delta m}{B/\mu_0} = \frac{\mu_0 nZe^2}{6m_\mathrm{e}}\langle r^2\rangle,$$

where n is the number of atoms per unit volume.

Amazingly enough, this is *exactly* the result given by quantum mechanics,[26] with the only difference that in it, $\langle r^2\rangle$ should be understood as the simultaneously statistical *and* quantum-mechanical average, taking into account the spread of the electrons' wavefunctions. Thus the susceptibility may be crudely estimated by taking $\langle r^2\rangle$ equal to the square of the Bohr radius $r_\mathrm{B} \equiv 4\pi\varepsilon_0\hbar^2/m_\mathrm{e}e^2$.[27] This substitution yields

$$|\chi_\mathrm{m}| \sim \frac{\mu_0 nZe^2 r_\mathrm{B}^2}{6m_\mathrm{e}} \equiv \frac{2\pi}{3}Z\left(nr_\mathrm{B}^3\right)\alpha^2, \qquad (*)$$

where, in the last expression, the parameter

$$\alpha \equiv \frac{e^2}{4\pi\varepsilon_0\hbar c} \equiv \frac{e^2\mu_0 c}{4\pi\hbar} \equiv \frac{e^2}{4\pi\hbar}\left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2} \approx \frac{1}{137} << 1$$

is the *fine-structure constant*, which describes the relative strength of electromagnetic interactions on the quantum-relativistic scale.[28] It is evident that even for the heaviest atoms ($Z \sim 10^2$) in their condensed-matter form (where $n \sim r_\mathrm{B}^{-3}$), this magnetic susceptibility is still small: $|\chi_\mathrm{m}| << 1$, i.e. the orbital diamagnetism is always weak.

(ii) As was discussed in Sec. 5.5 of the lecture notes, we may expect the full alignment of spontaneous magnetic dipoles to be achieved at $m_0B \sim k_\mathrm{B}T$, resulting in the induced magnetic moment $\Delta m = Zm_0$ per atom, and hence the saturation of the magnetization at the level $M_0 = nZm_0$. Hence it is reasonable to expect that in a weak magnetic field, the magnetization scales as[29]

$$M \sim \frac{m_0 B}{k_\mathrm{B}T}M_0 = \frac{nZm_0^2 B}{k_\mathrm{B}T}, \qquad \text{for } M << M_0,$$

corresponding to the paramagnetic (positive) susceptibility

[26] See, e.g., the solution of QM Problem 6.14.
[27] See, e.g., QM Secs. 1.1 and 3.6. As a mnemonic rule, the expression r_B may be obtained by equating the scales of the electrostatic ($e^2/4\pi\varepsilon_0 r_\mathrm{B}$) and quantum-kinetic ($\hbar^2/m_\mathrm{e}r_\mathrm{B}^2$) energies in Bohr's model of the hydrogen atom, without any numerical coefficients. (Note also that each of these fractions gives the atomic energy scale $E_\mathrm{H} \equiv (e^2/4\pi\varepsilon_0)^2 m_\mathrm{e}/\hbar^2 \approx 27.2$ eV, called the *Hartree energy*.)
[28] In this course, we will run into the fine-structure constant again in Sec. 10.4; it will also be repeatedly discussed in the QM part of the series. For its more exact value, see *Appendix UCA: Selected Units and Constants*.
[29] This assumption is confirmed by quantitative statistical analyses – see, e.g., SM Sec. 2.4.

$$\chi_{\mathrm{m}} \equiv \frac{M}{H} \approx \frac{M}{B/\mu_0} \approx \frac{\mu_0 n Z m_0^2}{k_{\mathrm{B}} T}.$$

This result obeys the so-called *Curie law*, $\chi_{\mathrm{m}} \propto 1/T$ (also valid for χ_{e} in paraelectrics).

In order to estimate this susceptibility, we may take $m_0 \sim \mu_{\mathrm{B}} \equiv e\hbar/2m_{\mathrm{e}}$, getting

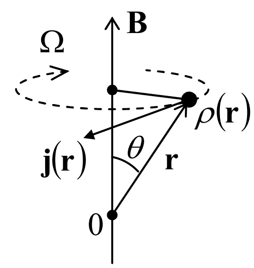
$$\chi_{\mathrm{m}} \sim \frac{\mu_0 n Z}{k_{\mathrm{B}} T}\left(\frac{e\hbar}{2m_{\mathrm{e}}}\right)^2 \equiv \pi Z\left(n r_{\mathrm{B}}^3\right)\alpha^2 \frac{E_{\mathrm{H}}}{k_{\mathrm{B}} T}.$$

For room temperatures ($T \sim 300$K), the ratio $E_{\mathrm{H}}/k_{\mathrm{B}}T$ is close to 10^3, so comparing this estimate with Eq. (*), we see that the spin paramagnetism is much stronger than the orbital diamagnetism,[30] so the latter effect prevails only in materials with zero net atomic spins – see Table 5.1 in the lecture notes.

Problem 5.19.* Use the classical picture of the orbital ("Larmor") diamagnetism, discussed in Sec. 5.5 of the lecture notes, to calculate its (small) contribution $\Delta\mathbf{B}(0)$ to the magnetic field $\mathbf{B}$ felt by an atomic nucleus, treating the electrons of the atom as a spherically symmetric cloud with an electric charge density $\rho(r)$. Express the result via the value $\phi(0)$ of the electrostatic potential of the cloud and use this expression for a crude numerical estimate of the ratio $\Delta B(0)/B$ for the hydrogen atom.

Solution: As was discussed in Sec. 5.5, a relatively low external magnetic field $\mathbf{B}$ causes the torque-induced precession of a classical particle's orbit around the field's direction, with the angular velocity

$$\boldsymbol{\Omega} = -\frac{q}{2m_0}\mathbf{B}.$$



In the (admittedly, somewhat eclectic but still very useful) model, in which the *quantum-mechanical* spread of electron positions at $\mathbf{B} = 0$ is represented by a time-independent *classical* charge density $\rho(\mathbf{r})$, the rotation induces loop currents (see the figure above) with density $\mathbf{j}(\mathbf{r}) = \mathbf{v}(\mathbf{r})\rho(\mathbf{r})$, where $\mathbf{v}(\mathbf{r}) = \boldsymbol{\Omega}\times\mathbf{r}$.[31] The magnetic field correction $\Delta\mathbf{B}$ due to these currents may be calculated using Eq. (5.14). For the origin-located nuclei (with $\mathbf{r} = 0$), and with $\mathbf{r}'$ replaced with $\mathbf{r}$ (just for notation simplicity), that formula yields

$$\Delta\mathbf{B}(0) = -\frac{\mu_0}{4\pi}\int \mathbf{j}(\mathbf{r})\times\frac{\mathbf{r}}{r^3}d^3r = -\frac{\mu_0}{4\pi}\int(\boldsymbol{\Omega}\times\mathbf{r})\times\mathbf{r}\frac{\rho(\mathbf{r})}{r^3}d^3r = \frac{\mu_0}{4\pi}\frac{q}{2m_0}\int(\mathbf{B}\times\mathbf{r})\times\mathbf{r}\frac{\rho(\mathbf{r})}{r^3}d^3r.$$

The vector $(\mathbf{B}\times\mathbf{r})\times\mathbf{r}$ lies in the mutual plane of the vectors $\mathbf{r}$ and $\mathbf{B}$ may be represented as a vector sum of two components: $-\mathbf{B}r^2\sin^2\theta$ directed against the vector $\mathbf{B}$ (where θ is the angle between the vectors $\mathbf{B}$ are $\mathbf{r}$ – see the figure above) and another one, normal to this vector. For any axially symmetric charge distribution $\rho(\mathbf{r})$, the integral of the latter component over the azimuthal angle vanishes. Hence,

[30] This estimate is invalid in metals with their conduction electrons forming a degenerate Fermi "sea". In this case, the diamagnetic susceptibility, calculated within a simple model of independent Fermi particles, called the *Landau diamagnetism*, equals exactly 1/3 of the so-called *Pauli paramagnetism* due to the spin orientation of the same electrons – see, e.g., the solutions of SM Problems 3.10 and 3.11.

[31] If you are uncomfortable with the last relation, please revisit a classical mechanics course, e.g., the discussion at the beginning of CM Sec. 4.1 of this series, in particular Eq. (4.9).

for any sign of the moving particles' charge (which determines the sign of not only q, but also of ρ), the vector $\Delta\mathbf{B}(0)$ is directed exactly against $\mathbf{B}$ and hence describes a diamagnetic response of the orbital motion. The expression for its relative magnitude,

$$\frac{\Delta B(0)}{B} = -\frac{\mu_0}{4\pi}\frac{q}{2m_0}\int \sin^2\theta \frac{\rho(\mathbf{r})}{r} d^3r = -\frac{\mu_0}{4\pi}\frac{q}{2m_0} 2\pi \int_0^\pi \sin^3\theta d\theta \int_0^\infty \rho(\mathbf{r}) r dr,$$

may be further simplified takes an especially simple form if the charge distribution is spherically symmetric: $\rho(\mathbf{r}) = \rho(r)$. Indeed, in this case, the integral over θ is may be readily worked out by the usual substitution $\xi \equiv \cos\theta$ (so $d\xi = -\sin\theta\, d\theta$ and $\sin^2\theta = 1 - \xi^2$):

$$\int_0^\pi \sin^3\theta d\theta = \int_{-1}^{+1} \left(1-\xi^2\right) d\xi = \frac{4}{3}\cdot$$

The final result,

$$\frac{\Delta B(0)}{B} = -\frac{\mu_0}{4\pi}\frac{q}{m_0}\frac{4\pi}{3}\int_0^\infty \rho(r) r dr, \qquad (*)$$

may be readily related to the value (at the same central point) of the electrostatic potential $\phi(\mathbf{r})$ induced by this charge distribution. Indeed, by applying Eq. (1.38) of the lecture notes to the point $\mathbf{r} = 0$, and again replacing $\mathbf{r}'$ with $\mathbf{r}$, for the spherically symmetric charge density we get the same integral:

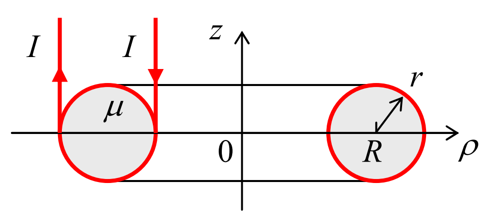
$$\phi(0) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r)}{r} d^3r = \frac{1}{4\pi\varepsilon_0} 4\pi \int_0^\infty \rho(r) r dr,$$

so the comparison with Eq. (*) yields

$$\frac{\Delta B(0)}{B} = -\frac{\varepsilon_0\mu_0}{3}\frac{q}{m_0}\phi(0) \equiv -\frac{1}{3}\frac{q}{m_0c^2}\phi(0).$$

This equality enables a simple estimate of the diamagnetic correction in the hydrogen atom: for electrons, with $q = -e$ and $m_0c^2 = m_ec^2 \approx 0.51$ MeV, $|\phi(0)|$ may be crudely estimated as $e/4\pi\varepsilon_0 r_B \equiv E_H/e$, where r_B is the Bohr radius, and $E_H \approx 27$ eV is the Hartree energy unit,[32] so

$$\left|\frac{\Delta B(0)}{B}\right| \sim \frac{1}{3}\frac{E_H}{m_ec^2} \approx 2\times 10^{-5}.$$



Problem 5.20. Calculate the self-inductance of a toroidal solenoid with a round cross-section of radius $r \sim R$ (see the figure on the right), with $N >> 1$, R/r wire turns uniformly distributed along the perimeter, and filled with a linear magnetic material of permeability μ.

Solution: In Sec. 5.2 of the lecture notes, the magnetic field inside such a solenoid without magnetic filling was calculated – see Eq. (5.41). According to the discussion at the beginning of Sec. 5.6, since all the field is concentrated inside the solenoid filled with the magnetic material, we may just multiply that result by the ratio μ/μ_0:

[32] See, e.g., either the solution of the previous problem or QM Sec. 1.1, in particular, Eqs. (1.9) and (1.13).

$$B = \frac{\mu NI}{2\pi\rho},$$

where ρ is the distance from the z-axis. Now we can calculate the magnetic flux piercing one wire loop:

$$\Phi_1 = \int_A B_n d^2 r = \frac{\mu NI}{2\pi}\int_{-r}^{+r} dz \int_{R-(r^2-z^2)^{1/2}}^{R+(r^2-z^2)^{1/2}} \frac{d\rho}{\rho} = \frac{\mu NI}{\pi}\int_0^r \ln\frac{R+\left(r^2-z^2\right)^{1/2}}{R-\left(r^2-z^2\right)^{1/2}} dz \equiv \frac{\mu NIr}{\pi}\int_0^1 \ln\frac{a+\left(1-\xi^2\right)^{1/2}}{a-\left(1-\xi^2\right)^{1/2}} d\xi,$$

where $a \equiv R/r > 1$ and $\xi \equiv z/r$. This is a table integral[33] equal to $\pi[a - (a^2 - 1)^{1/2}]$, so, finally,

$$\Phi_1 = \mu NI\left[R - \left(R^2 - r^2\right)^{1/2}\right].$$

Just as in the long solenoid discussed in Sec. 5.2 of the lecture notes, the whole flux Φ piercing all N turns of the whole wire is N times larger. As a result, the solenoid's inductance

$$L \equiv \frac{\Phi}{I} = \frac{\Phi_1 N}{I} = \mu N^2\left[R - \left(R^2 - r^2\right)^{1/2}\right]. \qquad (*)$$

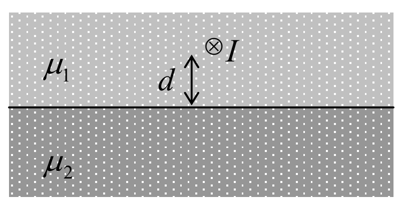
In the limit $r << R$, we may expand this expression into the Taylor series in small ratio r/R and, in the first nonvanishing approximation, get

$$L \approx \mu N^2 \frac{r^2}{2R} = \mu n^2 lA, \quad \text{with } n \equiv \frac{N}{l}, \quad l \equiv 2\pi R, \quad \text{and } A \equiv \pi r^2.$$

We see that in this limit, the result coincides with the inductance of a long straight solenoid, calculated in Sec. 5.6 – see Eq. (5.143). It is curious that in the opposite limit ($r = R$), Eq. (*) also acquires a very simple form,

$$L = \mu N^2 r.$$

Note that these results, as well as Eq. (5.142) of the lecture notes for the straight solenoid, may be used for relatively small values of N only if $\mu >> \mu_0$; otherwise, we have to account for the "stray" magnetic field spilling out between separated wire turns.



Problem 5.21. A long, straight, thin wire carrying current I runs parallel to a sharp plane boundary between two uniform, linear magnetic media – see the figure on the right. Calculate the magnetic field everywhere in the system, and the force (per unit length) exerted on the wire.

Solution: If the linear magnetic material surrounding the wire was purely uniform ($\mu(\mathbf{r})$ = const), its field lines would be circular, with the magnitude described by Eq. (5.20) of the lecture notes, adjusted as prescribed by Eq. (5.115). This result could be represented in the following vector form:

$$\mathbf{H}(\boldsymbol{\rho}) = \mathbf{n}_\varphi \frac{I}{2\pi|\boldsymbol{\rho} - \boldsymbol{\rho}_0|}, \qquad B(\boldsymbol{\rho}) = \mathbf{n}_\varphi \frac{\mu I}{2\pi|\boldsymbol{\rho} - \boldsymbol{\rho}_0|}, \qquad (*)$$

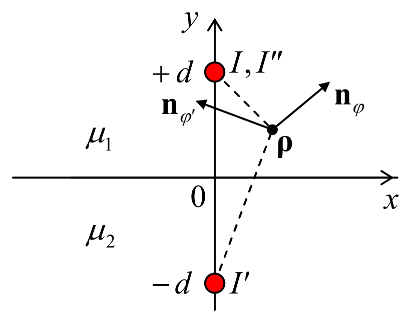
[33] See, e.g., MA (6.13).

where $\boldsymbol{\rho}$ is the 2D radius vector of the observation point within a plane normal to the wire (e.g., the plane of the figure above), $\boldsymbol{\rho}_0$ is that of the wire, and $\mathbf{n}_\varphi$ is the unit azimuthal vector in this plane, which is normal to the vector $\boldsymbol{\rho} - \boldsymbol{\rho}_0$. However, in our current case, with $\mu_1 \neq \mu_2$, such a solution would not satisfy the second of the boundary conditions (5.118)-(5.119),

$$H_\tau = \text{const}, \quad B_n = \text{const}, \tag{**}$$

at the interface between the two materials.

Inspired by the solution of similar electrostatic and dc-current problems in Chapters 2-4 of the lecture notes (see, in particular, the problem illustrated by Fig. 3.12), we may try to look for the solution of our current problem in the form of a sum of two contributions of the type (*): one from the actual current I (for observation points in the lower half-space, replaced by another value, I'') and the other from an image current I' passing at the same distance d from the interface, but on its opposite side – see the figure on the right. With the Cartesian coordinates selected as shown in this figure, the assumed solution (analogous to Eq. (3.66) of the similar electrostatics problem) is



$$\mathbf{H}(\boldsymbol{\rho}) = \begin{cases} \mathbf{n}_\varphi \dfrac{I}{2\pi|\boldsymbol{\rho} - \boldsymbol{\rho}_0|} + \mathbf{n}_{\varphi'} \dfrac{I'}{2\pi|\boldsymbol{\rho} - \boldsymbol{\rho}'|}, & \text{for } y \geq 0, \\ \mathbf{n}_\varphi \dfrac{I''}{2\pi|\boldsymbol{\rho} - \boldsymbol{\rho}_0|}, & \text{for } y \leq 0, \end{cases} \tag{***}$$

where $\boldsymbol{\rho}_0 = \{0, +d\}$, $\boldsymbol{\rho}' = \{0, -d\}$. At the interface, i.e. at $\boldsymbol{\rho} = \{x, 0\}$, this solution yields

$$H_x = \begin{cases} \dfrac{I}{2\pi} \dfrac{d}{x^2 + d^2} - \dfrac{I'}{2\pi} \dfrac{d}{x^2 + d^2}, & \text{for } y \geq 0, \\ \dfrac{I''}{2\pi} \dfrac{d}{x^2 + d^2}, & \text{for } y \leq 0, \end{cases}$$

$$B_y = \begin{cases} \dfrac{\mu_1 I}{2\pi} \dfrac{x}{x^2 + d^2} + \dfrac{\mu_1 I'}{2\pi} \dfrac{x}{x^2 + d^2}, & \text{for } y \geq 0, \\ \dfrac{\mu_2 I''}{2\pi} \dfrac{x}{x^2 + d^2}, & \text{for } y \leq 0. \end{cases}$$

These solutions satisfy the boundary conditions (**) if I' and I'' are selected so that

$$I - I' = I'', \qquad \mu_1(I + I') = \mu_2 I''.$$

Solving this simple system of two equations, we get

$$I' = -\frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} I, \qquad I'' = \frac{2\mu_1}{\mu_1 + \mu_2} I.$$

These relations, together with Eqs. (***), give the solution of the first task of the problem. Now we may use them to calculate the magnetic field at the position $\boldsymbol{\rho}_0$ of the actual wire (excluding its own, diverging contribution):

$$\mathbf{H}(\boldsymbol{\rho}_0) = \mathbf{n}_{\varphi'} \frac{I'}{2\pi|\boldsymbol{\rho}_0 - \boldsymbol{\rho}'|} = \mathbf{n}_{\varphi'} \frac{I'}{4\pi d} = -\mathbf{n}_{\varphi'} \frac{I}{4\pi d} \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}.$$

At this point, the unit vector $\mathbf{n}_{\varphi'}$ (see the figure above) is opposite to $\mathbf{n}_x$, so the field is horizontal, with

$$H_x(\boldsymbol{\rho}_0) = \frac{I}{4\pi d} \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}, \qquad \text{i.e. } B_x(\boldsymbol{\rho}_0) = \frac{I}{4\pi d} \mu_1 \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}.$$

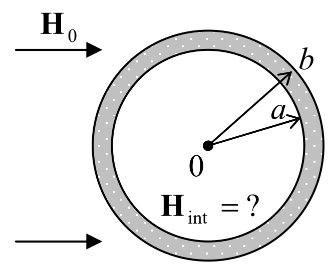
Per the basic Eq. (5.15), the force exerted on the wire is vertical, with

$$\frac{F_y}{l} = I_z B_x = IB_x = \frac{I^2}{4\pi d} \mu_1 \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}.$$

For the important particular case of a wire stretched in free space, parallel to a linear magnetic with the permeability μ, this formula yields

$$\frac{F_y}{l} = \frac{I^2}{4\pi d} \mu_0 \frac{\mu_0 - \mu}{\mu_0 + \mu}.$$

Hence the current, regardless of its sign, is attracted by a paramagnetic material (with $\mu > \mu_0$) and repulsed by a diamagnetic one.



Problem 5.22. Solve the magnetic shielding problem similar to that discussed in Sec. 5.6 of the lecture notes, but for a spherical rather than cylindrical shell, with the same central cross-section as shown in Fig. 5.16, partly reproduced on the right. Compare the efficiency of those two shields, for the same permeability μ of the shell, and the same b/a ratio.

Solution: Guided by the general axially symmetric solution (2.172) of the Laplace equation in the spherical coordinates, we may look for the scalar potential of the magnetic field in the form

$$\phi_m = \begin{cases} \left(-H_0 r + b_1'/r^2\right)\cos\theta, & \text{for } b \le r, \\ \left(a_1 r + b_1/r^2\right)\cos\theta, & \text{for } a \le r \le b, \\ -H_{\text{int}} r\cos\theta, & \text{for } r \le a\ , \end{cases}$$

which differs from Eq. (5.127) of the lecture notes (for the cylindrical shield) only by the power of the radius factor in the denominators. Plugging this solution into the boundary conditions, ϕ_m = const and $\mu\partial\phi_\text{m}/\partial r$ = const, for both interfaces ($r = b$ and $r = a$), we get the following system of four equations:

$$-H_0 b + b_1'/b^2 = a_1 b + b_1/b^2, \qquad \left(a_1 a + b_1/a^2\right) = -H_{\text{int}} a,$$

$$\mu_0\left(-H_0 - b_1'/2b^3\right)H_0 = \mu\left(a_1 - b_1/2b^3\right), \qquad \mu\left(a_1 - b_1/2a^3\right) = -\mu_0 H_{\text{int}},$$

for four unknown coefficients a_1, b_1, b_1', and H_{int}. Solving the system, we get, in particular:

$$\frac{H_{\text{int}}}{H_0} = \frac{\alpha_s - 1}{\alpha_s - (a/b)^3}, \qquad \text{where } \alpha_s \equiv \frac{(\mu + 2\mu_0)(\mu + \mu_0/2)}{(\mu - \mu_0)^2}. \qquad (*)$$

Comparing the magnetic material factors α for the spherical (α_s) and cylindrical (α_c) geometries (see Eq. (5.129) of the lecture notes),

$$\frac{\alpha_s}{\alpha_c} = \frac{(\xi+2)(\xi+1/2)}{(\xi+1)^2} \equiv \frac{\xi^2 + (5/2)\xi + 1}{\xi^2 + (4/2)\xi + 1}, \quad \text{where } \xi \equiv \frac{\mu}{\mu_0},$$

we see that $\alpha_s > \alpha_c$ for any paramagnetic shell material (with $\mu > \mu_0$), but for the most important case of soft ferromagnets with $\mu >> \mu_0$, this difference is minor: $\alpha_s \to \alpha_c \to 1$. More importantly, since $a < b$, the factor $(a/b)^3$ in the denominator of Eq. (*) is smaller than the factor $(a/b)^2$ in the denominator of Eq. (5.129). Hence, the spherical shields are more effective – if not always practically acceptable because of the sample placement convenience and fabrication costs. A popular compromise is to use short cylinders with thick lids that may be closed after the sample has been placed inside.

Problem 5.23. Calculate the magnetic field's distribution around a spherical permanent magnet with uniform magnetization $\mathbf{M}_0$ = const.

Solution: Due to the absence of stand-alone currents in the region of our interest, the field $\mathbf{H}$ in the problem may be represented by Eq. (5.120) of the lecture notes, $\mathbf{H} = -\nabla\phi_\text{m}$, where the scalar potential ϕ_m satisfies the Laplace equation both inside the sphere and outside it. Due to the axial symmetry of the problem with respect to the direction of the constant magnetization $\mathbf{M}_0$, which may be conveniently taken for the polar axis, the general solution of the Laplace equation may be represented in the form similar to Eq. (2.172) of the lecture notes. Setting the arbitrary constant a_0 to zero, and taking into account that due to zero divergence of the vector $\mathbf{H}$, the constant b_0 vanishes as well, this solution is reduced to

$$\phi_\text{m} = \sum_{l=1}^{\infty} \mathcal{P}_l(\cos\theta) \times \begin{cases} a_l r^l + b_l / r^{l+1}, & \text{for } r \leq R, \\ a_l' r^l + b_l' / r^{l+1}, & \text{for } R \leq r, \end{cases}$$

where R is the sphere's radius, and $\mathcal{P}_l(\xi)$ are the Legendre polynomials (2.169). Next, since the field is produced by the sphere itself, it should vanish at large distances from it, so all coefficients a_l' must equal zero. Similarly, the potential inside the cylinder has to be finite, so all coefficients b_l have to vanish as well. As a result, the above expression for ϕ_m is reduced to

$$\phi_\text{m} = \sum_{l=1}^{\infty} \mathcal{P}_l(\cos\theta) \times \begin{cases} a_l r^l, & \text{for } r \leq R, \\ b_l' / r^{l+1}, & \text{for } R \leq r. \end{cases}$$

In order to find the coefficients a_l and b_l', we have to write boundary conditions at the sphere's surface ($r = R$). From Eq. (5.118) for the tangential component of field $\mathbf{H}$, we get

$$\sum_{l=1}^{\infty} a_l R^l \mathcal{P}_l(\cos\theta) = \sum_{n=1}^{\infty} \frac{b_l'}{R^{l+1}} \mathcal{P}_l(\cos\theta). \qquad (*)$$

For the normal component of the field, we have to use the general Eq. (5.118), B_n = const (rather than Eq. (5.119), which is only valid for linear magnetics) because $\mathbf{B}$ is a linear function of $\mathbf{H}$ (equal to $\mu_0\mathbf{H}$)

only outside the sphere, at $r \geq R$. At $r \leq R$ we should use the general relation (5.108), $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, for our geometry giving $B_n = \mu_0(H_n + M_n) = \mu_0(-\partial\phi_m/\partial r + M_n)$. For the fixed, uniform polarization $\mathbf{M}_0$, $M_n = M_0\cos\theta \equiv M_0\mathcal{P}_1(\cos\theta)$, so the second boundary condition takes the form

$$\mu_0\left[-\sum_{l=1}^{\infty} l a_l R^{l-1}\mathcal{P}_l(\cos\theta) + M_0\mathcal{P}_1(\cos\theta)\right] = \mu_0\sum_{l=1}^{\infty}(l+1)\frac{b_l'}{R^{l+2}}\mathcal{P}_l(\cos\theta). \qquad (**)$$

Due to the linear independence of the functions $\mathcal{P}_l(\cos\theta)$, Eqs. (*) and (**) fall apart into a set of independent couples of linear equations for each pair $\{a_l, b_l'\}$, but only for $l = 1$, such a pair of equations is inhomogeneous, i.e. gives a nonvanishing solution for coefficients a_1, b_1'. (All other coefficients *may* equal zero, and due to the uniqueness of the Laplace equation's solution, they *have to* equal zero.) Solving the system of two linear equations for $l = 1$, we get

$$a_1 = \frac{M_0}{3}, \qquad b_1' = \frac{M_0 R^3}{3}.$$

From here, the scalar potential of the magnetic field is

$$\phi_m\Big|_{r\leq R} = \frac{M_0}{3} r\cos\theta, \qquad \phi_m\Big|_{r\geq R} = \frac{M_0 R^3}{3r^2}\cos\theta.$$

The first of these expressions describes a uniform magnetic field with $\mathbf{H} = -\mathbf{M}_0/3$ and $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}_0) = (2/3)\mu_0\mathbf{M}_0$. (Note that the fields $\mathbf{H}$ and $\mathbf{B}$ have different directions!) The second formula shows that at $r > R$, the field exactly coincides with that of a magnetic dipole with the moment

$$\mathbf{m} = \frac{4\pi}{3}R^3\mathbf{M}_0 = V\mathbf{M}_0,$$

i.e. the sum of all elementary dipoles within the sphere – the result that could be conjectured even without the formal solution of the problem.

Note that this problem and its solution are very similar to Problem 3.13 on a sphere with fixed electric polarization. Note also that alternatively, the result may be obtained by the method discussed in the next problem – though in this particular case, that would involve a more bulky calculation.

Problem 5.24. A limited volume V is filled with a magnetic material with field-independent magnetization $\mathbf{M}(\mathbf{r})$. Write explicit expressions for the magnetic field induced by the magnetization and its potential, and recast these expressions into the forms that are more convenient when $\mathbf{M}(\mathbf{r}) = \mathbf{M}_0 =$ const throughout the volume.

Solution: As was discussed in Sec. 5.6 of the lecture notes, Eq. (5.132) for the magnetic field,

$$\nabla\cdot\mathbf{H} = -\nabla\cdot\mathbf{M},$$

is mathematically identical to the Maxwell equation Eq. (1.27) for the electric field:

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

As was discussed in Sec. 1.2 of the lecture notes, the last expression is equivalent to Eq. (1.9):

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi}\int_V \frac{\rho(\mathbf{r}')}{\varepsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3r', \tag{*}$$

for the field created by the electric charge contained inside the volume V. Hence, if the only source of the field $\mathbf{H}$ in a system is the magnetization of some volume V, we may use this analogy to write

$$\mathbf{H}(\mathbf{r}) = \frac{1}{4\pi}\int_V \frac{\rho_\mathrm{m}(\mathbf{r}')}{\mu_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3r', \qquad \text{where } \rho_\mathrm{m}(\mathbf{r}') \equiv -\mu_0\nabla\cdot\mathbf{M}(\mathbf{r}'). \tag{**}$$

This means that in the free space outside of the volume V,

$$\mathbf{B}(\mathbf{r}) = \mu_0\mathbf{H}(\mathbf{r}) = \frac{1}{4\pi}\int_V \rho_\mathrm{m}(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3r' \equiv \frac{\mu_0}{4\pi}\int_V [-\nabla\cdot\mathbf{M}(\mathbf{r}')]\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3r'.$$

However, per Eq. (5.108), inside volume V, the field $\mathbf{B}$ is different:

$$\mathbf{B}(\mathbf{r}) = \mu_0[\mathbf{M}(\mathbf{r})+\mathbf{H}(\mathbf{r})] = \mu_0\mathbf{M}(\mathbf{r}) + \frac{\mu_0}{4\pi}\int_V [-\nabla\cdot\mathbf{M}(\mathbf{r}')]\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3r'.$$

Next, as we know from Sec. 1.3 of the lecture notes, Eq. (*) is equivalent to Eq. (1.38),

$$\phi(\mathbf{r}) = \frac{1}{4\pi}\int_V \frac{\rho(\mathbf{r}')}{\varepsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3r',$$

for the electrostatic potential defined by Eq. (1.33):

$$\mathbf{E} = -\nabla\phi.$$

This means that the magnetic field potential, defined by the similar Eq. (5.120),[34]

$$\mathbf{H} = -\nabla\phi_\mathrm{m},$$

may be calculated as

$$\phi_\mathrm{m}(\mathbf{r}) = \frac{1}{4\pi}\int_V \frac{\rho_\mathrm{m}(\mathbf{r}')}{\mu_0}\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3r' \equiv \frac{1}{4\pi}\int_V [-\nabla\cdot\mathbf{M}(\mathbf{r}')]\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3r'. \tag{***}$$

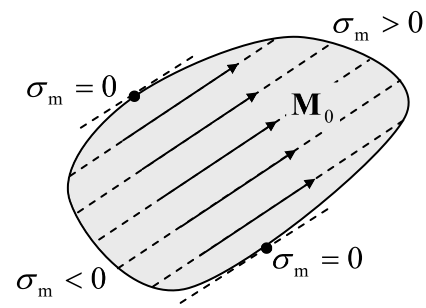
As was also discussed in Sec. 5.6, if $\mathbf{M}(\mathbf{r}) = \mathbf{M}_0$ = const inside the volume V, and $\mathbf{M}(\mathbf{r}) = 0$ outside it, the effective magnetic charge density $\rho_\mathrm{m} = -\mu_0\nabla\cdot\mathbf{M}$ is different from zero (and formally infinite) only on the surface S of the volume. Hence it may be characterized by the effective surface charge density σ_m, which may be calculated by the integration of ρ_m along the local coordinate n normal to the surface, from a point just inside the surface to a point just outside it (i.e. out of the volume V):

$$\sigma_\mathrm{m} \equiv \int_{n_s-0}^{n_s+0}\rho_\mathrm{m}\,dn = -\mu_0\int_{n_s-0}^{n_s+0}\nabla\cdot\mathbf{M}\,dn = -\mu_0\int_{n_s-0}^{n_s+0}\frac{\partial M_n}{\partial n}dn = \mu_0 M_n\Big|_{n_s-0} = \mu_0 M_0\cos\theta,$$

where θ is the angle between the direction of the vector $\mathbf{M}_0$ and the outer normal $\mathbf{n}$ to the surface.[35] This means that the effective surface charge is positive at the points where the vector $\mathbf{M}_0$ is directed out of the

[34] As a (hopefully, unnecessary) reminder, this expression is only valid in the absence of a vortex-like field induced by stand-alone currents – the condition satisfied in our case of the field induced by magnetization $\mathbf{M}(\mathbf{r})$.

[35] Admittedly, this is just another form of Eq. (5.134) of the lecture notes.



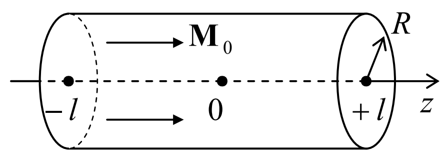
magnetic material, negative in the opposite case, and equals zero at the points where $\mathbf{M}_0$ is parallel to the surface – see the figure on the right.

In this case, Eq. (**) takes the form

$$\mathbf{H}(\mathbf{r}) = \frac{M_0}{4\pi}\int_S \cos\theta(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^2r', \qquad (****)$$

and Eq. (***), the form

$$\phi_m(\mathbf{r}) = \frac{M_0}{4\pi}\int_S \cos\theta(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^2r'.$$



Problem 5.25. Use the results of the previous problem to calculate the distribution of the magnetic field **H** along the axis of a straight permanent magnet of length $2l$ and a round cross-section of radius R, with a uniform magnetization $\mathbf{M}_0$ parallel to the axis – see the figure on the right.

Solution: Due to the axial symmetry of the problem, the magnetic field on the axis has to be directed along it: $\mathbf{H} = \mathbf{n}_z H_z$. This is why it is sufficient to use the z-component of Eq. (****) of the model solution of the previous problem with $\cos\theta = \pm 1$:

$$H_z(z) = \pm\frac{M_0}{4\pi}\int_S \frac{z-z'}{|\mathbf{r}-\mathbf{r}'|^3}d^2r'.$$

where the integration is extended only over the magnet's flat ends. Taking the magnet's center for $z = 0$ (so the positions of its ends are $z = \pm l$, as shown in the figure above), we get

$$H_z(z) = \frac{M_0}{4\pi}\left\{(z-l)\int_0^R \frac{2\pi\rho d\rho}{\left[(z-l)^2+\rho^2\right]^{3/2}} - (z+l)\int_0^R \frac{2\pi\rho d\rho}{\left[(z+l)^2+\rho^2\right]^{3/2}}\right\},$$

where ρ is the distance of the source point $\mathbf{r}'$ from the symmetry axis. These two integrals may be readily worked out by similar substitutions: $\xi \equiv (z \pm l)^2 + \rho^2$, so in both cases $2\pi\rho d\rho = \pi d\xi$:

$$H_z(z) = \frac{M_0}{4\pi}\left[(z-l)\int_{(z-l)^2}^{(z-l)^2+R^2}\frac{\pi d\xi}{\xi^{3/2}} - (z+l)\int_{(z+l)^2}^{(z+l)^2+R^2}\frac{\pi d\xi}{\xi^{3/2}}\right].$$

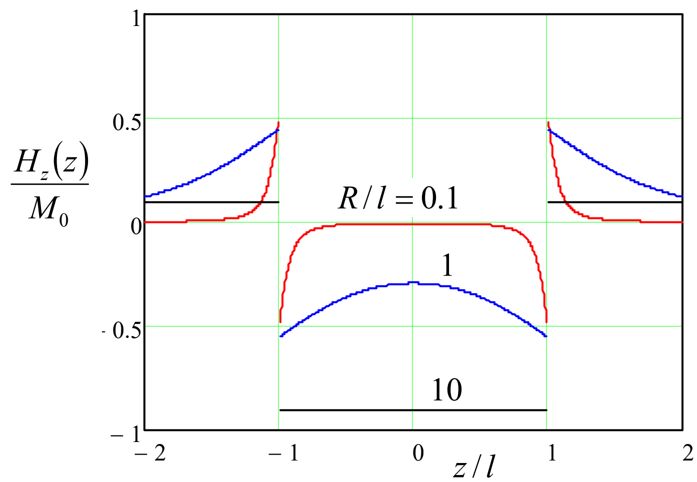
$$= \frac{M_0}{2}\left[-\frac{(z-l)}{\left[(z-l)^2+R^2\right]^{1/2}} + \text{sgn}(z-l) + \frac{(z+l)}{\left[(z+l)^2+R^2\right]^{1/2}} - \text{sgn}(z+l)\right].$$

This function is plotted in the figure below for three representative values of the R/l ratio. Note, first of all, the reversal of the direction of the field **H** at each end of the magnet – the result which should not be too surprising after looking at Eq. (5.134) of the lecture notes and the solution of the very similar Problem 23 for a sphere. Note that the magnitude of **H** is always below M_0 at all points, so the field $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ is still directed along the magnetization vector: $B_z > 0$ for all z.

At $R << l$, **H** is nearly a sum of Coulomb-like fields of two equal and opposite point "magnetic charges" $q_m = \pm\mu_0 M_0 A = \pm\mu_0 M_0 \pi R^2$. In the opposite limit, the field inside the magnet is nearly constant: $H_z \approx -M_0$, while outside it is very small, and drops further only at distances $|z| \sim R >> l$. In one more limit, $|z| >> R, l$, the result is reduced to

$$H_z = \frac{M_0 R^2 l}{z^3},$$

$$B_z = \mu_0 \frac{M_0 R^2 l}{z^3} \equiv \frac{\mu_0}{4\pi} \frac{2(M_0 2\pi R^2 l)}{z^3}.$$



This is just the field (5.99) of a magnetic dipole, with the dipole moment $\mathbf{m} = \mathbf{M}_0 V$, where $V = 2\pi R^2 l$ is the magnet's volume, on its axis (where $\mathbf{r} = \mathbf{n}_z z$, $\mathbf{m} = \mathbf{n}_z m$).

Problem 5.26. A flat end of a long straight permanent magnet, similar to that considered in the previous problem but with an arbitrary cross-section of area A, is stuck to a flat surface of a large sample of a linear magnetic material with a very high permeability $\mu >> \mu_0$. Calculate the normally directed force needed to detach them.

Solution: As was discussed in Sec. 5.6 of the lecture notes, the fields inside ultra-high-μ materials are well described by the approximation $\mathbf{H} = 0$, so according to the definition (5.120) of the scalar magnetic potential, we may take $\phi_m = 0$ inside its bulk and on its surface, forming the boundary condition for the distribution of ϕ_m outside of the sample. But this condition, and Eq. (5.121) governing the distribution, are completely similar to that for the distribution of the electrostatic potential ϕ outside a similarly shaped conductor.

As we know very well (see, e.g., Sec. 2.9), such a distribution, due to a stand-alone point electric charge q at a distance d from the plane surface of a conductor, is described by the conductor's replacement with the additional charge $q' = -q$ located at the same distance d from the surface's plane, but on the opposite side of it. Hence, the same is true for the magnetic field of each effective magnetic charge q_m outside a flat surface of a high-μ sample. This fact may be used to calculate the magnetic field distribution in many problems with various systems of charges q_m.

As was discussed in the same Sec. 5.6, in our current case of a flat end of a permanent magnet with a constant magnetization $\mathbf{M}_0$ normal to the end's surface, the effective magnetic charges form a uniform sheet of area A, with the areal density $\sigma_m = \mu_0 M_0$. Hence the magnetic image charge is distributed on a similar sheet, on the other side of the surface, with density $-\sigma_m$. But this situation is exactly similar to the attraction of two opposite poles of permanent-magnet rods – see Fig. 5.18 and its discussion. Hence the force of attraction between a permanent magnet and a linear high-μ magnetic material (and the force necessary for their detachment) is given by the same Eq. (5.136):

$$|F| = \frac{\mu_0 M_0^2 A}{2}.$$

This also means that the crude estimate of this force given in the lecture notes, $|F|/A \sim 4\times10^5$ Pa, is also valid for typical refrigerator magnets because the refrigerator walls are usually made of steel, and the magnetic permeability of most varieties of this material satisfy the requirement $\mu >> \mu_0$ well. (The main exception are the so-called *stainless steels* – alloys of iron and more than 10% chromium, which is almost non-magnetic, with $\mu \approx \mu_0$. However, such steels, broadly used in physical experiment because of their unique chemical and thermal properties, are relatively expensive and hence rarely used outside of special application fields.)

Problem 5.27. A permanent magnet with a uniform magnetization $\mathbf{M}_0$ has the form of a spherical shell with an internal radius R_1 and an external radius $R_2 > R_1$. Calculate the magnetic field inside the shell.

Solution: According to Eqs. (5.134) and (5.137) of the lecture notes (see also the solution of Problem 24), an abrupt leap of magnetization on the external surface of the shell is equivalent to the effective magnetic charge with the areal density $\sigma_m = \mu_0 M_0 \cos\theta$, where θ is the angle between the radius vector $\mathbf{r}$ of the point and the direction of the magnetization vector $\mathbf{M}_0$. As we know from the solution of Problem 2.28, a similar distribution, $\sigma = \sigma_0\cos\theta$, of the electric charge density on a spherical surface of radius R would induce, inside this surface, a uniform electric field $\mathbf{E}$ of magnitude

$$E = \frac{\sigma_0}{3\varepsilon_0}, \qquad \text{for } r < R,$$

directed opposite to the ray $\theta = 0$. But due to the similarity of the Poisson equations (1.27) and (5.133) for the electric and magnetic fields,[36] with the replacement

$$\frac{\rho}{\varepsilon_0} \leftrightarrow \frac{\rho_m}{\mu_0} \equiv -\nabla\cdot\mathbf{M}, \qquad \text{i.e. } \frac{\sigma}{\varepsilon_0} \leftrightarrow \frac{\sigma_m}{\mu_0},$$

we may immediately translate this result of electrostatics to magnetostatics, giving us the following uniform magnetic field induced by the external surface of our shell:

$$\mathbf{H}_2 = -\frac{\mathbf{M}_0}{3}, \qquad \text{for } r < R_2. \qquad (*)$$

For a uniform sphere of the radius R_2, this would be the final result; however, in our current case of an empty shell, the abrupt back leap of the magnetization on its inner surface gives an equal and opposite field component

$$\mathbf{H}_1 = \frac{\mathbf{M}_0}{3}, \qquad \text{for } r < R_1,$$

so inside the shell, the net field vanishes:

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 = 0, \quad \text{and } \mathbf{B} = \mu_0\mathbf{H} = 0, \qquad \text{for } r < R_1 < R_2.$$

Note, however, that the field inside the shell's material, i.e. at $R_1 < r < R_2$, does not vanish, being a sum of a purely dipole field of its inner surface and the uniform field (*) of the external surface.

[36] This similarity was already discussed in detail in the model solution of Problem 24.

Problem 5.28. A very broad film of thickness $2t$ is permanently magnetized normally to its plane, with a periodic checkerboard pattern, with the square of area $a\times a$:

$$\mathbf{M}\Big|_{|z|<t} = \mathbf{n}_z M(x,y), \qquad \text{with } M(x,y) = M_0 \operatorname{sgn}\left(\cos\frac{\pi x}{a}\cos\frac{\pi y}{a}\right).$$

Calculate the magnetic field's distribution in space.[37]

Solution: Due to the absence of stand-alone currents in the system, we may describe the magnetic field in it by using the scalar potential ϕ_m defined by Eq. (5.120) of the lecture notes:

$$\mathbf{H} = -\nabla\phi_\mathrm{m}. \tag{*}$$

According to Eq. (5.121), in each of the three uniform z-regions ($z < -t$, $-t < z < +t$, and $t < z$), the potential has to satisfy the Laplace equation. Taking into account the symmetry of the problem (which requires in particular that $\mathbf{H}(-z) = \mathbf{H}(z)$, so if $\phi_\mathrm{m}(0)$ is set to zero, then $\phi_\mathrm{m}(-z) = \phi_\mathrm{m}(z)$), and its $a\times a$ – periodicity in the x-y plane, it is natural to look for the solution in the following variable-separated form (very similar to that in Eq. (2.95) of the lecture notes):

$$\phi_\mathrm{m}(x,y,z) = \sum_{n,m=1}^{\infty} c_{nm}\cos\frac{\pi nx}{a}\cos\frac{\pi my}{a}\times\begin{cases}\operatorname{sgn}(z)\exp\{-\gamma_{nm}|z|\}/\exp\{-\gamma_{nm}t\}, & \text{for } |z|\geq t,\\ \sinh\gamma_{nm}z/\sinh\gamma_{nm}t, & \text{for } |z|\leq t.\end{cases} \tag{**}$$

(The denominators in the last operands are set up to have the first boundary condition at the film's surfaces, the potential's continuity, satisfied for any choice of the coefficients c_{nm}.) Indeed, the substitution of any term of this series to the Laplace equation shows that it is satisfied if the separation constants γ_{nm} obey the relation similar to Eq. (2.93):

$$\gamma_{nm} = \frac{\pi}{a}\left(n^2+m^2\right)^{1/2}. \tag{***}$$

According to Eq. (*), the distribution (**) corresponds to the following normal component of the magnetic field on the film's top surface:[38]

$$H_z(x,y,t\pm 0) = -\frac{\partial\phi_\mathrm{m}}{\partial z}\Big|_{z=t\pm 0} = \sum_{n,m=1}^{\infty} c_{nm}\gamma_{nm}\cos\frac{\pi nx}{a}\cos\frac{\pi my}{a}\times\begin{cases}1, & \text{for } z = t+0,\\ (-\cosh\gamma_{nm}t/\sinh\gamma_{nm}t), & \text{for } z = t-0.\end{cases}$$

On the other hand, Eq. (5.134) applied to the film's surface at $z = t$, gives us the following second boundary condition:

$$H_z(x,y,t+0) - H_z(x,y,t-0) = M(x,y),$$

so, for our particular magnetization pattern $M(x, y)$, the coefficients c_{nm} have to be found from the following system of equations:

$$\sum_{n,m=1}^{\infty} c_{nm}\gamma_{nm}\cos\frac{\pi nx}{a}\cos\frac{\pi my}{a}\left(1+\frac{\cosh\gamma_{nm}t}{\sinh\gamma_{nm}t}\right) = M_0\operatorname{sgn}\left(\cos\frac{\pi x}{a}\cos\frac{\pi y}{a}\right).$$

[37] This problem is of evident relevance for the *perpendicular magnetic recording* (PMR), which presently dominates high-density digital magnetic storage technology.

[38] The z-asymmetry of the solutions (**) takes care of the boundary condition at $z = -t$ automatically.

As usual, let us multiply both parts of this equation by a function from the same orthogonal set, in this case, $\cos(\pi n'x/a)\cos(\pi m'y/a)$ with arbitrary integer n' and m', and then integrate them over one of the x-y periods of the structure, say over the square $0 \le x, y \le a$. On the left-hand side, only terms with $n = n'$ and $m = m'$ survive, and we get[39]

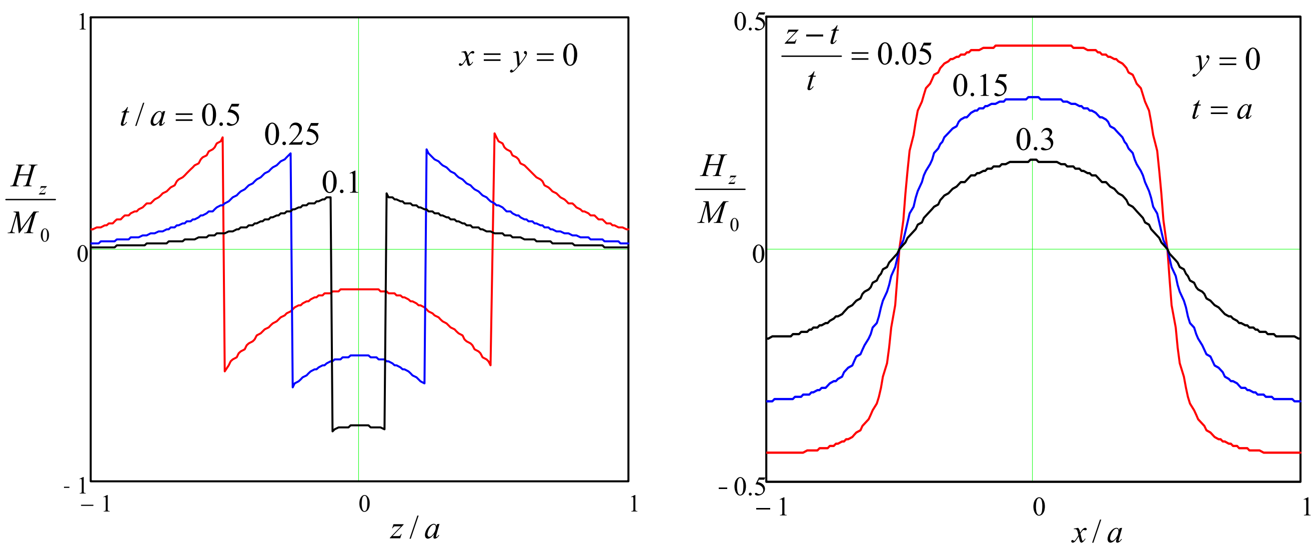
$$c_{nm}\gamma_{nm}\left(1+\frac{\cosh\gamma_{nm}t}{\sinh\gamma_{nm}t}\right)\frac{a^2}{4} = M_0\int_0^a dx\int_0^a dy\cos\frac{\pi nx}{a}\cos\frac{\pi my}{a}\times\operatorname{sgn}\left(\cos\frac{\pi x}{a}\cos\frac{\pi y}{a}\right)$$
$$\equiv M_0\frac{4a^2}{\pi^2 nm}\sin\frac{\pi n}{2}\sin\frac{\pi m}{2},$$

so the coefficients c_{nm} depend on a only implicitly, via γ_{nm}:

$$c_{nm} = \frac{16}{\pi^2 nm}M_0\frac{\sinh\gamma_{nm}t}{\gamma_{nm}(\sinh\gamma_{nm}t+\cosh\gamma_{nm}t)}\sin\frac{\pi n}{2}\sin\frac{\pi m}{2}. \qquad (****)$$

(Note that the last two factors make c_{nm} nonvanishing only if both integers n and m are odd, and also provide its sigh alternation for even n and m, speeding up the convergence of the series (**).)

Formulas (*)-(****) give a complete solution of the field distribution problem. For example, the left panel in the figure below shows the resulting H_z as a function of z at the center of one of the checkerboard squares (say, $x = y = 0$), for several values of the film thickness.



The jumps of the field by $\pm M_0$ at the film surfaces, and the resulting reversal of its sign, qualitatively similar to that in the previous problem, are clearly visible. From the point of view of digital magnetic recording application, these plots imply that to avoid a significant drop of the magnetic-field "readout signal", the film's half-thickness t should be larger than at least ~0.3a, where a^2 is the recorded bit area.

[39] The integration on the right-hand side is easy using the fact that within our integration area, the sgn function is equal to (+1) if both x and y are either between 0 and a/2, or between a/2 and a, and to (–1) in the complementary regions.

The right panel of the figure shows H_z as a function of x at $y = 0$, for several values of z, for a sufficiently thick film ($t = a$). Both panels show that in order to avoid a substantial signal loss, the magnetic field sensing device (in modern technology, a GMR sensor) has to be rather close to the film's surface. Indeed, the distance $(z - t) = 0.3t$ already causes more than a two-fold drop of the field's swing from its maximum value M_0.

Problem 5.29.* Based on the discussion of the quadrupole electrostatic lens in Sec. 2.4 of the lecture notes, suggest permanent-magnet systems that may similarly focus particles moving close to the system's axis, for the cases when each particle carries:

(i) an electric charge,

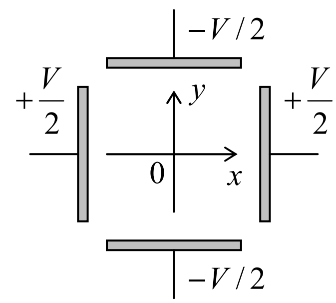
(ii) no net electric charge, but a spontaneous magnetic dipole moment **m** of a certain orientation.

Solutions:

(i) As was discussed in Sec. 2.4, a quadrupole lens is a cylindrical system exerting the force **F** with Cartesian components

$$F_x = -cx, \quad F_y = +cy \qquad (*)$$

(where c is a constant), on a particle moving along the z-axis. The electrostatic system of four hyperbolically-shaped electrodes, having this property in *all space* between them, is shown in Fig. 2.9 of the lecture notes.



However, it is clear that any cylindrical system of four electrodes with a similar symmetry of their voltage biases, for example, the one shown in the figure on the right, has the property (*) *near its center* (i.e. at $x, y \to 0$). Indeed, due to the mirror symmetry of the system relative to the planes $x = 0$ and $y = 0$, the electrostatic potential ϕ at the central line $x = y = 0$ should equal zero and have vanishing partial derivatives $\partial\phi/\partial x$ and $\partial\phi/\partial y$ (and, actually, all odd derivatives as well). Hence the Taylor expansion of the potential at that point has to start from quadratic terms, with equal but opposite coefficients for x and y:

$$\phi(x, y) = \kappa\left(x^2 - y^2\right) + O\left(\rho^4\right), \qquad \text{at } \rho \equiv \left(x^2 + y^2\right)^{1/2} \to 0,$$

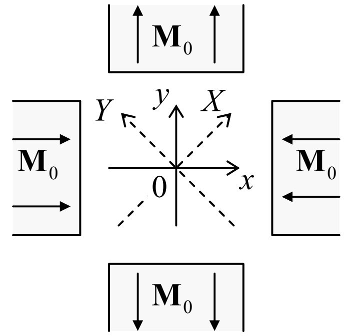
with $\kappa \sim V/a^2$, where a is the spatial scale of the system's cross-section. (As a sanity check, the leading term of this expansion[40] satisfies the Laplace equation $\nabla^2\phi = 0$.) Now calculating the force $\mathbf{F} = q\mathbf{E} = -q\nabla\phi$ exerted on a particle with an electric charge q, we get Eqs. (*) with $c = 2q\kappa$.

Now the analogy between the electrostatic field induced by surface electric charges, and the magnetic field induced by permanent magnets with the magnetization normal to their axis-facing surfaces (see, e.g., the discussion following Eq. (5.132) of the lecture notes and the model solutions of the four previous problems) implies that a similar distribution of the magnetic potential,

$$\phi_\mathrm{m}(x, y) = \kappa_\mathrm{m}\left(x^2 - y^2\right), \qquad \text{at } \rho \to 0, \qquad (**)$$

[40] Note that this term coincides with the exact potential distribution (2.75) in the hyperbolic-electrode system.

with $\kappa_\mathrm{m} \propto M_0$, may be obtained, for example, in the system shown in the figure on the right. The magnetic field in the free space between the magnets, corresponding to this potential, equals $\mathbf{B} = -\mu_0\nabla\phi_\mathrm{m}$, so its Cartesian components are



$$B_x = -2\mu_0\kappa_\mathrm{m} x, \quad B_y = 2\mu_0\kappa_\mathrm{m} y .$$

As a result, the Cartesian components of the magnetic Lorentz force $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$ exerted on a charged particle, moving along the z-axis, are

$$F_x = -qv_z B_y = -2\mu_0 q v_z \kappa_\mathrm{m} y, \qquad F_y = qv_z B_x = -2\mu_0 q v_z \kappa_\mathrm{m} x .$$

This result is different from Eqs. (*) not only by the signs but also by field components' proportionality to different coordinates. However, relative to the Cartesian coordinates $\{X, Y\}$ turned by angle $\pi/4$ relative to $\{x, y\}$, the force has the required properties. Indeed, writing the relations between the two pairs of coordinates, which are evident from the figure above,

$$X = \frac{x+y}{\sqrt{2}}, \quad Y = \frac{-x+y}{\sqrt{2}}; \qquad \text{so } x = \frac{X-Y}{\sqrt{2}}, \quad y = \frac{X+Y}{\sqrt{2}},$$

we may use them to recast Eq. (**) as

$$\phi_\mathrm{m} = \kappa_\mathrm{m}\left(x^2 - y^2\right) = \frac{\kappa_m}{2}\left[(X-Y)^2 - (X+Y)^2\right] \equiv -2\kappa_\mathrm{m} XY ,$$

so

$$B_X = 2\mu_0\kappa_\mathrm{m} Y, \quad B_Y = 2\mu_0\kappa_\mathrm{m} X , \qquad (***)$$

and the expressions for the Lorentz force components,

$$F_X = -qv_z B_Y = -2\mu_0 q v_z \kappa_\mathrm{m} X, \qquad F_Y = qv_z B_X = 2\mu_0 q v_z \kappa_\mathrm{m} Y ,$$

have the same functional form as in Eq. (*), with the coefficient $c = 2\mu_0 q v_z \kappa_\mathrm{m}$.

(ii) According to Eq. (5.102) of the lecture notes, the force exerted on a magnetic dipole $\mathbf{m}$ in the external magnetic field $\mathbf{B}$ is

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}). \qquad (****)$$

For the quadrupole magnetic field (***), this force is coordinate-independent, i.e. does not have the required focusing properties. A clue in the search for a suitable system may be obtained by rewriting Eq. (**) in the complex form:

$$\phi_\mathrm{m} = \kappa_\mathrm{m}\,\mathrm{Re}\left(\boldsymbol{z}^2\right), \qquad \text{with } \boldsymbol{z} \equiv x + iy \equiv \rho e^{i\varphi},$$

where φ is the polar angle on the $[x, y]$ plane. Now let us consider a different magnetic potential

$$\phi_\mathrm{m} = \mathrm{Re}\left(\lambda \boldsymbol{z}^3\right) \equiv |\lambda|\rho^3\cos(3\varphi + \theta),$$

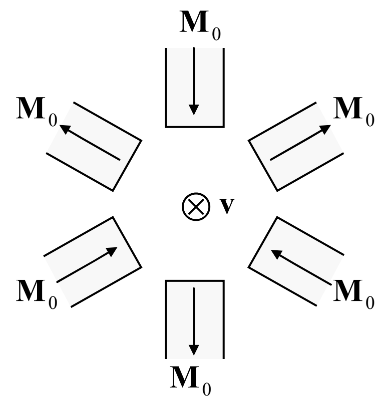
where λ is some complex constant, and θ is its argument. Such a function provides the extra power of $\boldsymbol{z}$, necessary to make the force (****) proportional to the particle's deviation from the system's axis. Indeed, calculating the radial component of the force (****) with $\mathbf{m}$ = const,[41] we get

[41] See, e.g., MA Eq. (10.2) – to be used twice.

$$F_\rho = 6\mu_0 |\lambda| \rho \left[-m_\rho \cos(3\varphi + \theta) + m_\varphi \sin(3\varphi + \theta) \right].$$

This expression shows that the component is indeed proportional to the distance ρ of the particle's trajectory from the system's axis, so it does provide beam focusing, with the focal distance depending both on the angle φ and the dipole moment's orientation.

In addition, according to the discussion in Sec. 2.4 of the lecture notes, any analytic complex function $\boldsymbol{f}(\boldsymbol{z})$, in particular $\lambda \boldsymbol{z}^3$, does satisfy the Laplace equation – and so does its real part. Hence, the above distribution $\phi_m(\rho, \varphi)$ may be obtained using some permanent magnet system. The form of this distribution implies that the system should be symmetric with respect to the rotation by $\Delta\varphi = 2\pi/3$, rather than by $\pi \equiv 2\pi/2$ for the quadrupole system shown in the figure above, i.e. needs 3 rather than 2 pairs of magnets of alternating polarity – for example as shown in the figure on the right.



Such *sextupole* (also called "hexapole") magnet systems are actually used in experiment, in particular, for focusing beams of neutrons with their non-zero dipole magnetic moments.

Chapter 6. Electromagnetism

Problem 6.1. Prove that the electromagnetic induction e.m.f. $\mathscr{V}_{\text{ind}}$ in a conducting loop may be measured as shown on two panels of Fig. 6.1 of the lecture notes:

(i) by measuring the current $I = \mathscr{V}_{\text{ind}}/R$ induced in the loop closed with an Ohmic resistor R, or

(ii) using a voltmeter inserted into the loop.

Solutions:

(i) The issue is not quite trivial, because the resistance is defined (see Sec. 4.2) as the ratio V/I, where V is a *charge-induced* voltage, in its turn defined as the difference of electrostatic potentials ϕ at the resistor's ends – see Eq. (2.25). On the other hand, for the vortex field $\mathbf{E}_{\text{ind}}$ of electromagnetic induction, such representation is generally impossible. Moreover, the Faraday law does not prescribe a unique spatial distribution of the field; rather it only specifies its contour integral (6.2) or, equivalently, the field's curl at each point – see Eq. (6.5). The exact particular distribution $\mathbf{E}_{\text{ind}}(\mathbf{r})$ depends not only on that the time evolution of the inducing magnetic field $\mathbf{B}(\mathbf{r}, t)$, but also on the geometry and properties of the system.

Note, however, that for the "usual" (charge-induced, potential) field $\mathbf{E}$, we may represent the voltage drop at a resistor as the integral

$$V = \int_{\text{A}}^{\text{B}} \mathbf{E} \cdot d\mathbf{r}$$

along any path connecting the resistor's terminals A and B. Now using the differential form (4.8) of the Ohm law, we may write

$$\int_{\text{A}}^{\text{B}} \mathbf{E} \cdot d\mathbf{r} = \int_{\text{A}}^{\text{B}} \frac{\mathbf{j}(\mathbf{r}) \cdot d\mathbf{r}}{\sigma(\mathbf{r})}.$$

The results of Sec. 4.3 show that the normalized law,

$$\mathbf{f}(\mathbf{r}) \equiv \frac{\mathbf{j}(\mathbf{r})}{I}, \qquad (*)$$

of the dc current distribution in a conductor is uniquely defined by Eq. (4.6), $\nabla \cdot \mathbf{j} = 0$, equivalent to $\nabla \cdot \mathbf{f} = 0$, with the boundary conditions $f_n = 0$ on the conductor's surface, and $f_\tau = 0$ on its interfaces with the external electrodes, so we may use Eq. (*) to express the above integral as

$$\int_{\text{A}}^{\text{B}} \mathbf{E} \cdot d\mathbf{r} = IR, \qquad \text{with } R \equiv \int_{\text{A}}^{\text{B}} \frac{\mathbf{f}(\mathbf{r}) \cdot d\mathbf{r}}{\sigma(\mathbf{r})}, \qquad (**)$$

the last equality giving an equivalent definition of the sample's resistance.

Now moving on to electromagnetic induction, we may argue that since the current distribution law $\mathbf{f}(\mathbf{r})$, and hence the resistance R, are governed by relations including $\mathbf{j}$ alone, they do not depend on the nature of the applied electric field. Hence in the particular case of a conductor, it is its geometry that dictates the field's distribution inside it:

$$\mathbf{E}_{\text{ind}} = \frac{\mathbf{j}(\mathbf{r})}{\sigma(\mathbf{r})} \equiv \frac{I}{\sigma(\mathbf{r})}\mathbf{f}(\mathbf{r}), \qquad (***)$$

leaving Eq. (6.2) to govern only the total magnitude I of the current. (This internal distribution, in turn, dictates that of the field $\mathbf{E}_{\text{ind}}(\mathbf{r})$ outside the conductor, following the conductivity hierarchy that was discussed in Sec. 4.3.)

Now let us consider a closed resistive loop that may be represented as the result of a merger of the endpoints A and B. Though such a closed loop does not have external electrodes, the condition $j_\tau = 0$ discussed above is still valid on an arbitrary cross-sectional surface normal, in each point, to the vector $\mathbf{j}$. (Such surface may be understood as a galvanic connection of two ultimately thin external electrodes.) Hence, all the above relations are valid for such a closed loop as well, so integrating Eq. (***) along it, and then using Eq. (**), we get

$$\mathscr{V}_{\text{ind}} \equiv \oint_C \mathbf{E}_{\text{ind}} \cdot d\mathbf{r} = I\oint_C \frac{\mathbf{f}(\mathbf{r})\cdot d\mathbf{r}}{\sigma(\mathbf{r})} \equiv IR,$$

thus giving the required proof of the relation $I = \mathscr{V}_{\text{ind}}/R$.

(ii) The prevailing *electrodynamic* species of voltmeters[42] are essentially sensitive galvanometers that measure the weak current I induced inside them by the voltage under measurement, recalibrated into the voltmeter readout as

$$V = R_{\text{int}} I,$$

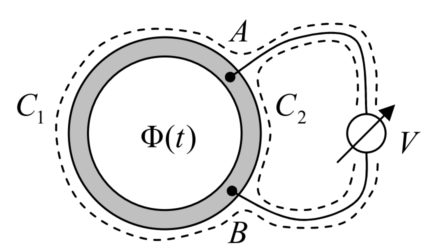
where R_{int} is the voltmeter's internal resistance. As it follows from the solution of Task (i), for our system (the voltmeter inserted into the loop) we may write

$$I = \frac{\mathscr{V}_{\text{ind}}}{R_{\text{int}} + R_{\text{ext}}},$$

where R_{ext} is the resistance of the loop itself. Combining the two displayed formulas, we get

$$V = \frac{R_{\text{int}}}{R_{\text{int}} + R_{\text{ext}}}\mathscr{V}_{\text{ind}}.$$

This result shows that if the internal resistance of the voltmeter is appropriately high, $R_{\text{int}} >> R_{\text{ext}}$, it indeed measures the induced e.m.f. faithfully. The situation, however, changes if a voltmeter does not interrupt the induction loop, but rather is connected in parallel with its part – see, for example, the next problem.



Problem 6.2. The flux Φ of the magnetic field that pierces a resistive ring is being changed in time, while the field outside of the ring is negligibly low. A voltmeter is connected to a part of the ring, as shown in the figure on the right. What would the voltmeter show?

Solution: A naïve application of the Faraday induction law (6.2) to contour C_1 (shown with the external dashed line in the figure

[42] The alternative, *electrostatic voltmeter* species, will be briefly discussed in SM Sec. 6.5. Those voltmeters are typically slow and not used for the measurement of such transient effects as electromagnetic induction.

above) would give

$$V = \oint_{C_1} \mathbf{E}_{\text{ind}} \cdot d\mathbf{r} = -\dot{\Phi}. \qquad \text{[WRONG!]}$$

The fallacy of this approach becomes evident if we notice that it might be equally applied to contour C_2, giving a totally different answer:

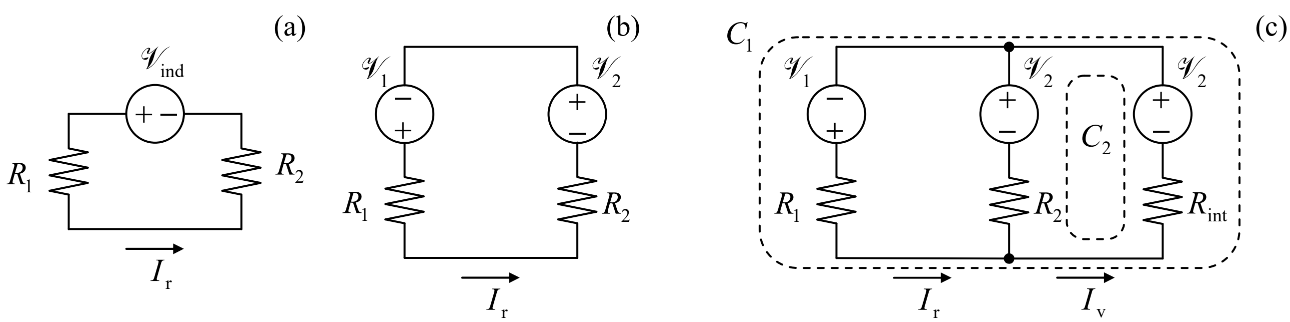
$$V = \oint_{C_2} \mathbf{E}_{\text{ind}} \cdot d\mathbf{r} = 0. \qquad \text{[WRONG!]}$$

The reason for these errors is the lack of account for the Ohmic voltage drops on the current-carrying parts of the ring.

The simplest way to get the correct answer is to use the *equivalent lumped circuit* language, which is broadly used in electrical engineering and was briefly discussed in Secs. 4.1 and 6.6 of the lecture notes. First, note that in the absence of the voltmeter, the system is similar to that analyzed in the previous problem, so according to its solution, the current I_r in the ring may be expressed as

$$I_{\text{r}} = \frac{\mathscr{V}_{\text{ind}}}{R} = \frac{\mathscr{V}_{\text{ind}}}{R_1 + R_2}, \qquad \text{with } \mathscr{V}_{\text{ind}} \equiv \oint_{\text{ring}} \mathbf{E}_{\text{ind}} \cdot d\mathbf{r},$$

where R_1 and R_2 are the resistances of the two parts of the ring between the contact points A and B, i.e. running along the contours C_1 and C_2, respectively. This result may be obtained from the equivalent circuit shown on panel (a) of the figure below, where each circle means a perfect voltage source, i.e. an imaginary two-terminal device that maintains the voltage equal to the specified e.m.f. (in this case, $\mathscr{V}_{\text{ind}}$) between its terminals, regardless of the external circuit – see the model solution of Problem 4.2.



Since the integral defining $\mathscr{V}_{\text{ind}}$ may be always broken into two parts, corresponding to the two fragments of the ring,

$$\mathscr{V}_{\text{ind}} = \mathscr{V}_1 + \mathscr{V}_2, \qquad \text{where } \mathscr{V}_1 \equiv \underset{(\text{along } C_1)}{\int_A^B} \mathbf{E}_{\text{ind}} \cdot d\mathbf{r}, \qquad \mathscr{V}_2 \equiv \underset{(\text{along } C_2)}{\int_B^A} \mathbf{E}_{\text{ind}} \cdot d\mathbf{r},$$

the equivalent circuit (a) may be transformed into another one, shown on panel (b). The latter circuit gives the same current as the former one (hence the term "equivalent") but has the advantage that it may be readily generalized to the case of the voltmeter connected between points A and B – see panel (c). The only nontrivial component of the last circuit is the additional source of the same e.m.f. $\mathscr{V}_2$ in the voltmeter branch, which is necessary to satisfy the Faraday induction law for the contour C_2:

$$\oint_{C_2} \mathbf{E}_{\text{ind}} \cdot d\mathbf{r} = \mathscr{V}_2 - \mathscr{V}_2 = 0.$$

Note that there is nothing unphysical in that compensating e.m.f. source, because the induced electric field $\mathbf{E}_{\text{ind}}$ is *not* localized in the magnetic field's region (only its curl is – see Eq. (6.5) of the lecture notes), and may extend to the whole space. (For example, if the ring and the magnetic field distribution in our current problem are axially symmetric, so should the field $\mathbf{E}_{\text{ind}}$,[43] and Eq. (6.2) is satisfied with the following solution

$$\mathbf{E}_{\text{ind}} = \mp \frac{|\dot{\Phi}|}{2\pi\rho}\mathbf{n}_\varphi,$$

where ρ is the distance from the system's axis, and the sign depends on the magnetic field's direction.) This field extends to infinity and induces the e.m.f. not only in the ring but also in the voltmeter's wires, so in contour C_2, they exactly compensate each other.

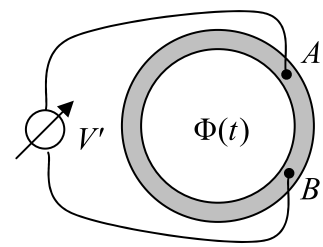
Now writing the usual Kirchhoff laws for the circuit (c) (see the discussion in Secs. 4.1 and 6.6) and solving the resulting simple system of equations for the voltmeter current I_v, we get a somewhat bulky result, which, in the most important limit $R_{\text{int}} >> R_{1,2}$ (indeed, only in this case, a galvanometer may be called a voltmeter) reduces to the simple form[44]

$$I_v = \frac{\mathscr{V}_{\text{ind}}}{R_{\text{int}}}\frac{R_2}{R_1 + R_2}, \quad \text{for } R_i >> R,$$

so the voltmeter readout is

$$V \equiv I_v R_{\text{int}} = \frac{R_2}{R_1 + R_2}\mathscr{V}_{\text{ind}} \le \mathscr{V}_{\text{ind}}.$$

Note that if the voltmeter was connected to the same points A and B, but physically located on the other side of the ring (see the figure on the right), the measurement result would be different:



$$V' = -\frac{R_1}{R_1 + R_2}\mathscr{V}_{\text{ind}},$$

the minus sign being valid if the positive and negative terminals of the voltmeter are still connected to the same points of the ring. This sign difference enables the popular and spectacular lecture demonstration, in which two voltmeters placed on opposite sides of an induction ring and connected *simultaneously* to the same points with the same polarity, show readouts of opposite signs.

Such experiments, as well as the above analysis, clearly show that at the Faraday induction (and generally in electromagnetism) such a notion as "voltage between points A and B" depends on the way of its measurement, and is not uniquely defined without such specification. This is a natural result of the vortex nature of the induced electric field, which (in contrast to the electrostatic field) cannot be represented as a gradient of a certain scalar function such as ϕ – whose difference the voltage would be.

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[43] Strictly speaking, this symmetry is violated by the voltmeter's wires; this perturbation may be minimized by running wire fragments mostly along concentric circles.

[44] Actually, the calculation may be much simplified by the replacement of the connection in series in each branch with the parallel connection of the same resistor and a perfect current generator – see the model solution of Problem 4.2.

Problem 6.3. A weak constant magnetic field **B** is applied to a small, axially-symmetric permanent magnet, with its dipole magnetic moment **m** directed along its axis and rapidly rotating about the same axis, with an angular momentum **L**. Calculate the electric field resulting from the field's application, and formulate the conditions of your result's validity.

Solution: According to Eq. (5.101) of the lecture notes, the magnetic field exerts the torque

$$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}$$

on the dipole. Per the basic law of rotational dynamics,[45] the torque makes the dipole's angular moment evolve as $\dot{\mathbf{L}} = \boldsymbol{\tau}$. Since, per our problem's conditions, the vectors **L** and **m** maintain the same direction (along the symmetry axis of the magnet), we may write $\mathbf{m} = \gamma\mathbf{L}$.[46] As a result, the dipole moment changes in time as

$$\dot{\mathbf{m}} = \gamma\dot{\mathbf{L}} = \gamma\boldsymbol{\tau} = \gamma\mathbf{m} \times \mathbf{B}\,. \qquad (*)$$

According to basic kinematics,[47] this equation describes an additional, relatively slow rotation (the so-called *torque-induced precession*) of the vector **m**, and hence of the magnet's symmetry axis, about the direction of the magnetic field's vector **B**, with the angular velocity

$$\boldsymbol{\Omega} = -\gamma\mathbf{B}\,, \qquad (**)$$

thus maintaining the angle θ between the vectors **m** and **B** at its initial value. Directing the z-axis along the magnetic field, we may describe this rotation as

$$\mathbf{m} = \mathbf{m}_z + \mathbf{m}_\rho(t), \quad \text{with } \mathbf{m}_z = \mathbf{n}_z m\cos\theta, \quad \mathbf{m}_\rho(t) = m\sin\theta\,\mathrm{Re}\left[\left(\mathbf{n}_x - i\mathbf{n}_y\right)\exp\left\{i\left(\Omega t + \varphi_0\right)\right\}\right], \qquad (***)$$

where φ_0 is the constant phase, which is also determined by the initial conditions (and the selected time origin).

Due to the Faraday induction, this rotation induces, in the space around the magnet, an electric field **E** changing in time periodically, with the same frequency Ω. To calculate the field, we may use the first of Eqs. (6.7) of the lecture notes, which, in the absence of an electrostatic potential ϕ, reduces to

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}.$$

According to Eq. (5.90), the vector potential of the magnetic dipole's field is

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \mathbf{r}}{r^3}\,, \qquad (****)$$

so the electric field at a fixed point **r** is

$$\mathbf{E} = -\frac{\mu_0}{4\pi}\frac{\dot{\mathbf{m}} \times \mathbf{r}}{r^3}.$$

Now using Eqs. (*)-(****), we finally get

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[45] See, e.g., CM Eq. (1.33).

[46] In atomic and nuclear physics, the scalar coefficient γ is frequently a fundamental constant (see, e.g., Eq. (5.95) of the lecture notes), called the *gyromagnetic ratio*. It will be repeatedly discussed in the QM part of this series.

[47] See, e.g., CM Sec. 4.1, in particular Eq. (4.8).

$$\mathbf{E} = \frac{\mu_0}{4\pi}\gamma\frac{(\mathbf{B}\times\mathbf{m})\times\mathbf{r}}{r^3} = \frac{\mu_0}{4\pi}\gamma B\frac{[\mathbf{n}_z\times\mathbf{m}_\rho(t)]\times\mathbf{r}}{r^3} = \frac{\mu_0}{4\pi}\gamma Bm\sin\theta\frac{\mathrm{Re}[(\mathbf{n}_y + i\mathbf{n}_x)\times\mathbf{r}\exp\{i(\Omega t+\varphi_0)\}]}{r^3}$$
$$= \frac{\mu_0}{4\pi}\gamma Bm\sin\theta\frac{(\mathbf{n}_x z - \mathbf{n}_z x)\cos(\Omega t+\varphi_0) + (\mathbf{n}_y z - \mathbf{n}_z y)\sin(\Omega t+\varphi_0)}{r^3}.$$

So, the oscillation of any Cartesian component of the induced electric field in time is exactly sinusoidal, and the amplitude of these oscillations drops with the distance as $1/r^2$. This result is only valid in the quasistatic approximation i.e. at $r << c/\Omega$; as will be extensively discussed in Chapter 8 of the lecture notes, at larger distances, Eq. (****) needs to be amended, leading to the electromagnetic radiation of waves with their electric and magnetic fields proportional to $1/r$ at $r >> c/\Omega$.[48]

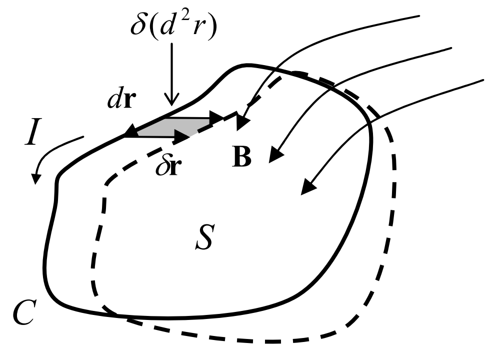
Another condition of quantitative validity of our result follows from the theory of torque-induced precession of such "symmetric tops":[49] the applied field **B** should be sufficiently low to keep the precession-induced addition $I\Omega$ to the basic angular momentum **L** negligibly small:

$$I|\Omega| << L, \qquad \text{i.e. } B << \frac{m}{\gamma^2 I},$$

where I is the principal moment of inertia of the magnet's rotation about any axis normal to its symmetry axis.

Problem 6.4. The similarity of Eq. (5.53) obtained in Sec. 5.3 without any use of the Faraday induction law, and Eq. (5.54) proved in Sec. 6.2 using it, implies that the law may be derived from magnetostatics. Prove that this is indeed true for a particular case of a current loop being slowly deformed in a fixed magnetic field $\mathbf{B}(\mathbf{r})$.

Solution: Consider a thin wire loop with current I, placed in a magnetic field $\mathbf{B}$ – see the figure on the right. According to Eq. (5.21) of the lecture notes, the magnetic force exerted by the field upon a small fragment dr of the wire is



$$d\mathbf{F} = I(d\mathbf{r}\times\mathbf{B}) \equiv -I(\mathbf{B}\times d\mathbf{r}),$$

where d**r** is the vector tangential to the loop's contour and directed along the current I. Now let the wire be slightly (and slowly) deformed so that this particular fragment is displaced by a small distance $\delta\mathbf{r}$. (Let me hope that the figure above makes the difference between the elementary vectors $d\mathbf{r}$ and $\delta\mathbf{r}$ absolutely clear.)

In order to keep the wire's acceleration (and hence the kinetic energy of the system) negligibly small, some other external forces should balance the force $d\mathbf{F}$, providing an equal and opposite force.

[48] See, in particular, Eqs. (8.124)-(8.126). Note that in quantum mechanics, the torque-induced precession as such does not lead to radiation; only quantum transitions between the quantized precession states (with different quantized values of $m_z = m\cos\theta$) lead to radiation with the frequency Ω – see, e.g., QM Chapter 9. However, for the values of L much larger than Planck's constant, the final conclusions of the classical and quantum theories coincide.

[49] See, e.g., CM Sec. 4.5 – in particular, Eqs. (4.73) and (4.85) and their discussion.

With the account of this opposite direction, the work of these external forces at the displacement $\delta\mathbf{r}$, and hence the change of the magnetic potential energy U of the system, is

$$\delta(dU) = -d\mathbf{F}\cdot\delta\mathbf{r} = I\delta\mathbf{r}\cdot(\mathbf{B}\times d\mathbf{r}).$$

Let us apply to this mixed product the operand rotation rule of the vector algebra[50] in such a way that the vector $\mathbf{B}$ comes out of the vector product:

$$\delta(dU) = I\mathbf{B}\cdot(d\mathbf{r}\times\delta\mathbf{r}). \qquad (*)$$

But the magnitude of the vector product in the parentheses is nothing more than the area $\delta(d^2r) \equiv \delta(dA)$ swept by the wire's fragment at the deformation (see the figure above again), while its direction coincides with the unit vector $\mathbf{n} = (d\mathbf{r}/dr)\times(\delta\mathbf{r}/\delta r)$ normal to this elementary area $d\mathbf{A}$. The scalar multiplication of the vector $\mathbf{B}$ by this unit vector is equivalent to taking the $\mathbf{B}$'s component B_n along this normal. Hence, integrating Eq. (*) over all the wire length, we get the following result for the variation of the total magnetic energy of the system:

$$\delta U = I\oint_C B_n \delta(d^2r).$$

If $\mathbf{B}$ is fixed, i.e. does not change at the loop's deformation, the variation sign may be moved out from the integral, and we get

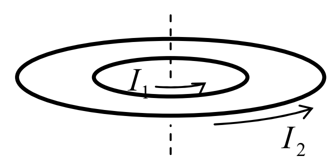
$$\delta U = I\delta\Phi, \qquad (**)$$

where Φ is the magnetic flux through the loop.

Now let the work $\delta\mathscr{W}$ necessary for this energy change come from a generator of an external voltage V_{ext}, inserted somewhere in the loop. In order for the system to stay in quasi-equilibrium during the slow deformation of the loop, this voltage should counter-balance the electromagnetic induction's e.m.f.: $V_{\text{ext}} = -\mathscr{V}_{\text{ind}}$. The work of this voltage at the transfer of charge $\delta Q = I\delta t$ during a small time interval δt is

$$\delta\mathscr{W} = V_{\text{ext}}\delta Q = -\mathscr{V}_{\text{ind}}\delta Q = -\mathscr{V}_{\text{ind}} I\delta t.$$

Requiring this work to be equal to the potential energy's change (**) it causes, we arrive at the Faraday induction law (6.2) – for this particular case only. As was discussed in sec. 6.1 of the lecture notes, the law is actually much broader.



Problem 6.5. Could Problem 5.2 (i.e. the semi-quantitative analysis of the mechanical stability of the system shown in the figure on the right) be solved using potential energy arguments?

Solution: Since in this problem, the currents are assumed to be fixed, the stable equilibrium of the system corresponds to the minimum of the Gibbs potential energy given by Eqs. (6.17) and (6.20) of the lecture notes. In our magnetic-material-free case ($\mu = \mu_0$), it reads[51]

[50] See, e.g., MA Eq. (7.6).

[51] Note that the sign in this expression (and hence the conclusions of this analysis) would be opposite if not currents in the loops but individual magnetic fluxes in each of them were fixed, as they may be if the loops are superconducting – see, e.g., Sec. 6.5 of the lecture notes.

$$U_G = -\frac{1}{2\mu_0}\int B^2 d^3 r\,. \qquad (*)$$

Using the linear superposition principle, we may represent the vector **B** at each point as the sum $\mathbf{B}_1(\mathbf{r}) + \mathbf{B}_2(\mathbf{r})$, where each field is proportional to the corresponding current. Plugging this expression for **B** into Eq. (*), we may rewrite it as

$$U_G = U_1 + U_2 + U_{\text{int}},$$

where $U_k \propto I_k^2$, and only the last term

$$U_{\text{int}} = -\frac{1}{\mu_0}\int \mathbf{B}_1(\mathbf{r})\cdot\mathbf{B}_2(\mathbf{r})d^3 r\,, \qquad (**)$$

describing the energy of magnetic interaction of the loops, depends on their mutual orientation.

This energy is evidently smallest (and hence corresponds to the system's equilibrium) if the directions of the fields $\mathbf{B}_1$ and $\mathbf{B}_2$ coincide in the regions where they are strongest, namely near the common center of the loops.[52] In particular, any turn of the inner loop by angle θ from the common axis decreases the magnitude of U_{int} by a factor $\sim\cos\theta$ and hence increases the energy, so the stable equilibrium corresponds to $\theta = 0$. (In the limit of a relatively small radius of the internal loop, when it may be well approximated by its dipole moment **m** (5.97) whose direction coincides with that of the field $\mathbf{B}_1$ at the center of the ring, this conclusion is directly confirmed by Eq. (5.100) of the lecture notes.)

Similarly, if the inner loop is displaced, from the plane of the outer loop, in either direction along the common axis, the "field overlap" integral in Eq. (**) decreases, so the stable equilibrium corresponds to its coplanar position.

However, the analysis of the system's stability (or rather instability :-) with respect to the lateral mutual displacement of the rings, by using the energy arguments, is somewhat less apparent than that using the interaction force arguments – see the model solution of Problem 5.2.

<u>Problem 6.6</u>. Use energy arguments to calculate the pressure exerted by the magnetic field **B** inside a long uniform solenoid of length l and a cross-section of area $A << l^2$, with $N >> l/A^{1/2} >> 1$ turns, on its "walls" (windings), and the forces exerted by the field on the solenoid's ends, for two cases:

(i) the current through the solenoid is fixed by an external source, and

(ii) after the initial current setting, the ends of the solenoid's wire, with negligible resistance, are connected, so it continues to carry a non-zero current.

Compare the results, and give a physical interpretation of the direction of these forces.

Solutions:

(i) If the current in the solenoid is maintained by an external current source, its equilibrium corresponds to the minimum of the total potential energy of the system (solenoid + source), i.e. of the

[52] Strictly speaking, the individual fields $\mathbf{B}_1$ and $\mathbf{B}_2$ are the largest in close vicinities of the corresponding wires, but since here the field lines are closed circles, i.e. the regions with equal and opposite values of the scalar product $\mathbf{B}_1\cdot\mathbf{B}_2$ are virtually equal in volume, they give a negligible net contribution to U_{int}.

Gibbs energy U_G of the solenoid. In a long solenoid (of length $l >> A^{1/2}$), with dense winding ($N >> l/A^{1/2} >> 1$), the internal field's **H** magnitude is given by Eq. (5.141):

$$H = \frac{NI}{l},$$

While outside the solenoid, it is virtually zero outside the solenoid. Hence we may use Eq. (6.20) to get

$$U_G = -\frac{B^2}{2\mu}V = -\frac{\mu H^2}{2}Al \equiv -\frac{\mu N^2 I^2 A}{2l} . \qquad (*)$$

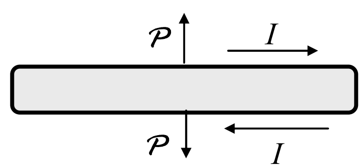
This expression shows that to minimize U_G (i.e. to maximize its magnitude), the field "tries" to *increase* the area A of the solenoid and *decrease* its length l. These effects may be quantified by the calculation, respectively, of the pressure on the "walls" (windings),[53]

$$\mathcal{P} \equiv -\frac{\partial U_G}{\partial V}\Big|_{l=\text{const}} = -\frac{1}{l}\frac{\partial U_G}{\partial A} = \frac{\mu N^2 I^2}{2l^2} \equiv \frac{B^2}{2\mu}, \qquad (**)$$

and the effective force applied to its ends:

$$F \equiv -\frac{\partial U_G}{\partial l} = -\frac{\mu N^2 I^2 A}{2l^2} \equiv -\frac{B^2}{2\mu}A . \qquad (***)$$

The directions of these forces may be readily interpreted using the basic Eq. (5.1). Indeed, the lateral force on a current-carrying "wall" of the solenoid is dominated by the oppositely-directed current in the opposite wall, corresponding to their *repulsion*. (This is especially apparent if the cross-section has an elongated shape – see the figure on the right.) On the other hand, the force in the direction along the solenoid's axis, exerted on each wire turn, is dominated by its adjacent turns, with the same current direction, corresponding to the *attraction* of the turns.



(ii) If the solenoid is disconnected from the current source, its equilibrium corresponds to the minimum of its own potential energy, U, which may be calculated using Eq. (6.15):

$$U = \frac{B^2}{2\mu}V \equiv \frac{\mu N^2 I^2 A}{2l} .$$

This expression differs from Eq. (*) only by the sign, and it may look that now the results (**) and (***) would be opposite. However, this is not so. Indeed, if the wire's resistance is negligible, it cannot sustain any voltage, in particular Faraday's e.m.f. (6.2) in the closed loop formed by the connection of the wire's ends, and hence the magnetic flux Φ in the loop, rather than the current I, remains constant at the solenoid's shape variations. (As was discussed in Sec. 6.4-6.5, if the wiring is superconducting, this flux conservation is perfect, but even in realistic large, high-μ magnets with "normal" wires, the flux relaxation time $\tau = L/R$ may be in hours, much longer than needed for force measurements.)

Now using Eq. (5.143) of the lecture notes for the self-inductance L of the solenoid,

$$L \equiv \frac{\Phi}{I} = \mu n^2 Al \equiv \frac{\mu N^2 A}{l},$$

[53] As will be shown in Sec. 9.8 of the lecture notes, the last expression is very general.

we may eliminate the current from the above expression for U:

$$U = \frac{LI^2}{2} \equiv \frac{\Phi^2}{2L} \equiv \frac{\Phi^2 l}{2\mu N^2 A}.$$

From here, we may readily calculate the pressure on the walls as

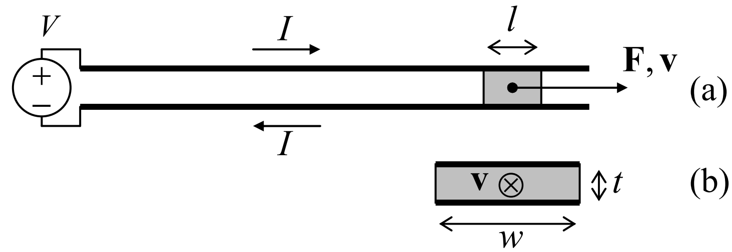
$$\mathcal{P} \equiv -\frac{\partial U}{\partial V}\Big|_{l=\text{const}} = -\frac{1}{l}\frac{\partial U}{\partial A} = \frac{\Phi^2}{2\mu N^2 A^2} \equiv \frac{B^2}{2\mu},$$

and the longitudinal force as

$$F \equiv -\frac{\partial U}{\partial l} = -\frac{\Phi^2}{2\mu N^2 A} \equiv -\frac{B^2}{2\mu}A,$$

getting exactly the same results as given by Eqs. (**) and (***).

Note that this problem, for a system with a uniform *magnetic* field (a long solenoid), is a virtually exact analog of Problems 3.27-28 for systems with a uniform *electric* field (planar capacitors), with a similar general conclusion: the (correctly calculated :-) forces do not depend on whether the system is connected to an external source or not, despite using different potential energies for the analyses of these cases. Let me hope the solutions of these three problems make the notion of the Gibbs potential energy, and the conditions of its application, even more clear.



Problem 6.7. The *electromagnetic railgun* is a projectile launch system consisting of two long parallel conducting rails and a sliding conducting projectile shorting the current I fed into the system by a powerful source – see panel (a) in the figure of the right. Calculate the force exerted on the projectile, using two approaches:

(i) by a direct calculation, assuming that the cross-section of the system has the simple shape shown on panel (b) of the figure above, with $t << w$, l, and

(ii) by using the energy balance (for simplicity, neglecting the Ohmic resistances in the system), and compare the results.

Solutions:

(i) Due to the condition $t << l$, w, the magnetic field induced by the shorting currents in the projective itself is negligible in comparison with that induced by the current in the rails. As was calculated in the solution of Problem 5.4, the latter magnetic field is directed normally to the rails, and its magnitude, in a particular cross-section of the projectile, is

$$B(x) = \mu\frac{I(x)}{w}.$$

Here x is the distance (along the rails) from one end of the projectile to its other end, so $I(0) = I$, and $I(l) = 0$. According to Eq. (5.8) of the lecture notes, the force $d\mathbf{F}$ exerted by this field on a small fragment dx

of the projectile, which carries the shorting current $dI = J(x)dx$ (where J is its linear density), is directed along the rails, always away from the current source (see the figure above), and has the magnitude

$$dF = B(x)t dI = J(x)B(x)t dx = \frac{\mu t}{w} J(x)I(x)dx ,$$

so the total force exerted on the projectile is

$$F = \int_{x=0}^{x=l} dF = \frac{\mu t}{w}\int_0^l J(x)I(x)dx .$$

Since the current's conservation requires that $dI(x)/dx = -J(x)$, we get the following simple result,

$$F = -\frac{\mu t}{w}\int_0^l \frac{dI(x)}{dx} I(x)dx \equiv -\frac{\mu t}{w}\int_{x=0}^{x=l} I(x)dI(x) \equiv -\frac{\mu t}{w}\left[\frac{I^2(x)}{2}\right]_{x=0}^{x=l} = \frac{\mu t}{w}\frac{I^2}{2}, \qquad (*)$$

valid regardless of the distribution of the shorting currents $J(x)$ inside the projective (which may be due to, for example, its nonuniform conductivity).

Note that according to another result of Problem 5.4 (in an evident way generalized to the case $\mu \neq \mu_0$), the first fraction in the last form of Eq. (*) is just the self-inductance L_0 of the rail system (or if you like, the mutual inductance between the rails) per unit length, so our result may be represented in a simpler form:

$$F = \frac{L_0 I^2}{2} . \qquad (**)$$

As we will see in a minute, this simplicity is not occasional.

(ii) At negligible resistance of all components of the system, the only voltage V between the rails at the current source's location is the Faraday induction e.m.f. $\mathscr{V}_{\text{ind}} = -d\Phi/dt$. So, the elementary work done by the source *against* the e.m.f. during an elementary time interval dt is $d\mathscr{W} = -\mathscr{V}_{\text{ind}}dQ = -\mathscr{V}_{\text{ind}}Idt = Id\Phi$. If the projectile is at a distance $X >> l$ from the current source, the flux $\Phi = LI$ may be well approximated as L_0XI, so we get

$$d\mathscr{W} = Id\Phi = Id(L_0XI) \equiv L_0I^2dX + L_0XIdI . \qquad (***)$$

On the other hand, the change of the magnetic energy (5.71) of the loop during the same time interval is

$$dU = d\left(\frac{LI^2}{2}\right) = d\left(\frac{L_0XI^2}{2}\right) \equiv \frac{1}{2}L_0I^2dX + L_0XIdI ,$$

so Eq. (***) may be rewritten as

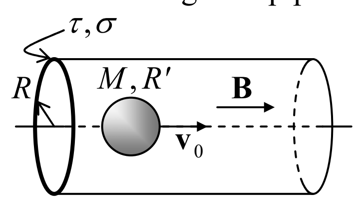
$$d\mathscr{W} = dU + \frac{1}{2}L_0I^2dX .$$

Due to the conservation of the total energy of the system, the last term has to describe the mechanical work $d\mathscr{W}_{\text{m}} = FdX$ of the magnetic field on the projectile, so for the force F, we again get Eq. (**), showing that this expression is independent of the particular cross-section of the system.

In the modern (so far, experimental) railgun systems, the (pulsed) driving current, typically supplied by a capacitor battery's discharge, may be as high as $\sim 5\times10^6$ A, so Eq. (**), with $L_0 \sim \mu_0 \sim 10^{-6}$ H/m, shows that the force may be above 10^7 N, providing a projectile's acceleration approaching $\sim 10^6$

m/s^2 (i.e. ~10^5 g!). Even much higher accelerations may be achieved in railguns with plasma projectiles, which are being explored as possible reaction triggers in inertial-confinement nuclear fusion systems.

Problem 6.8. A uniform static magnetic field **B** is applied along the axis of a long thin pipe of radius R and wall thickness $\tau \ll R$, made of a material with Ohmic conductivity σ. A sphere of mass M and radius $R' \ll R$, made of a linear magnetic material with permeability $\mu \gg \mu_0$, is launched, with an initial velocity v_0, to fly ballistically along the pipe's axis – see the figure on the right. Use the quasistatic approximation to calculate the distance the sphere would pass before it stops. Formulate the conditions of validity of your result.



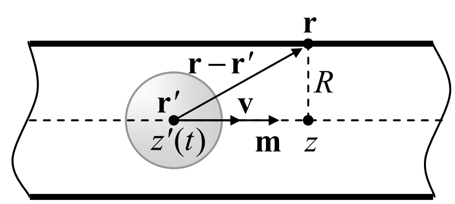
Solution: According to Eqs. (5.125) of the lecture notes, taken in the limit $\mu \gg \mu_0$, the magnetic field distortion created by such a sphere outside it coincides with the field of a point magnetic dipole with the moment[54]

$$\mathbf{m} = \frac{4\pi R'^3}{\mu_0}\mathbf{B}, \qquad (*)$$

positioned in the sphere's center. In the quasistatic approximation, the vector potential created by the dipole at a fixed point **r** may be found from Eq. (5.90) with the duly shifted origin:

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times[\mathbf{r}-\mathbf{r}'(t)]}{|\mathbf{r}-\mathbf{r}'(t)|^3},$$

where **r**' is the radius vector of the effective dipole, i.e. of the sphere's center. According to this formula, for our axially symmetric geometry (see the figure on the right), the vector potential is purely azimuthal, $\mathbf{A} = A\mathbf{n}_\varphi$, and its magnitude at the pipe's wall is



$$A\big|_{\text{wall}} = \frac{\mu_0}{4\pi}\frac{mR}{\left\{[z-z'(t)]^2+R^2\right\}^{3/2}}.$$

Now we may use Eq. (5.65) to calculate the magnetic flux through a cross-section of the pipe:

$$\Phi(z,t) \equiv \oint_{R'} \mathbf{A}\big|_{\text{wall}}\cdot d\mathbf{r} = 2\pi R A\big|_{\text{wall}} = \frac{\mu_0 m R^2}{2}\frac{1}{\left\{[z-z'(t)]^2+R^2\right\}^{3/2}}.$$

According to Faraday's induction law (6.2), the rate of change of this flux in time,

$$\frac{\partial\Phi(z,t)}{\partial t} = \frac{\mu_0 m R^2}{2}\frac{\partial}{\partial z'}\left(\frac{1}{\left\{[z-z'(t)]^2+R^2\right\}^{3/2}}\right)\frac{dz'}{dt} = \frac{\mu_0 m R^2}{2}\left(\frac{3[z-z'(t)]}{\left\{[z-z'(t)]^2+R^2\right\}^{5/2}}\right)v, \qquad (**)$$

[54] Note that the moment is automatically aligned with the applied magnetic field, so all following results are robust with respect to the sphere's rotation – unless it is so fast that a finite re-magnetization speed does not allow **m** to follow its quasi-stationary value (*).

(where $v \equiv dz'/dt$ is the velocity of the sphere) causes an equal but opposite e.m.f. $\mathscr{V}_{\text{ind}}$. Due to the axial symmetry of our system, the induced electric field's magnitude does not depend on the azimuthal angle:

$$\mathbf{E}_{\text{ind}}(\mathbf{r},t) = \mathbf{n}_\varphi E_{\text{ind}}(z,t), \qquad \text{with } E_{\text{ind}}(z,t) = \frac{\mathscr{V}_{\text{ind}}(z,t)}{2\pi R} = -\frac{\partial \Phi(z,t)}{\partial t}\frac{1}{2\pi R}.$$

In the thin wall of the pipe, this field creates circular eddy currents with the linear density

$$\mathbf{J}(z,t) = \mathbf{n}_\varphi J(z,t), \qquad \text{with } J(z,t) = j(z,t)\tau = \sigma E_{\text{ind}}(z,t)\tau = -\frac{\sigma\tau}{2\pi R}\frac{\partial \Phi(z,t)}{\partial t}.$$

These loop currents, in turn, induce an additional axially symmetric magnetic field $\mathbf{B}_J$. For any point z'' at the system's axis, this field is directed along the axis, and its magnitude may be calculated by integration, using Eq. (5.23) of the lecture notes with the proper replacement $z \to z'' - z$:

$$B_J(z'',t) = \frac{\mu_0 R^2}{2}\int_{-\infty}^{+\infty}\frac{J(z,t)dz}{\left[(z''-z)^2 + R^2\right]^{3/2}}.$$

According to Eq. (5.102) of the lecture notes, this axially-oriented field exerts on the effective magnetic dipole $\mathbf{m} = \mathbf{n}_z m$ of the sphere located at point $z'(t)$, the *drag force* $\mathbf{F} = \mathbf{n}_z F$ with :[55]

$$F = m\frac{\partial B_J(z'',t)}{\partial z''}\bigg|_{z''=z'(t)} = m\frac{\mu_0 R^2}{2}\int_{-\infty}^{+\infty}\frac{\partial}{\partial z''}\frac{1}{\left[(z''-z)^2+R^2\right]^{3/2}}\bigg|_{z''=z'(t)} J(z,t)dz$$

$$= m\frac{\mu_0 R^2}{2}\int_{-\infty}^{+\infty}\frac{3[z-z'(t)]}{\left\{[z-z'(t)]^2+R^2\right\}^{5/2}}J(z,t)dz = -m\frac{\mu_0 R}{4\pi}\sigma\tau\int_{-\infty}^{+\infty}\frac{3[z-z'(t)]}{\left\{[z-z'(t)]^2+R^2\right\}^{5/2}}\frac{\partial\Phi(z,t)}{\partial t}dz.$$

Plugging in Eq. (**), we get[56]

$$F = -\frac{9}{8\pi}(\mu_0 m)^2\sigma\tau v R^3\int_{-\infty}^{+\infty}\frac{[z-z'(t)]^2}{\left\{[z-z'(t)]^2+R^2\right\}^5}dz \equiv -\frac{9}{8\pi}(\mu_0 m)^2\sigma\tau v\frac{1}{R^4}I, \quad \text{with } I \equiv \int_{-\infty}^{+\infty}\frac{\xi^2}{(\xi^2+1)^5}d\xi,$$

where $\xi \equiv [z - z'(t)]/R$. The dimensionless integral I may be worked out, for example, using the following variable replacement: $\xi = \tan\alpha$, so $d\xi = d\alpha/\cos^2\alpha$, while $1/(\xi^2 + 1) = \cos^2\alpha$, giving:

$$I = \int_{-\pi/2}^{+\pi/2}\sin^2\alpha\cos^6\alpha\, d\alpha \equiv \int_{-\pi/2}^{+\pi/2}(1-\cos^2\alpha)\cos^6\alpha d\alpha \equiv \int_{-\pi/2}^{+\pi/2}(\cos^6\alpha - \cos^8\alpha)d\alpha.$$

Now the function under the integral may be represented as a sum of expressions proportional to power-free trigonometric functions, using a repeated application of the well-known trigonometric identities[57]

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[55] Here the condition $R' << R$, specified in the assignment, is essential.

[56] An alternative way to obtain this expression for the drag force $\mathbf{F}$ is to use the above result for $\mathbf{E}_{\text{ind}}(z, t)$ and Eq. (4.39) of the lecture notes for the Joule power density, $\wp = \sigma E_{\text{ind}}^2$, to calculate the total power $\mathscr{P}$ as an integral of $\wp$ over pipe wall's volume. After that, we may argue that $\mathscr{P}$ should be equal to the rate, $-\mathbf{F}\cdot\mathbf{v}$, of the sphere's mechanical energy loss. The drawback of this approach is that it does not reveal the physical origin of the drag force as explicitly as the calculation made above.

[57] See, e.g., MA Eqs. (3.2d) and (3.3).

$$\cos^2 a = \frac{1}{2} + \frac{1}{2}\cos 2a, \quad \text{and} \ \cos^3 a = \frac{3}{4}\cos a + \frac{1}{4}\cos 3a .$$

The final result of this (a bit tedious but elementary) calculation is

$$\cos^6 \alpha - \cos^8 \alpha = \frac{5}{128} + \frac{1}{32}\cos 2\alpha - \frac{1}{32}\cos 4\alpha - \frac{1}{32}\cos 6\alpha - \frac{1}{128}\cos 8\alpha .$$

At the integration over our interval $-\pi/2 \leq \alpha \leq +\pi/2$, all terms of this expression but the first one vanish, so the integral is simply

$$I = \frac{5\pi}{128},$$

and taking into account Eq. (*), the drag force F may be rewritten as[58]

$$F = -\eta v, \qquad \text{with} \ \eta \equiv \frac{9}{8\pi}(\mu_0 m)^2 \frac{\sigma\tau}{R^4}\frac{5\pi}{128} \equiv \frac{45\pi^2}{64}\left(R'^3 B\right)^2 \frac{\sigma\tau}{R^4}.$$

Since the force is proportional to the sphere's velocity, the equation of its motion along the pipe's axis,

$$M\dot{v} = -\eta v,$$

is linear, so it may be readily integrated to give an exponential law, $v(t) = v_0 \exp\{-t/\mathcal{T}\}$, with the time constant $\mathcal{T} = M/\eta$. From here, the distance passed by the sphere tends to

$$d = \int_0^\infty v(t)dt = v_0 \int_0^\infty \exp\left\{-\frac{t}{\mathcal{T}}\right\}dt = v_0\mathcal{T} = \frac{Mv_0}{\eta} = \frac{64}{45\pi^2}\frac{Mv_0 R^4}{\sigma\tau R'^6 B^2}. \qquad (***)$$

Just to get some feeling of this result, a practicable 3-Tesla magnetic field would stop a 1-cm-radius steel ball (with a mass ~0.3 kg), launched with the initial speed of 1 m/s into a copper pipe of a 10-cm radius with 1-mm-thick walls after it has passed just $d \approx 87$ cm.

There are two major conditions of validity of Eq. (***). First, our solution assumed that the induced current j is uniformly distributed over the pipe wall's thickness τ. This assumption is valid only if τ is much smaller than not only R but also the skin depth, $\delta_s = (2/\mu_0\sigma\omega)^{1/2}$, at the main frequency components, $\omega \sim v/R \leq v_0/R$, of the transient process of the current's rise and fall, i.e. if

$$\tau << \left(\frac{R}{\mu\sigma v_0}\right)^{1/2}. \qquad (****)$$

The second condition is that the magnetic flux Φ_J of the field created by the induced current J through the cross-section πR^2 of the pipe remains much smaller than Φ, so it may be neglected (as it was) at the calculation of the derivative (**). The above results show that this is true at a condition stronger than Eq. (****):

$$\mu_0\sigma\tau v_0 << 1.$$

Physically, this is the condition that the L/R time constant of the pipe (equal to $\mu_0\sigma\tau R/2$) is much shorter than the time scale R/v_0 of the current's rise and fall. In our numerical example, even this

[58] η is a usual notation for such a *drag coefficient* – see, e.g., CM Sec. 5.1.

stronger condition is well satisfied. (An additional task for the reader: think about possible ways to take the L/R time constant into account if it is not negligible.)

Problem 6.9. A planar thin-wire loop with inductance L, resistance R, and area A is launched to fly ballistically from field-free space into a region where the magnetic field **B** is constant. Calculate the final change of the kinetic energy of the loop, assuming that the time of its entry into the field region is much shorter than the relaxation time constant L/R and that the loop cannot rotate.

Solution: As was stated in Sec. 6.1 of the lecture notes, and proved in the model solution of Problem 1, the current I in a loop with an Ohmic resistance R obeys the relation $IR = \mathscr{V}_{\text{ind}}$, where $\mathscr{V}_{\text{ind}} = -d\Phi/dt$ is the Faraday induction e.m.f. (2), and Φ is the full magnetic flux piercing the loop's area. In a loop with a substantial self-inductance L, we need to take into account the contributions to Φ made not only by the external magnetic field $\mathbf{B}_{\text{ext}}$ but also by the current itself – see Eq. (5.68) and its discussion:[59]

$$\Phi = \Phi_{\text{ext}} + LI, \qquad \text{with } \Phi_{\text{ext}} \equiv \int_A (\mathbf{B}_{\text{ext}})_n d^2r \cdot$$

As a result, the flux (and hence the current) may be calculated from the following equation:

$$\frac{d\Phi}{dt} = -IR = \frac{\Phi_{\text{ext}} - \Phi}{L} R, \qquad \text{i.e. } \tau \frac{d\Phi}{dt} + \Phi = \Phi_{\text{ext}}, \qquad \text{where } \tau \equiv \frac{L}{R} \cdot \tag{*}$$

In our current problem, during the short time interval $\Delta t \ll \tau$ of the loop's entry into the field, the external flux Φ_{ext} leaps from 0 to $BA\cos\theta$, where θ is the angle between the field **B** and the vector normal to the loop's plane. As Eq. (*) shows, during this time, the flux Φ cannot change substantially, so the leap of Φ_{ext} has to be compensated by that of the current I from zero to

$$I_0 = -\frac{BA}{L}\cos\theta \cdot$$

This current gives the loop the magnetic moment **m** (5.97), with the magnitude $m = I_0 A = -BA^2\cos\theta/L$ and the direction normal to the loop's plane. Hence the energy of its interaction with the field may be found from Eq. (5.100), with the additional factor ½ due to the field-induced character of the moment **m**:[60]

$$U_0 = -\frac{\mathbf{m}\cdot\mathbf{B}}{2} = \frac{B^2A^2}{2L}\cos^2\theta \cdot$$

This positive potential energy can only come from the initial kinetic energy T of the loop, causing its reduction:

$$\Delta T = -U_0 = -\frac{B^2A^2}{2L}\cos^2\theta \cdot \tag{**}$$

(If the initial value of T is smaller than this entry cost, the loop is reflected back from the field region.) Inside the constant-field region, the flux Φ_{ext} through a non-rotating loop stays constant, and the linear differential equation (*) may be readily integrated, giving the well-known current and flux relaxation law:

[59] A different sign before the term LI in Eq.(6.78) was due to the specific choice convenient for the discussion of the Josephson effect – see the accompanying footnote.
[60] Note that this result is independent of the mutual orientation of the loop's plane and its velocity.

$$\Phi = \Phi_{\text{ext}}\left(1 - \exp\left\{-\frac{t}{\tau}\right\}\right), \quad I = I_0 \exp\left\{-\frac{t}{\tau}\right\}.$$

This gradual relaxation leads to the dissipation (irreversible reduction) of the potential energy U, with no accompanying change of its kinetic energy T. (Indeed, at the motion of any closed current loop in a constant magnetic field, the net Lorentz force (5.8) vanishes, so the loop's kinetic energy cannot change.) Hence, Eq. (**) gives the final answer to our problem.

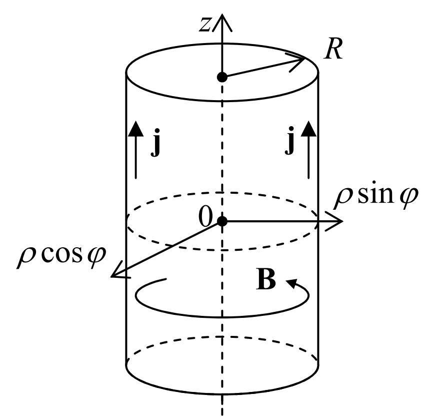
Problem 6.10. AC current of frequency ω is passed through a long uniform wire with a round cross-section of a radius R comparable with the skin depth δ_s. In the quasistatic approximation, find the current's distribution across the cross-section, and analyze it in the limits $R << \delta_s$ and $\delta_s << R$. Calculate the effective ac resistance of the wire (per unit length) in these two limits.

Solution: The current distribution at the skin effect obeys an equation similar to Eq. (6.23) for the magnetic field **B**. Indeed, let us stop halfway through the derivation of that equation – see the second form of Eq. (6.23),

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{1}{\sigma}\nabla \times \mathbf{j},$$

and take the curl of both sides. According to the Maxwell equation $\mathbf{j} = \nabla\times\mathbf{H} = \nabla\times(\mathbf{B}/\mu)$, the left-hand side of the resulting relation is equal to $\mu\partial\mathbf{j}/\partial t$, and since in the quasistatic approximation, $\nabla\cdot\mathbf{j} = 0$, its right-hand side equals $\nabla^2\mathbf{j}/\sigma$,[61] so we get

$$\frac{\partial \mathbf{j}}{\partial t} = \frac{1}{\sigma\mu}\nabla^2 \mathbf{j}.$$



Since in our axially symmetric situation, $\mathbf{j} = \mathbf{n}_z j(\rho, t)$, a similar equation is valid for the scalar function $j(\rho, t)$, so spelling out the Laplace operator in the cylindrical coordinates,[62] for the Fourier amplitude of this function, we get the ordinary differential equation

$$-i\omega j_\omega = \frac{1}{\sigma\mu}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dj_\omega}{d\rho}\right).$$

By replacing ρ with the dimensionless coordinate $\xi \equiv \kappa\rho$, with $\kappa^2 \equiv i\omega\sigma\mu \equiv -2i/\delta_s^2$, the equation may be recast as

$$\frac{d^2 j_\omega}{d\xi^2} + \frac{1}{\xi}\frac{dj_\omega}{d\xi} + j_\omega = 0.$$

This is the Bessel equation (2.130) with $n = 0$, and hence its general solution is

$$j_\omega(\rho) = c_1 J_0(\xi) + c_2 Y_0(\xi) \equiv c_1 J_0(\kappa\rho) + c_2 Y_0(\kappa\rho). \qquad (*)$$

The argument $\xi = \kappa\rho$ is now a complex function, but as was mentioned just after Eq. (2.158) of the lecture notes, this does not affect the general properties of the Bessel functions discussed in Sec. 2.7. In

[61] See, e.g., MA Eq. (11.3).

[62] See, e.g., MA Eq. (10.3) with $\partial/\partial z = \partial/\partial\varphi = 0$.

particular, Eqs. (2.132), (2.135), (2.142), (2.143), and (2.152) all remain valid for an arbitrary complex argument. Hence, according to Eq. (2.152), the function $Y_0(\xi)$ diverges at $|\xi| \to 0$, so the constant c_2 in Eq. (*) should equal 0. The other constant, c_1, may be calculated by requiring the total current through the wire's cross-section to have a certain complex amplitude I_ω:

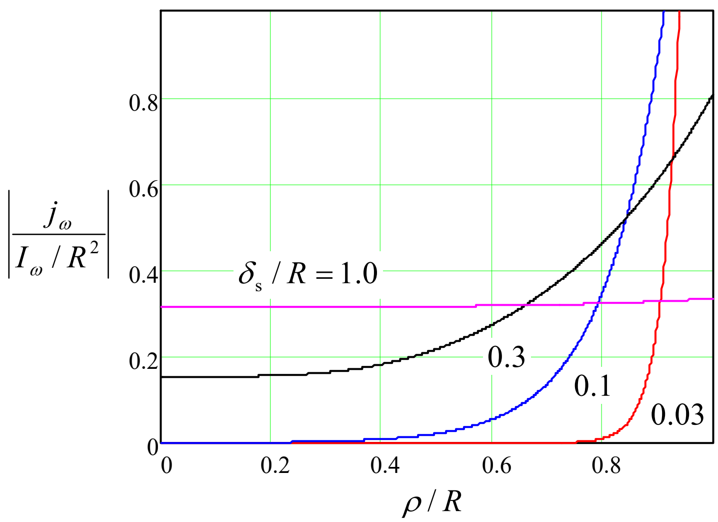
$$I_\omega = 2\pi \int_0^R j_\omega \rho d\rho = 2\pi c_1 \int_0^R J_0(\kappa\rho)\rho d\rho = \frac{2\pi c_1}{\kappa^2} \int_0^{\kappa R} J_0(\xi)\xi d\xi .$$

This integral may be readily calculated by using Eq. (2.143) with $n = 1$:

$$I_\omega = \frac{2\pi c_1}{\kappa^2} [\xi J_1(\xi)]_0^{\kappa R} = \frac{2\pi c_1 R}{\kappa} J_1(\kappa R),$$

so, finally,

$$j_\omega(\rho) = \frac{\kappa J_0(\kappa\rho)}{2\pi R J_1(\kappa R)} I_\omega .$$



Plots in the figure on the right show the radial distribution of $|j_\omega|$, i.e. of the real amplitude of the current's density, for several values of the ratio δ_s/R. In the limit $R << \delta_s$, i.e. $|\kappa|\rho \leq |\kappa|R << 1$, we may use the Taylor series (2.132) to expand the functions $J_0(\xi)$ and $J_1(\xi)$ near the origin. In the lowest nonvanishing approximation, $J_0(\xi) \approx 1$ and $J_1(\xi) \approx \xi/2$, so

$$j_\omega(\rho) \approx \frac{\kappa I_\omega}{2\pi R(\kappa R/2)} \equiv \frac{I_\omega}{\pi R^2} = \text{const},$$

i.e. the current is uniformly distributed over the wire's cross-section. (As the figure above shows, actually, the inequality $R << \delta_s$ should not be strong to ensure a virtually uniform current distribution.)

In the opposite limit, $\delta_s << R$, i.e. $1 << |\kappa|\rho \leq |\kappa|R$, we may use the asymptotic expansion (2.135) to get

$$J_0(\kappa\rho) \approx \left(\frac{2}{\pi\kappa\rho}\right)^{1/2} \cos\left(\kappa\rho - \frac{\pi}{4}\right) = \left(\frac{2}{\pi\kappa\rho}\right)^{1/2} \cos\left[(1+i)\frac{\rho}{\delta_s} - \frac{\pi}{4}\right],$$

$$J_1(\kappa R) \approx \left(\frac{2}{\pi\kappa R}\right)^{1/2} \cos\left(\kappa R - \frac{\pi}{4} - \frac{\pi}{2}\right) = \left(\frac{2}{\pi\kappa R}\right)^{1/2} \sin\left[(1+i)\frac{\rho}{\delta_s} - \frac{\pi}{4}\right].$$

Now using the well-known formulas MA Eq. (3.5) (which may serve as the definition of the hyperbolic functions), and the asymptotic relations $\sinh b \approx \cosh b \approx (1/2)\exp\{b\}$ at $b >> 1$, we get

$$J_0(\kappa\rho) \approx \left(\frac{1}{2\pi\kappa\rho}\right)^{1/2} \exp\left\{\frac{\rho}{\delta_s}\right\} \exp\left\{-i\left(\frac{\rho}{\delta_s} - \frac{\pi}{4}\right)\right\}, \qquad J_1(\kappa R) \approx \left(\frac{1}{2\pi\kappa R}\right)^{1/2} i \exp\left\{\frac{R}{\delta_s}\right\} \exp\left\{-i\left(\frac{R}{\delta_s} - \frac{\pi}{4}\right)\right\},$$

so, finally,

$$j_\omega(\rho) \approx -I_\omega \frac{i\kappa}{2\pi(R\rho)^{1/2}} \exp\left\{-\frac{R-\rho}{\delta_s}\right\} \exp\left\{-i\frac{R-\rho}{\delta_s}\right\}.$$

At $(R - \rho) \sim \delta_s << R$, this solution coincides with that following from Eqs. (6.32)-(6.34) of the lecture notes, with the replacement $x \to R - \rho$. This is natural because in this limit, on the current decay scale δ_s $<< R$, the surface's curvature is negligible.

Next, the simplest way to define the effective ac resistance $\mathcal{R}(\omega)$ of the wire is to write the same relation for the average Joule power of electric energy loss,

$$\overline{\mathscr{P}} = \frac{\mathcal{R}(\omega)\left| I_\omega^2 \right|}{2},$$

as follows from Eq. (4.40) of the lecture notes in the dc limit. (The factor ½ is due to the averaging over the period of sinusoidal oscillations of the current.)[63] For this power, per unit length of the wire, the time averaging of Eq. (4.39) gives the following expression:

$$\frac{\overline{\mathscr{P}}}{l} = \frac{1}{2\sigma}\int_A \left| j_\omega(\rho) \right|^2 d^2\rho,$$

so we can calculate the effective resistance per unit length as

$$\frac{\mathcal{R}(\omega)}{l} = \frac{1}{\sigma}\int_A \left| \frac{j_\omega(\rho)}{I_\omega} \right|^2 d^2\rho \equiv \frac{1}{\sigma} 2\pi \int_0^R \left| \frac{j_\omega(\rho)}{I_\omega} \right|^2 \rho d\rho.$$

Plugging in the current distribution limits calculated above and integrating, we get

$$\frac{\mathcal{R}(\omega)}{l} \approx \frac{1}{\pi\sigma R} \times \begin{cases} 1/R, & \text{for } R << \delta_s, \\ 1/2\delta_s, & \text{for } \delta_s << R. \end{cases}$$

In the first of these expressions, we may readily recognize Eq. (4.21) for the dc resistance of the whole wire, with the cross-section area $A = \pi R^2$. The second of these results is the same as the dc resistance of a thin hollow tube, with radius R, thickness δ_s, and hence the cross-section area $A' = 2\pi R\delta_s$. Both results fit the physical picture of the current distribution – see the figure above.

Note that since $\delta_s \propto \omega^{-1/2}$, in the high-frequency limit ($\delta_s/R \to 0$), the ac resistance of metallic wires grows as $\omega^{1/2}$,[64] providing serious problems for telecommunication technologies – see the discussion in Sec. 7.9.

<u>Problem 6.11</u>. A long round cylinder of radius R, made of a uniform Ohmic conductor with conductivity σ and magnetic permeability μ, is placed into a uniform ac magnetic field $\mathbf{H}_{ext} = \mathbf{H}_0\cos\omega t$ directed along its symmetry axis. Calculate the spatial distribution of the magnetic field's amplitude and,

[63] Another (perhaps less obvious) definition giving the same result is $\mathcal{R}(\omega)/l \equiv \text{Re}[E_\omega(R)/I_\omega]$, where $E_\omega(R)$ is the complex amplitude of the ac electric field on the wire's surface.

[64] The fraction $\mathcal{R}(\omega\to\infty)/\mathcal{R}(\omega\to 0) = R/2\delta_s \propto \omega^{1/2}$ is sometimes called the *Rayleigh resistance ratio*.

in particular, its value on the cylinder's axis. Spell out the last result in the limits of relatively small and large R.

Solution: This is just an axially symmetric version of the skin-effect problem solved in Sec. 6.3 of the lecture notes. Due to the symmetry, the magnetic fields (both **B** and **H**) at any point are directed along the cylinder's axis,

$$\mathbf{H}(\mathbf{r},t) = \mathbf{n}_z \operatorname{Re}\left[H_\omega(\rho)e^{-i\omega t}\right],$$

where ρ is the distance for the axis (while the induced electric field and hence the eddy current density have only azimuthal components). As a result, the Laplace operator of the only component of the magnetic field takes a simple form:[65]

$$\nabla^2\mathbf{H} = \mathbf{n}_z \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dH}{d\rho}\right),$$

and Eq. (6.23) yields the following equation for its complex amplitude inside the cylinder:

$$-i\omega H_\omega = \frac{1}{\sigma\mu}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dH_\omega}{d\rho}\right), \qquad \text{for } \rho \le R.$$

This is exactly the same equation as was obtained for the current density's amplitude j_ω in the solution of the previous problem, and we may borrow its solution (also giving a finite value at $\rho = 0$):

$$H_\omega(\rho) = cJ_0(\kappa\rho),$$

where J_0 is the zero-order Bessel function of the first kind, while

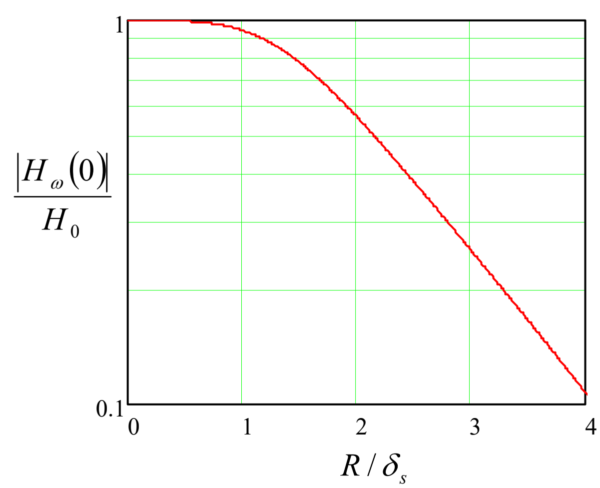
$$\kappa \equiv (-i\omega\mu\sigma)^{1/2} \equiv -\frac{1-i}{\delta_s}.$$

The numerical plots of $|H_\omega|$ as a function of ρ are also similar to those for $|j_\omega|$ in the model solution of the previous problem; however, their vertical scaling is different because in our current case, the constant c should be calculated from the boundary condition $H_\omega(R) = H_0$, giving

$$H_\omega(\rho) = H_0\frac{J_0(\kappa\rho)}{J_0(\kappa R)}.$$

Now by using the fact that $J_0(0) = 1$ (see, e.g., either Eq. (2.132) or Fig. 2.18 of the lecture notes, with $n = 0$), for the field on the cylinder's axis, we get simply

$$H_\omega(0) = H_0\frac{1}{J_0(\kappa R)}.$$



The figure on the right shows (in the appropriate semi-log scale) the modulus of this value, as a function of R normalized by the skin depth δ_s. For small values of the R/δ_s ratio, the result may be well approximated by using the leading two terms of the expansion (2.132):

[65] See, e.g., MA Eqs. (10.3) and (10.6) with $\mathbf{f} = f(\rho)\mathbf{n}_z$.

$$\frac{H_\omega(0)}{H_0} = \frac{1}{J_0(\kappa R)} \approx \frac{1}{1-(\kappa R/2)^2} \equiv \frac{1}{1+iR^2/2\delta_s^2},$$

so at $R/\delta_s \to 0$,

$$\frac{|H_\omega(0)|}{H_0} \approx \frac{1}{\left[1+\left(R^2/2\delta_s^2\right)^2\right]^{1/2}} \approx 1-\frac{1}{8}\left(\frac{R}{\delta_s}\right)^4 \to 1.$$

In this approximation, the applied field is virtually uniform everywhere in the cylinder because the diamagnetic effect of the eddy currents is small.

In the opposite limit, we may reuse the large-argument asymptote of function J_0, which was employed in the model solution of the previous problem, to get

$$\frac{|H_\omega(0)|}{H_0} \approx |2\pi\kappa R|^{1/2} \exp\left\{-\frac{R}{\delta_s}\right\} = 2\left(\pi\frac{R}{\delta_s}\right)^{1/2} \exp\left\{-\frac{R}{\delta_s}\right\} \to 0, \qquad \text{for } \frac{R}{\delta_s} \to \infty.$$

In this limit, the eddy currents shield most of the cylinder's bulk from the applied external magnetic field, which therefore penetrates only into a relatively thin skin layer, exactly as into a conducting semi-space – see Sec. 6.3 of the lecture notes.

Problem 6.12.* Define and calculate an appropriate spatial-temporal Green's function for Eq. (6.25) of the lecture notes, and then use this function to analyze the dynamics of propagation of the external magnetic field that is suddenly turned on at $t = 0$ and then kept constant:

$$H(x<0,t) = \begin{cases} 0, & \text{at } t<0, \\ H_0, & \text{at } t>0, \end{cases}$$

into an Ohmic conductor occupying the semi-space $x > 0$ – see Fig. 6.2.

Hint: Try to use a function proportional to $\exp\{-(x-x')^2/2(\delta x)^2\}$, with a suitable time dependence of the parameter δx and a properly selected pre-exponential factor.

Solution: As was discussed in Sec. 2.10 of the lecture notes, the Green's function approach uses the linearity of the corresponding differential equation, giving an explicit expression of the linear superposition principle. For a *time-independent* but *inhomogeneous* equation, such as the Poisson equation (1.41), the superposition is applied to the sum of elementary right-hand sides distributed over the space – see Eq. (2.203). In that case, the Green's function is purely spatial.[66] On the other hand, for a *time-dependent* but *homogeneous* equation, such as Eq. (6.25):

$$\frac{\partial H}{\partial t} = \frac{1}{\mu\sigma}\frac{\partial^2 H}{\partial x^2}, \qquad \text{for } x \geq 0,$$

the superposition principle may be applied to the spatial distribution of the initial conditions:

[66] Similarly, Green's functions of ordinary inhomogeneous differential equations describing the time dynamics alone are purely temporal – see, e.g., CM Sec. 5.1.

$$H(x,t)=\int_0^{\infty}G(x,t;x',t')H(x',t')dx'. \qquad (*)$$

With this definition, the Green's function G, considered as a function of x and t, is just the particular solution of Eq. (6.25), corresponding to delta-functional initial conditions:

$$G(x,t;x',t')=H(x,t), \qquad \text{for } H(x,t')=\delta(x-x'), \quad \text{with } x'>0. \qquad (**)$$

First, let us calculate the Green's function for the case when Eq. (6.25) is valid in *infinite* space ($-\infty < x < +\infty$), by using the provided *Hint*. By selecting the pre-exponential coefficient to keep the function properly normalized,[67]

$$\int_{-\infty}^{+\infty}G(x,t;x',t')dx=\int_{-\infty}^{+\infty}G(x,t';x',t')dx=\int_{-\infty}^{+\infty}H(x,t')dx=\int_{-\infty}^{+\infty}\delta(x-x')dx=1,$$

we get the well-known expression[68]

$$G(x,t;x',t')=\frac{1}{(2\pi)^{1/2}\,\delta x(t,t')}\exp\left\{-\frac{(x-x')^2}{2[\delta x(t,t')]^2}\right\}.$$

Now plugging this formula into Eq. (6.25), we see that it is indeed the solution of the equation, provided that $\delta x(t, t')$ satisfies the following ordinary differential equation:

$$\frac{\partial}{\partial t}\delta x=\frac{1}{\mu\sigma\delta x}, \qquad \text{i.e. } \frac{\partial}{\partial t}\left[(\delta x)^2\right]=\frac{2}{\mu\sigma}=\text{const}.$$

With the requirement $\delta x \to 0$ at $t \to t'$, following from the initial condition (**), the solution of this equation is obviously

$$(\delta x)^2=\frac{2(t-t')}{\mu\sigma}, \qquad \text{so } \delta x(t,t')=\left[\frac{2(t-t')}{\mu\sigma}\right]^{1/2}.$$

Since the Green's function depends on the difference $(t - t')$ only, in all formulas below I will take $t' = 0$, and use shorthand notations $G(x, t; x') \equiv G(x, t; x', 0)$ and $\delta x(t) \equiv \delta x(t, 0)$. As Eq. (**) tells us, the physical sense of $\delta x(t)$ is the gradually increasing width of the spatial distribution of the magnetic field induced by a delta-functional spatial "pulse" of the external field at $t = 0$.

Now we may use the equation's linearity, and the mirror-image themes discussed in Chapters 2-5 of the lecture notes, to construct the Green's function for the semi-space $x > 0$, which obeys the boundary condition $G(0, t; x') = 0$:

$$G(x,t;x')=\frac{1}{(2\pi)^{1/2}\,\delta x(t)}\left[\exp\left\{-\frac{(x-x')^2}{2[\delta x(t)]^2}\right\}-\exp\left\{-\frac{(x+x')^2}{2[\delta x(t)]^2}\right\}\right], \qquad \text{with } \delta x(t)=\left(\frac{2t}{\mu\sigma}\right)^{1/2}, \qquad (***)$$

[67] This requirement follows from the fact that integrating both parts of Eq. (6.25) from $x = -\infty$ to $x = +\infty$, and assuming that the field's gradient has a final spatial extent, i.e. that $\partial H/\partial x \to 0$ at $x \to \pm\infty$, we get

$$\frac{d}{dt}\int_{-\infty}^{+\infty}H(x,t)dx=0..$$

[68] It describes, for example, the *Gaussian* (or "normal") *probability distribution* with variance equal to $(\delta x)^2$.

and hence automatically generates solutions (*) that satisfy this solution.

Next, due to the eventual decay of the eddy currents in time, the constant applied magnetic field cannot be prevented from its eventual penetration into the conductor: $H(x, t) \to H_0$. Since the Green's function (***) tends to zero at $t \to 0$, we better reduce our problem to finding some function $H'(x, t)$ with the same trend. Evidently, this may be done by representing the field as the difference

$$H(x,t) \equiv H_0 - H'(x,t),$$

with the initial value $H'(x, 0) = H_0$. For this auxiliary field, Eqs. (*) and (***) yield

$$H'(x,t) = H_0 \int_0^{+\infty} G(x,t;x')dx' = \frac{H_0}{(2\pi)^{1/2}\delta x(t)} \int_0^{\infty} \left[\exp\left\{ -\frac{(x-x')^2}{2[\delta x(t)]^2} \right\} - \exp\left\{ -\frac{(x+x')^2}{2[\delta x(t)]^2} \right\} \right] dx'. \quad (****)$$

Note that the integration limits automatically exclude the non-physical mirror-image delta function at $x' < 0$, used to construct the Green's function (***).

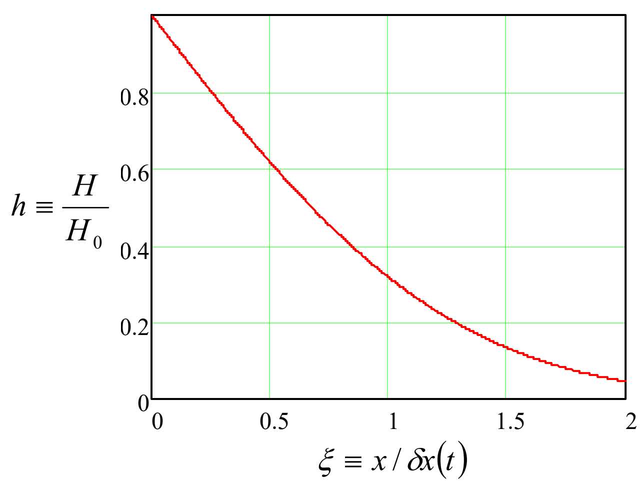
This integral may be readily expressed via the so-called *error function*

$$\operatorname{erf}(\zeta) \equiv \frac{2}{\pi^{1/2}} \int_0^{\zeta} \exp\{-\xi^2\} d\xi;$$

however, for most practical purposes (such as plotting, asymptote analysis, etc.), the explicit integral form (****) is preferable. Note that by introducing the following normalized variables: field $h \equiv H/H_0$ and coordinate $\xi \equiv x/\delta x(t) = x/(2t/\mu\sigma)^{1/2}$, we may rewrite our result in the form of a universal function

$$h(\xi) = 1 - \frac{1}{(2\pi)^{1/2}} \int_0^{\infty} \left[\exp\left\{ -\frac{(\xi-\xi')^2}{2} \right\} - \exp\left\{ -\frac{(\xi+\xi')^2}{2} \right\} \right] d\xi',$$

plotted in the figure on the right, so the field's profile in the actual (dimensional) space just stretches with time as $\delta x(t) \propto t^{1/2}$.



<u>Problem 6.13</u>. Solve the previous problem using the variable separation method, and compare the results.

Solution: Let us look for the solution of the same Eq. (6.25),

$$\mu\sigma \frac{\partial H}{\partial t} = \frac{\partial^2 H}{\partial x^2}, \qquad \text{for } x \geq 0,$$

in the usual variable-separated form, but (just as in the solution of the previous problem) singling out the eventual uniform distribution of the field:

$$H(x,t) = H_0 - \sum_k T_k(t) X_k(x), \qquad (*)$$

so the second term of Eq. (*) is equal to H_0 at $t = 0$ and has to vanish at $t \to \infty$. Plugging a particular term of this series into the equation, and dividing both sides by $T_k X_k$, we get

$$\mu\sigma \frac{1}{T_k}\frac{dT_k}{dt} = \frac{1}{X_k}\frac{d^2 X_k}{dx^2} = \text{const} \equiv -k^2 .$$

Solving the resulting simple differential equations for T_k and X_k, we get[69]

$$T_k = a_k \exp\left\{-\frac{k^2}{\mu\sigma}t\right\}, \qquad X_k = b_k \sin kx ,$$

with (at this stage) arbitrary a_k and b_k. Since the segment on which our equation is valid ($0 \le x \le \infty$) is semi-infinite, the spectrum of the eigenvalues k is continuous, so plugging these solutions into Eq. (*), we need to replace the summation over k with the corresponding integration:

$$H(x,t) = H_0 - \int_{-\infty}^{+\infty} c_k \sin kx \exp\left\{-\frac{k^2}{\mu\sigma}t\right\} dk, \qquad \text{where } c_k \equiv a_k b_k . \qquad (**)$$

What remains is to find the function c_k from the initial condition:

$$\int_{-\infty}^{+\infty} c_k \sin kx \, dk = H_0, \qquad \text{for } x > 0. \qquad (***)$$

As usual, we may calculate c_k from this equation by multiplying both sides by a similar fundamental function but with an arbitrary argument, in our case $\sin k'x$, and integrating them over x. Changing the integration order on the left-hand side, we get

$$\int_{-\infty}^{+\infty} dk \; c_k \int_0^{\infty} dx \sin kx \sin k'x = H_0 \int_0^{\infty} \sin k'x \, dx .$$

Since the expression under the inner integral on the left-hand side is an even function of x, the integral may be transformed as

$$\int_0^{\infty} \sin kx \sin k'x \, dx = \frac{1}{2}\int_{-\infty}^{\infty} \sin kx \sin k'x \, dx$$
$$= \frac{1}{4}\int_{-\infty}^{\infty} [\cos(k-k')x - \cos(k+k')x] \, dx = \frac{1}{4}\text{Re}\left[\int_{-\infty}^{\infty} e^{i(k-k')x} dx - \int_{-\infty}^{\infty} e^{i(k+k')x} dx\right].$$

Per MA Eq. (14.4), these integrals are equal to, respectively, $2\pi\delta(k \mp k')$, so our equation yields

$$\frac{\pi}{2}\left(c_{k'} - c_{-k'}\right) = H_0 \int_0^{\infty} \sin k'x \, dx \equiv -H_0 \left.\frac{\cos k'x}{k'}\right|_{x=0}^{x\to\infty} .$$

As Eq. (**) shows, c_k has to be an odd function of k, so $(c_{k'} - c_{-k'}) = 2c_{k'}$. Also, the upper limit on the right-hand side of the last equality may be ignored because the solution should not be sensitive to a

[69] Other fundamental function candidates, $\exp\{+k^2t/\mu\sigma\}$ for $T_k(t)$ and $\cos kx$ for $X_k(x)$, do not satisfy our boundary and initial conditions $H(x, \infty) = H(0, t) = H_0$.

very slow quenching of the right-hand side of Eq. (***) at $x \to \infty$. As a result, changing the index notation from k' to k, we get

$$c_k = \frac{H_0}{\pi k},$$

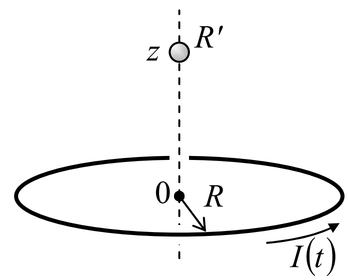
so Eq. (**) yields

$$H(x,t) = H_0\left[1 - \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin kx}{k}\exp\left\{-\frac{k^2}{\mu\sigma}t\right\}dk\right] \equiv H_0\left[1 - \frac{2}{\pi}\int_{0}^{+\infty}\frac{\sin kx}{k}\exp\left\{-\frac{k^2}{\mu\sigma}t\right\}dk\right].$$

Introducing dimensionless variables $\xi \equiv x/(2t/\mu\sigma)^{1/2}$, and $\kappa \equiv k(2t/\mu\sigma)^{1/2}$ (so that $kx = \kappa\xi$), and $h \equiv H/H_0$, we may rewrite this result in a parameter-free form

$$h = h(\xi) \equiv 1 - \frac{2}{\pi}\int_{0}^{+\infty}\frac{\sin \kappa\xi}{\kappa}\exp\left\{-\frac{\kappa^2}{2}\right\}d\kappa, \qquad (****)$$

which shows that the field distribution profile in the conductor is universal, and all its time evolution is reduced to the increase of its spatial scale $(2t/\mu\sigma)^{1/2}$. Indeed, the numerical plot of this result exactly coincides with that shown in the model solution of the previous problem, despite the rather different analytical formulas. This fact is not quite surprising, because it is straightforward (but still recommended for the reader) to show that Eq. (****) is just as the Fourier-integral expansion of the result derived in the previous problem.



Problem 6.14. Calculate the average force exerted by ac current $I(t)$ of amplitude I_0, flowing in a planar round coil of radius R, on a conducting sphere with a much smaller radius R' (which is still much larger than the skin depth δ_s at the ac current's frequency), located on the loop's axis, at distance z from its center – see the figure on the right.

Solution: In close vicinity of the sphere, due to the condition $R' \ll R$, the current's field may be considered locally uniform. On the other hand, the condition $\delta_s \ll R'$ allows us to ignore the effects of ac field penetration into the sphere, and thus to use the coarse-grain (ideal-diamagnetic) фззкщчшьфешщт $B(t) = 0$ inside the sphere. As a result, the instantaneous distribution of the applied ac field outside the sphere is similar to that of a dc field around a superconducting sphere – see Eq. (6.58) and Fig. 6.3 of the lecture notes. As was discussed there, this field is a sum of the locally-uniform field $\mathbf{H}(t)$ created by the current at the point of the sphere's location, and a purely dipole field corresponding to the magnetic dipole moment given by Eq. (5.125) with $\mu = 0$:

$$\mathbf{m}(t) = -2\pi R'^3\mathbf{H}(t).$$

The negative sign means that the dipole moment is directed opposite to the applied field $\mathbf{H}(t)$ – which is natural for the ideal diamagnetism.

The field $\mathbf{H}(t) = \mathbf{B}(t)/\mu_0$ is given by Eq. (5.23) of the lecture notes:

$$\mathbf{H}(t) = \mathbf{n}_z\frac{I(t)}{2}\frac{R^2}{\left(R^2 + z^2\right)^{3/2}},$$

The energy of interaction between an external field and a *fixed* magnetic dipole is given by Eq. (5.100) of the lecture notes, $U = -\mathbf{m}\cdot\mathbf{B}$; however, in our case, the dipole is not fixed but is induced by the field ($\mathbf{m} \propto \mathbf{B} = \mu_0\mathbf{H}$), so we need to multiply this expression by the usual factor ½, getting

$$U(t) = -\frac{\mathbf{m}(t)\cdot\mathbf{B}(t)}{2} = -\frac{\mathbf{m}(t)\cdot\mu_0 H(t)}{2} = \pi\mu_0 R'^3 H^2(t) = \frac{\pi}{4}\mu_0 I^2(t)\frac{R'^3R^4}{(R^2+z^2)^3}.$$

The time average of this energy over the period of the sinusoidal ac current of amplitude I_0 is

$$\overline{U} = \frac{\pi}{8}\mu_0 I_0^2 \frac{R'^3R^4}{(R^2+z^2)^3}.$$

The energy is positive and decreases with distance $|z|$, so the average force exerted on the sphere,

$$\overline{F} = -\frac{\partial\overline{U}}{\partial z} = \frac{3\pi}{4}\mu_0 I_0^2 \frac{R'^3R^4}{(R^2+z^2)^4}z,$$

has the same sign as z, i.e. corresponds to repulsion.

As a matter of principle, such repulsion of a sphere located above the loop (i.e. with $z > 0$) may be used for balancing the sphere's weight mg if it is lower than the maximum value of the force,

$$\overline{F}_{\max} = \overline{F}\Big|_{z=z_{\text{opt}}=R/\sqrt{7}} = \frac{3\pi}{4}\frac{7^{1/2}}{8^4}\mu_0 I_0^2\left(\frac{R'}{R}\right)^3 \approx 1.522\times10^{-3}\,\mu_0 I_0^2\left(\frac{R'}{R}\right)^3,$$

because one (the highest) of the two stationary points z corresponding to the equilibrium condition $\overline{F} = mg$ is stable with respect to vertical perturbations. However, this point is unstable with respect to lateral displacements of the sphere – or of a small levitated object of any shape (see, e.g., the solution of Problem 5.2); as a result, levitation requires more complex geometries.

A more practical problem of such levitation is the energy losses in the skin layer (see Eq. (6.36) of the lecture notes); the losses may be avoided using superconductivity, but this approach involves costly deep refrigeration. As a result, we still have not yet achieved the old dream of magnetically-levitated ground transportation systems.

<u>Problem 6.15</u>. A small planar wire loop carrying a fixed current I is located relatively far from a planar surface of a superconductor. Within the coarse-grain (ideal-diamagnet) description of the Meissner-Ochsenfeld effect, calculate:

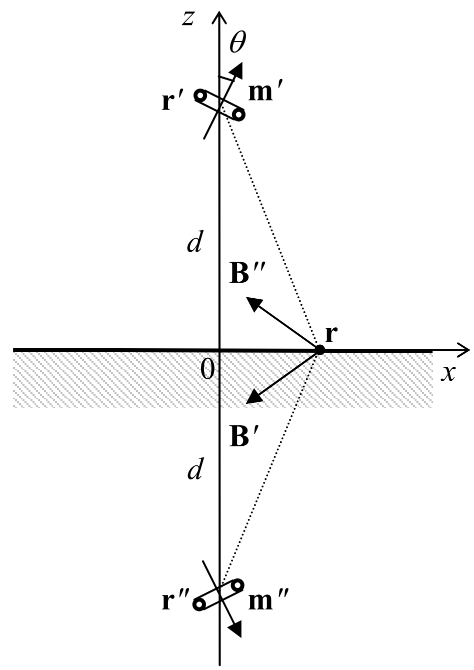
(i) the energy of the loop-superconductor interaction,
(ii) the force and torque acting on the loop, and
(iii) the distribution of supercurrents on the superconductor's surface.

Solution: As was discussed in Sec. 6.4 of the lecture notes, in the ideal-diamagnet (coarse-grain) approximation, the field outside the superconductor has to satisfy the boundary condition (6.59):

$$B_n = 0. \qquad (*)$$

The field of a wire loop in unlimited free space, at distances much larger than the loop size, may be described in the magnetic-dipole approximation. An elementary generalization of Eq. (5.99) of the lecture notes to an arbitrary position **r**' of the loop's center is

$$\mathbf{B}'(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3(\mathbf{r}-\mathbf{r}')[(\mathbf{r}-\mathbf{r}')\cdot\mathbf{m}'] - \mathbf{m}'|\mathbf{r}-\mathbf{r}'|^2}{|\mathbf{r}-\mathbf{r}'|^5}, \qquad (**)$$



where **m**' is its magnetic moment – in our case, with the magnitude $m' = IA$, where A is the loop area. Note that a simultaneous flipping of signs of the same Cartesian components of the vectors **m** and (**r** – **r**') reverses the sign of this component of the field, but not its other components. From this symmetry, it is clear that the boundary condition (*) may be satisfied by representing the total magnetic field outside the superconductor as a sum of this field **B**' and a similar field **B**" of a mirror-image magnetic dipole **m**" that is located and oriented as shown in the figure on the right. Selecting the coordinate axes as shown in this figure (with the x-axis in the same vertical plane as the vector **m**'), we may write

$$x'' = x' = 0, \quad y'' = y' = 0, \quad z'' = -z' = -d,$$

$$m''_x = m'_x = m\sin\theta, \quad m''_y = m'_y = 0, \quad m''_z = -m'_z = -m\cos\theta.$$

Note that the image dipole orientation is exactly opposite to that in Problem 3.8, due to the different boundary conditions. Hence we may reuse the solution of that problem, just minding the signs and the fundamental constants, to get the following results.

(i) The interaction energy,

$$U_{\text{int}} = -\frac{1}{2}\mathbf{m}'\cdot\mathbf{B}''(\mathbf{r}') = -\frac{1}{2}\left(m'_x B''_x + m''_z B''_z\right)_{x=y=0,z=d} = \frac{\mu_0}{8\pi}\frac{m^2}{(2d)^3}\left(1+\cos^2\theta\right),$$

corresponds to the *repulsion* of the dipole from the superconductor, at any dipole's orientation. This is very natural, because a superconductor pushes out the external field lines, and hence repels their source. Such repulsion is the basis of all magnetic levitation projects (including the "maglev" trains) using superconductivity – see the brief discussion in the model solution of the previous problem.

(ii) The force and the torque may be readily calculated from U_{int}:

$$F = F_z = -\frac{\partial U_{\text{int}}}{\partial d} = \frac{\mu_0}{8\pi}\frac{3m^2}{8d^4}\left(1+\cos^2\theta\right), \qquad \tau = \tau_y = -\frac{\partial U_{\text{int}}}{\partial\theta} = \frac{\mu_0}{8\pi}\frac{m^2}{8d^3}\sin 2\theta.$$

Note that the system's stable equilibrium with respect to rotation is at $\theta = \pm\pi/2$, with the magnetic moment parallel to the surface.

(iii) The linear density of the surface supercurrent may be calculated from its universal relation (6.38) with the magnetic field just outside the surface:

$$\mathbf{J} = \mathbf{n}_z \times \mathbf{H}_\tau \equiv \mathbf{n}_z \times \mathbf{B}_\tau / \mu_0,$$

where $\mathbf{B}_\tau$ is the tangential component of the magnetic field (in our coordinates, $\mathbf{B}_\tau = B_x\mathbf{n}_x + B_y\mathbf{n}_y$) near the surface. By spelling out Eq. (**) in Cartesian components, we may readily find the net field of the two dipoles, $\mathbf{B} = \mathbf{B}' + \mathbf{B}''$ in an arbitrary point $\mathbf{r} = x\mathbf{n}_x + y\mathbf{n}_y$ of the surface ($z = 0$):

$$\mathbf{B}_\tau = \mathbf{B}'_\tau + \mathbf{B}''_\tau = 2\mathbf{B}'_\tau = \frac{\mu_0 m}{2\pi}\frac{\mathbf{n}_x\left[\left(2x^2 - y^2 - d^2\right)\sin\theta - 3dx\cos\theta\right] + \mathbf{n}_y 3y\left(x\sin\theta - d\cos\theta\right)}{(x^2+y^2+d^2)^{5/2}},$$

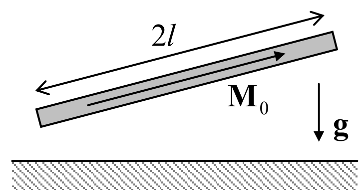
so, finally,

$$\mathbf{J} = \frac{m}{2\pi}\frac{\mathbf{n}_x 3y\left(-x\sin\theta + d\cos\theta\right) + \mathbf{n}_y\left[\left(2x^2 - y^2 - d^2\right)\sin\theta - 3dx\cos\theta\right]}{(x^2+y^2+d^2)^{5/2}}.$$

In the case when the current loop plane is parallel to the superconductor's surface ($\sin\theta = 0$, $\cos\theta = \pm 1$), the current lines are concentric circles, with $\rho \equiv (x^2 + y^2)^{1/2}$ = const, and its magnitude,

$$|J| = \frac{m}{2\pi}\frac{3d\rho}{\left(\rho^2 + d^2\right)^{5/2}},$$

does not depend on the azimuthal angle. The direction of this vortex current is opposite to that of that in the loop, thus fulfilling its function of shielding the superconductor's interior from the penetration of the loop's field.

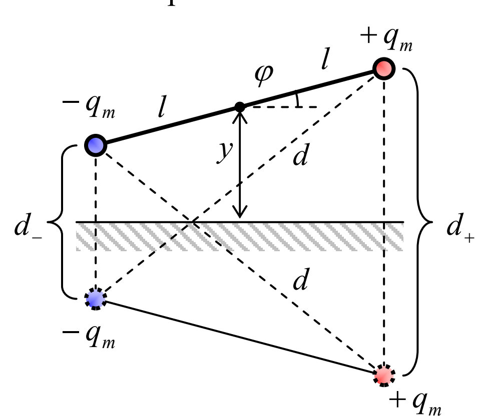


Problem 6.16. A straight uniform magnet of length $2l$, cross-section area $A << l^2$, and mass m, with a permanent longitudinal magnetization M_0, is placed over a horizontal surface of a superconductor – see the figure on the right. Within the ideal-diamagnet description of superconductivity, find the stable equilibrium position of the magnet.

Solution: As was discussed in Sec. 5.6 of the lecture notes, the magnetic field induced by a long and thin ($A << l^2$) permanent magnet, with the magnetization $\mathbf{M}_0$ parallel to its length, is equivalent to that of two point magnetic charges $q_m = \pm\mu_0 M_0 A$ located at its ends, each producing a Coulomb-like radial magnetic field

$$\mathbf{B}(\mathbf{r}) = \frac{q_\mathrm{m}}{4\pi}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = \pm\frac{\mu_0 M_0 A}{4\pi}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3},$$

where $\mathbf{r}'$ is the charge's (i.e., the magnet end's) position. In the coarse-grain (ideal-diamagnet) picture of the Meissner-Ochsenfeld effect, the magnetic field cannot penetrate into the superconductor. Thus the boundary problem for a magnetic point charge near the superconductor's surface, in this approximation, is generally similar to that of an electric point charge near a conductor's surface and may be solved exactly as was discussed in Sec. 2.9, i.e. by the introduction of its mirror image. However, the image charge sign should be similar (rather than opposite) to the initial charge, because it should compensate the normal (rather than tangential) component of the field at the surface – see Eq. (6.59).



Hence, in our case, we get the system of two actual and two image magnetic charges shown in the figure on the right. Introducing the vertical coordinate y of the magnet's center and the angle φ of its slope as shown in that figure, we may readily express the distances $d_\pm$ and d between the charges as functions of these two generalized coordinates:[70]

$$d_\pm = 2(y \pm l \sin\varphi),$$
$$d^2 = (2y)^2 + (2l\cos\varphi)^2 \equiv 4(y^2 + l^2\cos^2\varphi).$$

Continuing to use the analogy with electrostatics, in particular with Eq. (2.192) with the replacements $q \to q_\mathrm{m}$ and $\varepsilon_0 \to \mu_0$,[71] we may write the following expression for the potential energy of the system as[72]

$$U = \frac{1}{4\pi\mu_0}\frac{q_\mathrm{m}^2}{2}\left(\frac{1}{d_+} + \frac{1}{d_-} - \frac{2}{d}\right) + mgy = \frac{q_\mathrm{m}^2}{8\pi\mu_0}\left[\frac{1}{2(y + l\sin\varphi)} + \frac{1}{2(y - l\sin\varphi)} - \frac{1}{(y^2 + l^2\cos^2\varphi)^{1/2}}\right] + mgy$$
$$\equiv \frac{q_\mathrm{m}^2}{8\pi\mu_0}\left\{\left[\frac{y}{y^2 - l^2\sin^2\varphi} - \frac{1}{(y^2 + l^2\cos^2\varphi)^{1/2}}\right] + \frac{y}{a^2}\right\}, \qquad \text{with } a^2 \equiv \frac{q_\mathrm{m}^2}{8\pi\mu_0}\bigg/mg \equiv \frac{\mu_0 M_0^2 A^2}{8\pi}\bigg/mg.$$

For any fixed y within the geometrically possible region $y > \pm l\sin\varphi$, the function $U(\varphi)$ has two similar minima at $\varphi = 0$ and π, i.e. at $\sin^2\varphi = 0$ and $\cos^2\varphi = 1$. Perhaps the easiest way to see that is to expand the expression in the square brackets (which alone depends on the angle) in the Taylor series at small φ, keeping only two leading terms:

$$\frac{y}{y^2 - l^2\sin^2\varphi} - \frac{1}{(y^2 + l^2\cos^2\varphi)^{1/2}} \equiv \frac{1}{y(1 - l^2\sin^2\varphi/y^2)} - \frac{1}{(y^2 + l^2)^{1/2}\left[1 - l^2\sin^2\varphi/(y^2 + l^2)\right]^{1/2}}$$
$$\approx \left[\frac{1}{y} - \frac{1}{(y^2 + l^2)^{1/2}}\right] + l^2\varphi^2\left[\frac{1}{y^3} - \frac{1}{2(y^2 + l^2)^{3/2}}\right].$$

Since the second term in the last square brackets is always smaller than the first one, the potential energy always grows with φ^2.

Hence, the stable stationary position of the magnet is horizontal.[73] Its stationary height y over the superconducting surface should be calculated from the requirement $dU/dy = 0$ (at $\varphi = 0$ or π). For arbitrary values of the only dimensionless parameter a/l of the problem, the equation for y, resulting from this condition,

$$\frac{1}{y^2} - \frac{y}{(y^2 + l^2)^{3/2}} = \frac{1}{a^2}, \qquad (*)$$

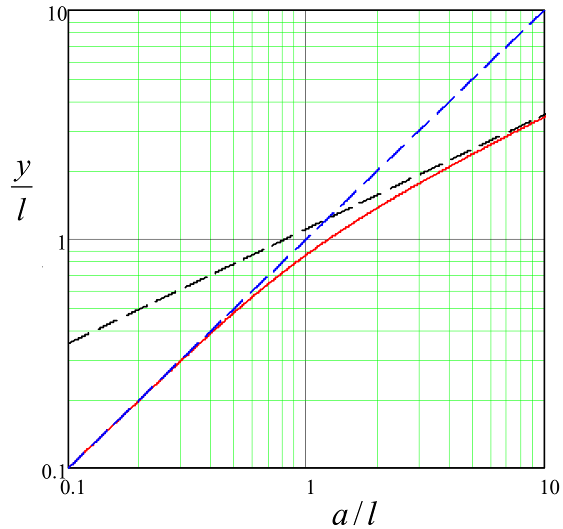
[70] The magnet's potential energy (and hence its equilibrium) is evidently unaffected by its arbitrary translational displacement along the surface and arbitrary rotation about the vertical axis.

[71] See Eqs. (5.137)-(5.139) and the accompanying discussion.

[72] We may ignore the potential energies of interaction of the magnetic charge pairs belonging to the same magnet, because these energies do not depend on the magnet's position, i.e. on coordinates y and φ. (The longitudinal forces resulting from these interactions are compensated by the mechanical stress forces in the rigid magnets.)

[73] Note that this conclusion is in agreement with the solution of Task (ii) of the previous problem. Indeed, in the particular case $y >> l$ (see the balance of this solution), the magnet may be approximated as a magnetic dipole – just as a small current loop.

may be solved only numerically. However, this equation may be readily used for plotting the reciprocal function, $a(y)$, and then just transposing the plot – see the solid red curve in the figure on the right. Moreover, Eq. (*) is easily solvable analytically in two limits, giving:

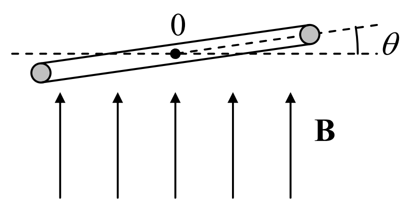


$$y \to \begin{cases} a, & \text{for } a << l, \\ (3/2)^{1/4}(la)^{1/2}, & \text{for } l << a, \end{cases} \qquad (**)$$

– see the two dashed lines in the figure on the right. The exact solution describes a smooth crossover between these asymptotes, taking place at $a/l \approx 1.2$.

Physically, the first line of Eq. (**) corresponds to the limit of relatively weak magnetic interaction, when gravity presses the magnet very close to the superconductor surface, and only the closest magnetic poles ("magnetic charges") interact substantially. On the other hand, if the magnetic interaction is strong enough, the magnet's repulsion by the superconducting surface brings it up so high ($y >> l$) that its field near the surface is close to that of a dipole with moment $m = VM_0 = 2lAM_0$. As a result, the second line of Eq. (**) may be calculated from Eqs. (5.99) and (5.100) of the lecture notes with this value of m and an additional factor of ½ due to the induced nature of the dipole's image.

<u>Problem 6.17</u>. A plane superconducting wire loop of area A and inductance L may turn, without static friction, about a horizontal axis 0 (in the figure on the right, normal to the plane of the drawing) passing through its center of mass. Initially, the loop had been horizontal (with $\theta = 0$) and carried supercurrent I_0 in such a direction that its magnetic dipole vector was directed down. Then a uniform magnetic field **B**, directed vertically up, was applied. Using the ideal-diamagnet description of the Meissner-Ochsenfeld effect, find all possible equilibrium positions of the loop, analyze their stability, and give a physical interpretation of the results.



Solution: Due to the magnetic flux quantization (see (6.62) of the lecture notes and its discussion), the total flux through the loop,[74]

$$\Phi = \Phi_{\text{ext}} - LI = AB\cos\theta - LI,$$

has to be constant and equal to its value,

$$\Phi = -LI_0,$$

before the external field's application. Combining these two relations, we get

[74] Here, just as Eq. (6.78), the sign before the term LI is selected in such a way that the magnetic moment **m** created by current $I > 0$ is directed opposite to the field **B**. However, this choice (if consistent) does not affect the energy U and hence the final results.

$$I = I_0 + I_B, \qquad \text{with } I_B = \frac{\Phi_{\text{ext}}}{L} = \frac{AB\cos\theta}{L}.$$

This means that the vertical component m_z of the magnetic moment (5.97) of the loop may be represented as the sum of two parts: $m_0 = -AI_0\cos\theta$ created by the pre-existing current, and $m_B = -AI_B \cos\theta$ created by the current I_B induced by the external field. Such decomposition of m is important because while the former part's contribution to the potential energy U of the loop may be calculated using Eq. (5.100):

$$U_0 = -m_0 B = ABI_0 \cos\theta,$$

the latter part's contribution should be multiplied by the usual factor ½:[75]

$$U_B = -\frac{1}{2} m_B B = \frac{1}{2} AI_B \cos\theta \, B = \frac{A^2B^2\cos^2\theta}{2L}.$$

As a result, the total potential energy of the loop is[76]

$$U = U_0 + U_B = ABI_0\cos\theta + \frac{A^2B^2\cos^2\theta}{2L}.$$

Since this is a system with just one degree of freedom (θ), its equilibrium positions correspond to the extrema of the function $U(\theta)$, in which $dU/d\theta = 0$, while their stability is determined by the sign of the second derivative of this function: if $d^2U/d\theta^2 \geq 0$, then the stationary position is stable. For our function, the equilibrium requirement gives the following equation:

$$\frac{dU}{d\theta} \equiv -ABI_0\sin\theta - \frac{A^2B^2\sin\theta\cos\theta}{L} \equiv -AB\sin\theta\left(I_0 + \frac{AB\cos\theta}{L}\right) = 0. \qquad (*)$$

In relatively weak fields, when

$$AB < LI_0, \qquad (**)$$

the expression in the parentheses of the right-hand side of Eq. (*) is positive for all values of θ, so this equation has only two physically distinguishable roots, both with $\sin\theta = 0$:

$$\theta_0 = 0, \qquad \text{and } \theta_1 = \pi,$$

both corresponding to the horizontal position of the loop. Now, calculating the second derivative,

$$\frac{d^2U}{d^2\theta} = \frac{d}{d\theta}\left(-ABI_0\sin\theta - \frac{A^2B^2\sin\theta\cos\theta}{L}\right) \equiv \frac{d}{d\theta}\left(-ABI_0\sin\theta - \frac{A^2B^2\sin 2\theta}{2L}\right)$$
$$= -ABI_0\cos\theta - \frac{A^2B^2\cos 2\theta}{L} \equiv -AB\left(I_0\cos\theta + \frac{AB\cos 2\theta}{L}\right),$$

we see that if the low-field condition (**) is satisfied, the second term in the last parentheses cannot affect the sign of the whole expression, so it is negative for θ_0 and positive for θ_1. This means that the

[75] Cf. Problem 9 whose first part is very similar to this one, because on a time scale $\Delta t << L/R$, a resistive loop behaves similarly to a superconducting one.

[76] Since the rotation axis passes through the loop's center of mass, gravity does not give any contribution to U.

initial horizontal position of the loop, with $\theta = \theta_0 = 0$, is unstable, while the flipped state, with $\theta = \theta_1 = \pi$, is its only stable position.

However, in higher fields, when the condition (**) is not valid, the second term in the parentheses (at $\theta = \theta_1 = \pi$, equal to AB/L), overrules the first term (at $\theta = \theta_1 = \pi$, equal to $-I_0$), so the flipped horizontal state becomes unstable as well. Instead, at such fields, Eq. (*) acquires two other roots,

$$\theta_{2,3} = \pm\cos^{-1}\left(-\frac{LI_0}{AB}\right).$$

Since

$$\left.\frac{d^2U}{d^2\theta}\right|_{\theta=\theta_{3,4}} = -AB\left(I_0\cos\theta_{2,3} + \frac{AB\cos 2\theta_{2,3}}{L}\right) = -AB\left[I_0\cos\theta_{2,3} + \frac{AB}{L}\left(2\cos^2\theta_{2,3} - 1\right)\right]$$
$$= AB\left\{I_0\frac{LI_0}{AB} - \frac{AB}{L}\left[2\left(\frac{LI_0}{AB}\right)^2 - 1\right]\right\} \equiv \frac{1}{L}\left(A^2B^2 - L^2I_0^2\right),$$

both states are stable as soon as they exist (at $AB > LI_0$).

The physics of this curious behavior is as follows. In weak fields, the role of the induced current I_B is small, and the applied field just tries to flip the loop, so its magnetic moment vector $\mathbf{m} \approx \mathbf{m}_0$ would be aligned with its vector $\mathbf{B}$. However, in high magnetic fields, the effects related to I_B dominate, and in the limit $B >> LI_0/A$, the field tries to turn the loop's *plane* parallel to the vector $\mathbf{B}$: $\theta_{2,3} \to \pm\pi/2$ ($\cos\theta_{2,3} \to 0$). This fact may be interpreted using the discussion at the end of Sec. 6.2 of the lecture notes, even though it is quantitatively valid only for cylindrical geometry. As was shown there, the external magnetic field tends to fill all available space. In the case of a superconducting ring with $I_0 \to 0$, the Meissner-Ochsenfeld effect pushes the field out of a volume of the order of $A^{3/2}\cos\theta$, so the system "tries" to minimize this volume by making $\cos\theta \to 0$.

Problem 6.18. Use the London equation to analyze the penetration of a uniform external magnetic field into a thin ($t \sim \delta_L$) planar superconducting film whose plane is parallel to the field.

Solution: According to the definition $\mathbf{B} = \nabla\times\mathbf{A}$ of the vector potential, in this 1D situation, the field's magnitude B is just the spatial derivative of the vector potential's magnitude, and hence obeys an equation similar to the 1D version of Eq. (6.56) of the lecture notes:

$$\frac{d^2B}{dx^2} = \frac{1}{\delta_L^2}B,$$

where the x-axis is normal to the film. The general solution of this linear equation is a linear combination of the functions $\exp\{\pm x/\delta_L\}$, or alternatively, of the functions $\sinh(x/\delta_L)$ and $\cosh(x/\delta_L)$. In our current problem, the magnetic field's distribution should be symmetric relative to the plane passing through the middle of the film's thickness. This is why, by selecting this plane for $x = 0$, we get

$$B(x) = \text{const}\times\cosh\frac{x}{\delta_L}.$$

The constant in this relation may be found from the usual macroscopic (not the coarse-grain!) boundary condition (5.117) applied to film surfaces: $H(\pm t/2) = H_0 = B_0/\mu_0$, i.e. $B(\pm t/2) = (\mu/\mu_0)B_0$. (For most superconductors, the difference between the ratio (μ/μ_0) and 1 is negligible, but I still keep it spelled out for clarity.) As a result, we get

$$B(x) = B_0 \frac{\mu}{\mu_0} \frac{\cosh(x/\delta_\mathrm{L})}{\cosh(t/2\delta_\mathrm{L})}.$$

This result shows that if the film is thin ($t << \delta_\mathrm{L}$), the field is almost uniformly distributed over its thickness. In the opposite limit, the field only penetrates into two δ_L-scale layers at the film's surfaces and is shielded from the film's interior by the equal and opposite supercurrents flowing in these layers.

Problem 6.19. Use the London equation to calculate the distribution of the supercurrent density j inside a long straight superconducting wire with a circular cross-section of radius $R \sim \delta_\mathrm{L}$, carrying current I.

Solution: In this cylindrical and axially symmetric geometry, the solution of the London equation (see Eq. (6.56) of the lecture notes) may be looked for in the form $\mathbf{A}(\mathbf{r}) = A(\rho)\mathbf{n}_z$, where the z-axis coincides with that of the wire, and ρ is the distance of the observation point $\mathbf{r}$ from that axis. With the well-known expression for the Laplace operator in cylindrical coordinates,[77] the equation becomes[78]

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dA}{d\rho}\right) = \frac{1}{\delta_L^2} A.$$

With the introduction of the natural dimensionless argument $\xi \equiv \rho/\delta_\mathrm{L}$, this equation becomes

$$\frac{d^2 A}{d\xi^2} + \frac{1}{\xi}\frac{dA}{d\xi} - A = 0,$$

which is the modified Bessel equation (2.155) with $\nu = 0$. Its general solution is

$$A = a_1 I_0(\xi) + a_2 K_0(\xi),$$

so the supercurrent density j (which, within the London gauge, is proportional to A – see Eq. (6.55) of the lecture notes) is

$$j(\rho) = c_1 I_0\left(\frac{\rho}{\delta_\mathrm{L}}\right) + c_2 K_0\left(\frac{\rho}{\delta_\mathrm{L}}\right).$$

Since the modified Bessel function of the second kind, $K_0(\xi)$, diverges at $\xi \to 0$ (see, e.g., either Eq. (2.157) or just the right panel of Fig. 2.22) while j has to stay finite at all points, the coefficient c_2 has to equal 0. The remaining coefficient c_1 may be readily related to the total current:

$$\int_A j(\rho) d^2\rho = 2\pi \int_0^R j(\rho)\rho d\rho = 2\pi c_1 \int_0^R I_0\left(\frac{\rho}{\delta_\mathrm{L}}\right)\rho d\rho \equiv 2\pi \delta_L^2 c_1 \int_0^{R/\delta_\mathrm{L}} I_0(\xi)\xi d\xi = I.$$

[77] See, e.g, MA Eq. (10.3) with $\partial/\partial\varphi = \partial/\partial z = 0$.

[78] Note that this equation, and the balance of the solution, are very similar to those in Problem 10, with the "only" exception that the complex parameter κ given by Eq. (6.30) is now replaced with the real parameter $1/\delta_\mathrm{L}$.

The last dimensionless integral may be calculated using the recurrence relation (2.143) (valid for any Bessel functions), in our current case with $n = 1$. As a result, we get

$$c_1 = I \Big/ 2\pi\delta_L^2 \int_0^{R/\delta_\mathrm{L}} I_0(\xi)\xi d\xi = I \Big/ 2\pi\delta_L^2 [\xi I_1(\xi)]_0^{R/\delta_\mathrm{L}} = \frac{I}{2\pi\delta_\mathrm{L} R I_1(R/\delta_\mathrm{L})},$$

so, finally,

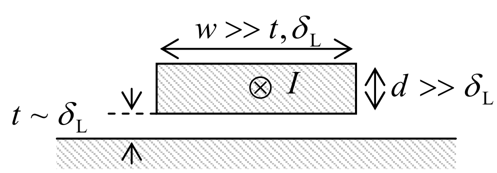
$$j = \frac{I}{2\pi\delta_\mathrm{L} R}\frac{I_0(\rho/\delta_\mathrm{L})}{I_1(R/\delta_\mathrm{L})}.$$

If the wire is very thin ($R << \delta_\mathrm{L}$), we may use the first of Eqs. (2.157) to write $I_0(\rho/\delta_\mathrm{L}) \approx 1$, $I_1(R/\delta_\mathrm{L}) \approx R/2\delta_\mathrm{L}$, so $j \approx I/\pi R^2 =$ const, i.e. the supercurrent is uniformly distributed over the wire's cross-section. In the opposite limit $\delta_\mathrm{L} << R$, we may use the first of the asymptotic formulas (2.158) to reduce our result to

$$j \approx \frac{I}{2\pi\delta_\mathrm{L} R}\left(\frac{R}{\rho}\right)^{1/2}\frac{\exp\{\rho/\delta_L\}}{\exp\{R/\delta_L\}} \equiv \frac{I}{2\pi\delta_\mathrm{L}(R/\rho)^{1/2}}\exp\left\{-\frac{R-\rho}{\delta_L}\right\}.$$

This expression shows that appreciable supercurrent only flows in a δ_L-thin sheet at the wire's surface – exactly like in the plane-surface problem – see Eq. (6.57).

Problem 6.20. Use the London equation to calculate the inductance (per unit length) of a long uniform superconducting strip placed close to the surface of a similar superconductor – see the figure on the right, which shows the structure's cross-section.



Solution: DC supercurrent does not require any electric field to sustain it, so it is free to self-distribute over the strip's cross-section to minimize the total energy of the system. In particular, in our case when t, $\delta_\mathrm{L} << w$, all the current in the strip has to be localized at its surface facing the bulk superconductor (which plays the role of a *magnetic ground plane*), to minimize the magnetic field outside the gap t, because any substantial field in those areas would decrease only on distances of the order of w and hence give contributions with a very large spatial weight $\sim w^2$, into the magnetic energy per unit length:

$$\frac{U}{l} = \int \frac{B^2(\mathbf{r})}{2\mu(\mathbf{r})} d^2 r. \qquad (*)$$

Next, for a gap of a constant thickness t, the linear density J of the supercurrent should be constant across the gap's width w, because otherwise the magnetic flux (also per unit length),

$$\frac{\Phi}{l} = \int B(x) dx \propto J, \qquad (**)$$

of the field $B(x)$ it creates in the gap would change across the width. (Here x is the Cartesian coordinate across the gap.) This is impossible because the thick superconducting electrodes prevent the magnetic field lines' from escaping the gap. So, $J = I/w =$ const.

The last (but not least) fact we need is that to prevent the magnetic field in the gap from spreading into the ground plane, its surface has to carry an equal and opposite supercurrent J. (This fact is the basis of *dc current transformers*, which are important components of superconductor electronic circuits.[79])

Now the problem may be readily solved using Eq. (6.57) of the lecture notes. Indeed, the integral in Eq. (**) is the sum of three integrals: one over the gap (giving the contribution $Bt = \mu_0 Ht = \mu_0 Jt = \mu_0 It/w$), and two similar integrals through the London penetration layers of the top electrode and the ground plane:

$$\int_0^\infty B(x)dx = B(0)\int_0^\infty \exp\left\{-\frac{x}{\delta_L}\right\}dx = \mu H(0)\delta_L = \mu J\delta_L = \frac{\mu I\delta_L}{w}.$$

As a result, for the inductance L per unit length, we get

$$\frac{L}{l} \equiv \frac{\Phi}{Il} = \frac{1}{I}\int B(x)dx = \frac{1}{I}\left(\frac{\mu_0 It}{w} + 2\frac{\mu I\delta_L}{w}\right) \equiv \mu_0\frac{t_{ef}}{w}, \qquad \text{with } t_{ef} \equiv t + 2\frac{\mu}{\mu_0}\delta_L. \qquad (***)$$

This expression is similar to the solution of Problem 5.4(iv), except for the effective increase of the gap between the strips due to the magnetic field's penetration into the superconductors. In practical superconductor integrated circuits, the insulating gap thicknesses t may be from a few nanometers to ~100 nm, while δ_L is of the order of 100 nm, so this penetration effect is quite noticeable (and even may be used for the London depth measurement).

Note also that in contrast to the coarse-grain (ideal-diamagnet) description of superconductors, which is equivalent to taking $\mu = 0$ inside these materials, the London equation takes supercurrents into account explicitly as "stand-alone" currents, so the magnetic permeability participating in the above result describes only the effects of *localized* atomic currents. In typical superconductors, these effects are relatively weak ($\mu \approx \mu_0$),[80] so the effective gap width t_{ef} is very close to just $(t + 2\delta_L)$.

One more important note: it might be tempting to derive Eq. (***) by calculating the magnetic field energy (*) and requiring it to equal $LlI^2/2$. However, such a calculation gives a wrong result for t_{ef}, without the factor of 2 before the second term. The reason for this discrepancy is somewhat subtle: since the Cooper pairs move in superconductors without dissipation, we cannot neglect their *kinetic energy*. Indeed, according to the London equation (6.55), the effective velocity of the Cooper pair condensate is

$$\mathbf{v} \equiv \frac{\mathbf{j}}{qn_p} = -\frac{q}{m}\mathbf{A}.$$

For the Meissner-Ochsenfeld effect in a 1D geometry, described by Eq. (6.57), the vector potential definition $\nabla\times\mathbf{A} \equiv \mathbf{B}$ yields $A^2 = \delta_L^2 B^2 \equiv (m/\mu q^2 n_p)B^2$, so the kinetic energy of Cooper pairs per unit volume,

$$\frac{T}{V} = n_p\frac{mv^2}{2} = n_p\frac{m}{2}\left(\frac{q}{m}\right)^2 A^2 = n_p\frac{m}{2}\left(\frac{q}{m}\right)^2\frac{m}{\mu q^2 n_p}B^2 \equiv \frac{B^2}{2\mu},$$

is exactly equal to that of the magnetic field. Naturally, its account properly doubles the contribution to the total energy from the London penetration layers.

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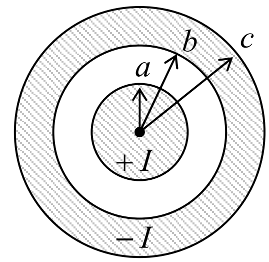
[79] See, e.g., A. Kadin, *Introduction to Superconducting Circuits*, Wiley, 1999.

[80] In particular, because ferromagnetism generally suppresses superconductivity, so the two effects rarely coexist in the same material.



Problem 6.21. Calculate the inductance (per unit length) of a superconducting cable with the round cross-section shown in the figure on the right, in the following limits:

(i) $\delta_L << a, b, c - b$, and

(ii) $a << \delta_L << b, c - b$.

Solutions: Due to the system's axial symmetry, the direction of the magnetic field is normal to the direction **ρ** from the system's axis, and its magnitude is a function of $\rho \equiv |\mathbf{\rho}|$ alone. Hence the *magnetic* energy (per unit length of the cable) may be calculated just as

$$\frac{U}{l} = \frac{1}{2\mu_0}\int B^2(\rho)d^2\rho = \frac{\pi}{\mu_0}\int B^2(\rho)\rho d\rho \ ,$$

where the integral should be extended to all distances at which B is substantial. Note that in order to use this formula for the evaluation of the inductance in the usual sense of the word (i.e. the mutual inductance between the internal and external conductors), the algebraic sum of their currents (i.e. the net current in the cable) has to be zero, so according to the Ampère law (5.37), the field vanishes at $\rho > c$.

(i) In the limit $\delta_L << a, b, c - b$, we can neglect the magnetic field's penetration into both conductors of the cable,[81] and use Eq. (5.20) of the lecture notes for the field between them:

$$B(\rho) = \frac{\mu_0 I}{2\pi\rho}.$$

An elementary integration gives

$$\frac{U}{l} = \frac{\pi}{\mu_0}\int_a^b B^2(\rho)\rho d\rho = \frac{\pi}{\mu_0}\left(\frac{\mu_0 I}{2\pi}\right)^2\int_a^b \frac{d\rho}{\rho} = \frac{\mu_0 I^2}{4\pi}\ln\frac{b}{a},$$

so, according to Eq. (5.71), the self-inductance per unit length is

$$\frac{L}{l} \equiv \frac{U/(I^2/2)}{l} = \frac{\mu_0}{2\pi}\ln\frac{b}{a}.$$

Comparing this result with Eq. (5.79) of the lecture notes, we see that this inductance is somewhat lower than that of a normal-conductor cable (at relatively low frequencies), due to the negligible magnetic field penetration into the conductors. Note, however, that the above expression is valid for normal-conductor cables as well, provided that the frequency of the current is so high that the skin-depth δ_s (6.33) is much smaller than all transverse dimensions of the cable.

(ii) As Eq. (6.56) shows, in the limit $a << \delta_L$, the supercurrent is uniformly distributed across the inner conductor, with density $j = I/\pi a^2$, just as in a uniform normal conductor. Thus we may repeat the calculations of Sec. 5.3, which have led to Eq. (5.79), with the integration from 0 to b only, and get

$$\frac{L_m}{l} \equiv \frac{U/(I^2/2)}{l} = \frac{1}{2\pi}\left(\mu_0 \ln\frac{b}{a} + \frac{\mu}{4}\right),$$

[81] This essentially means that in this limit, we are using the coarse-grain (ideal-diamagnetic) description of the Meissner-Ochsenfeld effect.

i.e. the same answer as for a normal-metal cable with $(c - b) \to 0$ – see Eq. (5.80). However, this is not the end of the story. As was discussed in the solution of the previous problem, in superconductors, the magnetic energy has to be summed up with the kinetic energy T of the Cooper-pair condensate. Moreover, in our current case of a very thin inner conductor ($a << \delta_L$, i.e. $A_a \equiv \pi a^2 << \delta_L^2$), T is much larger than U. Indeed, for the uniformly distributed supercurrent, the kinetic energy per unit length is

$$\frac{T}{l} = A_a n_p \frac{mv^2}{2} = A_a n_p \frac{m}{2}\left(\frac{j}{qn_p}\right)^2 = A_a n_p \frac{m}{2}\left(\frac{I}{qn_p A_a}\right)^2 = \frac{m}{q^2 n_p A_a}\frac{I^2}{2},$$

so the so-called *kinetic inductance* associated with this effect,

$$\frac{L_k}{l} \equiv \frac{T/(I^2/2)}{l} = \frac{m}{n_p q^2 A_a} = \mu\frac{\delta_L^2}{A_a} = \mu\frac{\delta_L^2}{\pi a^2} >> \mu,$$

is much larger than the magnetic inductance $L_m/l \sim \mu_0$.

<u>Problem 6.22</u>. Use the London equation to analyze the magnetic field shielding by a superconducting thin film of thickness $t << \delta_L$, by calculating the penetration of the field induced by current I in a thin wire that runs parallel to a wide planar thin film, at a distance $d >> t$ from it, into the space behind the film.

Solution: This problem is conceptually very close to that of Problem 2.39 on electrostatic screening, so the reader is advised to review its solution first.[82] The only substantial novelty of the magnetic field is the vector character of its potential **A**. However, as was discussed in Sec. 5.2 of the lecture notes,[83] the potential created by the current I in a straight wire has only one Cartesian component, directed along the wire, and it is clear that a superconducting film parallel to it cannot change this alignment. (The proof of the last fact is simple: if **A** had a transverse component, would it be directed clockwise or counter-clockwise?) So, in our geometry, the London equation (6.56) is valid for the magnitude A of the only (say, z-) component of $\mathbf{A} = A\mathbf{n}_z$.

Hence, we may return to the same reasoning as was used in the model solution of Problem 2.39. Namely, due to the imposed condition $t << \delta_L$, the gradient of the vector potential A in the film is dominated by its component along the axis (say, y) normal to the film's plane, so the London equation (6.56) is well approximated by its 1D version

$$\frac{\partial^2 A}{\partial y^2} = \frac{1}{\delta_L^2}A,$$

even if A is relatively slowly (on distances of the order of $d >> t$) changing in the plane of the film. Integrating both sides of this equation over the film's thickness t, we get

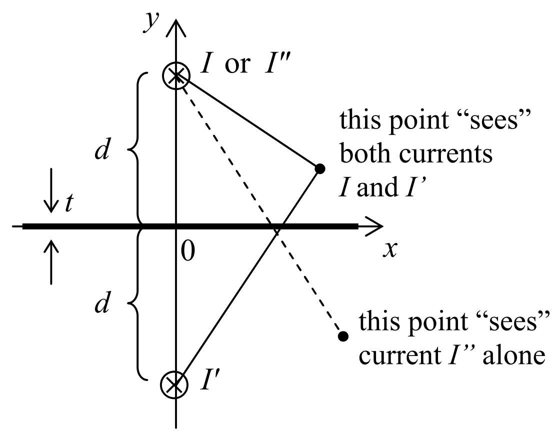
$$\frac{\partial A_+}{\partial y} - \frac{\partial A_-}{\partial y} = \frac{1}{\delta_L^2}\int_{-t/2}^{+t/2} A dy, \qquad (*)$$

[82] The current problem is more realistic, because (as was discussed in Sec. 6.4) the London penetration depth δ_L is typically of the order of 10^{-7}-10^{-6} m, so superconducting films of thickness $t <<\ \delta_L$ are quite realistic.
[83] See, e.g., the discussion leading to Eq. (5.49).

where the indices ± refer to opposite film's surfaces. This relation shows that if $t << \delta_L$, the film can create only a small change (of the order of At/δ_L^2) of the derivative $\partial A/\partial y$ through its thickness, and hence just a negligible variation (of the order of At^2/δ_L^2) of the vector potential inside it. As a result, Eq. (*) yields two boundary conditions for the potential's distribution outside the film:

$$A_+ = A_- \equiv A, \qquad \frac{\partial A_+}{\partial y} - \frac{\partial A_-}{\partial y} = \frac{t}{\delta_L^2} A. \qquad (**)$$

Now, inspired by the success of numerous charge-image analyses in Chapters 2-5, we may try to describe the magnetic field (and hence its potential A) in the semi-space where the actual current I resides (say, $y > 0$) as a superposition of the fields induced by that current and its image I' running in parallel, at the same distance d from the film, but on its opposite side – see the figure on the right.



In addition, to describe the field's remnants at the other side of the film (at $y < 0$), we may use the effective current I'' co-located with the actual current. As a result, by using Eq. (5.51) of the lecture notes for each current's potential in unlimited space, we may look for the potential distribution in the form[84]

$$A(x,y) = -\frac{\mu_0}{2\pi} \times \begin{cases} I \ln\left[x^2 + (y-d)^2\right]^{1/2} + I' \ln\left[x^2 + (y+d)^2\right]^{1/2}, & \text{for } y > 0, \\ I'' \ln\left[x^2 + (y-d)^2\right]^{1/2}, & \text{for } y < 0. \end{cases} \qquad (***)$$

Plugging this solution into the boundary conditions (**), we see that they are indeed satisfied (so Eq. (***) is indeed the unique solution of our boundary problem) if the effective currents I' and I'' obey the following relations:

$$I + I' = I'', \qquad (I - I') - I'' = \frac{td}{\delta_L^2} I'',$$

which are similar to those for image charges in the corresponding electrostatic screening problem. As a result, the solution of this simple system of equations for I' and I'' is also similar. In particular, for the ratio I''/I, which is an adequate measure of the field's penetration behind the film, we get

$$\frac{I''}{I} = \frac{1}{1 + td/2\delta_L^2}.$$

As in electrostatics, the most interesting feature of this result is that the ratio starts to drop as soon as t is increased to $\sim\delta_L^2/d << \delta_L$, i.e. the field may be well (though not exponentially well) screened even by a film much thinner than the penetration depth – if d is large enough. Another formulation of the same fact, which may be more revealing in some problems, is that a thin ($t << \delta_L$) superconductor film has a certain characteristic *lateral* size scale

[84] As was discussed in Sec. 3.3 of the lecture notes, absolutely the same solution (though with different constants q' and q'') describes the electric field penetration into a half-space filled with a linear dielectric.

$$\delta_\perp \equiv \frac{2\delta_L^2}{t},$$

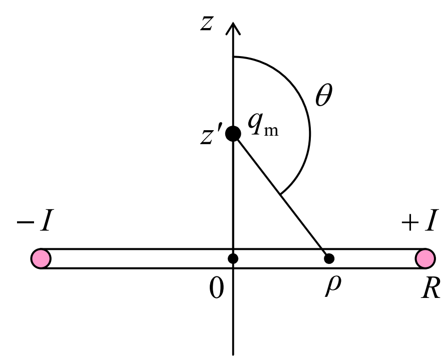
usually called the *perpendicular penetration length*, which characterizes its diamagnetic properties. In particular, it is easy to use Eqs. (5.28) and (6.55) to show (the additional task highly recommended to the reader) that a supercurrent run through a strip of width w is uniformly distributed over its width only if $w \ll \delta_\perp$. This is one of the reasons why in applications, superconducting strips are typically run very close to a superconducting ground plane, which enforces the uniform current distribution even in wide thin films – see, e.g., Problem 20 above.

Problem 6.23. Assuming that the magnetic monopole does exist and has a magnetic charge q_m, calculate the change ΔI of current in a superconducting loop due to a passage of a single monopole through its area. Evaluate ΔI for a monopole with the charge's value conjectured by P. Dirac, $q_m = nq_0 \equiv n(2\pi\hbar/e)$ with an integer n, and compare the result with the magnetic flux quantum Φ_0 (6.62). Review your result for a similar passage of a single quasi-monopole magnetic charge formed at one of the ends of a permanent-magnet needle – see Fig. 5.19 of the lecture notes and the accompanying discussion.

Hint: To simplify calculations, you may consider the monopole's passage along the symmetry axis of a round ring of radius R, made of a superconducting wire with a cross-section's area A satisfying the conditions $\delta_L^2 \ll A \ll R^2$.

Solution: As was discussed in Sec. 5.6 of the lecture notes (see Eq. (5.138) and its derivation), the magnetic charges are usually defined so that if a single charge q_m is located at point $\mathbf{r}'$, it creates the magnetic field

$$\mathbf{B}(\mathbf{r}) = \frac{q_m}{4\pi|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}'). \qquad (*)$$



Let us direct the z-axis along the line of a slow passage of the monopole, i.e. along the symmetry axis of the considered superconducting ring, with the origin on its plane – see the figure on the right. Then, taking into account the given condition $A \ll R^2$, the flux of the field (**) through the ring's area may be calculated as[85]

$$\Phi_{\text{ext}} \equiv \int_S B_n d^2r = 2\pi\int_0^R B_z \rho d\rho = \frac{q_m}{2}\int_0^R \frac{(-z')}{\left(\rho^2+z'^2\right)^{3/2}}\rho d\rho$$

$$= -\frac{q_m}{4}z'\int_0^R \frac{d\left(\rho^2+z'^2\right)}{\left(\rho^2+z'^2\right)^{3/2}} = -\frac{q_m}{2}z'\frac{1}{\left(\rho^2+z'^2\right)^{1/2}}\Bigg|_{\rho=0}^{\rho=R} \equiv \frac{q_m}{2}\left[\frac{z'}{\left(R^2+z'^2\right)^{1/2}} - \operatorname{sgn}(z')\right]. \qquad (**)$$

Due to the strong condition $A \gg \delta_L^2$, one might assume that due to the Meissner-Ochsenfeld effect, the full magnetic flux (6.78) through the ring,

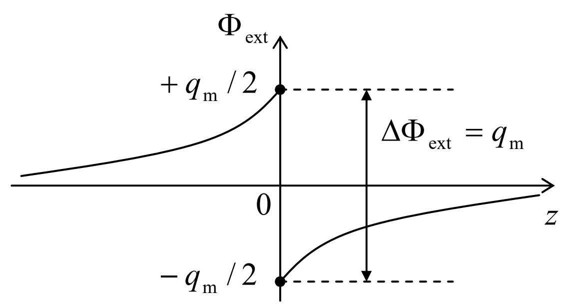
$$\Phi = \Phi_{\text{ext}} - LI$$

[85] Here the observation's point distance ρ from the axis $z = 0$ is typeset in Roman font, to avoid any chance of confusion with the electric charge density ρ, typeset in Italics.

(where I is the supercurrent in the ring, with the positive direction indicated in the figure above, and L is the ring's self-inductance) should be constant during the monopole's passage, the current's change (referred to its initial value at $-z' >> R$ when $\Phi_{ext} = 0$) should be

$$\Delta I = \frac{\Phi_{ext}}{L} = \frac{q_m}{2L}\left[\frac{z'}{\left(R^2 + z'^2\right)^{1/2}} - \operatorname{sgn}(z')\right]. \qquad (***)$$

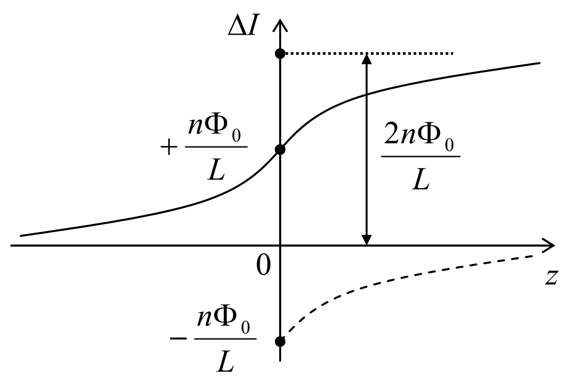
Here we have arrived at a contradiction: the function $\Phi_{ext}(z')$, and hence the calculated current $I(z')$, are discontinuous at $z' = 0$, experiencing jumps $\Delta\Phi_{ext} = -q_m$ and hence $\Delta I = -q_m/L$ – see the figure on the right. This is not some technical error in our calculation. Indeed, if the magnetic charge is slightly below the ring's plane, its field lines piercing the ring's area are directed up, so their flux is positive and approaches ½ of the total flux $\Phi_{full} = q_m$ of all field lines originating from the point charge. As soon as the charge crosses the plane, the signs of the field and the flux reverse, with Φ_{ext} keeping its magnitude – exactly as Eq. (***) predicts.[86] So, our math is right but for physics, such an instant jump of such a measurable variable as current is unacceptable.



This problem is actually much broader: in 1931, Paul Dirac noticed that a magnetic monopole's passage even through usual matter would provide an instant phase shift of any quantum-mechanical wavefunction met on its way. In order to make this disrupting effect less drastic, Dirac suggested that the monopole charges q_m may be multiples of

$$q_0 \equiv \frac{2\pi\hbar}{e},$$

because in this case, the shifts of the wavefunction phases would be multiples of 2π, with no observable consequences for ordinary matter. For our case of a ring made of a usual superconductor with the effective supercurrent carrier charges equal to $-2\boldsymbol{e}$ rather than $-e$, the jump $\Delta\Phi_{ext}$ induced by such q_m (with the replacement $e \to 2e$) would be a multiple of $2\Phi_0$, where Φ_0 is the flux quantum (6.62), making the new quantum state of the superconducting Bose-Einstein condensate in the ring compatible with the same current I. Assuming that during this hypothetical instantaneous jump, the current would be continuous, we arrive at the function $I(z')$ shown with the solid line in the figure on the right, where the dashed line shows the unphysical result (****). The resulting change of the current by $2n\Phi_0/L$ (where n is some unknown integer) would be sufficiently large to be measured, for example, by a SQUID magnetometer – see Sec. 6.5.



[86] By the way, from this reasoning, it is clear that the flux jump's magnitude $\Delta\Phi = -q_m$ does not depend on the loop's shape and on the exact trajectory of the magnetic monopole. (The reader is challenged to use Eq. (*) for a more formal proof of this fact, by employing the same approach as in the solution of Problem 1.7.) This jump also remains the same if the monopole moves with a very high (relativistic) velocity.

Actually, magnetic monopole searches using such superconducting loop setups were carried out in the 1980s by several groups, and one observed event (compatible with $n = 4$) was even claimed – but then never confirmed by either that or any other research team.

Note that the above contradiction does not appear if the ring's plane is pierced by an end of a thin ferromagnetic needle, despite the fact that the magnetic field it creates around is also described by Eq. (*), in this case, with $q_m = \pm\mu_0 M_0 A$ – see Fig. 5.19a of the lecture notes and the accompanying discussion. Indeed, as was shown in the solution of Problem 5.25, the magnetic field inside the thin needle is nearly uniform: $\mathbf{H} = -\mathbf{M}_0$, $\mathbf{B} = \mu_0\mathbf{H} = -\mu_0\mathbf{M}_0$. As a result, in this case, the jump of the flux (**) due to the needle's field outside it is exactly compensated by the equal and opposite jump of the flux due to the field inside the needle. This means that the function $\Delta I(z')$ behaves as the last figure above shows (with the gradual change by $q_m/L = \pm\mu_0 M_0 A/L$) without any hypothetical assumptions. The passage, through the same ring, of the second end of the needle, which carries an equal and opposite magnetic charge (see Eq. (5.138) again), induces the opposite change of the current, returning it to the initial value.

Problem 6.24. Use the Ginzburg-Landau equations (6.54) and (6.63) to calculate the largest ("critical") value of supercurrent in a uniform superconducting wire with a cross-section area much smaller than δ_L^2.

Solution: As was discussed in the solution of Problem 19, in a wire with a cross-section so small, the current density, and hence the gauge-invariant combination $(\hbar\nabla - iq\mathbf{A})\psi$ participating in the Ginzburg-Landau equations, are uniformly distributed along the cross-section, so Eqs. (6.54) yields

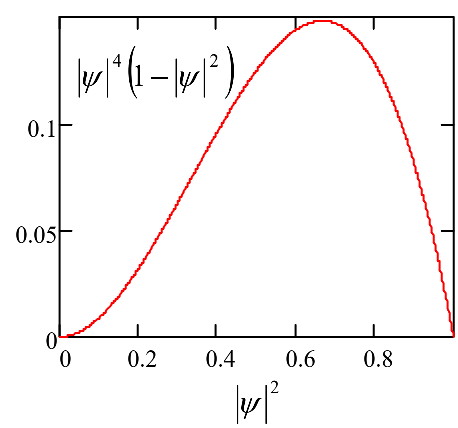
$$j \equiv j_z = \frac{\hbar q n_p}{m}|\psi|^2\left(\frac{\partial\varphi}{\partial z} - \frac{q}{\hbar}A\right), \tag{*}$$

where A is the (only) Cartesian component of the vector potential, directed along the wire's length, which is taken for the z-axis. As was discussed at the derivation of Eq. (6.55) of the lecture notes, in this geometry, we may (just for the notation simplicity) select the gauge $\varphi = \text{const}$, so Eq. (*) yields

$$A = -\frac{m}{q^2 n_p|\psi|^2}j \equiv -\frac{\mu\delta_L^2}{|\psi|^2}j\,.$$

Plugging this expression into the second form of Eq. (6.63), taken in the same gauge, we get the following equation:

$$\xi^2\left(\frac{\mu q\delta_L^2}{\hbar}j\right)^2 = |\psi|^4\left(1-|\psi|^2\right). \tag{**}$$



At $j \to 0$, its solution gives the unperturbed value (equal to 1) of the modulus of the wavefunction ψ of the Cooper pair condensate, but an increase of current suppresses it – see the figure on the right, which shows the right-hand side of Eq. (**) as a function of $|\psi|^2$. As an elementary differentiation shows, the function reaches its maximum at

$$|\psi|^2 = \frac{2}{3}, \quad \text{with } \left[|\psi|^4\left(1-|\psi|^2\right)\right]_{\max} = \left(\frac{2}{3}\right)^2\left(1-\frac{2}{3}\right) \equiv \frac{4}{27} \approx 0.148\,.$$

From here, the maximum ("critical") value of the supercurrent density is

$$j_c = \left(\frac{4}{27}\right)^{1/2} \frac{\hbar}{\mu|q|\delta_L^2\xi}. \qquad (***)$$

Increasing j above this value destroys the superconductivity, driving the wire into its "normal" state, with $\psi = 0$. Note that an external magnetic field **H**, applied to a bulk superconductor, has a qualitatively similar effect. Indeed, as Eqs. (6.38) and (6.57) of the lecture notes show, it induces a surface supercurrent with a linear density $J \sim H$,[87] distributed within a layer of thickness δ_L, i.e. with the areal density $j \sim H/\delta_L$. When this density approaches the critical value (***), i.e. at

$$H \sim H_c \equiv \frac{\hbar}{\sqrt{2}\mu|q|\delta_L\xi}, \qquad \text{i.e. } B \sim B_c \equiv \mu H_c \equiv \frac{\hbar}{\sqrt{2}|q|\delta_L\xi} \equiv \frac{\Phi_0}{2\sqrt{2}\pi\delta_L\xi}. \qquad (****)$$

the superconductivity in the surface layer starts to be suppressed. As the detailed solutions of the Ginzburg-Landau equations show,[88] the behavior in higher fields may be rather complicated and depends on the relation between the superconductor's characteristic length δ_L and ξ. In the so-called *type-I superconductors* with $\delta_L < \xi/\sqrt{2}$, the superconductivity (at least in cylindrical geometries) is suppressed, at $H = H_c$, simultaneously in the whole sample. However, in the *type-II superconductors* that have $\xi/\sqrt{2} < \delta_L$, the magnetic field may penetrate into the superconductor in the form of separate Abrikosov vortices – see the discussion in Sec. 6.5 of the lecture notes.[89] Moreover, in the case $\xi/\sqrt{2} << \delta_L$, such vortices may penetrate the superconductor even at much lower fields – see Eq. (6.32) of the lecture notes and also the next problem.

Problem 6.25. Use the discussion of a long straight Abrikosov vortex, in the limit $\xi << \delta_L$, in Sec. 6.5 of the lecture notes, to prove Eqs. (6.71)-(6.72) for its energy per unit length and the first critical field.

Solution: As it follows from the model solutions of Problems 20-21, the total energy of the vortex is the sum of its magnetic field energy:

$$U = \frac{1}{2\mu}\int B^2 d^3r\,,$$

and the kinetic energy of the Cooper-pair condensate:

$$T = \frac{m}{2}\int n_p v^2 d^3r \equiv \frac{m}{2}\int n_p\left(\frac{j}{qn_p}\right)^2 d^3r \equiv \frac{m}{2q^2n_p}\int j^2 d^3r \equiv \frac{\mu\delta_L^2}{2}\int j^2 d^3r\,,$$

[87] As Eq. (6.38) shows, in a cylindrical geometry (such as shown in Fig. 5.15 or Fig. 6.2) the relation $J = H_{ext}$ is exact; however, in the general case, there is an additional geometric factor in their relation.

[88] See, e.g., Chapters 1 and 5 in M. Tinkham, *Introduction to Superconductivity*, 2nd ed., McGraw-Hill, 1996.

[89] For such superconductors, the above result for the critical current is valid for wires with cross-section areas smaller than not only δ_L^2 but also ξ^2.

where, for the last step, Eq. (6.46) of the lecture notes was used. Plugging into these expressions, respectively, Eqs. (6.68) and (6.70), we get the following energies per unit length of the vortex:

$$\frac{U}{l}=\frac{1}{2\mu}\int B^2 d^2r=\frac{1}{2\mu}2\pi\int_{\sim\xi}^{\infty}B^2(\rho)\rho d\rho=\frac{\pi}{\mu}\left(\frac{\Phi_0}{2\pi\delta_L^2}\right)^2\int_{\sim\xi}^{\infty}K_0^2\left(\frac{\rho}{\delta_\mathrm{L}}\right)\rho d\rho\equiv\frac{\Phi_0^2}{4\pi\mu\delta_L^2}\int_{\varepsilon}^{\infty}K_0^2(\zeta)\zeta d\zeta\ ,$$

$$\frac{T}{l}=\frac{\mu\delta_\mathrm{L}^2}{2}\int j^2 d^2r=\frac{\mu\delta_\mathrm{L}^2}{2}2\pi\int_{\sim\xi}^{\infty}j^2(\rho)\rho d\rho=\pi\mu\delta_\mathrm{L}^2\left(\frac{\Phi_0}{2\pi\mu\delta_L^2}\right)^2\int_{\sim\xi}^{\infty}K_1^2\left(\frac{\rho}{\delta_\mathrm{L}}\right)\rho d\rho\equiv\frac{\Phi_0^2}{4\pi\mu\delta_L^2}\int_{\varepsilon}^{\infty}K_1^2(\zeta)\zeta d\zeta\ ,$$

so the total energy of the vortex per unit length, i.e. its tension, is

$$\mathscr{T}\equiv\frac{U+T}{l}=\frac{\Phi_0^2}{4\pi\mu\delta_\mathrm{L}^2}\int_{\varepsilon}^{\infty}\left[K_0^2(\zeta)+K_1^2(\zeta)\right]\zeta d\zeta\ ,$$

where $\varepsilon \sim \xi/\delta_\mathrm{L} << 1$. (This lower cutoff is due to the fact, that Eqs. (6.68) and (6.70) are only valid at distances much larger than the coherence distance ξ – see the discussion in Sec. 6.5 of the lecture notes.) Such integrals are well-known:[90]

$$\int K_n^2(\zeta)\zeta d\zeta=\frac{\zeta^2}{2}\left[K_n^2(\zeta)-K_{n-1}(\zeta)K_{n+1}(\zeta)\right];$$

taking into account that, according to Eq. (2.150), $K_{-1}(\zeta) = -K_1(\zeta)$, for our particular case they give

$$\int_{\varepsilon}^{\infty}\left[K_0^2(\zeta)+K_1^2(\zeta)\right]\zeta d\zeta=\left\{\frac{\zeta^2}{2}\left[K_0^2(\zeta)+2K_1^2(\zeta)-K_0(\zeta)K_2(\zeta)\right]\right\}_{\varepsilon}^{\infty}.$$

As Eq. (2.158) and Fig. 2.22 of the lecture notes show, on the upper limit all these functions vanish. Now using the second of Eqs. (2.157) for their approximate values on the lower limit, we get

$$\int_{\varepsilon}^{\infty}\left[K_0^2(\zeta)+K_1^2(\zeta)\right]\zeta d\zeta\approx\frac{\varepsilon^2}{2}K_0(\varepsilon)K_2(\varepsilon)\approx\ln\frac{1}{\varepsilon}\approx\ln\frac{\delta_\mathrm{L}}{\xi}\ ,$$

thus proving Eq. (6.71):

$$\mathscr{T}\approx\frac{\Phi_0^2}{4\pi\mu\delta_\mathrm{L}^2}\ln\frac{\delta_\mathrm{L}}{\xi}\ .$$

A more detailed theory,[91] involving an explicit solution of the Ginzburg-Landau equation (6.53) in the vortex core (i.e. at distances at $\rho \sim \xi << \delta_\mathrm{L}$ from its axis), confirms this result, offering just a small correction: $\ln(\delta_\mathrm{L}/\xi) \rightarrow \ln(\delta_\mathrm{L}/\xi) + 0.50$.

In order to calculate the first critical field H_{c1}, let us use Eq. (6.17) of the lecture notes for the Gibbs potential energy, corrected for the kinetic energy T of the Cooper-pair condensate by the replacement $U \rightarrow U + T \equiv \mathscr{T}l$, for a single vortex inside a superconductor:

$$\frac{U_\mathrm{G}}{l}=\mathscr{T}-\int \mathbf{H}_\mathrm{ext}\cdot\mathbf{B}d^2r\ ,$$

[90] This equality, valid for any Bessel functions, follows from the recurrence relations (2.142).
[91] C. R. Hu, *Phys. Rev. B*6, 1956 (1972).

where the integral may be limited by the cross-section of the vortex. In the cylindrical geometry we are considering, the field $\mathbf{B}$ of the vortex is directed along the external field $\mathbf{H}_{\text{ext}}$, which is constant. Hence we may take $\mathbf{H}_{\text{ext}}$ out of the integral, and the remaining integral of $\mathbf{B}$ is just the quantized magnetic flux Φ_0 of the vortex; hence

$$\frac{U_G}{l} = \mathscr{T} - H_{\text{ext}}\Phi_0 .$$

It is energy-favorable for a vortex to enter the superconductor if this value is lower than U_G in the absence of the vortex (with our choice of the arbitrary constant, this value is zero), i.e. if $H_{\text{ext}} > H_{c1}$, where

$$H_{c1} \equiv \frac{\mathscr{T}}{\Phi_0} \approx \frac{\Phi_0}{4\pi\mu\delta_L^2}\ln\frac{\delta_L}{\xi},$$

thus proving Eq. (6.72) of the lecture notes.

Note that the condition $H_{\text{ext}} > H_{c1}$ makes the vortex entrance *possible* but does not *guarantee* it, because it may be prevented by a substantial potential barrier at the surface, which is fully suppressed only when the field reaches the (much higher) value H_c estimated in the previous problem:

$$H_c \equiv \frac{\hbar}{\sqrt{2}\mu|q|\delta_L\xi} \equiv \frac{\Phi_0}{2\pi\sqrt{2}\mu|q|\delta_L\xi} \sim \frac{\delta_L}{\xi}H_{c1} >> H_{c1} .$$

However, certain experimental tricks (such as using slightly non-cylindrical geometries, e.g., some surface roughening) may enable reproducible vortex entry and exit at fields very close to H_{c1}.

Problem 6.26.* Use the Ginzburg-Landau equations (6.54) and (6.63) to prove the Josephson relation (6.76) for a small superconducting weak link, and express its critical current I_c via the Ohmic resistance R_n of the same weak link in its normal state.

Solution:[92] Let the weak link size scale a be much smaller than both the Cooper pair size ξ and the London penetration depth δ_L. The first of these relations ($a << \xi$) makes the left-hand side of the second form of Eq. (6.63), which scales as ξ^2/a^2, much larger than 1, i.e. much larger than its right-hand side. On the other hand, according to the discussion in Sec. 6.4 of the lecture notes, the second relation ($a << \delta_L$) allows us to neglect magnetic field effects, and hence drop the term ($-q\mathbf{A}$) from the parenthesis in Eq. (6.63), reducing it to just our familiar Laplace equation, but now for the complex wavefunction:

$$\nabla^2\psi = 0 .$$

Since weak coupling of the superconducting electrodes cannot change $|\psi|$ in their bulk, this differential equation should be solved with the following simple boundary conditions:

$$\psi(\mathbf{r}) \to \begin{cases} |\psi|e^{i\varphi_1}, & \text{for } \mathbf{r} \to \mathbf{r}_1, \\ |\psi|e^{i\varphi_2}, & \text{for } \mathbf{r} \to \mathbf{r}_2, \end{cases}$$

[92] This simple and elegant calculation belongs to L. Aslamazov and A. Larkin, *JETP Lett*. **9**, 87 (1969).

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are some points well inside the corresponding superconductors, i.e. at distances much larger than a from the weak link. It is straightforward to verify that the (unique) solution of this boundary problem may be expressed as follows,

$$\psi(\mathbf{r}) = |\psi| e^{i\varphi_1} f(\mathbf{r}) + |\psi| e^{i\varphi_2} [1 - f(\mathbf{r})],$$

via the *real* function $f(\mathbf{r})$ that satisfies the Laplace equation and the boundary conditions

$$f(\mathbf{r}) \to \begin{cases} 1, & \text{for } \mathbf{r} \to \mathbf{r}_1, \\ 0, & \text{for } \mathbf{r} \to \mathbf{r}_2. \end{cases}$$

The function $f(\mathbf{r})$ depends on the weak link's geometry and may be rather complicated, but we do not need to know it to get the most important result. Indeed, plugging this solution into Eq. (6.54), with the term $-q\mathbf{A}$ again ignored as being negligibly small, we get

$$\mathbf{j} = -\frac{\hbar q n_\mathrm{p}}{m} \nabla f \sin\varphi .$$

Integrating this relation over any cross-section S of the weak link, we arrive at B. Josephson's result (6.76),

$$I = I_\mathrm{c} \sin\varphi .$$

with the following critical current:

$$I_\mathrm{c} = -\frac{\hbar q n_p(T)}{m} \int_S (\nabla f)_n d^2 r . \qquad (*)$$

This result may be readily expressed via the resistance of the same weak link in the "normal" (non-superconducting) state, say at $T > T_\mathrm{c}$. Indeed, as we know from Sec. 4.3, the distribution of the electrostatic potential ϕ at normal conduction also obeys the Laplace equation, with boundary conditions that may be taken in the form

$$\phi(\mathbf{r}) \to \begin{cases} V, & \text{for } \mathbf{r} \to \mathbf{r}_1, \\ 0, & \text{for } \mathbf{r} \to \mathbf{r}_2. \end{cases}$$

Comparing the boundary problem for the function $\phi(\mathbf{r})$ with that for the function $f(\mathbf{r})$, we get $\phi = Vf$. This means that the gradient ∇f that participates in Eq. (*) is just $(-\mathbf{E}/V) = (-\mathbf{j}/\sigma V)$. Hence the integral in that formula is just $-I/\sigma V = -1/\sigma R_\mathrm{n}$, where R_n is the resistance of the weak link in its normal state. As a result, we get the expression

$$I_\mathrm{c} = \frac{\hbar q n_\mathrm{p}}{m\sigma} \frac{1}{R_\mathrm{n}},$$

showing that the $I_\mathrm{c} R_\mathrm{n}$ product is independent of the link's geometry, though it does depend on temperature – vanishing, together with n_p, at $T \to T_\mathrm{c}$.

The microscopic theory of superconductivity (which is beyond the framework of this series) confirms this conclusion and shows that well below the critical temperature, $I_\mathrm{c} R_\mathrm{n}$ of such short weak links is of the order of $\Delta(0)/e$ – typically, a few mV.

<u>Problem 6.27</u>. Use Eqs. (6.76) and (6.79) of the lecture notes to calculate the coupling energy of a Josephson junction and the full potential energy of the SQUID shown in Fig. 6.4c.

Solutions: The elementary work of an external voltage V applied to a lumped two-terminal device that carries current $I = dQ/dt$ (where Q is the total electric charge moving through the device) may be calculated as $d\mathscr{W} = VdQ = VIdt$. Combining this expression with Eqs. (6.76) and (6.79), we get

$$d\mathscr{W} = VIdt = \frac{\hbar}{2e}\frac{d\varphi}{dt} I_{\rm c} \sin\varphi \, dt \equiv \frac{\hbar I_{\rm c}}{2e} \sin\varphi \, d\varphi \equiv -\frac{\hbar I_{\rm c}}{2e} d(\cos\varphi).$$

Hence the work needed to drive the junction's phase to a certain value φ, i.e. its coupling ("Josephson") energy $U_{\rm J}$ is[93]

$$U_{\rm J}(\varphi) = \int^{\varphi} d\mathscr{W} = -E_{\rm J}\cos\varphi + \text{const}, \qquad \text{where } E_{\rm J} \equiv \frac{\hbar I_{\rm c}}{2e}.$$

Note that the function $U_{\rm J}(\varphi)$ is highly nonlinear, and quadratic only at $|\varphi| << 1$ (when Eq. (6.76) is reduced to $I \approx I_{\rm c}\varphi$):[94]

$$U_{\rm J}(\varphi) \approx E_{\rm J}\frac{\varphi^2}{2} + \text{const} = \frac{E_{\rm J}}{I_{\rm c}^2}\frac{I^2}{2} + \text{const}.$$

In a SQUID, the Josephson energy $U_{\rm J}$ should be added to the magnetic energy $U_{\rm m} = LI^2/2$ of the superconducting loop, so the energy of the system may be represented as

$$U = \frac{LI^2}{2} - E_{\rm J}\cos\varphi + \text{const} = \frac{(\Phi - \Phi_{\rm ext})^2}{2L} - \frac{\hbar I_{\rm c}}{2e}\cos\frac{2\pi\Phi}{\Phi_0} + \text{const}, \qquad (*)$$

where at the second step, the first of Eqs. (6.78) was used. If the magnetic flux Φ is understood as the generalized coordinate of the system, this function $U = U(\Phi)$ represents its effective potential energy (despite the "kinetic energy" terminology for its component $U_{\rm J}$), so its minima correspond to stable states of the system.[95] The figure below shows this potential energy profile, plotted for several values of the external flux, and two characteristic values of the normalized $LI_{\rm c}$ product, $\lambda \equiv 2\pi LI_{\rm c}/\Phi_0$.

An elementary analysis of Eq. (*) shows that the fixed-point requirement $\partial U/\partial\Phi = 0$ yields the stationary relation $\Phi(\Phi_{\rm ext})$ given by Eqs. (6.77)-(6.78) and shown in Fig. 6.6 of the lecture notes. The negative-slope branches of those plots correspond to the maxima of the energy U (with $\partial^2U/\partial\Phi^2 < 0$) and hence are unstable. As the plots show, a SQUID with $\lambda < 1$ always has just one stable state Φ, at $\lambda << 1$ closely following $\Phi_{\rm ext}$. On the contrary, a SQUID with $\lambda > 1$ may have several (at $\lambda >> 1$, many) stable states, corresponding to quasi-quantization of the flux.

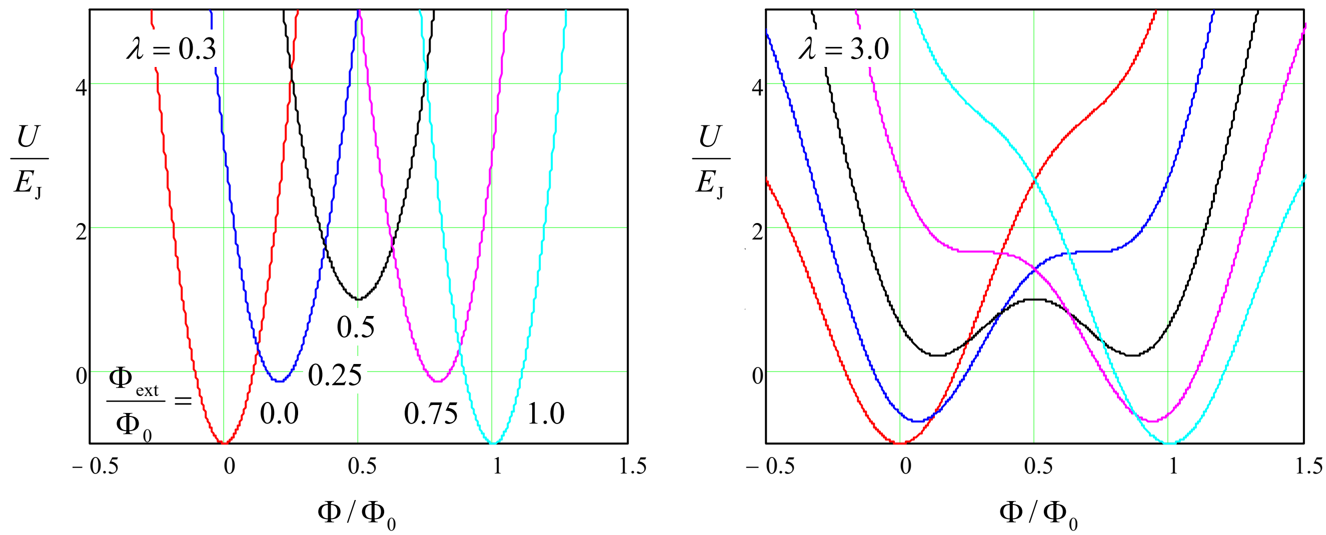
The flexibility of this $U(\Phi)$ profile to the change of parameters $\Phi_{\rm ext}$ and λ is broadly used in superconductor electronics; for example, it was used for the first experimental observation of the

[93] Physically, U_J is just a form of the kinetic energy of the Cooper-pair condensate, similar to that discussed in the solutions of Problems 6.19, 6.20, and 6.22, but for the specific case of a Josephson junction, with its highly nonlinear dynamics.

[94] By the way, the last expression shows that the non-magnetic (*kinetic* or *Josephson*) *inductance* of the junction for small currents is $L_{\rm J} \equiv E_J/I_{\rm c}^2 \equiv \hbar/2eI_{\rm c}$. This formula is broadly used in superconductor electronics.

[95] Since U is the potential energy of the system in a fixed magnetic external field, which plays the role of the generalized external force, its more fair name is the Gibbs potential energy – see, e.g., Sec. 6.2.

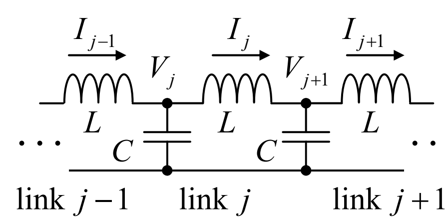
quantum interference of two macroscopic flux states,[96] while the bistability possible at $\lambda > 1$ is used in virtually all modern digital devices and circuits based on the Josephson effect.[97]



Finally, note that for a full analysis of the system's dynamics, the potential energy (*) has to be augmented by the electrostatic energy $CV^2/2$ of the Josephson junction. According to Eq. (6.79), this energy equals

$$\frac{C}{2}\left(\frac{\hbar}{2e}\right)^2\left(\frac{d\Phi}{dt}\right)^2,$$

and hence may be considered the kinetic energy T of the SQUID. The sum $(T + U)$ is the full energy of the SQUID, which is conserved in the absence of dissipation. In many applications, however, such dissipation, mostly due to the Ohmic current of "normal" (unpaired) electrons through the Josephson junction, flowing at $V \neq 0$, is essential for a fair description of the system's dynamics.



Problem 6.28. Analyze the possibility of wave propagation in a long uniform chain of lumped inductances and capacitances – see the figure on the right.

Hint: Readers without prior experience in electromagnetic wave analysis may like to use a substantial analogy between this effect and mechanical waves in a 1D chain of elastically coupled particles.[98]

Solution: Applying the 1st Kirchhoff law (the "node rule", see Eq. (6.88a) of the lecture notes) to the j^{th} link of the chain, numbered as the figure above shows, we get

$$I_{j-1} - I_j - C\dot{V}_j = 0, \qquad (*)$$

[96] J. Freedman *et al.*, *Nature* **406**, 43 (2000).
[97] See, e.g., A. Kadin, *Introduction to Superconducting Circuits*, Wiley (1999).
[98] See, e.g., CM Sec. 6.3.

where the last term on the left-hand side is the time derivative of the electric charge $Q_j = CV_j$ of the j^{th} capacitor. On the other hand, the 2$^{\text{nd}}$ Kirchhoff law, i.e. the "loop rule" (6.88b), applied to the contour containing the j^{th} inductance and two adjacent capacitors, yields

$$V_j - V_{j+1} - L\dot{I}_j = 0, \qquad (**)$$

where Eq. (6.85) was used for the voltage drop on the inductance L that carries current $I_j(t)$.

Generally, the system of differential equations (*) and (**) for a long chain, i.e. for many functions $V_j(t)$ and $I_j(t)$, may have rather complex solutions, but for our task, it is sufficient to look for the case when they are sinusoidal functions of time, with a constant phase shift (say, α) between adjacent links:[99]

$$I_j = \mathrm{Re}\left[I_\omega \exp\{i(\alpha j - \omega t)\}\right], \qquad V_j = \mathrm{Re}\left[V_\omega \exp\{i(\alpha j - \omega t)\}\right]. \qquad (***)$$

Plugging this solution into Eqs. (*) and (**), and then canceling the common factor $\exp\{i(\alpha j\text{-}\omega t)\}$, we get the following system of two linear homogeneous equations for the complex amplitudes I_ω and V_ω:

$$\begin{aligned} I_\omega\left(e^{-i\alpha} - 1\right) + i\omega C V_\omega &= 0, \\ V_\omega\left(1 - e^{i\alpha}\right) + i\omega L I_\omega &= 0. \end{aligned}$$

The equations are compatible if the system's determinant equals zero:

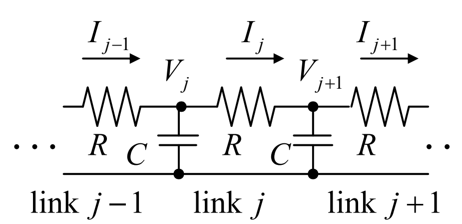
$$\begin{vmatrix} e^{-i\alpha} - 1 & i\omega C \\ i\omega L & 1 - e^{i\alpha} \end{vmatrix} = 0.$$

This condition yields the following *dispersion relation* between ω and α:[100]

$$\omega^2 = \omega_{\max}^2 \sin^2\frac{\alpha}{2}, \quad \text{where } \omega_{\max} \equiv \frac{2}{(LC)^{1/2}}.$$

So, the LC-chain indeed supports the process described by Eqs. (***) with a real parameter α, i.e. sinusoidal traveling waves,[101] if their frequency is below $\omega_{\max}$. (If the chain is excited at a higher frequency, the solution (***) is still valid but with purely imaginary values of α, and describes an exponential decay of the excitation magnitude as the link number j grows, i.e. as the observation point moves further from the excitation point.)

Problem 6.29. A sinusoidal e.m.f. of amplitude V_0 and frequency ω is applied to an end of a long chain of similar lumped resistors and capacitors, shown in the figure on the right. Calculate the law of decay of the ac voltage amplitude along the chain.



Solution: Applying the 1$^{\text{st}}$ Kirchhoff law (the "node rule",

[99] See, e.g., CM Eq. (6.26) and its discussion.
[100] A similar dispersion relation for mechanical elastic waves plays an important role in the theory of solids – see, e.g., CM Sec. 6.3 and SM Sec. 2.5.
[101] For a chain with a spatial period d, the exponents in Eq. (***) may be rewritten in a more usual form as $\exp\{i(kz - \omega t)\}$, with $k \equiv \alpha/d$, where the z-axis is directed along the chain.

see Eq. (6.88a) of the lecture notes) to the j^{th} link of the system, numbered as the figure above shows, we get the same equation as in the previous problem:

$$I_{j-1} - I_j - C\dot{V}_j = 0 . \qquad (*)$$

On the other hand, the 2nd Kirchhoff law (the "loop rule", Eq. (6.88b) of the lecture notes), applied to the contour containing the j^{th} resistor and two adjacent capacitors, yields a somewhat different result:

$$V_j - V_{j+1} - RI_j = 0 . \qquad (**)$$

Taking into account that the chain is driven by a sinusoidal e.m.f. of frequency ω, we may look for a solution of the system of Eqs. (*) and (**) in the same form as in the previous problem:

$$I_j = \text{Re}\left[I_0 \exp\{-\alpha j - i\omega t\}\right], \qquad V_j = \text{Re}\left[V_0 \exp\{-\alpha j - i\omega t\}\right]. \qquad (***)$$

Plugging this solution into Eqs. (*) and (**), and then canceling the common factor $\exp\{-\alpha j - i\omega t\}$, we get the following system of two linear, homogeneous algebraic equations for two complex amplitudes I_0 and V_0,

$$I_0\left(e^{\alpha} - 1\right) + i\omega C V_0 = 0,$$
$$V_0\left(1 - e^{-\alpha}\right) + RI_0 = 0$$

The equations are compatible if the system's determinant equals zero:

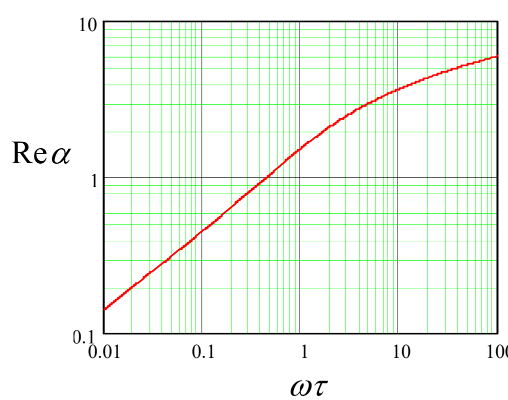
$$\begin{vmatrix} e^{\alpha} - 1 & i\omega C \\ R & 1 - e^{-\alpha} \end{vmatrix} = 0, \qquad \text{i.e. } 2(\cosh\alpha - 1) - i\omega RC = 0.$$

This equation yields[102] the following expression for the constant α, which is now a complex number:

$$\alpha = \cosh^{-1}(1 + 2i\omega\tau) \equiv \cosh^{-1}\left[\left(1 + \omega^2\tau^2\right)^{1/2} + \omega\tau\right] + i\cos^{-1}\frac{1}{\left(1 + \omega^2\tau^2\right)^{1/2} + \omega\tau} .$$

where $\tau \equiv RC/4$. If we are not interested in the phase shift described by the second term of this expression, we may use the first term to write the following law of decay of the real amplitude V_j of the ac oscillations in j^{th} link in a semi-infinite chain:

$$V_j = V_0 \exp\{-(\text{Re}\,\alpha)j\}$$
$$= V_0 \exp\left\{-\cosh^{-1}\left[\left(1 + \omega^2\tau^2\right)^{1/2} + \omega\tau\right]j\right\}$$



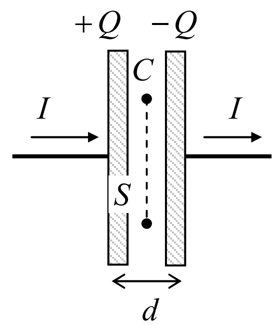
The figure on the right shows the dependence of the decay factor Reα on frequency – or rather on the dimensionless combination $\omega\tau$. At relatively low frequencies ($\omega\tau << 1$), Re$\alpha \approx (2\omega\tau)^{1/2} << 1$, i.e. the characteristic depth (expressed in the number of links) $d \equiv 1/\text{Re}\,\alpha$ of the oscillations' penetration into

[102] The simplest way to get the last form of this formula is to rewrite the first one as $\cosh\alpha \equiv \cosh(\text{Re}\,\alpha + i\text{Im}\,\alpha) \equiv \cosh(\text{Re}\,\alpha)\cos(\text{Im}\,\alpha) + i\sinh(\text{Re}\,\alpha)\sin(\text{Im}\,\alpha) = 1 + 2i\omega\tau$, require the real and imaginary parts of both sides to be separately equal, and then solve the resulting simple system of two equations for Reα and Imα.

the chain, is much larger than 1. However, as the frequency ω is increased well beyond $\sim 1/\tau$, i.e. beyond $\sim 1/RC$, we get Reα >> 1, i.e. the ac excitation essentially does not go beyond the first link of the chain. Due to this property, electrical engineers call such a circuit, used most typically with just a few links, the *low-pass RC filter*.

Problem 6.30. As was discussed in Sec. 6.7 of the lecture notes, the displacement current concept allows one to extend the Ampère law to time-dependent processes as

$$\oint_C \mathbf{H} \cdot d\mathbf{r} = I_S + \frac{\partial}{\partial t}\int_S D_n d^2 r \,.$$



We also have seen that this generalization makes the integral $\int \mathbf{H} \cdot d\mathbf{r}$ over an external contour, such as the one shown in Fig. 6.10, independent of the choice of the surface S limited by the contour. However, it may look like the situation is different for a contour drawn inside a capacitor – see the figure on the right. Indeed, if the contour's size is much larger than the capacitor's thickness, the magnetic field $\mathbf{H}$ created by the linear current I on the contour's line is virtually the same as that of a continuous wire, and hence the integral $\int \mathbf{H} \cdot d\mathbf{r}$ along the contour apparently does not depend on its area, while the magnetic flux $\int D_n d^2 r$ does, so the equation displayed above seems invalid. (The current I_S piercing this contour evidently equals zero.) Resolve this paradox, for simplicity considering an axially-symmetric system.

Solution: Let us consider a circular contour C of radius $\rho < R$, where R is the radius of circular parallel plates of the capacitor. If $\rho >> d$, we can use Eq. (5.20) of the lecture notes for the magnetic field $\mathbf{H}_I$ induced by the current I in the wire, so $\int \mathbf{H}_I \cdot d\mathbf{r}$ along the contour C is indeed equal to I. However, at $dI/dt \neq 0$, the system also has currents flowing in the capacitor plates in the radial direction, necessary to change the surface density $\sigma = Q/\pi R^2$ of the full charges $\pm Q$ of the plates in time. For example, the left plate's charge residing outside a circle of radius ρ equals $\pi(R^2 - \rho^2)\sigma$, so the linear density $J(\rho)$ of the surface currents in the plate may be found from the charge conservation requirement:

$$2\pi\rho J(\rho) = \frac{d}{dt}\left[\pi\left(R^2 - \rho^2\right)\sigma\right] \equiv \pi\left(R^2 - \rho^2\right)\frac{d\sigma}{dt}, \qquad \text{giving } J = \frac{\left(R^2 - \rho^2\right)}{2\rho}\frac{d\sigma}{dt}.$$

The magnetic field between the plates obeys Eq. (6.38), so it is also tangential to the circular contour C, opposite in direction to that of straight wire, and has the magnitude

$$H_J(\rho) = J(\rho) = \frac{\left(R^2 - \rho^2\right)}{2\rho}\frac{d\sigma}{dt} = \frac{\left(R^2 - \rho^2\right)}{2\pi\rho R^2}\frac{dQ}{dt} = \frac{\left(R^2 - \rho^2\right)}{2\pi\rho R^2} I.$$

As a result, the net integral of the field is

$$\oint_C \mathbf{H} \cdot d\mathbf{r} = \oint_C \mathbf{H}_I \cdot d\mathbf{r} + \oint_C \mathbf{H}_J \cdot d\mathbf{r} = I - \oint_C H_J dr = I - 2\pi\rho\frac{\left(R^2 - \rho^2\right)}{2\pi\rho R^2} I \equiv \frac{\rho^2}{R^2} I,$$

so it exactly matches the integral of the displacement current over the surface area $\pi\rho^2$:

$$\frac{\partial}{\partial t}\int_S D_n d^2r = \frac{\partial}{\partial t}\int_S \sigma d^2r = \frac{d\sigma}{dt}\int_S d^2r = \frac{d\sigma}{dt}\pi\rho^2 = \frac{1}{\pi R^2}\frac{dQ}{dt}\pi\rho^2 = \frac{\rho^2}{R^2}I,$$

thus resolving the paradox.

Problem 6.31. A straight, uniform, long wire with a circular cross-section of radius R, made of an Ohmic conductor with conductivity σ, carries dc current I. Calculate the flux of the Poynting vector through its surface and compare it with the Joule rate of energy dissipation.

Solution: As was discussed in Sec. 4.3 of the lecture notes, in such a uniform, long wire, the dc current is distributed uniformly, with a constant density

$$j = \frac{I}{\pi R^2}.$$

Per the differential form of the Ohm law, the electric field's distribution inside the wire is also uniform,

$$E = \frac{j}{\sigma} = \frac{I}{\pi\sigma R^2},$$

and is directed, as the current density, along the wire's length.

The magnetic field distribution in such a wire was (easily) calculated in Sec. 5.2; for the wire's surface, the second of Eqs. (5.38) yields

$$H = \frac{I}{2\pi R}.$$

Since the direction of the vector **H** is tangential to the wire's cross-section (see Fig. 5.5 of the lecture notes), i.e. perpendicular to the vector **E**, the Poynting vector (6.114) on the wire's surface is directed normally to it, into the wire's bulk, and has the magnitude

$$S = EH = \frac{I}{\pi\sigma R^2}\frac{I}{2\pi R} \equiv \frac{I^2}{2\pi^2\sigma R^3}.$$

Hence the power flow into a wire's fragment of length l is

$$\mathscr{P}_{\text{Poynting}} = 2\pi RlS = \frac{I^2}{\pi\sigma R^2}l.$$

On the other hand, according to the Joule law (4.39), the total power of energy dissipation in such a fragment is

$$\mathscr{P}_{\text{Joule}} = \not{p}V = \frac{j^2}{\sigma}\pi R^2 l = \left(\frac{I}{\pi R^2}\right)^2\frac{1}{\sigma}\pi R^2 l \equiv \frac{I^2}{\pi\sigma R^2}l.$$

So, we have arrived at a simple result, $\mathscr{P}_{\text{Joule}} = \mathscr{P}_{\text{Poynting}}$, thus confirming the natural picture of the continuous power flow in the system – from the current source to the electromagnetic field outside the wire, then through the wire's surface into its bulk, and then to heat.

Chapter 7. Electromagnetic Wave Propagation

Problem 7.1. Find the temporal Green's function of a medium whose complex permittivity $\varepsilon(\omega)$ obeys the Lorentz oscillator model given by Eq. (7.32) of the lecture notes, by using:

(i) the Fourier transform of the underlying Eq. (7.30), and
(ii) the direct solution of that equation.

Hint: For the Fourier-transform approach, you may like to use the Cauchy integral.[1]

Solutions:

(i) The general relation between $\varepsilon(\omega)$ and the Green's function $G(\theta)$ is given by Eq. (7.26b) of the lecture notes:

$$\varepsilon(\omega)-\varepsilon_0=\int_0^\infty G(\theta)e^{i\omega\theta}d\theta\,. \qquad (*)$$

Though the Green's function $G(\theta)$ does not have a physical sense for $\theta < 0$, mathematically nothing prevents us from extending the integral (*) to the whole axis $-\infty < \theta < +\infty$, by taking $G(\theta) = 0$ for $\theta < 0$. Now we can write the reciprocal transform:

$$G(\theta)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\varepsilon(\omega)-\varepsilon_0\right]e^{-i\omega\theta}d\omega\,.$$

For the particular form (7.32) of the function $\varepsilon(\omega)$, this equality becomes

$$G(\theta)=\frac{nq^2}{2\pi m}\int_{-\infty}^{+\infty}\frac{1}{(\omega_0^2-\omega^2)-2i\omega\delta}e^{-i\omega\theta}d\omega\,. \qquad (**)$$

In order to calculate this integral, let us first represent the denominator as a product $-(\omega-\boldsymbol{\omega}_+)(\omega-\boldsymbol{\omega}_-)$, where $\boldsymbol{\omega}_\pm$ are the values of the complex variable $\boldsymbol{\omega}$ at which the denominator turns to zero, i.e. the roots of the corresponding quadratic equation. An elementary calculation yields

$$\boldsymbol{\omega}_\pm=\pm\omega_0'-i\delta, \qquad \text{where } \omega_0'\equiv\left(\omega_0^2-\delta^2\right)^{1/2}.$$

Now we may rewrite the fraction under the integral in Eq. (**) as a sum of two simple singularities (*poles*):

$$\frac{1}{(\omega_0^2-\omega^2)-2i\omega\delta}=-\frac{1}{(\omega-\boldsymbol{\omega}_+)(\omega-\boldsymbol{\omega}_-)}=\frac{a}{\omega-\boldsymbol{\omega}_+}+\frac{b}{\omega-\boldsymbol{\omega}_-}.$$

The constants a and b may be calculated from the requirement for the two last expressions to be identical; this gives

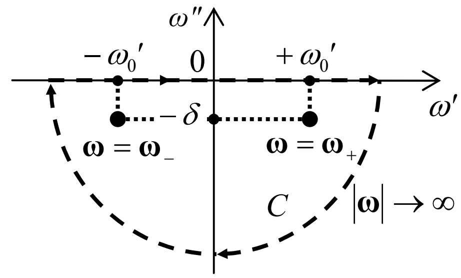
$$a=-b=-\frac{1}{2\omega_0'}\,.$$

As a result, Eq. (**) turns into

[1] See, e.g., MA Eq. (15.2).

$$G(\theta)=\frac{q^2}{4\pi m\omega_0'}\left[-\int_{-\infty}^{+\infty}e^{-i\omega\theta}\frac{d\omega}{\omega-\boldsymbol{\omega}_+}+\int_{-\infty}^{+\infty}e^{-i\omega\theta}\frac{d\omega}{\omega-\boldsymbol{\omega}_-}\right].$$

Since at $\theta > 0$, the factor $e^{-i\boldsymbol{\omega}\theta}$, considered a function of the complex argument $\boldsymbol{\omega} \equiv \omega' + i\omega''$, tends to zero at $|\boldsymbol{\omega}| \to \infty$ and $\omega'' < 0$, each integral in the last displayed formula may be replaced by the one over the closed contour C on the plane [ω', ω''], shown with the dashed line in the figure on the right. This integral may be worked out using the Cauchy integral formula:



$$\oint_C f(\boldsymbol{\omega})\frac{d\boldsymbol{\omega}}{\boldsymbol{\omega}-\boldsymbol{\Omega}}=2\pi i f(\boldsymbol{\Omega}).$$

The formula is valid if the function $f(\boldsymbol{\omega})$ is analytical (as our $e^{-i\boldsymbol{\omega}\theta}$ is), and the pole $\boldsymbol{\Omega}$ is located within the area encircled by the contour C. This is correct in our case because both poles $\boldsymbol{\omega}_\pm$ are in the negative semi-plane – see the figure above. However, the Cauchy theorem implies the positive (counterclockwise) integration's direction along a closed contour, while our direction is negative, so we have to change the signs before both integrals. As a result, we get

$$G(\theta)=\frac{nq^2}{4\pi m\omega_0'}2\pi i\left(e^{-i\boldsymbol{\omega}_+\theta}-e^{-i\boldsymbol{\omega}_-\theta}\right)\equiv\frac{nq^2}{m\omega_0'}e^{-\delta\theta}\sin\omega_0'\theta\,.$$

(ii) In Sec. 7.2 of the lecture notes, Eq. (7.32) was derived from the Lorentz oscillator model: a set of independent classical oscillators obeying the differential equation (7.30):

$$\frac{d^2x}{dt^2}+2\delta\frac{dx}{dt}+\omega_0^2x=\frac{q}{m}E(t)\,.$$

Its temporal Green's function $g(\theta)$ is just the solution of this differential equation for $t = \theta$ and $E = \delta(\theta)$, while the Green's function for the electric polarization of the media with n independent similar oscillators per unit volume is $G(\theta) = nqg(\theta)$, so we may write

$$\frac{d^2G}{d\theta^2}+2\delta\frac{dG}{d\theta}+\omega_0^2G=\frac{nq^2}{m}\delta(\theta)\,.$$

At $\theta > 0$, this equation is homogeneous,

$$\frac{d^2G}{d\theta^2}+2\delta\frac{dG}{d\theta}+\omega_0^2G=0\,, \qquad (***)$$

and has a simple, well-known solution[2]

$$G(\theta)=\left(c_c\cos\omega_0'\theta+c_s\sin\omega_0'\theta\right)e^{-\delta\theta}, \qquad (****)$$

where $\omega_0' \equiv (\omega_0^2 - \delta^2)^{1/2}$ is the same as in the first approach.

[2] If this is not evident, please consult CM Sec. 5.1.

In order to find the constants c_c and c_s, we need to formulate the appropriate initial conditions for Eq. (***). For that, let us integrate both parts of the initial (inhomogeneous) equation for $G(\theta)$ over a small time interval of width $2\Delta \to 0$, centered to point $\theta = 0$:

$$\left(\frac{dG}{d\theta}\Big|_{\theta=+\Delta} - \frac{dG}{d\theta}\Big|_{\theta=-\Delta}\right) + 2\delta\left(G\big|_{\theta=+\Delta} - G\big|_{\theta=-\Delta}\right) + \omega_0^2 \int_{-\Delta}^{+\Delta} G d\theta = \frac{nq^2}{m}.$$

Due to the causality principle, both the Green's function and its derivative should equal zero for all moments $t < t'$, i.e. $\theta < 0$. Moreover, $G(\theta)$ should be a continuous function, because is proportional to the oscillator's coordinate, which cannot change instantly even if the acting force is infinite at some moment. On the other hand, under such an infinite force pulse, the derivative $dG/d\theta$ may experience a finite jump,[3] so $dG/d\theta|_{\theta=+\Delta}$ may be different from zero. Since in the limit $\Delta \to 0$, all other terms on the left-hand side vanish, we get the following initial conditions for the homogeneous equation (***)

$$G(0) = 0, \qquad \frac{dG}{d\theta}(0) = \frac{nq^2}{m}.$$

Applying them to the above general solution of this equation, we get $c_c = 0$, $c_s = nq^2/m\omega_0'$, so

$$G(\theta) = \frac{nq^2}{m\omega_0'} e^{-\delta\theta} \sin \omega_0'\theta ,$$

i.e. (luckily :-) the same result as in the first approach.

Problem 7.2. The electric polarization of some material responds to an electric field step[4] in the following way:

$$P(t) = \varepsilon_1 E_0\left(1 - e^{-t/\tau}\right), \qquad \text{if} \quad E(t) = E_0 \times \begin{cases} 0, & \text{for } t < 0, \\ 1, & \text{for } 0 < t, \end{cases}$$

where $\tau > 0$ and ε_1 are some constants. Calculate the complex permittivity $\varepsilon(\omega)$ of this material, and discuss a possible simple physical model giving such dielectric response.

Solution: Let us calculate the dielectric response of an arbitrary system described by Eq. (7.23) of the lecture notes to such an electric field step:

$$P(t) = \int_0^\infty E(t-\theta)G(\theta)d\theta = E_0 \int_0^\infty \begin{Bmatrix} 0, & \text{for } t < \theta \\ 1, & \text{for } \theta < t \end{Bmatrix} G(\theta)d\theta = E_0 \int_0^t G(\theta)d\theta .$$

Comparing this response with that given in the problem's assignment, we see that the temporal Green's function of this system has to obey the following equation:

$$\int_0^t G(\theta)d\theta = \varepsilon_1\left(1 - e^{-t/\tau}\right), \qquad \text{for } t > 0.$$

---

[3] If in doubt about this point, please recall any of the standard collision problems in classical mechanics, solved assuming a finite momentum transfer between two bodies during an infinitesimal time interval.

[4] This function $E(t)$ is of course proportional to the well-known Heaviside step function $\theta(t)$ – see, e.g., MA Eq. (14.3). I am not using this notion here just to avoid confusion between two different uses of the Greek letter θ.

Differentiating both sides of this relation over t, we get

$$G(t) = \varepsilon_1 \frac{d}{dt}\left(1 - e^{-t/\tau}\right) \equiv \frac{\varepsilon_1}{\tau} e^{-t/\tau}, \qquad \text{for } t > 0.$$

Now it is straightforward to use Eq. (7.26b) to calculate

$$\varepsilon(\omega) = \varepsilon_0 + \int_0^\infty G(\theta) e^{i\omega\theta} d\theta = \varepsilon_0 + \frac{\varepsilon_1}{\tau}\int_0^\infty e^{-\theta/\tau}\, e^{i\omega\theta} d\theta = \varepsilon_0 + \frac{\varepsilon_1}{1 - i\omega\tau}.$$

Comparing this expression with Eq. (7.32) that follows from the Lorentz oscillator model (7.30), we see that they coincide in the low-frequency limit $\omega << \omega_0$ if we take

$$\varepsilon_1 = \frac{nq^2}{\kappa} \quad \text{and} \quad \tau = \frac{\eta}{\kappa}, \quad \text{with} \quad \kappa \equiv m\omega_0^2 \quad \text{and} \quad \eta \equiv 2m\delta_0.$$

In other words, such a dielectric response is pertinent, for example, to a medium whose independent electric dipoles $p = qx$ are due to the heavily overdamped charge displacements $x(t)$,[5] obeying the following equation of motion:

$$\eta\dot{x} + \kappa x = qE(t),$$

i.e. having negligible inertia.

Problem 7.3. Calculate the complex permittivity $\varepsilon(\omega)$ of a material whose temporal Green's function, defined by Eq. (7.23) of the lecture notes, is

$$G(\theta) = G_0\left(1 - e^{-\theta/\tau}\right),$$

with some positive constants G_0 and τ. What is the difference between this dielectric response and the apparently similar one considered in the previous problem?

Solution: With the given $G(\theta)$, Eq. (7.26) of the lecture notes yields

$$\varepsilon(\omega) = \varepsilon_0 + G_0\int_0^\infty \left(1 - e^{-\theta/\tau}\right) e^{i\omega\theta} d\theta = \varepsilon_0 + G_0\left[\left(\frac{1}{i\omega} - \frac{e^{-\theta/\tau}}{i\omega - 1/\tau}\right) e^{i\omega\theta}\right]_{\theta=0}^{\theta=\infty}.$$

We see that the exponential factor is uncertain at the upper limit, making the first term uncertain. This technical problem may be treated in the following standard way: adding to ω (before the integration) a small imaginary part $i\mu$, with $\mu > 0$,[6] and then (after the integration) taking the limit $\mu \to 0$:

$$\varepsilon(\omega) = \varepsilon_0 + G_0 \lim_{\mu\to 0} \int_0^\infty \left(1 - e^{-\theta/\tau}\right) e^{i(\omega + i\mu)\theta} d\theta = \varepsilon_0 - \frac{G_0}{i\omega(1 - i\omega\tau)} \equiv \varepsilon_0 - \frac{G_0}{\omega^2\tau + i\omega}. \quad (*)$$

[5] See, e.g., CM Sec. (4.1).

[6] This sign correctly reflects the properties of any stable physical system, by providing a gradual suppression of its response as the time interval between the cause and effect tends to infinity.

Concerning the second question, the temporal Green's function of a system is, by definition, its response to a delta-functional pulse (in our case, of the electric field), rather than to a step-like excitation specified in the previous problem, so the Green's functions and hence all dielectric properties of the systems considered in this and in the previous problem are rather different. Indeed, the function given by Eq. (*) also approaches the complex permittivity (7.32) of the Lorentz oscillator model but at very high rather than very low frequencies.

Problem 7.4. Use the oscillator model of an atom, given by Eq. (7.30) of the lecture notes, to calculate its average potential energy in a uniform, sinusoidal ac electric field, and use the result to calculate the potential profile created for the atom by a standing electromagnetic wave with the electric field amplitude $E_\omega(\mathbf{r})$.

Solution: For any isotropic linear model of a charged particle's motion, its oscillations induced by a sinusoidal electric field $\mathbf{E}(t) = E_x(t)\mathbf{n}_x$ with

$$E_x(t) = \text{Re}\left(E_\omega e^{-i\omega t}\right) \equiv \frac{1}{2}\left(E_\omega e^{-i\omega t} + \text{c.c.}\right),$$

are also sinusoidal and directed along the field: $\mathbf{r}(t) = x(t)\mathbf{n}_x$, with

$$x(t) = \text{Re}\left(x_\omega e^{-i\omega t}\right) \equiv \frac{1}{2}\left(x_\omega e^{-i\omega t} + \text{c.c.}\right)$$

This motion creates the oscillating dipole moment $\mathbf{p}(t) = q\mathbf{r}(t)$, also directed along the applied field, and hence the potential energy of its interaction with the applied field may be calculated using Eq. (3.15b) of the lecture notes:[7]

$$U(t) = -\frac{1}{2}\mathbf{p}(t)\cdot\mathbf{E}(t) = -\frac{q}{8}\left(E_\omega e^{-i\omega t} + \text{c.c.}\right)\left(x_\omega e^{-i\omega t} + \text{c.c.}\right) = -\frac{q}{8}\left(E_\omega x_\omega e^{-2i\omega t} + E_\omega x_\omega^* + \text{c.c.}\right).$$

Time averaging of the energy over the period of the applied field kills the terms proportional to $\exp\{\pm 2i\omega t\}$, giving

$$\overline{U} = -\frac{q}{8}\left(E_\omega x_\omega^* + \text{c.c.}\right) \equiv -\frac{q}{4}\text{Re}\left(E_\omega x_\omega^*\right).$$

In particular, for the oscillator model (7.30), the complex amplitude x_ω is given by Eq. (7.31):

$$x_\omega = \frac{q}{m}\frac{E_\omega}{\left(\omega_0^2 - \omega^2\right) - 2i\omega\delta_0},$$

so the average energy is

$$\overline{U} = -\frac{q^2}{4m}E_\omega E_\omega^* \,\text{Re}\left[\frac{1}{\left(\omega_0^2 - \omega^2\right) - 2i\omega\delta_0}\right] = \frac{q^2}{4m}E_\omega E_\omega^* \frac{\omega^2 - \omega_0^2}{\left(\omega^2 - \omega_0^2\right)^2 + \left(2\omega\delta_0\right)^2}. \qquad (*)$$

This expression shows that the potential energy is negative at $\omega < \omega_0$ and positive in the opposite case. In a more general model (7.33) corresponding to several oscillators with different resonance

[7] Note that a non-zero net charge of the atom (if present) also gives another, polarization-independent contribution (1.31), $U_0(\mathbf{r}, t) = q\phi(\mathbf{r}, t)$, to its potential energy in the applied field, but for a sinusoidal field, the electric potential ϕ is also a sinusoidal function of time, so the time average of this part of the interaction vanishes.

frequencies ω_j, such behavior repeats near each of them – see, e.g., the red line in Fig. 7.5 of the lecture notes.[8]

Proceeding to the second part of the problem, we should take into account that according to Eqs. (5.10) and (7.6)-(7.8), the interaction of non-relativistic charged particles, moving with velocity $v << c$, with any electromagnetic wave is dominated by the wave's electric field. In addition, the linear size of typical atoms and molecules is of the order of 10^{-10} m and hence much smaller than the wavelength λ, all the way up to extremely high frequencies corresponding to X-rays. Hence, for a standing wave, we may use Eq. (*), derived for a uniform electric field, with E_ω replaced with $E_\omega(\mathbf{r})$, where $\mathbf{r}$ is the atom's position. This expression shows that in a standing wave with $\omega < \omega_0$, the particle is repulsed from electric field maxima and hence is pushed to an adjacent field's minimum ("node"), while in the opposite case ($\omega > \omega_0$) it is pushed to the adjacent maximum of the standing wave.

As Eq. (*) shows, the effect is especially large if the oscillator's damping is low ($\delta << \omega_0$), as typical for atoms in gases, and the field's frequency is close to one of the atom's eigenfrequencies ω_0, but still out of the narrow interval $\omega_0 \pm \delta_0$. Note, however, that the effect exists even for free particles, which may be described by the Lorentz oscillator model with $\omega_0 = \delta_0 = 0$ when Eq. (*) is reduced to

$$\overline{U} = \frac{q^2}{4m\omega^2} E_\omega E_\omega^*, \tag{**}$$

showing that the particle is pushed toward the electric field's *minimum*.[9]

On the other hand, in the limit $\omega, \delta_0 << \omega_0$, in which Eq. (7.35) is valid, Eq. (*) yields

$$\overline{U} = -\frac{q^2}{4m\omega_0^2} E_\omega E_\omega^* \equiv -\frac{\varepsilon(0)-\varepsilon_0}{n} E_\omega E_\omega^* < 0, \tag{***}$$

so the atom is attracted to the regions where the field has its *maximum*. (In the last expression, $\varepsilon(0)$ is the dc electric susceptibility of a dilute medium with n particles per unit volume; in this form, Eq. (***) also holds for atomic clusters and small but macroscopic dielectric particles.) This effect is widely used in experimental practice for *optical trapping* of the particles, for example, small biological samples at the focus of paraxial (e.g., Gaussian) laser beams – the so-called *optical tweezers*.[10]

Problem 7.5. The solution of the previous problem shows that a standing electromagnetic wave may exert a time-averaged force on an otherwise free non-relativistic charged particle. Reveal the physics of this force by writing and solving the equations of motion of such a particle in:

(i) a linearly-polarized monochromatic plane traveling wave, and
(ii) a similar but standing wave.

Solutions:

[8] Just for the reader's reference, quantum mechanics confirms the same behavior, near each quantum transition frequency, for a charged particle confined in a potential well of arbitrary shape – not only of the quadratic shape corresponding to a harmonic oscillator – see, e.g., QM Problem 6.18.
[9] Exploring the physics of this force is the subject of the next problem.
[10] See, e.g., P. Jones *et al.*, *Optical Tweezers: Principles and Applications*, Cambridge U. Press, 2016.

(i) Directing the z-axis along the wave's propagation direction, and the x-axis along its electric field, we may use Eqs. (7.6)-(7.8) and (7.11) of the lecture notes to express the wave's fields as

$$\mathbf{E}(\mathbf{r},t) = \mathbf{n}_x \operatorname{Re}\left[E_\omega e^{i(kz-\omega t)}\right], \qquad \mathbf{B}(\mathbf{r},t) = \mu_0 \mathbf{H}(\mathbf{r},t) = \mu_0 \frac{\mathbf{n}_z \times \mathbf{n}_x}{Z_0} \operatorname{Re}\left[E_\omega e^{i(kz-\omega t)}\right] \equiv \frac{\mathbf{n}_y}{c} \operatorname{Re}\left[E_\omega e^{i(kz-\omega t)}\right],$$

so the Lorentz force (5.10) exerted by the wave on a particle with charge q, located at $\mathbf{r} = \{x, y, z\}$, is

$$\mathbf{F}(\mathbf{r},t) = q[\mathbf{E} + \mathbf{v} \times \mathbf{B}] = q\mathbf{n}_x \operatorname{Re}\left[E_\omega e^{i(kz-\omega t)}\right] + q\frac{\mathbf{v} \times \mathbf{n}_y}{c} \operatorname{Re}\left[E_\omega e^{i(kz-\omega t)}\right]. \qquad (*)$$

The electric force, expressed by the first term on the right-hand side, clearly has no time average, while the second term does. In the non-relativistic limit, when $v/c << 1$, the magnetic force expressed by the second term is much smaller than the electric one, so we may first calculate $\mathbf{v}$ from the 2$^{\text{nd}}$ Newton law in which the magnetic force is ignored:[11]

$$m\dot{\mathbf{v}} = q\mathbf{n}_x \operatorname{Re}\left[E_\omega e^{i(kz-\omega t)}\right].$$

Integrating both parts of this simple equation, we get

$$\mathbf{v} = \mathbf{n}_x \frac{q}{m\omega} \operatorname{Re}\left[iE_\omega e^{i(kz-\omega t)}\right],$$

where we have neglected the possible free motion of the particle by inertia, which is unrelated to the wave's effect. (In any realistic system, such ballistic motion eventually stops because of the energy loss due to some inevitable interaction of the particle with its environment.)

Now plugging this result into the second term of Eq. (*), we may calculate the magnetic force:

$$\mathbf{F}_\text{m}(\mathbf{r},t) = \frac{q^2}{m\omega c} \mathbf{n}_x \times \mathbf{n}_y \operatorname{Re}\left[iE_\omega e^{i(kz-\omega t)}\right] \operatorname{Re}\left[E_\omega e^{i(kz-\omega t)}\right]$$
$$= \frac{q^2}{m\omega c} \mathbf{n}_z \frac{1}{2}\left[iE_\omega e^{i(kz-\omega t)} + \text{c.c.}\right] \frac{1}{2}\left[E_\omega e^{i(kz-\omega t)} + \text{c.c.}\right].$$

The multiplication of the square brackets gives four terms, but only two of them (both proportional to $E_\omega E_\omega^*$) have their time dependences canceled, thus potentially contributing to the average force:

$$\overline{\mathbf{F}}(\mathbf{r}) = \overline{\mathbf{F}}_\text{m}(\mathbf{r}) = \frac{q^2}{4m\omega c} \mathbf{n}_z E_\omega E_\omega^* \left(ie^{ikz} e^{-ikz} + \text{c.c}\right) = \frac{q^2}{4m\omega c} \mathbf{n}_z E_\omega E_\omega^* (i + \text{c.c.}) = 0\,.$$

Hence, in this case, the full Lorentz force has no average. This result is in accordance with the solution of the previous problem, in which the average interaction energy is constant (and hence has no gradient) if $E_\omega E_\omega^*$ is constant – as it is in a traveling plane wave.

(ii) The situation changes if the wave is a standing one. Taking its fields, for example, in the form given by Eqs. (7.61)-(7.62) of the lecture notes, with the same linear polarization as above,

[11] It is straightforward (and hence left for the reader as an additional exercise) to use the last term of Eq. (*) to show that the magnetic force leads to much smaller z-oscillations of the particle with frequency 2ω.

$$\mathbf{E}(\mathbf{r},t) = 2\mathbf{n}_x \operatorname{Re}\left(iE_\omega e^{-i\omega t}\right)\sin kz, \qquad \mathbf{B}(\mathbf{r},t) = \mu_0\mathbf{H}(\mathbf{r},t) = 2\frac{\mathbf{n}_y}{c}\operatorname{Re}\left(E_\omega e^{-i\omega t}\right)\cos kz\,,$$

we see that the time-averaged electric component of the Lorentz force equals zero again. Using the same approach as in the previous task to find the magnetic component of the force, we get

$$m\dot{\mathbf{v}} = 2q\mathbf{n}_x \operatorname{Re}\left(iE_\omega e^{-i\omega t}\right)\sin kz, \qquad \text{giving } \mathbf{v} = -\mathbf{n}_x \frac{2q}{m\omega}\operatorname{Re}\left(E_\omega e^{-i\omega t}\right)\sin kz\,,$$

so

$$\mathbf{F}_\mathrm{m}(\mathbf{r},t) = q\mathbf{v}\times\mathbf{B} = -\frac{4q^2}{m\omega c}\mathbf{n}_x\times\mathbf{n}_y \operatorname{Re}\left(E_\omega e^{-i\omega t}\right)\sin kz \operatorname{Re}\left(E_\omega e^{-i\omega t}\right)\cos kz$$

$$\equiv -\frac{4q^2}{m\omega c}\mathbf{n}_z \frac{1}{2}\left(E_\omega e^{-i\omega t} + \text{c.c.}\right)\frac{1}{2}\left(E_\omega e^{-i\omega t} + \text{c.c}\right)\sin kz\cos kz.$$

Here again, only two of the four terms resulting from the parentheses multiplication have their time dependences canceled and hence contribute to the average force, but now they add up rather than subtract:

$$\overline{\mathbf{F}}(\mathbf{r}) = -\frac{q^2}{m\omega c}\mathbf{n}_z\left(E_\omega E_\omega^* + E_\omega^* E_\omega\right)\sin kz\cos kz \equiv -\frac{q^2}{m\omega c}\mathbf{n}_z E_\omega E_\omega^* \sin 2kz\,. \qquad (**)$$

This force vanishes only at all points where $2kz = n\pi$ (with $n = 0, -1, -2, \ldots$, because our result is valid only for $z \leq 0$ – see Fig. 7.8), but only at such points with even n, in which the *electric* field vanishes, it leads to a stable equilibrium.[12]

Let us compare this result with that following from Eq. (**) of the model solution of the previous problem:

$$\overline{U} = \frac{q^2}{4m\omega^2}E_\omega E_\omega^*.$$

Now we should recall that to obtain this result, we took $\mathbf{E}$ in the form $\mathbf{n}_x\operatorname{Re}(E_\omega e^{-i\omega t})$, so to comply with our current assumptions, we have to make the following replacement: $E_\omega \to 2iE_\omega \sin kz$, getting

$$\overline{U} \to \overline{U}(\mathbf{r}) = \frac{q^2}{4m\omega^2}(2iE_\omega \sin kz)(2iE_\omega \sin kz)^* = \frac{q^2}{m\omega^2}E_\omega E_\omega^* \sin^2 kz\,.$$

This average potential energy, depending only on z, yields an average force with just one Cartesian component:

$$\overline{\mathbf{F}}(\mathbf{r}) \equiv -\nabla\overline{U}(\mathbf{r}) = -\mathbf{n}_z\frac{q^2}{m\omega^2}E_\omega E_\omega^*\frac{d}{dz}\left(\sin^2 kz\right) = -\mathbf{n}_z\frac{q^2}{m\omega c}E_\omega E_\omega^*\sin 2kz,$$

i.e. exactly the same result as Eq. (**) of this solution.

Hence, we may conclude that the average force exerted on a charged particle by a wave is essentially the average *magnetic* component of the Lorentz force, but it would not arise without the particle's motion induced by the (much larger) *electric* component of the force.

[12] The last statement may be readily verified by exploring the equation of motion along axis z – an additional useful exercise for the reader.

Problem 7.6. Use the first of Eqs. (7.54) of the lecture notes to relate the integral $\int_0^\infty \varepsilon''(\Omega)\Omega d\Omega$ to the plasma frequency for the Lorentz oscillator model of a system of non-interacting particles.

Solution: As Eq. (7.33) shows, in the Lorentz oscillator model with low damping ($\delta_j << \omega_j$), the energy loss function $\varepsilon''(\Omega)$ is nonvanishing only in very small vicinities of the resonant frequencies ω_j. Hence, if the wave frequency ω is well above them all, the first of the dispersion relations (7.54) is reduced to

$$\varepsilon'(\omega) = \varepsilon_0 - \frac{2}{\pi\omega^2}\int_0^{+\infty} \varepsilon''(\Omega)\Omega d\Omega,, \qquad \text{for } \omega >> \omega_j.$$

But as was noted in Sec. 7.2 of the lecture notes, the condition $\omega >> \omega_j$ reduces the same Eq. (7.33), with the sum rule

$$\sum_j f_j = 1, \qquad (*)$$

to Eq. (7.36):

$$\varepsilon(\omega) = \varepsilon'(\omega) = \varepsilon_0\left(1 - \frac{\omega_\text{p}^2}{\omega^2}\right) \equiv \varepsilon_0 - \varepsilon_0\frac{\omega_\text{p}^2}{\omega^2}.$$

Comparing these two expressions for $\varepsilon'(\omega)$, we see that they coincide if

$$\int_0^\infty \varepsilon''(\Omega)\Omega d\Omega = \frac{\pi}{2}\varepsilon_0\omega_\text{p}^2.$$

Since this result is conditioned by Eq. (*), it is also sometimes called the *sum rule*. Quantum mechanics shows[13] that its validity extends well beyond the classical Lorentz model.

Problem 7.7. Prove that Eq. (6.42) of the lecture notes cannot be correct for all frequencies, and suggest its correction making the result compatible with both the causality principle and the physical model (6.39).

Solution: Let us assume that Eq. (6.41),[14]

$$j_\omega = \sigma(\omega)E_\omega, \qquad \text{with } \sigma(\omega) = i\frac{q^2 n}{m\omega}, \qquad (*)$$

is valid for all frequencies, and apply it to the case then the electric field is an ultimately short pulse applied at $t = 0$:

$$E(t) = c\delta(t),$$

where c is a constant. Then, by expanding this expression into a Fourier series,

$$E(t) = \int_{-\infty}^{+\infty} E_\omega e^{-i\omega t} d\omega,$$

[13] See, e.g., QM Chapter 6.

[14] Since here we are not discussing magnetic field effects, we may ignore the vector character of $\mathbf{j}_\omega$ and $\mathbf{E}_\omega$ – say, by considering the Cartesian components of these vectors in the field's direction.

using the reciprocal transform to calculate E_ω,

$$E_\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} E(t)e^{i\omega t}dt = \frac{c}{2\pi}\int_{-\infty}^{+\infty}\delta(t)e^{i\omega t}dt = \frac{c}{2\pi},$$

plugging the result into Eq. (*),

$$j_\omega = i\frac{q^2 n}{m\omega}E_\omega = i\frac{q^2 n}{m\omega}\frac{c}{2\pi},$$

and calculating the current density as a function of time,[15] we get

$$\begin{aligned} j(t) &= \int_{-\infty}^{+\infty} j_\omega e^{-i\omega t}d\omega = i\frac{cq^2 n}{2\pi m}\int_{-\infty}^{+\infty}\frac{e^{-i\omega t}d\omega}{\omega} \equiv i\frac{cq^2 n}{2\pi m}\int_{-\infty}^{+\infty}\frac{\cos\omega t\, d\omega}{\omega} + \frac{cq^2 n}{2\pi m}\int_{-\infty}^{+\infty}\frac{\sin\omega t\, d\omega}{\omega} \\ &= 0 + \frac{cq^2 n}{2\pi m}\int_{-\infty}^{+\infty}\frac{\sin\omega t\, d\omega}{\omega} = \frac{cq^2 n}{2\pi m}\mathrm{sgn}(t)\int_{-\infty}^{+\infty}\frac{\sin\xi\, d\xi}{\xi} = \frac{cq^2 n}{2\pi m}\times\begin{cases}\pi, & \text{for } t>0,\\ -\pi, & \text{for } t<0.\end{cases}\end{aligned} \qquad (**)$$

So, Eq. (*) predicts a nonvanishing current to flow *before* the electric field's pulse has been applied – an evident nonsense from the causality point of view, in any physical situation when the electric current is caused by the electric field.

The Kramers-Kronig dispersion relations offer an easy and natural fix for the situation. Indeed, if we take the complex function $f(\omega)$ in Eq. (7.52) equal to $\sigma(\omega)$ rather than $\varepsilon(\omega) - \varepsilon_0$, then instead of the second of Eqs. (7.53), we get

$$\sigma''(\omega) = -\frac{1}{\pi}P\int_{-\infty}^{+\infty}\sigma'(\Omega)\frac{d\Omega}{\Omega-\omega}.$$

Evidently, if we take

$$\sigma'(\omega) = \pi\frac{q^2 n}{m}\delta(\omega), \qquad (***)$$

we immediately get

$$\sigma''(\omega) = \frac{q^2 n}{m\omega},$$

i.e. the same imaginary part of the complex conductivity as in Eq. (*). However, its additional real part (***) gives the following additional contribution Δj to the current:

$$\Delta j_\omega = \sigma'(\omega)E_\omega = \pi\frac{q^2 n}{m}\delta(\omega)\frac{c}{2\pi} \equiv c\frac{q^2 n}{2m}\delta(\omega),$$

$$\Delta j(t) = \int_{-\infty}^{+\infty}\Delta j_\omega e^{-i\omega t}d\omega = c\frac{q^2 n}{2m}\int_{-\infty}^{+\infty}\delta(\omega)e^{-i\omega t}d\omega \equiv c\frac{q^2 n}{2m},$$

so the full current becomes

$$j_\Sigma(t) \equiv j(t) + \Delta j(t) = \frac{cq^2 n}{m}\times\begin{cases}1, & \text{for } t>0,\\ 0, & \text{for } t<0,\end{cases}$$

[15] The last step of this calculation uses MA Eq. (6.12).

thus fixing the causality problem without changing the imaginary part of $\sigma(\omega)$, and hence the free-particle model (6.39) behind it. Indeed, Eq. (***) has a clear physical sense, describing the infinite final velocity of the particles, and hence the infinite current they carry, for a zero-frequency (dc) electric field applied to a free particle described by this simple model.

Problem 7.8. Calculate, sketch, and discuss the dispersion relation for electromagnetic waves propagating in a medium described by the Lorentz oscillator model (7.32), for the case of negligible damping.

Solution: The given assumption, $\delta_0 = 0$, reduces Eq. (7.32) to a simpler form:

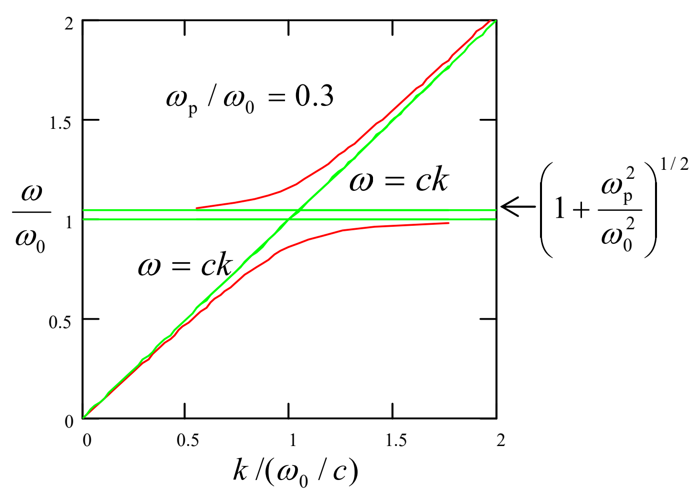
$$\varepsilon(\omega) = \varepsilon_0 + n\frac{q^2}{m}\frac{1}{(\omega_0^2 - \omega^2)} \equiv \varepsilon_0\left(1 + \frac{\omega_\mathrm{p}^2}{\omega_0^2 - \omega^2}\right), \qquad (*)$$

where ω_0 is the resonance frequency of the oscillators, and ω_p is the plasma frequency defined by Eq. (7.36) of the lecture notes. (For similar oscillators, that expression is reduced to Eq. (7.37) with $e \to q$:

$$\omega_\mathrm{p}^2 = \frac{nq^2}{\varepsilon_0 m};$$

note that the parameters ω_p and ω_0 are completely independent because ω_p is proportional to the oscillator density n, while ω_0 is a property of a single oscillator.) By plugging Eq. (*) into the general dispersion relation (7.28) for a uniform linear medium, assuming that our medium is not magnetically active: $\mu(\omega) \equiv \mu_0$, and recalling that $\varepsilon_0\mu_0 \equiv 1/c^2$, we get

$$k(\omega) = \frac{\omega}{c}\left(1 + \frac{\omega_\mathrm{p}^2}{\omega_0^2 - \omega^2}\right)^{1/2}.$$



The figure on the right shows a typical plot of this function (for a particular ratio $\omega_\mathrm{p}/\omega_0$). It reminds the standard level-anticrossing diagram[16] but coincides with it only in the limit $\omega_\mathrm{p} << \omega_0$, i.e. for very low density of the oscillators. For a non-zero ratio $\omega_\mathrm{p}/\omega_0$, there is a substantial frequency gap,[17]

$$\omega_0 < \omega < \omega_0', \quad \text{with } \omega_0' \equiv \left(\omega_0^2 + \omega_\mathrm{p}^2\right)^{1/2},$$

in which $\varepsilon(\omega) < 0$, and hence the electromagnetic wave propagation is impossible. At frequencies far from (both below and above) the gap, the dispersion relation is close to that of waves in free space, $k \approx \omega/c$, testifying that here the coupling between the wave and the medium oscillators is weak. However, at both approaches to the forbidden frequency gap, the dispersion relation plots strongly bend, indicating substantial energy sharing between the wave and the oscillators. In quantum mechanics, the elementary excitations of such a strongly coupled field are

[16] See, e.g., CM Sec. 6.1 and/or QM Secs. 2.7 and 5.1.

[17] Note that according to Eq. (*), the ratio ω_0'/ω_0 has a very simple physical sense: $(\omega_0'/\omega_0)^2 = \varepsilon(0)/\varepsilon_0 \equiv \varepsilon(0)/\varepsilon(\infty)$.

called *polaritons*, because of the electric-polarization origin of Eq. (7.32). (Note that the term "polaritons" is currently used in a much broader context than the Lorentz oscillator model – for virtually any excitations resulting from the interaction of traveling electromagnetic waves with a set of resonant dipoles.)

Problem 7.9. As was briefly discussed in Sec. 7.2 of the lecture notes,[18] a wave pulse of a finite but relatively large spatial extension $\Delta z >> \lambda \equiv 2\pi/k$ may be formed as a *wave packet* – a sum of sinusoidal waves with wave vectors $\mathbf{k}$ within a relatively narrow interval. Consider an electromagnetic plane wave packet of this type, with the electric field distribution

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\int_{-\infty}^{+\infty} \mathbf{E}_k e^{i(kz-\omega_k t)} dk, \qquad \text{with } k = \omega_k \left[\varepsilon(\omega_k)\mu(\omega_k)\right]^{1/2},$$

propagating along the z-axis in an isotropic, linear, and dissipation-free (but not necessarily dispersion-free) medium. Express the full energy of the packet (per unit area of the wave's front) via the complex amplitudes $\mathbf{E}_k$, and discuss its dependence on time.

Solution: For an isotropic linear medium we may use the first of Eqs. (6.113):[19]

$$u = \frac{\mathbf{E}\cdot\mathbf{D}}{2} + \frac{\mathbf{H}\cdot\mathbf{B}}{2},$$

so the total energy of the pulse (per unit area of the wave's front) may be calculated as

$$\frac{U}{A} = \int_{-\infty}^{+\infty} u dz = \frac{1}{2}\int_{-\infty}^{+\infty} \mathbf{E}\cdot\mathbf{D} dz + \frac{1}{2}\int_{-\infty}^{+\infty} \mathbf{H}\cdot\mathbf{B} dz. \qquad (*)$$

Just as it was done in Sec. 7.2 of the lecture notes for the derivation of Eq. (7.42), let us use the Fourier expansions of all these fields over real k, at arbitrary time t:

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\int_{-\infty}^{+\infty} \mathbf{E}_k e^{i\psi_k} dk \equiv \frac{1}{2}\left[\int_{-\infty}^{+\infty} \mathbf{E}_k e^{i\psi_k} dk + \text{c.c.}\right], \qquad \text{with the total phases } \psi_k \equiv kz - \omega_k t,$$

$$\mathbf{D}(\mathbf{r},t) = \mathrm{Re}\int_{-\infty}^{+\infty} \mathbf{D}_{k'} e^{i\psi_{k'}} dk' \equiv \frac{1}{2}\left[\int_{-\infty}^{+\infty} \mathbf{D}_{k'} e^{i\psi_{k'}} dk' + \text{c.c.}\right] = \frac{1}{2}\left[\int_{-\infty}^{+\infty} \varepsilon(\omega_{k'})\mathbf{E}_{k'} e^{i\psi_{k'}} dk' + \text{c.c.}\right],$$

and absolutely similarly for $\mathbf{H}$ and $\mathbf{B}$. Plugging these expressions into the first integral of Eq. (*), and changing the integration order, we get

$$\int_{-\infty}^{+\infty} \mathbf{E}\cdot\mathbf{D} dz = \frac{1}{4}\int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dk' \left[\varepsilon(\omega_{k'})\mathbf{E}_k\cdot\mathbf{E}_{k'}\int_{-\infty}^{+\infty} e^{i(\psi_k+\psi_{k'})} dz + \varepsilon^*(\omega_{k'})\mathbf{E}_k\cdot\mathbf{E}^*_{k'}\int_{-\infty}^{+\infty} e^{i(\psi_k-\psi_{k'})} dz + \text{c.c.}\right]. \qquad (**)$$

By using MA Eq. (14.4), we may readily work out both internal integrals, which do not depend on $\mathbf{E}_k$:

[18] For even more detail, see CM Sec. 5.3 and especially QM Sec. 2.2.

[19] Note that the second of Eqs. (6.13) is valid only for dispersion-free media.

$$\int_{-\infty}^{+\infty} e^{i\left(\psi_k \pm \psi_{k'}\right)} dz \equiv \exp\{-i(\omega_k \pm \omega_{k'})t\} \int_{-\infty}^{+\infty} e^{i\left(k \pm k'\right)z} dz$$

$$= 2\pi \exp\{-i(\omega_k \pm \omega_{k'})t\} \delta(k \pm k') = 2\pi \times \begin{cases} \exp\{-2i\omega_k t\}\delta(k+k'), & \text{for the upper sign,} \\ \delta(k-k'), & \text{for the lower sign,} \end{cases}$$

so the integral over k' in Eq. (**) is also elementary, and that equality is reduced to

$$\int_{-\infty}^{+\infty} \mathbf{E} \cdot \mathbf{D} dz = \frac{\pi}{2} \int_{-\infty}^{+\infty} \left[\varepsilon(\omega_k) \mathbf{E}_k \cdot \mathbf{E}_{-k} \exp\{-2i\omega_k t\} + \varepsilon^*(\omega_k) E_k E_k^* + \text{c.c.} \right] dk ,$$

because due to our definition of the partial wave frequencies, $\omega_{-k} = \omega_k$, and hence $\varepsilon(\omega_{-k}) = \varepsilon(\omega_k)$. According to the discussion in Sec. 7.2 of the lecture notes, in a lossless medium, ω_k is real, and $\varepsilon(\omega_k)$ is also a real, even function of frequency – and hence of k, so $\varepsilon^*(\omega_k) = \varepsilon(\omega_k)$. As a result, by spelling out the complex conjugate term, we get

$$\int_{-\infty}^{+\infty} \mathbf{E} \cdot \mathbf{D} dz = \frac{\pi}{2} \int_{-\infty}^{+\infty} \varepsilon(\omega_k) \left[\mathbf{E}_k \cdot \mathbf{E}_{-k} \exp\{-2i\omega_k t\} + \mathbf{E}_k^* \cdot \mathbf{E}_{-k}^* \exp\{2i\omega_k t\} + 2E_k E_k^* \right] dk .$$

Up to this point, the calculations are valid for any linear superposition of sinusoidal waves. If the packet is narrow as mentioned in the assignment, the first two terms in the square brackets are rapidly oscillating functions of time, with nearly zero averages, for all essential values of k. Neglecting these terms, we get

$$\int_{-\infty}^{+\infty} \mathbf{E} \cdot \mathbf{D}\, dz = \pi \int_{-\infty}^{+\infty} \varepsilon(\omega_k) E_k E_k^*\, dk .$$

Carrying out an absolutely similar calculation for the magnetic energy, we get

$$\int_{-\infty}^{+\infty} \mathbf{H} \cdot \mathbf{B}\, dz = \pi \int_{-\infty}^{+\infty} \mu(\omega_k) H_k H_k^* dk .$$

Now taking into account that Eqs. (7.6) and (7.7), generalized in accordance with Eq. (7.28):

$$\mathbf{H}_k = \frac{\mathbf{n}_z \times \mathbf{E}_k}{Z(\omega_k)} \operatorname{sgn}(k), \qquad \text{with } Z(\omega_k) = \left[\frac{\mu(\omega_k)}{\varepsilon(\omega_k)} \right]^{1/2},$$

we get

$$\mu(\omega_k) H_k H_k^* = \mu(\omega_k) \frac{E_k E_k^*}{[Z(\omega_k)]^2} = \varepsilon(\omega_k) E_k E_k^*,$$

so both contributions to the energy are equal and Eq. (*) yields

$$\frac{U}{A} = 2\pi \int_0^{+\infty} \varepsilon(\omega_k) E_k E_k^* dk .$$

This expression shows that the packet's energy does not depend on time, even though its shape may be deformed (most typically, spreads out with time) very significantly by the frequency dispersion of the medium.[20] This result certainly fits our intuitive notion of the wave packet's motion in the absence of attenuation.

Problem 7.10. Prove the Lorentz reciprocity relation (6.121) for a linear isotropic medium.

Solution: As was stated in Sec. 6.8 of the lecture notes, this formula relates the complex amplitudes (rather than the instantaneous values!) of two separate monochromatic electromagnetic fields of the same frequency ω, which are induced in a linear medium by stand-alone currents with complex amplitudes, respectively, $\mathbf{j}_1(\mathbf{r})$ and $\mathbf{j}_2(\mathbf{r})$. As was discussed in Sec. 7.1, in such media, the complex amplitudes of the variables obey all the relations pertinent to the instantaneous values of the fields, with the replacement[21]

$$\frac{\partial}{\partial t} \rightarrow -i\omega.$$

In particular, the Maxwell equations (6.99a) take, for such amplitudes, the following form:

$$\nabla \times \mathbf{E} - i\omega \mathbf{B} = 0, \qquad \nabla \times H + i\omega \mathbf{D} = \mathbf{j}.$$

Let us scalar-multiply all terms of the first of these equations, written for the fields labeled 1, by $\mathbf{H}_2$, and those of the second equation, by $\mathbf{E}_2$:

$$\mathbf{H}_2 \cdot \nabla \times \mathbf{E}_1 - i\omega \mathbf{H}_2 \cdot \mathbf{B}_1 = 0, \qquad \mathbf{E}_2 \cdot \nabla \times \mathbf{H}_1 + i\omega \mathbf{E}_2 \cdot \mathbf{D}_1 = \mathbf{E}_2 \cdot \mathbf{j}_1.$$

Flipping the field indices and the signs of all terms, we get

$$-\mathbf{H}_1 \cdot \nabla \times \mathbf{E}_2 + i\omega \mathbf{H}_1 \cdot \mathbf{B}_2 = 0, \qquad -\mathbf{E}_1 \cdot \nabla \times \mathbf{H}_2 - i\omega \mathbf{E}_1 \cdot \mathbf{D}_2 = -\mathbf{E}_1 \cdot \mathbf{j}_2.$$

Now let us add up each side of all these four equations. Grouping similar terms, we get

$$\begin{aligned} &(\mathbf{H}_2 \cdot \nabla \times \mathbf{E}_1 - \mathbf{E}_1 \cdot \nabla \times \mathbf{H}_2) + (\mathbf{E}_2 \cdot \nabla \times \mathbf{H}_1 - \mathbf{H}_1 \cdot \nabla \times \mathbf{E}_2) \\ &+ i\omega(-\mathbf{H}_2 \cdot \mathbf{B}_1 + \mathbf{H}_1 \cdot \mathbf{B}_2) + i\omega(\mathbf{E}_2 \cdot \mathbf{D}_1 - \mathbf{E}_1 \cdot \mathbf{D}_2) = \mathbf{E}_2 \cdot \mathbf{j}_1 - \mathbf{E}_1 \cdot \mathbf{j}_2. \end{aligned} \qquad (*)$$

But for an isotropic linear medium, the complex amplitudes of each field are related, at each spatial point, as $\mathbf{B}_{1,2} = \mu(\omega)\mathbf{H}_{1,2}$ and $\mathbf{D}_{1,2} = \varepsilon(\omega)\mathbf{E}_{1,2}$, so the last two parentheses on the left-hand side of this equation vanish – even for nonuniform systems where $\varepsilon(\omega)$ and $\mu(\omega)$ depend on coordinates.[22] Next, according to the vector calculus,[23] each of the first two parentheses on the left-hand side of Eq. (*) is just the divergence of the vector product of the corresponding fields, so the equation reduces to

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[20] Again, see QM Sec. 2.2 for a detailed discussion and examples of this effect.

[21] Here, as everywhere in my series, I am following the "physical" convention, representing a single complex component of a sinusoidal function of time as $f\exp\{-i\omega t\}$, with the negative sign in the exponent. The transfer to the "engineering" convention, with the positive sign in the exponent, is obviously equivalent to the change of the sign before ω. Since Eq. (6.121) that we want to prove does not include ω as all, the choice of the convention makes no difference for it.

[22] Moreover, even the reader whose tensor calculus is somewhat rusty can easily see why these cancellations (and hence the final result of this calculation) are valid even for an anisotropic medium with symmetric tensors $\varepsilon_{jj'}(\omega)$ and $\mu_{jj'}(\omega)$. In this case, for example, $\mathbf{E}_2 \cdot \mathbf{D}_1 \equiv \Sigma_{j,j'}\varepsilon_{jj'}(\omega)(E_2)_j(E_1)_{j'} = \Sigma_{j,j'}\varepsilon_{j'j}(\omega)(E_2)_j(E_1)_{j'} \equiv \mathbf{E}_1 \cdot \mathbf{D}_2$.

[23] See, e,g., MA Eq. (11.7), read backward.

$$\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \cdot \nabla \times \mathbf{H}_1) = \mathbf{E}_2 \cdot \mathbf{j}_1 - \mathbf{E}_1 \cdot \mathbf{j}_2 .$$

By integrating this expression over the volume V inside an arbitrary closed surface S and using the divergence theorem,[24] we get

$$\oint_S (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1)_n \, d^2 r = \int_V (\mathbf{E}_2 \cdot \mathbf{j}_1 - \mathbf{E}_1 \cdot \mathbf{j}_2) d^3 r , \qquad (**)$$

i.e. the Lorentz reciprocity relation (6.121). As will be shown in the next chapter, this relation may be further simplified if both field sources are space-localized.

Let me leave it to the reader to prove that in the quasistatic case, Eq. (**) may be used to derive the Rayleigh-Lorentz-Carson reciprocity relation, whose alternative proof was the subject of Problem 4.3.

Problem 7.11.* A plane wave of frequency ω is normally incident, from free space, on a plane surface of a collision-free plasma with the electron density growing slowly and linearly with the distance from the surface: $n = \gamma z$ for $z \geq 0$, where $\gamma > 0$ is a small constant. Calculate the functional form of the resulting standing wave's "tail" inside the plasma.

Solution: Reviewing the derivation of the first of Eqs. (7.3) of the lecture notes at the beginning of Sec. 7.1, we see that if the medium's properties are spatially dependent, these equations acquire additional terms, which couple them:

$$\left(\nabla^2 - \varepsilon\mu \frac{\partial^2}{\partial t^2}\right)\mathbf{E} = (\nabla\mu) \times \frac{\partial \mathbf{H}}{\partial t}, \qquad \left(\nabla^2 - \varepsilon\mu \frac{\partial^2}{\partial t^2}\right)\mathbf{H} = -(\nabla\varepsilon) \times \frac{\partial \mathbf{E}}{\partial t}. \qquad (*)$$

However, if the parameters ε and μ depend only on the distance from the surface, and their changes on one wavelength of the normally incident radiation are relatively small, the right-side terms of these equations only produce effects of the second order in small $\nabla\varepsilon$ and $\nabla\mu$,[25] and may be neglected in comparison with the much more dramatic effects provided by the spatial dependence of the product $\varepsilon\mu$ on their left-hand sides.

Hence, if the plasma is magnetically inactive and its density is a slow function of z, we may still describe the propagation of a normally incident wave in it by the usual 1D wave equation

$$\left(\frac{\partial^2}{\partial z^2} - \varepsilon\mu \frac{\partial^2}{\partial t^2}\right)\mathbf{E} = 0,$$

where ε is a slow function of z, while $\mu = \mu_0 =$ const. According to Eqs. (7.36)-(7.37) and their discussion, in a collision-free plasma we may take

$$\varepsilon(\omega) = \varepsilon_0\left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\right) \equiv \varepsilon_0\left(1 - \frac{ne^2}{\varepsilon_0 m_\mathrm{e}\omega^2}\right).$$

For our current problem where

[24] If you still do not remember it, see, e.g., MA Eq. (12.2).

[25] For example, at $\nabla\varepsilon \to 0$ and $\nabla\mu = 0$, the first-order correction due to the right-hand side of the second of Eqs. (*) results only in a proportionally small change of **H**, which has no way to sneak back into the equation for **E**.

$$n = \begin{cases} 0, & \text{for } z \leq 0, \\ \gamma z, & \text{for } 0 \leq z, \end{cases}$$

this gives the following ordinary differential equation for the complex amplitude of a plane monochromatic wave propagating along the z-axis,

$$\left[\frac{d^2}{dz^2} + k^2(z)\right] E_\omega = 0, \quad \text{where } k^2(z) \equiv \frac{\omega^2}{c^2} \times \begin{cases} 1, & \text{for } z \leq 0, \\ 1 - \dfrac{\gamma e^2}{\varepsilon_0 m_e \omega^2} z, & \text{for } 0 \leq z. \end{cases} \qquad (**)$$

As was discussed in Sec. 7.3 of the lecture notes, its solution for negative z is the usual standing wave of a constant amplitude

$$E_\omega = 2E_0 \sin[k_0(z - \delta_-)],$$

where E_0 is the amplitude of the incident wave, $k_0 \equiv (\varepsilon_0\mu_0)^{1/2}\omega \equiv \omega/c$ is the wave number in free space, and δ_- is a phase shift describing the wave's penetration into a surface layer of the plasma – see, e.g., Fig. 7.4 of the lecture notes. The solution of Eq. (**) for $z > 0$ and an *arbitrary* positive γ is proportional to one of the so-called *Airy functions*, namely Ai(ξ), where in our current case,

$$\xi \equiv \left(\frac{\gamma e^2}{\varepsilon_0 c^2 m_e}\right)^{1/3} (z - \delta_+), \qquad \text{with } \delta_+ = \frac{\varepsilon_0 m_e \omega^2}{\gamma e^2}.$$

(As Eq. (**) shows, δ_+ has the simple physical meaning of the z-point where the propagation constant $k^2(z)$ changes its sign.) However, since in this series, the Airy functions are discussed, for the first time, only in the QM course (Sec. 2.4), and our task is limited to sufficiently small γ, the problem may be solved without using them.

For that, let us notice that if $z > \delta_+$, then $k^2(z)$ is negative, so $k(z) = \pm i\kappa(z)$, with

$$\kappa^2(z) \to \frac{\gamma e^2}{\varepsilon_0 m_e c^2}(z - \delta_+) > 0. \qquad (***)$$

As we know from Eqs. (7.64) and (7.68), if $k^2 < 0$ of a medium is constant, the wave inside it decreases as $\exp\{-\kappa z\}$. Hence, we may expect that if κ does depend on z but slowly enough, then at $\kappa(z)(z - \delta_+)$ >> 1,

$$E_\omega(z) \propto \exp\{-f(z)\}, \qquad \text{with } \frac{df}{dz} = \kappa(z) + \varphi(z), \quad \text{i.e. } f(z) = \int^z \kappa(z')dz' + \int^z \varphi(z')dz',$$

where the correction $\varphi(z)$ would be relatively small: $|\varphi(z)| \ll \kappa(z)$. Indeed, plugging the assumed solution into Eq. (**), we see that it is indeed satisfied if, in the first approximation in $\varphi(z)$,

$$2\kappa\varphi - \frac{d\kappa}{dz} = 0, \quad \text{i.e. } \varphi = \frac{1}{2\kappa}\frac{d\kappa}{dz} \equiv \frac{d}{dz}\ln(\kappa^{1/2}),$$

so our solution becomes[26]

$$E_\omega(z) \propto \exp\{-f(z)\} = \exp\left\{-\int^z \kappa(z')dz' - \ln(\kappa^{1/2})\right\} \equiv \frac{1}{\kappa^{1/2}}\exp\left\{-\int^z \kappa(z')dz'\right\}.$$

[26] This is the particular case (for $k^2 < 0$) of the famous *WKB approximation*. This method is one of the main theoretical tools of quantum mechanics and hence will be discussed, in much more detail, in QM Sec. 2.4.

In our particular case (***), this solution reads

$$E_\omega(z) \propto \frac{1}{(z-\delta_+)^{1/4}} \exp\left\{-\frac{2}{3}\left(\frac{\gamma e^2}{\varepsilon_0 m_e c^2}\right)^{1/2}(z-\delta_+)^{3/2}\right\} \to 0, \quad \text{at } (z-\delta_+) \to \infty.$$

This is indeed the correct asymptotic behavior of the Airy function Ai(ξ). Note that this solution depends on the wave's frequency only via the position $\delta_+ \propto \omega^2$ of the effective reflection boundary.

<u>Problem 7.12</u>.* Analyze the effect of a time-independent uniform magnetic field $\mathbf{B}_0$, parallel to the direction $\mathbf{n}$ of an electromagnetic wave propagation, on the wave's dispersion in plasma, within the same simple model that was used in Sec. 7.2 of the lecture notes for the derivation of Eq. (7.38). (Limit your analysis to relatively weak waves, whose magnetic field is much smaller than $\mathbf{B}_0$.)

Hint: You may like to represent the incident wave as a linear superposition of two circularly polarized waves, with opposite polarization directions.

Solution: For clarity, let us first repeat the derivation of Eq. (7.38), valid for $\mathbf{B}_0 = 0$, for a plasma of free noninteracting particles, now spelling out the spatial orientations of the field and of the field-induced particle motion. For a free particle ($\omega_0 = 0$, $\delta_0 = 0$), Eq. (7.30) may be rewritten as

$$m\ddot{\mathbf{r}} = q\mathbf{E}(t), \qquad (*)$$

so for a monochromatic, linearly polarized wave with $\mathbf{E}(t) = \mathbf{n}_x\text{Re}[E_\omega\exp\{-i\omega t\}]$ (where $\mathbf{n}_x$ is the polarization direction), we may look for the solution in a similar form, $\mathbf{r}(t) = \mathbf{n}_x\text{Re}[r_\omega\exp\{-i\omega t\}]$. Plugging these expressions into Eq. (*), we get the following equation for the complex amplitude of the particle's forced oscillations:

$$-m\omega^2 r_\omega = qE_\omega, \qquad \text{giving } r_\omega = -\frac{q}{m\omega^2}E_\omega.$$

Now calculating the complex amplitude,

$$P_\omega = np_\omega = nqr_\omega = -\frac{nq^2}{m\omega^2}E_\omega,$$

of the resulting electric polarization of the plasma, $\mathbf{P}(t) = \mathbf{n}_x\text{Re}[P_\omega\exp\{-i\omega t\}]$, we may use Eq. (3.33) to calculate its frequency-dependent permittivity,

$$\varepsilon(\omega) \equiv \frac{D_\omega}{E_\omega} \equiv \varepsilon_0 + \frac{P_\omega}{E_\omega} = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right),$$

with the plasma frequency ω_p expressed similarly to Eq. (7.37):

$$\omega_p^2 \equiv \frac{nq^2}{\varepsilon_0 m},$$

i.e. reproduce Eq. (7.36) of the lecture notes. As was discussed in Sec. 7.2 of the lecture notes, this permittivity immediately leads to the plasma dispersion law (7.38),

$$k(\omega) = \omega\left[\varepsilon(\omega)\mu_0\right]^{1/2} = \frac{\omega}{c}\left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\right)^{1/2},$$

which, in particular, forbids the propagation of waves with frequencies $\omega < \omega_\mathrm{p}$ in the plasma.

In the case of nonvanishing dc field $\mathbf{B}_0$, we have to generalize Eq. (*), taking into account the full Lorentz force (5.10) acting on the particle:

$$m\ddot{\mathbf{r}} = q\left[\mathbf{E}(t) + \dot{\mathbf{r}} \times \mathbf{B}_0\right]. \qquad (**)$$

This equation is still linear in $\mathbf{r}(t)$, and hence the plasma's response still may be characterized by a (frequency-dependent) permittivity. However, Eq. (**) shows that if the field $\mathbf{B}_0$ is directed along the wave's propagation (say, the z-axis), i.e. $\mathbf{B}_0 = \mathbf{n}_z B_0$, any motion in the direction x of the wave's electric field causes a magnetic force in the y-direction, and vice versa, so the electric polarization acquires not only the x-, but also the y-component. This means that the plasma's response becomes anisotropic, and its permittivity for linearly polarized waves has to be described with a 2×2 tensor rather than a scalar.

However, the calculation may be simplified using the provided *Hint*, i.e. representing the incident wave, with an arbitrary polarization, as a sum of two circularly polarized ones – see the discussion at the end of Sec. 7.1 of the lecture notes. For example, a wave linearly polarized in the x-direction may be represented as

$$\begin{aligned}\mathbf{E}(t) &= \mathbf{n}_x \operatorname{Re}\left[E_\omega \exp\{-i\omega t\}\right] = \sum_\pm \operatorname{Re}\left[\mathbf{n}_x \frac{E_\omega}{2}\exp\{-i\omega t\} \pm i\mathbf{n}_y \frac{E_\omega}{2}\exp\{-i\omega t\}\right] \\ &\equiv \sum_\pm \operatorname{Re}\left[\left(\mathbf{n}_x \pm i\mathbf{n}_y\right)E_\omega^\pm \exp\{-i\omega t\}\right], \qquad \text{with } E_\omega^\pm = \frac{E_\omega}{2}.\end{aligned} \qquad (***)$$

Indeed, looking for the solution of Eq. (**) in a similar form, i.e. as a sum of two simultaneous circular motions but with not necessarily equal complex amplitudes:

$$\mathbf{r}(t) = \sum_\pm \operatorname{Re}\left[\left(\mathbf{n}_x \pm i\mathbf{n}_y\right) r_\omega^\pm \exp\{-i\omega t\}\right],$$

we see that since

$$\left(\mathbf{n}_x \pm i\mathbf{n}_y\right)\times\mathbf{B}_0 = \left(\mathbf{n}_x \pm i\mathbf{n}_y\right)\times\mathbf{n}_z B_0 \equiv \left(\mathbf{n}_x\times\mathbf{n}_z \pm i\mathbf{n}_y\times\mathbf{n}_z\right)B_0 = \left(-\mathbf{n}_y \pm i\mathbf{n}_x\right)B_0 \equiv \pm iB_0\left(\mathbf{n}_x \pm i\mathbf{n}_y\right),$$

Eq. (**) naturally separates into two independent and different relations for their complex amplitudes:

$$-m\omega^2 r_\omega^\pm = q\left[E_\omega^\pm - i\omega r_\omega^\pm\left(\pm iB_0\right)\right],$$

giving

$$r_\omega^\pm = -\frac{q}{m\omega^2 \mp q\omega B}E_\omega^\pm \equiv -\frac{q}{m\omega\left(\omega \pm \omega_\mathrm{c}\right)}E_\omega^\pm,$$

where $\omega_\mathrm{c} \equiv -qB_0/m$ is the well-known *cyclotron frequency* of a particle's motion in the magnetic field.[27] This result is very natural: it shows that the longitudinal magnetic field speeds up the particle's rotation

[27] For a detailed discussion of this motion (in both the non-relativistic and relativistic cases), see Sec. 9.6 of the lecture notes.

in one transverse direction and slows its rotation in the opposite direction (induced by the wave's electric field) but does not entangle these two circular motion modes.

Now calculating the resulting electric permittivity of the plasma exactly as was done above, we see that for each circularly polarized wave, it may be still characterized by a scalar function of frequency, but for the waves with opposite polarizations, these functions are different:

$$\varepsilon^{\pm}(\omega) \equiv \varepsilon_0 + \frac{P_\omega^{\pm}}{E_\omega^{\pm}} = \varepsilon_0 \left[1 - \frac{\omega_\mathrm{p}^2}{\omega(\omega \pm \omega_\mathrm{c})}\right],$$

and so are their dispersion relations:

$$k^{\pm}(\omega) = \omega\left[\varepsilon^{\pm}(\omega)\mu_0\right]^{1/2} = \frac{\omega}{c}\left[1 - \frac{\omega_\mathrm{p}^2}{\omega(\omega \pm \omega_\mathrm{c})}\right]^{1/2}. \qquad (****)$$

The last relation shows, in particular, that the magnetic field increases the cut-off frequency ω_min for wave propagation, at which the wave vector vanishes, for one circular polarization (depending on the sign of the particle's charge) and decreases it for the counterpart polarization – though never fully suppresses this threshold:

$$\omega_\mathrm{min}^{\pm} = \left(\omega_\mathrm{p}^2 + \frac{\omega_\mathrm{c}^2}{4}\right)^{1/2} \pm \frac{|\omega_\mathrm{c}|}{2} \to \begin{cases} |\omega_\mathrm{c}|, \\ \omega_\mathrm{p}^2 / |\omega_\mathrm{c}|, \end{cases} \quad \text{for } |\omega_\mathrm{c}|/\omega_\mathrm{p} \to \infty.$$

Moreover, even for the frequencies above both values of ω_min, the phase and group velocity for each of the two modes is different, so an incident wave of any polarization (even a linear one), upon entering the plasma, splits into two independently propagating circularly polarized waves. (For example, for the Earth ionosphere in the planet's own magnetic field, ω_p is typically below 10^8 s^{-1}, while $|\omega_\mathrm{c}| \approx 10^7$ s^{-1}, so the wave splitting effects may be quite noticeable.) In the case of small splitting, when

$$\Delta k \equiv k^+ - k^- \approx \frac{\omega_\mathrm{c}\omega_\mathrm{p}^2}{c\omega^2} << k^{\pm}, \qquad \text{for } |\omega_\mathrm{c}| << \omega, \omega_\mathrm{p},$$

its effect may be conveniently represented as a slow rotation of the linear polarization plane by an angle $\theta \propto z$. Indeed, after propagation by distance z, we may write $k^{\pm}z = kz \pm \theta$, where k is the wave vector in the absence of the magnetic field and $\theta \equiv (\Delta k)\, z/2$, so the initially x-polarized wave (***) becomes

$$\mathbf{E}(z,t) = \sum_{\pm} \mathrm{Re}\left[(\mathbf{n}_x \pm i\mathbf{n}_y)\frac{E_\omega}{2}\exp\{i(k^{\pm}z - \omega t)\}\right] = \sum_{\pm} \mathrm{Re}\left[(\mathbf{n}_x \pm i\mathbf{n}_y)\frac{E_\omega}{2}e^{\pm i\theta}\exp\{i(kz - \omega t)\}\right].$$

Representing this wave as an explicit sum of two Cartesian components,

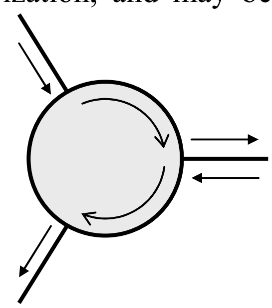
$$\mathbf{E}(z,t) = \mathrm{Re}\left\{\frac{E_\omega}{2}\left[\mathbf{n}_x\left(e^{i\theta} + e^{-i\theta}\right) + i\mathbf{n}_y\left(e^{i\theta} - e^{-i\theta}\right)\right]\exp\{i(kz - \omega t)\}\right\},$$

we see that the ratio of their complex amplitudes is

$$\frac{E_{\omega y}}{E_{\omega x}} = \frac{i\left(e^{i\theta} - e^{-i\theta}\right)}{e^{i\theta} + e^{-i\theta}} \equiv \tan\theta,$$

i.e. remains real. According to Eq. (7.17) of the lecture notes, this means that the wave remains linearly polarized at each point, but the polarization plane slowly rotates by the angle θ proportional both to the traveled distance z and to $\Delta k \propto \omega_c$, i.e. to the applied magnetic field. This is the *Faraday effect* (or "Faraday rotation") – one of several *magneto-optical phenomena*.

It is also an example of the so-called *non-reciprocal effects*, which violate the Lorentz reciprocity relation (6.121). Indeed, as was discussed at the end of Sec. 6.8 of the lecture notes, that relation predicts, in particular, that the electromagnetic waves propagate through a medium similarly in both directions. In the case of the Faraday rotation, however, at the wave's direction reversal (say, at its reflection from a mirror), the direction of the magnetic field "as seen by the wave" changes. As a result, the sign of ω_c in Eq. (****) changes, so the direction of the rotation of the wave's quasi-linear polarization reverses from the wave's "point of view", and hence remains the same as observed from the lab frame. In other words, in the lab frame, the rotation goes in the same direction regardless of the wave's direction, accumulating at the wave's roundtrip, so with the appropriate choice of the plasma layer's thickness, the returning wave may have the alternative (y-) linear polarization, and may be readily separated from the incident one by using a polarization filter.



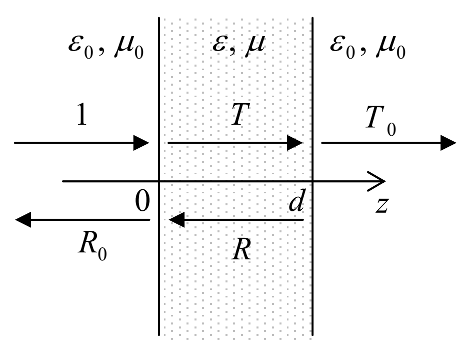
This effect is used for the implementation of very important non-reciprocal microwave and optical devices – *circulators*, operating as shown in the figure on the right. (Typically they use dc-magnetic-field-biased ferromagnetic materials rather than gaseous plasmas.) The circulators allow, for example, to stabilize the operation of *reflection amplifiers* based on the effective negative resistance of nonlinear active electron devices.

Problem 7.13 A monochromatic plane electromagnetic wave is normally incident, from free space, on a uniform slab with electric permittivity ε and magnetic permeability μ, with the slab's thickness d comparable with the wavelength.

(i) Calculate the power transmission coefficient $\mathcal{T}$, i.e. the fraction of the incident wave's power, that is transmitted through the slab.

(ii) Assuming that ε and μ are frequency-independent and positive, analyze in detail the frequency dependence of $\mathcal{T}$. In particular, how does the function $\mathcal{T}(\omega)$ depend on the slab's thickness d and the wave impedance $Z \equiv (\mu/\varepsilon)^{1/2}$ of its material?

Solutions:



(i) Since the wave may be partly reflected not only from the front surface but also from the back surface of the slab (see the figure on the right), we have to look for the solution of the 1D wave equation in the form of five traveling waves:

$$E = E_\omega e^{-i\omega t} \times \begin{cases} e^{ik_0 z} + R_0 e^{-ik_0 z}, & \text{for } z \leq 0, \\ Te^{ikz} + R e^{-ikz}, & \text{for } 0 \leq z \leq d, \\ T_0 e^{ik_0 z}, & \text{for } d \leq z. \end{cases}$$

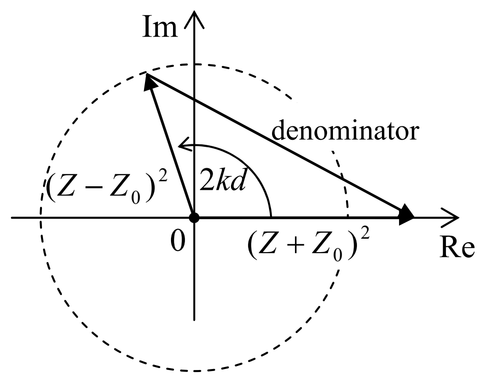
The four (generally, complex) coefficients R, R_0, T, and T_0 participating in these expressions may be found by plugging them into the four boundary conditions that express the continuity of the electric field E and the magnetic field $H = \pm E/Z$ at both surfaces ($z = 0$ and $z = d$). These conditions give us the following system of four linear equations:

$$1 + R_0 = T + R, \qquad \frac{1 - R_0}{Z_0} = \frac{T - R}{Z},$$
$$Te^{ikd} + Re^{-ikd} = T_0 e^{ik_0 d}, \qquad \frac{Te^{ikd} - Re^{-ikd}}{Z} = \frac{T_0 e^{ik_0 d}}{Z_0}.$$

Solving this system, we get, in particular,

$$T_0 = \frac{4ZZ_0}{(Z + Z_0)^2 - (Z - Z_0)^2 \exp\{2ikd\}} \exp\{i(k - k_0)d\}, \qquad (*)$$

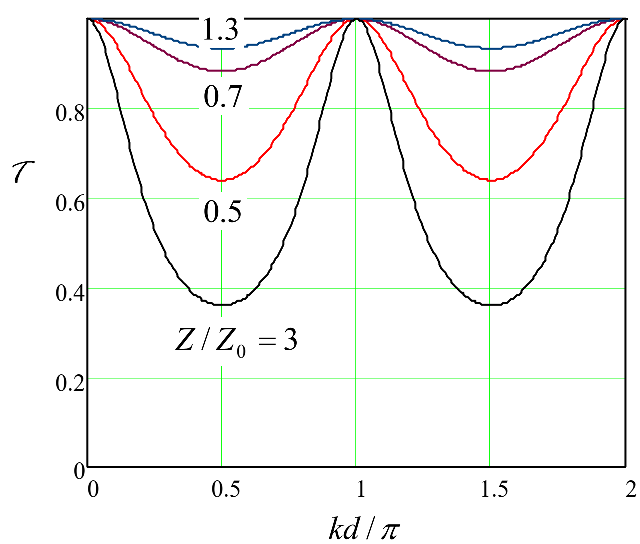
and since, for our system, the wave impedances in the regions in front of the slab and behind it are equal, the power transmission coefficient may be now calculated just as $\mathcal{T} = |T_0|^2$.



(ii) The simplest way to analyze the dependence of T_0 and $\mathcal{T}$ on the system's parameters (at constant and positive ε and μ, and hence constant and positive Z and Z_0), is to consider the diagram in the figure on the right, which shows the denominator in Eq. (*), a complex-variable function, as the difference of two 2D vectors, of lengths $(Z + Z_0)^2$ and $(Z - Z_0)^2 < (Z + Z_0)^2$, respectively, with the angle $2kd$ between them.

It is clear, first of all, that the denominator, and hence $|T_0|$ and $\mathcal{T}$, oscillate as functions of both the wave frequency ω and the slab's thickness d, with the period $\Delta(kd) = \pi$ (see the figure on the right), with the transmitted power having equal maxima at[28]

$$kd = \pi m, \quad m = 1, 2, \ldots .$$



This is the well-known *constructive interference* condition, fulfilled when the slab's thickness d equals an integer number of π/k, i.e. of the de Broglie half-wavelengths $\lambda/2$.[29]

The fact that would be harder to predict is that in each maximum, the transmission is perfect:

[28] Formally, the value $m = 0$ should be in this list as well, but it corresponds to physically trivial cases of either $d = 0$ (no slab at all), or $k = 0$, i.e. $\omega = 0$ (no wave at all).

[29] A similar effect in quantum mechanics is called *resonant tunneling* – see, e.g., QM Sec. 2.5.

$$\left|T_0\right|_{\max} = \left|T_0\right|_{kd=\pi m} = \frac{4ZZ_0}{(Z+Z_0)^2-(Z-Z_0)^2} \equiv 1,$$

regardless of the Z/Z_0 ratio. This ratio, however, does affect the sharpness of these resonances and the depth of the transmission minima: the farther Z/Z_0 from 1, the lower the transmission minimum at such *destructive interference*,

$$\left|T_0\right|_{\min} = \left|T_0\right|_{kd=\pi(m+1/2)} = \frac{4ZZ_0}{(Z+Z_0)^2+(Z-Z_0)^2} \equiv \frac{2(Z/Z_0)}{(Z/Z_0)^2+1}, \qquad \text{so } \mathcal{T}_{\min} = \frac{4(Z/Z_0)^2}{\left[(Z/Z_0)^2+1\right]^2} \le 1.$$

Problem 7.14. A plane electromagnetic wave with a free-space wave number k_0 is normally incident on a planar conducting film of thickness $d \sim \delta_s << 1/k_0$. Calculate the power transmission coefficient of the system and analyze the result in the limits of small and large values of the ratio d/δ_s.

Solution: This problem may be solved by re-deriving (or just using:-) the result of the previous problem, which may be rewritten as

$$\mathcal{T} = \left|T_0\right|^2, \quad \text{where } T_0 = \frac{4ZZ_0}{(Z+Z_0)^2\exp\{-ikd\}-(Z-Z_0)^2\exp\{ikd\}}\exp\{-ik_0 d\}. \qquad (*)$$

By reviewing the calculations that have led to this result, we may see that it is valid even if the wave impedance Z and the wave number k inside the film are complex numbers evaluated at the wave's frequency ω – see Eqs. (7.28) of the lecture notes:

$$Z = Z(\omega) \equiv \left[\frac{\mu(\omega)}{\varepsilon(\omega)}\right]^{1/2}, \qquad k = k(\omega) \equiv \omega\left[\varepsilon(\omega)\mu(\omega)\right]^{1/2}.$$

The problem's assignment implies that the film is non-magnetic, so $\mu(\omega) = \mu_0$, and that its conductance is Ohmic and non-dispersive, so we may account for it by using Eq. (7.46) with real $\sigma(\omega) = \sigma = \text{const}$:

$$\varepsilon(\omega) = \varepsilon_{\text{opt}}(\omega) + i\frac{\sigma}{\omega}.$$

Since, per Eq. (6.33), the skin depth δ_s equals $(2/\mu_0\sigma\omega)^{1/2}$, the given condition $\delta_s << 1/k_0 \equiv 1/\omega(\varepsilon_0\mu_0)^{1/2}$ means that $\sigma/\omega >> \varepsilon_0$, so unless we are in very close vicinity of some optical resonance (which should have been specified in the assignment), $\sigma/\omega >> \varepsilon_{\text{opt}}(\omega)$ as well, and we may neglect the first contribution to the complex permittivity, taking[30, 31]

$$\varepsilon(\omega) = i\frac{\sigma}{\omega}, \qquad \text{so } Z = \left(\frac{\mu_0\omega}{i\sigma}\right)^{1/2}, \text{ and } k = \left(i\sigma\omega\mu_0\right)^{1/2} \equiv \frac{1+i}{\delta_s}, \qquad (**)$$

[30] Actually, this means that for the thin film description, we are using the quasistatic approximation (see Sec. 6.3 of the lecture notes), i.e. are neglecting the displacement currents in the film in comparison with the actual (Ohmic) currents in it. However, using Eq. (*) for $\mathcal{T}$, which follows from the full system of Maxwell equations, means that we are still keeping the due account of the displacement currents in the space around the film.

[31] You may notice that the last expression for k differs from Eq. (6.30) for $\kappa_\pm$ only by the multiplication by the imaginary unity. (The difference is due to a different definition of $\kappa_\pm$ – see Eq. (6.29).)

$$\frac{Z}{Z_0}=\frac{(\mu_0\omega/i\sigma)^{1/2}}{(\mu_0/\varepsilon_0)^{1/2}}\equiv\frac{1}{(2i)^{1/2}}\left[\omega(\varepsilon_0\mu_0)^{1/2}\right]\left(\frac{2}{\mu_0\sigma\omega}\right)^{1/2}=\frac{k_0\delta_\mathrm{s}}{1+i}.$$

Plugging these expressions into Eq. (*) and taking into account that the condition $k_0\delta_\mathrm{s} << 1$ means, in particular, that $|Z| << Z_0$, and that since $k_0d << 1$ as well, i.e. $\exp\{-k_0d\} \approx 1$, we get

$$T_0\approx\frac{4ZZ_0}{\left(Z_0^2+2ZZ_0\right)\exp\{-ikd\}-\left(Z_0^2-2ZZ_0\right)\exp\{ikd\}}=\left\{\cos kd+\frac{1-i}{2k_0\delta_\mathrm{s}}\sin kd\right\}^{-1}.$$

Calculating the power transparency $\mathcal{T}$ from the last (innocently looking :-) expression, we have to be careful because according to the last of Eqs. (**), the argument kd of the involved trigonometric functions is a complex number. After the separation of their real and imaginary parts,[32] we get

$$\begin{aligned}T_0&=\left\{\cos\left[(1+i)\frac{d}{\delta_\mathrm{s}}\right]+\frac{1-i}{2k_0\delta_\mathrm{s}}\sin\left[(1+i)\frac{d}{\delta_\mathrm{s}}\right]\right\}^{-1}\\&=\left\{\left[\cosh\frac{d}{\delta_\mathrm{s}}\left(\cos\frac{d}{\delta_\mathrm{s}}+\frac{1}{2k_0\delta_\mathrm{s}}\sin\frac{d}{\delta_\mathrm{s}}\right)+\frac{1}{2k_0\delta_\mathrm{s}}\sinh\frac{d}{\delta_\mathrm{s}}\cos\frac{d}{\delta_\mathrm{s}}\right]\right.\\&\quad\left.-i\left[\sinh\frac{d}{\delta_\mathrm{s}}\left(\sin\frac{d}{\delta_\mathrm{s}}-\frac{1}{2k_0\delta_\mathrm{s}}\cos\frac{d}{\delta_\mathrm{s}}\right)+\frac{1}{2k_0\delta_\mathrm{s}}\cosh\frac{d}{\delta_\mathrm{s}}\sin\frac{d}{\delta_\mathrm{s}}\right]\right\}^{-1},\end{aligned}$$

so the final result is given by a somewhat bulky formula

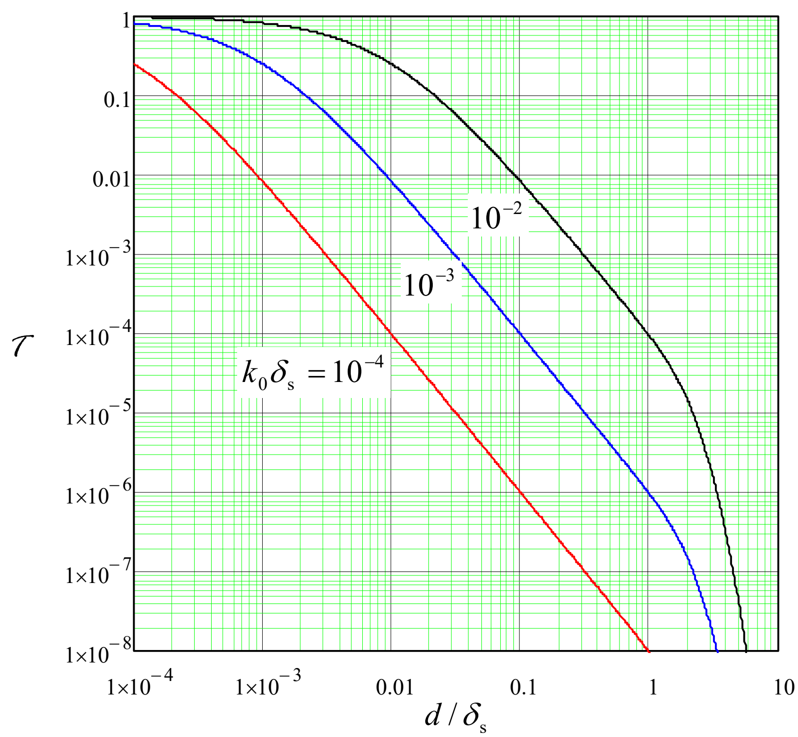
$$\begin{aligned}\mathcal{T}=|T_0|^2=&\left\{\left[\cosh\frac{d}{\delta_\mathrm{s}}\left(\cos\frac{d}{\delta_\mathrm{s}}+\frac{1}{2k_0\delta_\mathrm{s}}\sin\frac{d}{\delta_\mathrm{s}}\right)+\frac{1}{2k_0\delta_\mathrm{s}}\sinh\frac{d}{\delta_\mathrm{s}}\cos\frac{d}{\delta_\mathrm{s}}\right]^2\right.\\&\left.+\left[\sinh\frac{d}{\delta_\mathrm{s}}\left(\sin\frac{d}{\delta_\mathrm{s}}-\frac{1}{2k_0\delta_\mathrm{s}}\cos\frac{d}{\delta_\mathrm{s}}\right)+\frac{1}{2k_0\delta_\mathrm{s}}\cosh\frac{d}{\delta_\mathrm{s}}\sin\frac{d}{\delta_\mathrm{s}}\right]^2\right\}^{-1}.\end{aligned}$$

It may be readily simplified in the limits of relatively large and small values of the film's thickness:

$$\mathcal{T}\to\begin{cases}\dfrac{(k_0\delta_\mathrm{s})^2}{2}\exp\left\{-\dfrac{2d}{\delta_\mathrm{s}}\right\}\to 0, & \text{for }\ 2\ln\dfrac{1}{k_0\delta_\mathrm{s}}<<\dfrac{d}{\delta_\mathrm{s}},\\ \left(1+\dfrac{d}{k_0\delta_\mathrm{s}^2}\right)^{-2}, & \text{for }\ \dfrac{d}{\delta_s}<<1.\end{cases}\qquad(***)$$

Plots in the figure below show the calculated $\mathcal{T}$ as a function of the d/δ_s ratio for several values of the parameter $k_0\delta_\mathrm{s}$, typical for the transmission of microwave radiation through thin metallic films. The crossover between the asymptotic behaviors (***), taking place at $d \sim \delta_s$, is clearly visible.

[32] See, e.g., MA Eq. (3.5).



The plots also show that it makes sense to break out the second of the asymptotes, for $d/\delta_s << 1$, into two sub-limits:

$$\mathcal{T} \to \begin{cases} \left(\dfrac{k_0\delta_s^2}{d}\right)^2 << 1, & \text{for } k_0\delta_s << \dfrac{d}{\delta_s} << 1, \\ 1 - \dfrac{2d}{k_0\delta_s^2} \to 1, & \text{for } \dfrac{d}{\delta_s} << k_0\delta_s << 1. \end{cases}$$

The first of these functions describes the quasi-linear part of the log-log plots in the figure above; the smaller $k_0\delta_s$, the broader this region. The second of them may be rewritten in a different form by using Eq. (6.33) for the skin depth, and Eq. (7.8) for the free-space wave impedance:

$$\mathcal{T} = 1 - \frac{\sigma d\mu_0\omega}{k_0} \equiv 1 - \frac{Z_0}{R_\square}, \qquad \text{for } Z_0/R_\square << 1, \qquad (****)$$

(where $R_\square \equiv 1/\sigma d$),[33] and derived in a much simpler way. Indeed, if $R_\square >> Z_0$, the film almost does not disturb the incident wave (so in the crudest approximation, $\mathcal{T} \to 1$), and the electric field **E** applied to the film is just that of the incident wave – at the normal incidence we are exploring now, it is parallel to the film. According to the first form of Eq. (4.39), this field leads to the following Joule power loss per unit area of the film:

$$\frac{d\mathscr{P}}{dA} = \not{p}d = \sigma E^2 d = \frac{E^2}{R_\square}.$$

[33] As was already mentioned several times, starting from the model solution of Problem 4.12, the $R_\square$ so defined is called the sheet resistance (or the "resistance per square") of the thin film.

On the other hand, according to Eq. (7.9), the power of the incident wave (also per unit area of its front) is $S = E^2/Z_0$, so the lost fraction of the power is $(d\mathscr{P}/dA)/S = Z_0/R_\square$, immediately leading to Eq. (****) for the transmitted fraction of the power.

Note that the second of Eqs. (***) may be also rewritten in terms of the $Z_0/R_\square$ ratio (in this more general case, not limited to its small values):

$$\mathcal{T} = \frac{1}{(1 + Z_0/2R_\square)^2}, \qquad \text{for } d << \delta_s .$$

The derivation of this expression from scratch, utilizing the smallness of d in comparison with δ_s from the very beginning, will be the subject of the next problem.

Problem 7.15. One of the results of the previous problem's solution was the following expression for the coefficient of power transmission of a plane electromagnetic wave through a thin conducting film of thickness $d << \delta_s$, λ, at normal incidence:

$$\mathcal{T} = \frac{1}{(1 + Z_0/2R_\square)^2},$$

where $R_\square \equiv 1/\sigma d$ is the sheet resistance ("resistance per square") of the film. Derive this formula in a simpler way, utilizing the smallness of d from the very beginning. Also, calculate the power reflection coefficient $\mathcal{R}$, compare it with $\mathcal{T}$, and comment.

Solution: Just as it was done in Sec. 7.4 of the lecture notes, after placing the origin of the z-axis (directed along the wave's propagation and hence normal to the film) at the film's location, we may look for the complex amplitudes of the fields (all proportional to $e^{-i\omega t}$) in the following form,[34]

$$E_\omega = E_0 \times \begin{cases} \left(e^{ikz} + Re^{-ikz}\right), & \text{for } z \leq 0, \\ Te^{ikz}, & \text{for } 0 \leq z, \end{cases} \qquad H_\omega = \frac{E_0}{Z_0} \times \begin{cases} \left(e^{ikz} - Re^{-ikz}\right), & \text{for } z < 0, \\ Te^{ikz}, & \text{for } 0 < z. \end{cases}$$

These expressions satisfy the Maxwell equations outside the film, so in order to find the coefficients R and T, we should only formulate and use appropriate boundary conditions. The first of them is, as before, the continuity of the only (tangential) component of the electric field at $z = 0$, giving

$$1 + R = T . \qquad (*)$$

However, the magnetic field is not continuous because of the Ohmic current induced, by the wave, in the film. Repeating the Ampère-law arguments that give, in particular, Eq. (6.38) of the lecture notes, for our current case, we get

$$H_{z<0} - H_{z>0} = J,$$

where J is the linear density of the current. Due to condition $d << \delta_s$, the current is uniformly distributed across the film's thickness, and may be calculated just as

[34] Note that by writing these expressions we have already used the condition $d << \lambda$, by neglecting the distance between the two regions, i.e. film's thickness d, on the scale of wavelength $\lambda \equiv 2\pi/k$.

$$J = jd = \sigma E\big|_{z=0} d \equiv \frac{E\big|_{z=0}}{R_\square},$$

resulting in the following second boundary condition:

$$\frac{1-R}{Z_0} - \frac{T}{Z_0} = \frac{T}{R_\square}. \qquad (**)$$

Now readily solving the simple system of linear equations (*) and (**), we get

$$T = \frac{1}{1+Z_0/2R_\square}, \quad R = \frac{Z_0/2R_\square}{1+Z_0/2R_\square},$$

so for the power transmission coefficient, $\mathcal{T} \equiv |T|^2$, we indeed get the formula cited in the assignment, while the power reflection coefficient is:[35]

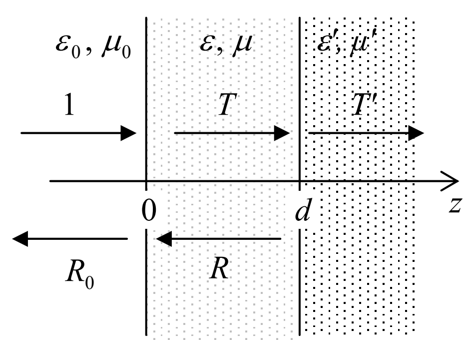
$$\mathcal{R} \equiv |R|^2 = \frac{(Z_0/2R_\square)^2}{(1+Z_0/2R_\square)^2}.$$

Most noticeably, a very resistive film with $R_\square >> Z_0$ affects the power transmission more significantly than its reflection:

$$\mathcal{T} \approx 1 - \frac{Z_0}{R_\square}, \qquad \mathcal{R} \approx \left(\frac{Z_0}{2R_\square}\right)^2, \qquad \text{for } \frac{Z_0}{R_\square} << 1.$$

One may wonder why the same ac current $\mathbf{J}(t)$ flowing in the film does not radiate equally to both sides; the reason is that it acts on the background of the incident wave's field $\mathbf{E}(t)$. Since both processes, and hence the incident wave and the wave induced by the current have the same frequency, and are in a firm phase relation (*coherent* with each other), their powers do not add up – see a more detailed discussion of this issue in Chapter 8.

<u>Problem 7.16</u>. A plane wave of frequency ω is normally incident, from free space, on a plane surface of a material with real electric permittivity ε' and magnetic permeability μ'. To minimize the wave's reflection from the surface, it may be covered with a layer, of thickness d, of another transparent material – see the figure on the right. Calculate the optimal values of ε, μ, and d.



Solution: This problem is an evident generalization of Problem 13, and may be approached very similarly, by considering five plane waves with the complex amplitudes indicated in the figure. The system of four linear equations relating these amplitudes, resulting from the macroscopic boundary conditions on the two interfaces, is also a straightforward generalization of that derived and solved in Problem 13:

[35] Note that for any finite Z_0/R_s ratio, the sum ($\mathcal{T} + \mathcal{R}$) is below 1, with the balance corresponding to the power absorbed in the film due to the Joule dissipation – see Eq. (4.39) of the lecture notes.

$$1+R_0=T+R, \qquad \frac{1-R_0}{Z_0}=\frac{T-R}{Z},$$

$$Te^{ikd}+Re^{-ikd}=T'e^{ik'd}, \qquad \frac{Te^{ikd}-Re^{-ikd}}{Z}=\frac{T'e^{ik'd}}{Z'},$$

where Z_0, Z, and Z' are the wave impedances (7.7) of the corresponding materials, and $k = \omega(\varepsilon\mu)^{1/2}$. However, now the equations have more parameters, so their solutions are more bulky, and may be harder to analyze. The simplest way toward such analysis is to solve the system for the reflected wave's amplitude R_0, eliminating the variables T and R first, and then the product $T'e^{ik'd}$ treated as one variable.[36] The result is

$$R_0=\frac{(Z'-Z_0)Z\cos kd-i\left(Z^2-Z_0Z'\right)\sin kd}{(Z'+Z_0)Z\cos kd-i\left(Z^2+Z_0Z'\right)\sin kd}. \qquad (*)$$

As the simplest sanity check, for $Z = Z_0$, i.e. a free-space "layer", this result reduces to

$$R_0\Big|_{Z=Z_0}=\frac{Z'-Z_0}{Z'+Z_0}e^{2ikd},$$

i.e. to the expression that differs from the first of Eqs. (7.68), with the proper notation replacements $Z_- \rightarrow Z_0$ and $Z_+ \rightarrow Z'$, only by the reflected wave's additional phase shift $2kd$ – just as we should have for its $2d$-long roundtrip inside the layer. As another check, for the constructive interference case, i.e. $kd = \pi m$ with $m = 0, 1, 2,\ldots$, i.e. for $\sin kd = 0$ (in particular for $d = 0$, i.e. for the surface with no coating at all), Eq. (*) reduces to Eq. (7.68) for *any* Z.

The result we are seeking may be obtained in the middle between those resonance values of d, i.e. at the destructive interference of the waves reflected from the interfaces. In this case, $kd = \pi(m + ½)$, so $\cos kd = 0$, and Eq. (*) yields

$$R_0\Big|_{kd=2\pi\left(m+½\right)}=\frac{Z^2-Z_0Z'}{Z^2+Z_0Z'}.$$

Now, by selecting

$$Z=Z_{\text{opt}}\equiv\left(Z_0Z'\right)^{1/2}, \qquad (**)$$

we get $R_0 = 0$, i.e. the system does not reflect – at least at the discrete frequencies providing the above resonant condition of kd. It is easy to use Eq. (*) to verify that the frequency bandwidth around such value, within which the reflection is relatively low,[37] is the largest for $m = 0$, i.e. at

$$kd=\frac{\pi}{2}, \qquad \text{i.e. } d=\frac{\pi}{2k}\equiv\frac{\lambda}{4},$$

where $\lambda = 2\pi/k$ is the wavelength inside the "coating" layer.

---

[36] Another possible approach to this problem is to use the *transfer matrix* method. For the solution of this simple problem, its introduction and usage would take more space than the given solution, but for more complex cases, such as multilayer systems, the method is invaluable – not only in electromagnetism but also in quantum mechanics and statistical physics – see, e.g., QM Sec. 2.5 and SM 4.5.

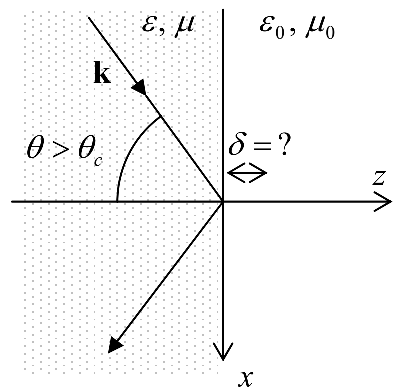
[37] Even larger bandwidths may be reached using multilayer coatings, which are becoming more and more popular as their fabrication technologies improve.

Nowadays, such quarter-wavelength *antireflective coating* is broadly used in optics – from billion-dollar telescopes all the way down to consumer glasses. Note that most optical materials are not magnetic, $\mu \approx \mu' \approx \mu_0$, so $Z \propto \varepsilon^{1/2} \propto n$, Eq. (**) is more frequently represented as[38]

$$n = n_{\text{opt}} = (n')^{1/2},$$

where n is the index of refraction – see Eq. (7.84). However, this form of the full result (**) should not obscure the important physical fact that it relates the wave *impedances* of the materials, and has nothing to do with their wave propagation *speed*, which is most frequently associated with the refractive index in many undergraduate (and even some graduate) textbooks.

Problem 7.17. A monochromatic plane wave is incident from inside a medium with $\varepsilon\mu > \varepsilon_0\mu_0$ on its planar surface, at an incidence angle θ larger than the critical angle $\theta_c = \sin^{-1}(\varepsilon_0\mu_0/\varepsilon\mu)^{1/2}$. Calculate the depth δ of the evanescent wave penetration into the free space, and analyze its dependence on θ. Does the result depend on the wave's polarization?



Solution: Selecting the x- and z-axes within the plane of incidence (see the figure on the right), we can claim that in our problem $\partial/\partial y = 0$, so the 3D wave equation that describes the electric and magnetic fields at $z > 0$ (see, e.g, Eq. (7.3) of the lecture notes) is reduced to its 2D form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) f = 0,$$

where f is any field's component, and $c = 1/(\varepsilon_0\mu_0)^{1/2}$ is the speed of light in free space. Let us look for the solution of this differential equation in a natural 2D-wave form

$$f = f_\omega \exp\{i(k_x x + k_z z - \omega t)\}. \qquad (*)$$

In order to satisfy the boundary conditions at $z = 0$ for an arbitrary t, the transverse component k_x of the evanescent wave vector has to match those of the incident and reflected waves:

$$k_x = k \sin\theta = \omega(\varepsilon\mu)^{1/2} \sin\theta, \qquad (**)$$

– cf. Eq. (7.80) of the lecture notes. On the other hand, in the free-space region, the wave vector's magnitude should be equal to $k_0 \equiv \omega/c$, so

$$k_x^2 + k_z^2 = k_0^2. \qquad (***)$$

At $\theta > \theta_c \equiv \sin^{-1}(\varepsilon_0\mu_0/\varepsilon\mu)^{1/2} = \sin^{-1}(k_0/k)$, we have $k_x > k_0$, so Eq. (***) may be only satisfied by a purely imaginary $k_z = i/\delta$. Plugging this expression into Eq. (*), we get

[38] For a typical optical glass, $n' \approx 1.5$, so the optimal coating material would have $n_{\text{opt}} = (n')^{1/2} \approx 1.22$. However, there are no convenient solid transparent materials with such refractive index, so the most popular practical choice is magnesium fluoride (Mg_2F) having $n \approx 1.38$ and hence giving $|R_0|^2 \approx 0.015$ instead of $|R_0|^2 \approx 0.04$ for uncoated glass.

$$f = f_\omega \exp\{i(k_x x - \omega t)\}\exp\{-z/\delta\},$$

so δ is exactly the penetration depth we need. As a result, Eqs. (**) and (***) yield

$$\delta = \frac{1}{\left(k_x^2 - k_0^2\right)^{1/2}} = \frac{1}{\omega\left(\varepsilon\mu\sin^2\theta - \varepsilon_0\mu_0\right)^{1/2}} = \frac{c\sin\theta_\mathrm{c}}{\omega\left(\sin^2\theta - \sin^2\theta_\mathrm{c}\right)^{1/2}} \equiv \frac{\lambda_0}{2\pi}\frac{\sin\theta_\mathrm{c}}{\left(\sin^2\theta - \sin^2\theta_\mathrm{c}\right)^{1/2}},$$

where $\lambda_0 = 2\pi/k_0$ is the free-space wavelength.

The result shows that δ is the smallest at the "grazing-angle" incidence, $\theta \to \pi/2$:

$$\delta \to \frac{\lambda_0}{2\pi}\frac{\sin\theta_\mathrm{c}}{\left(1 - \sin^2\theta_\mathrm{c}\right)^{1/2}} \equiv \frac{\lambda_0}{2\pi}\tan\theta_\mathrm{c},$$

and diverges as θ approaches θ_c. Finally, note that this result for δ is a result of "kinematic" analysis, and hence (just like the Snell and reflection laws) does not depend on the incident wave's polarization.

Problem 7.18. Calculate the critical angle θ_c for a wave of frequency ω, incident from free space upon a planar surface of a plasma with electron density n, and discuss the implications of the result for ultraviolet and X-ray optics.

Solution: The critical angle of a plane interface between two media may be calculated from Eq. (7.85) of the lecture notes,

$$\sin\theta_c = \left(\frac{\varepsilon_+\mu_+}{\varepsilon_-\mu_-}\right)^{1/2},$$

with the indices ± referring to the media, respectively, behind and in front of the interface – see Fig. 7.10. For our particular case, we should use Eqs. (7.36)-(7.37) for ε_+, while taking $\varepsilon_- = \varepsilon_0$ and $\mu_+ = \mu_- = \mu_0$. The result is

$$\sin\theta_c = \left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\right)^{1/2}, \qquad \text{with } \omega_\mathrm{p}^2 = \frac{ne^2}{\varepsilon_0 m_\mathrm{e}}.$$

For the analysis, it is helpful to rewrite this relation in the equivalent form

$$\cos\theta_\mathrm{c} = \frac{\omega_\mathrm{p}}{\omega}.$$

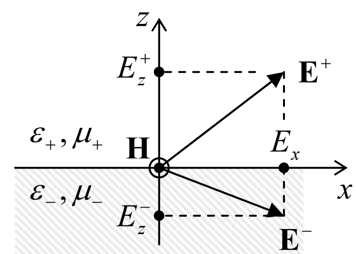
It shows that the real angle θ_c exists only for the frequencies ω above the plasma frequency ω_p. (At $\omega < \omega_\mathrm{p}$ the wave is totally reflected from the plasma at any angle of incidence because it cannot propagate in it at all – see Fig. 7.6 and its discussion in Sec. 7.2 of the lecture notes.) As the wave's frequency grows, $\cos\theta_\mathrm{c}$ decreases, i.e. the angle θ_c grows. Eventually, at $\omega >> \omega_\mathrm{c}$ the angle becomes very close to $\pi/2$. Expanding $\cos\theta_\mathrm{c}$ near this point into the Taylor series: $\cos\theta_\mathrm{c} = \pi/2 - \theta_\mathrm{c} + \ldots$, and keeping only the two leading terms of the expansion, we get

$$\theta_c \approx \frac{\pi}{2} - \frac{\omega_\mathrm{p}}{\omega}, \qquad \text{for } \omega >> \omega_\mathrm{p}.$$

This result means that the total reflection of waves from the plasma may be obtained even at very high frequencies, at the so-called *grazing angles* on incidence. This effect enables the fabrication of reflective components of ultraviolet and X-ray optical systems (most importantly, of EUV lithography tools) from metals, whose ω_p is barely above the visible light range – see the estimate in Sec. 7.2.

Problem 7.19. Analyze the possibility of propagation of surface electromagnetic waves along a planar boundary between plasma and free space. In particular, calculate and analyze the dispersion relation of the waves.

Hint: Assume that the magnetic field of the wave is parallel to the boundary and perpendicular to the wave's propagation direction. (After solving the problem, justify this choice.)



Solution: With the suggested assumption, the problem may be readily solved for a planar interface between two linear, isotropic media with even arbitrary frequency dispersion of both $\varepsilon_\pm(\omega)$ and $\mu_\pm(\omega)$.[39] Indeed, with the coordinate frame selection shown in the figure on the right, where the x-axis is directed along the wave propagation, the fields of a monochromatic surface wave may be represented in the form

$$\begin{aligned}\mathbf{E}^\pm(\mathbf{r},t) &= \mathrm{Re}\left[\left(E_x^\pm \mathbf{n}_x + E_y^\pm \mathbf{n}_y + E_z^\pm \mathbf{n}_z\right)\exp\{i(kx-\omega t)\mp \kappa_\pm z\}\right],\\ \mathbf{H}^\pm(\mathbf{r},t) &= \mathrm{Re}\left[H^\pm \mathbf{n}_y \exp\{i(kx-\omega t)\mp \kappa_\pm z\}\right].\end{aligned} \qquad (*)$$

Here the x-component k of the wave vector is taken the same for both media to satisfy the boundary conditions (see below) simultaneously at all points of the interface, while its z-components are assumed imaginary to ensure that the wave fields decay into the media bulk.[40] Plugging Eqs. (*) into the source-free (homogeneous) macroscopic Maxwell equations (7.2), we see that they indeed represent a valid solution, provided that the following relations are satisfied:

$$\kappa_\pm^2 = k^2 - \omega^2 \varepsilon_\pm(\omega)\mu_\pm(\omega), \qquad (**)$$

$$E_x^\pm = \mp \frac{\kappa_\pm}{i\omega\varepsilon_\pm(\omega)} H^\pm, \qquad E_y^\pm = 0, \qquad E_z^\pm = -\frac{k}{\omega\varepsilon_\pm(\omega)} H^\pm. \qquad (***)$$

These relations show, in particular, that if both media are lossless (i.e. if $\varepsilon_\pm$ and $\mu_\pm$ are both real), the exponents $\kappa_\pm$ are real as well, and the phases of oscillations of the Cartesian components E_x and E_z are shifted by $\pm\pi/2$. The last fact means that the vector $\mathbf{E} = \{E_x, 0, E_z\}$ in each medium rotates, its end following an ellipse with the semi-axes proportional to the coefficients κ and k, respectively.

In order to find these coefficients, Eq. (**) alone is insufficient. An additional equation may be obtained from applying the boundary conditions (3.37), (3.56), and (5.117):

[39] As was discussed in Sec. 7.2 of the lecture notes, any non-zero Ohmic conductivity $\sigma(\omega)$ of a medium may be incorporated into its complex electric permittivity $\varepsilon(\omega)$ – see Eq. (7.46).
[40] This means that in contrast to the situation at the total internal reflection (see its discussion in Sec. 7.4), the interface wave is evanescent in both media rather than in just one of them.

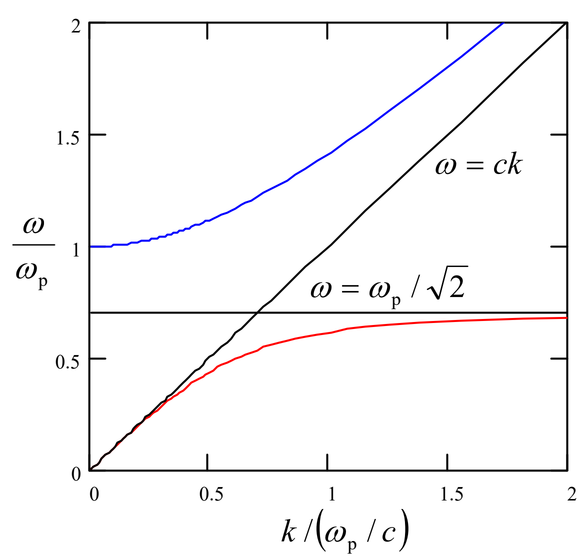
$$E_x^+ = E_x^- \equiv E_x, \qquad \varepsilon_+(\omega)E_z^+ = \varepsilon_-(\omega)E_z^-, \qquad H^+ = H^-.$$

These homogeneous linear equations[41] are compatible with Eqs. (***) only if

$$\frac{\kappa_+}{\kappa_-} = -\frac{\varepsilon_+(\omega)}{\varepsilon_-(\omega)}. \qquad (****)$$

This is the key relation for the interface/surface waves. It shows that such waves cannot propagate on the interface between two "usual" wave media, with real and positive dielectric constants. However, as was discussed in detail in Sec. 7.2, dilute gases feature frequency ranges with negative electric permittivity immediately above resonance (in quantum mechanics, quantum-transition) frequencies – see, e.g., Fig. 7.5.[42] Moreover, according to Eq. (7.36), in a collision-free plasma, the function $\varepsilon(\omega)$ is real and negative at all frequencies below the plasma frequency ω_p given by Eq. (7.37). For the waves on the surface of such a plasma with $\mu_- = \mu_0$, Eqs. (**), (****), and (7.36) yield

$$k = \frac{\omega}{c}\left(\frac{\omega_p^2 - \omega^2}{\omega_p^2 - 2\omega^2}\right)^{1/2}, \quad \text{for } \omega < \omega_p.$$



This dispersion relation is shown with the red line in the figure on the right, while the blue line shows the relation (also shown in Fig. 7.6 in the lecture notes) for the usual ("3D") waves in the same plasma.

At low frequencies ($\omega \ll \omega_p$), the surface waves[43] have almost the same properties as the usual electromagnetic waves in free space: $k \approx \omega/c$, $v_{gr} \approx v_{ph} \approx c$. (This is natural, because in this case, according to Eq. (****) with $\varepsilon_+(\omega) = \varepsilon_0$ and $\varepsilon_-(\omega) \approx -\varepsilon_0\omega_p^2/\omega^2 \to -\infty$, $\kappa_+ \ll \kappa_-$, i.e. the wave propagates mostly is free space, only weakly "guided" by the plasma surface.) As the frequency grows, the wave vector diverges at $\omega \to \omega_p/\sqrt{2}$, so the surface plasmons cannot propagate at higher frequencies. In this limit, $\varepsilon_-(\omega) \approx -\varepsilon_0 = -\varepsilon_+(\omega)$, so $\kappa_+ \approx \kappa_-$, i.e. the wave is equally shared by the plasma and free space.

Note that the conceptual importance of such plasmons extends beyond plasma physics and engineering. For example, let us return to the simplest (and very common) situation shown in Fig. 2.26 of the lecture notes, and ask ourselves: if a charge q has been brought close to a metallic surface from afar very fast, how long would it take for the familiar stationary picture of the charge image to become quantitatively valid? There is of course the relativistic limitation $\Delta t \gg d/c$, but the limitation imposed by surface plasmon propagation, $\Delta t \gg 1/\omega_p$, where is the plasma frequency of the conduction electrons

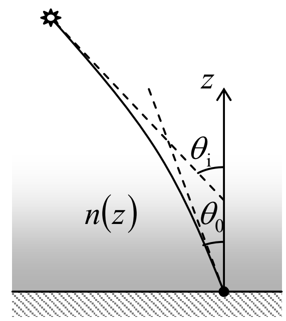
[41] Actually, one of these relations is redundant.

[42] Note that if the interface waves can propagate, i.e. if $\varepsilon_+(\omega)$ and $\varepsilon_-(\omega)$ have different signs, the signs of z-components of the electric field in the two bordering media are opposite, as shown in the figure above. As a result, the electric field vectors in the media rotate in opposite directions – see, e.g., the nice animation available online at http://en.wikipedia.org/wiki/Surface_plasmon_polariton#Dispersion_relation.

[43] The quantized excitations of these waves are called either *surface plasmons* or (due to their similarity to the polaritons discussed in Problem 7.8) *surface plasmon polaritons*.

in the metal, may be much more significant. As was mentioned in Sec. 7.2 of the lecture notes, for good metals, $\omega_p \sim 10^{16}\ \text{s}^{-1}$, so the time needed for the image charge formation (read: for the surface charges to approach their stationary distribution) is of the order of a femtosecond.

Finally, let us revisit the initial assumption $H_z = 0$. Due to the symmetry of the homogeneous Maxwell equations (6.100) to the simultaneous swap $\mathbf{E} \leftrightarrow \mathbf{H}$, $\mathbf{D} \leftrightarrow -\mathbf{B}$, it is evident that similar surface waves with $H_z \neq 0$ but $E_z = 0$, are also possible, but only if $\mu(\omega) < 0$ in one of the media, which is not common. Note also that conceptually similar (though quantitatively different) mechanical surface waves may propagate at sharp interfaces between different elastic media.[44]



Problem 7.20. Light from a very distant source arrives to an observer through a plane layer of nonuniform medium with a certain 1D gradient of its refraction index $n(z)$, at angle θ_0 – see the figure on the right. What is the genuine direction θ_i to the source, if $n(z) \to 1$ at $z \to \infty$? (This problem is evidently important for high-precision astronomical measurements at the Earth's surface.)

Solution: If the layer thickness' scale d is much larger than light's wavelength λ, we may represent it locally (on a distance Δz such that $\lambda << \Delta z << d$) as a plane wave – see, e.g., Eq. (7.79) of the lecture notes. Hence, if n changes only in the z-direction, we may repeat the arguments at the beginning of Sec. 7.4 to conclude that the wave vector's component in the direction perpendicular to this axis, equal to $k\sin\theta \propto n\sin\theta$, cannot change at the wave's refraction between any two elementary sub-layers. Hence, the product $n(z)\sin\theta(z)$ has to stay constant throughout the whole layer. Applying this conclusion to the initial direction θ_i (at $z >> d$, where $n(z) \to 1$) and the observer's location ($z = 0$, where $n(z) = n(0) \equiv n_0$), we finally get a very simple answer:

$$\sin\theta_i = n_0 \sin\theta_0, \qquad \text{i.e. } \theta_i = \sin^{-1}(n_0 \sin\theta_0), \tag{*}$$

i.e. just the Snell law, regardless of the details of the function $n(z)$.

As Table 3.1 of the lecture note shows, for the air on the Earth's surface, $n_0 \approx \kappa^{1/2} \approx 1.0003$, so the angular corrections $R \equiv \theta_i - \theta_0$ to the actual positions of stars and planets (called *atmospheric refraction*) may be rather substantial – about one angular minute at $\theta_0 \sim \pi/4$. Still, it is typically much smaller than θ_0, so expanding $\sin\theta_i$ in the Taylor series in small R and keeping only the two leading terms, we may approximate Eq. (*) as

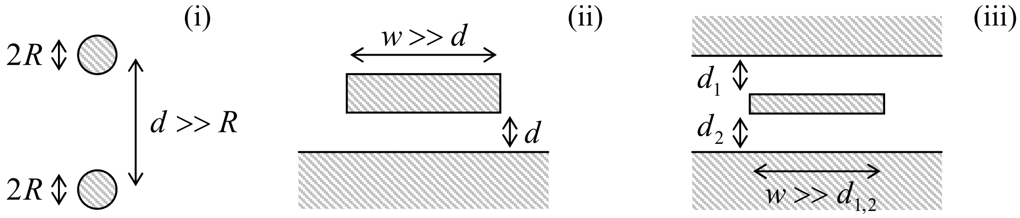
$$\sin\theta_0 + R\cos\theta_0 \approx n_0 \sin\theta_0, \qquad \text{giving } R \approx (n_0 - 1)\tan\theta_0 \tag{**}$$

– the form suggested by W. Snellius himself.

Note, however, that not only Eq. (**) but even the more general Eq. (*) should be applied to practical astronomical observations at angles $\theta_0 \to \pi/2$ with care because here the Earth's curvature, ignored in the derivation of that result, becomes important. With the account of these effects, the atmospheric refraction at the horizon is as large as ~35 angular minutes. i.e. is of the same order as the visible diameters of the Sun and the Moon.

[44] See, e.g., CM Sec. 7.7, and references therein.

Problem 7.21. Calculate the TEM impedance Z_W of uniform transmission lines with well-conducting electrodes and the cross-sections shown in the figure below:



(i) two parallel round wires separated by distance $d >> R$,
(ii) a *microstrip line* of width $w >> d$,
(iii) a *stripline* with $w >> d_1 \sim d_2$,

in all cases using the coarse-grain boundary conditions on conductor surfaces. Assume that the conductors are embedded into a linear dielectric with constant ε and μ.

Solutions:

(i) From the similar limit of the solution of Problem 2.13, with $R_1 = R_2 = R$, generalized to the dielectric filling by replacing ε_0 with ε, the mutual capacitance per unit length is

$$C_0 = \frac{\pi\varepsilon}{\ln(d/R)}.$$

The inductance L_0 per unit length may be found either from magnetostatics or (even easier) from the general relation (7.114):[45]

$$L_0 = \frac{\varepsilon\mu}{C_0} = \frac{\mu}{\pi}\ln\frac{d}{R}.$$

Combining these expressions, we get

$$Z_W \equiv \left(\frac{L_0}{C_0}\right)^{1/2} = \frac{Z}{\pi}\ln\frac{d}{R}, \qquad \text{where } Z \equiv \left(\frac{\mu}{\varepsilon}\right)^{1/2}.$$

This formula is close in structure to that for the coaxial cable – see Eq. (7.120) of the lecture notes, with an extra factor of 2 due to two (rather than one) thin conductors, and with the distance d between the wires playing the role similar to the diameter of the outer conductor of the cable.

As was mentioned in Sec. 7.6 of the lecture notes, such wire pairs, lighter and cheaper than the coaxial cable, have been the basis of all early electric communication technologies (such as telegraphy), and are still used, in the form of twisted pairs, in landline telephony at frequencies up to a few hundred kHz, and in relatively short Ethernet cables at frequencies up to a few hundred MHz.

(ii) Neglecting the fringe field effects (whose relative contribution scales as $d/w << 1$), we may use Eq. (3.55) for the line's capacitance per unit length:

$$C_0 = \frac{\varepsilon w}{d}.$$

[45] Note again that this relation is valid only at the coarse-grain boundary condition (6.38), acceptable when the magnetic field penetration into the conducting electrodes is negligible (in our current case, when $\delta_s << R$).

The inductance per unit length may be found, again, either from magnetostatics[46] or from the same Eq. (7.114):

$$L_0 = \frac{\varepsilon\mu}{C_0} = \frac{\mu d}{w}.$$

Combining these expressions, we get

$$Z_W = Z\frac{d}{w} << Z.$$

This expression shows that the microstrip lines may have a very low impedance. Such low-impedance lines are broadly used in electronics, e.g., on printed circuit boards. Due to the low fringe fields, they may ensure low mutual interference ("crosstalk") even if spaced rather closely – at distances of the order of $w >> d$.

(iii) The stripline geometry, which is broadly used in multi-layer printed circuit boards and for on-chip interconnects in microelectronics, provides even lower crosstalk between adjacent transmission lines. Its parameters may be readily found by calculating C_0 as a parallel connection of two plane capacitors, and L_0 from that result and Eq. (7.114):

$$C_0 = \varepsilon w\left(\frac{1}{d_1} + \frac{1}{d_2}\right), \qquad L_0 = \frac{\varepsilon\mu}{C_0} = \frac{\mu}{w}\left(\frac{1}{d_1} + \frac{1}{d_2}\right)^{-1}, \qquad Z_W \equiv \left(\frac{L_0}{C_0}\right)^{1/2} = Z\frac{1}{w}\left(\frac{1}{d_1} + \frac{1}{d_2}\right)^{-1}.$$

If one of the gaps is much narrower than the other one (say, $d_1 << d_2$), the electromagnetic field is concentrated in it, and the result is reduced to that for the microstrip line.

Problem 7.22. Modify the solution of Task (ii) of the previous problem for a superconductor microstrip line, taking into account the magnetic field's penetration into both the strip and the ground plane.

Solution: In this case, the expression for L_0 has to be replaced with the solution of Problem 6.20:

$$L_0 = \frac{\mu}{w}\left(d + 2\delta_L\right),$$

while the capacitance C_0 per unit length remains the same as in the solution of the previous problem, because the electric field penetrates into conductors (including superconductors) by a much smaller, typically negligible screening length – see Sec. 2.1 of the lecture notes. As a result, the TEM wave propagation speed in such a line is lower than that (v) of plane waves:[47]

[46] As was discussed in the solution of Problem 6.20 in the context of superconducting electrodes, if the width d of the gap between the conductors and the depth of the field penetration into the line electrodes (say, the skin-effect depth δ_s) are much smaller than w, the field in the gap is uniform, $B = \mu I/w$. If, in addition, the field penetration depth is also much less than d, then the magnetic field is localized in the gap, and the magnetic flux (per unit length) is $\Phi/l = Bd = \mu Id/w$, immediately giving the above formula for L_0.

[47] This difference is especially large ($v'/v \sim 0.1$) in the so-called *long* (or "distributed") *Josephson junctions* with $d \sim 1$ nm and $\delta_L \sim 100$ nm, enabling very interesting (pseudo-relativistic) effects of interactions between the slow TEM waves and magnetic-field-induced "waves" of supercurrent – see, e.g., Section 6.4 of the monograph by M. Tinkham (see the last section of *References*).

$$v' = \frac{1}{(L_0 C_0)^{1/2}} = \frac{1}{(\varepsilon\mu)^{1/2}}\left(\frac{d}{d+2\delta_\mathrm{L}}\right)^{1/2} = \frac{v}{\left(1+2\delta_\mathrm{L}/d\right)^{1/2}} < v.$$

Note that in this case, the "general" relation (7.114) is not satisfied because its derivation is based on the assumption of similar boundary conditions for the magnetic and electric fields on the conductor surfaces.

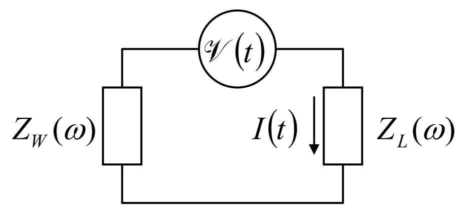
Problem 7.23.* What lumped ac circuit would be equivalent to the TEM-line system shown in Fig. 7.19 of the lecture notes, with an incident wave's power $\mathscr{P}_\mathrm{i}$? Assume that the wave reflected from the lumped load does not return to it.

Solution: It is intuitively clear that the circuit has to include:

(i) the lumped load with the impedance $Z_L(\omega)$,

(ii) a lumped circuit element representing passive properties of the transmission line, with the ac impedance equal to the line's impedance $Z_W(\omega)$, and

(iii) some ac generator representing the incident wave.



Following these arguments, let us try to use the circuit shown in the figure on the right, where $\mathscr{V}(t)$ is an e.m.f. representing the incident wave. An elementary calculation of the complex amplitude of the current $I(t)$ in the load yields $I_\omega = \mathscr{V}_\omega/[Z_L(\omega) + Z_W(\omega)]$, so the voltage drop across the load is $V_\omega = I_\omega Z_L(\omega) = \mathscr{V}_\omega Z_L(\omega)/[Z_L(\omega) + Z_W(\omega)]$. Now the calculation similar to that carried out at the derivation of Eq. (7.42) yields the following expression for the average power absorbed in the load:

$$\mathscr{P}_L = \frac{1}{2} V_\omega I_\omega^* = \frac{1}{2}\frac{|\mathscr{V}_\omega|^2}{|Z_L(\omega) + Z_W(\omega)|^2}\,\mathrm{Re}\,Z_L(\omega). \qquad (*)$$

On the other hand, we can calculate the same power in the real system (Fig. 7.19) by subtracting, from the incident power $\mathscr{P}_\mathrm{i}$, the reflected wave's power $\mathscr{P}_\mathrm{r} = \mathscr{P}_\mathrm{i}|R|^2$, where R is given by Eq. (7.118):

$$\mathscr{P}_L = \mathscr{P}_\mathrm{i}\left(1-|R|^2\right) = \mathscr{P}_\mathrm{i}\left(1 - \frac{|Z_L(\omega) - Z_W(\omega)|^2}{|Z_L(\omega) + Z_W(\omega)|^2}\right) \equiv \mathscr{P}_\mathrm{i}\frac{|Z_L(\omega) + Z_W(\omega)|^2 - |Z_L(\omega) - Z_W(\omega)|^2}{|Z_L(\omega) + Z_W(\omega)|^2}.$$

In the most important case of a loss-free line, its impedance Z_W is real, and this expression reduces to

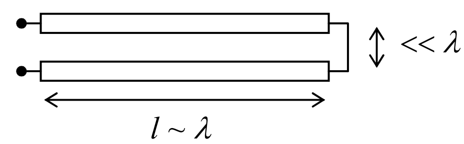
$$\mathscr{P}_L = \mathscr{P}_i \frac{4Z_W(\omega)}{|Z_L(\omega) + Z_W(\omega)|^2}\,\mathrm{Re}\,Z_L(\omega). \qquad (**)$$

Comparing Eqs. (*) and (**), we see that they give similar results (and hence the lumped circuit is equivalent to the wave system shown in Fig. 7.19), if the effective ac e.m.f. has the amplitude

$$|\mathscr{V}_\omega| = \left[8\mathscr{P}_i Z_W(\omega)\right]^{1/2}.$$

(Note the somewhat counter-intuitive numerical coefficient.)

Such representation of a transmission line, carrying an incident wave, by its impedance and an e.m.f. is valid even if the lumped load is nonlinear and/or time-dependent, and is broadly used for the analysis of not only linear but also nonlinear and parametric microwave devices. Note, however, that this result is only valid under the assumption mentioned in the assignment, that the wave reflected from the load does not return to it. This condition (generally beneficent for applications, e.g., for the amplifier stability) may be satisfied using such non-reciprocal devices as circulators – see the discussion in the model solution of Problem 12.



Problem 7.24. Find the lumped ac circuit equivalent to a loss-free TEM transmission line of length $l \sim \lambda$, with a small cross-section area $A << \lambda^2$, as "seen" (measured) from one end, if the line's conductors are galvanically connected ("shortened") at the other end – see the figure on the right. Discuss the result's dependence on the signal frequency.

Solution: Neglecting losses in the transmission line, we can describe any "global" variable, e.g., the voltage $V(z, t)$ or the current $I(z, t)$ in it, by a sum of two sinusoidal waves: one traveling to the shortened end, and another one reflected from it. Taking the shortened end's position for $z = 0$, for the complex amplitudes of these variables we can write

$$V_\omega(z \leq 0) = V_0\left(e^{ikz} + Re^{-ikz}\right), \qquad I_\omega(z \leq 0) = \frac{V_0}{Z_W}\left(e^{ikz} - Re^{-ikz}\right).$$

– cf. Eqs. (7.64)-(7.65) for plane waves. Now requiring the ratio V_ω/I_ω to vanish at the shorted end (at $z = 0$), we get $R = -1$, so at the other end of the line (at $z = -l$)

$$Z(\omega) \equiv \frac{V_\omega}{I_\omega}\Big|_{z=-l} = Z_W \frac{e^{-ikl} - e^{ikl}}{e^{-ikl} + e^{ikl}} \equiv -iZ_W \tan kl\,. \qquad (*)$$

This result shows that in contrast to the TEM line impedance $Z_W = (L_0/C_0)^{1/2}$, which does not depend on frequency, the effective ac impedance $Z(\omega)$ of the shorted line segment is a strong function of ω. In particular, at low frequencies $\omega << v/l$ (i.e. at $kl << 1$) this expression reduces to

$$Z(\omega) \approx -iZ_W kl = -i\left(\frac{L_0}{C_0}\right)^{1/2} \omega\left(L_0 C_0\right)^{1/2} l \equiv -i\omega L_0 l = -i\omega L\ ,$$

where $L = L_0 l$ is the segment's inductance. This is a well-known expression for the complex impedance of a lumped inductance. (The negative sign is due to the fact that we are using the "physics" convention $\exp\{-i\omega t\}$ for the time dependence of all variables, vs the $\exp\{+i\omega t\}$ common in electrical engineering.)

However, as kl is increased, the impedance given by Eq. (*) grows faster than that of a lumped inductance and diverges at $kl = \pi/2$, i.e. at wavelength $l = \pi/2k \equiv \lambda/4$. This divergence means that in the absence of power losses, a nonvanishing voltage V_ω may be sustained by vanishing current. Such resonant behavior is similar to that of an ac circuit consisting of a lumped inductance and a lumped capacitance connected in parallel; it repeats at all frequencies where

$$k_n l = \frac{\pi}{2} + n\pi\ , \qquad (**)$$

i.e. $(n + 1/2)\lambda_n/2 = l$. These so-called *parallel resonances* are interleaved, at wave number values

$$k'_n l = n\pi\,, \tag{***}$$

with resonances of a different kind, where $Z(\omega) = V_\omega/I_\omega$ vanishes, and hence a finite ac current may be sustained with little, if any, ac voltage. These are so-called *series resonances*, similar to those in lumped in-series *LC* circuits.

Such *distributed resonators* are broadly used in electrical engineering at high frequencies where the implementation of genuinely lumped circuits may lead to unacceptably high energy losses.

Problem 7.25. Represent the fundamental H_{10} wave in a rectangular waveguide (see Fig. 7.22 of the lecture notes) as a sum of two plane waves, and discuss the physics behind such a representation.

Solution: Let us use the last of Eqs. (7.138) of the lecture notes to write the following expression for the instant electric field of this wave:

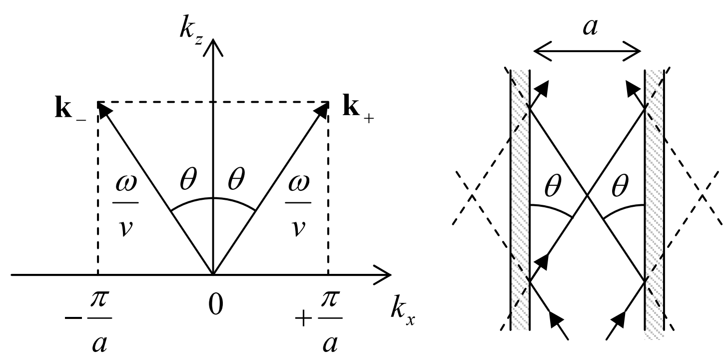
$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[i\frac{ka}{\pi} ZH_l \sin\frac{\pi x}{a} \exp\{i(k_z z - \omega t)\}\right]\mathbf{n}_y$$

$$\equiv \mathrm{Re}\left[E_\omega \exp\left\{ i\left(\frac{\pi}{a}x + k_z z - \omega t\right)\right\} - E_\omega \exp\left\{ i\left(-\frac{\pi}{a}x + k_z z - \omega t\right)\right\}\right]\mathbf{n}_y,$$

with $E_\omega \equiv (ka/2\pi)ZH_l$. The last displayed expression shows that inside the waveguide, the mode is nothing more than the sum of two plane waves, linearly polarized along the y-axis, with equal amplitudes E_ω, and wave vectors $\mathbf{k}_\pm$ with the Cartesian components $k_x = \pm\pi/a$, $k_y = 0$, and the k_z defined by Eqs. (7.122) and (7.132) (the latter one, with $n = 1$ and $m = 0$), so

$$k_x^2 + k_y^2 + k_z^2 = k_\pm^2 = k^2 = \frac{\omega^2}{v^2},$$

– see the left panel of the figure on the right.



This figure shows that the plane wave propagation angle θ satisfies the following relation:

$$\sin\theta = \frac{|k_x|}{k} = \frac{\pi}{ka} = \frac{\lambda_0/2}{a},$$

where $\lambda_0 = 2\pi/k$ is the TEM wavelength. This angle asymptotically approaches zero at very high frequencies ($\omega << \omega_c$) but grows as the frequency is reduced toward the cutoff value ω_c, so at $\omega \to \omega_c$, $\theta \to \pi/2$, i.e. $k_z \to 0$. Note that the cutoff frequency corresponds to the half-wavelength of the plane wave being exactly equal to the wide side of the waveguide, so longer waves cannot propagate in it.

The fact that we are dealing with two usual plane waves may be confirmed by the electromagnetic field amplitude ratio calculation. Indeed, the similar decomposition of the H_{10} mode's magnetic field, given by Eqs. (7.131) and (7.137), also yields two plane waves:

$$\mathbf{H}(\mathbf{r},t) = \frac{1}{2}H_l\,\mathrm{Re}\left[\left(-\frac{k_z a}{\pi}\mathbf{n}_x + \mathbf{n}_z\right)\exp\left\{ i\left(\frac{\pi}{a}x + k_z z - \omega t\right)\right\} + \left(\frac{k_z a}{\pi}\mathbf{n}_x + \mathbf{n}_z\right)\exp\left\{ i\left(-\frac{\pi}{a}x + k_z z - \omega t\right)\right\}\right],$$

each with the amplitude

$$\left|H_\omega\right| = \frac{H_l}{2}\left[\left(\frac{k_z a}{\pi}\right)^2 + 1\right]^{1/2} = H_l \frac{ka}{2\pi} = \frac{1}{Z}\left|E_\omega\right| \ ,$$

whose ratio to E_ω is given by the plane wave's impedance $Z = (\mu/\varepsilon)^{1/2}$, regardless of the ratio ω/ω_c.

Now, let us discuss why this simple decomposition of waves in the waveguide into plane waves is possible. If these two plane waves propagated in an unlimited isotropic medium, we could notice, first of all, that an insertion of conducting walls normal to the y-axis at *any* locations (say, at $y = 0$ and $y = b$, see Fig. 7.22) would not perturb the field between them. Indeed, as we know from Chapter 2, the electric field normal to the conductor's surface is screened from penetration into its bulk by surface charges with density $\sigma(x,y) = \varepsilon E_z(x,y)$, without any perturbation of the applied field. Similarly, the magnetic fields of these two plane waves are tangential to the wall surfaces and are shielded from penetration into their bulk by the skin-effect currents of the linear density (6.38), without any field's perturbation outside of the walls.

The situation with the other two walls (at $x = 0$ and $x = a$, see Fig. 7.22) is more involved since at an arbitrary location of such a wall, the sum of two plane waves would have some tangential component of the electric field, and some normal component of the magnetic field, which cannot interact with the wall without incident wave's perturbation. However, as the above formulas for **E** and **H** show, at the specific positions $x_n = n\pi/k_x$, with any integer n, the component sums vanish, and the wall insertion leaves the field between the walls intact.

These results may be also interpreted by saying that the fundamental H_{10} wave is formed by a plane wave repeatedly reflected from the side walls of the waveguide – see the right panel of the figure above.

<u>Problem 7.26</u>.* For the coaxial cable (see, e.g., Fig. 7.20 of the lecture notes), find the lowest non-TEM mode and calculate its cutoff frequency.

Solution: The analysis may repeat that of the hollow circular waveguide (Fig. 7.23a) in Sec. 7.7 of the lecture notes, up to the derivation of the Bessel equation (7.140) for the radial factor $\mathcal{R}(\rho)$ of the longitudinal field f – depending on the mode, either E_z or H_z. However, in the coaxial cable, the axial points with $\rho = 0$ are inaccessible for the field, and hence, instead of the simple solution given by Eq. (7.141), we have to look for its radial part in the form of a linear superposition of the Bessel functions of the first and the second kind:

$$f_{nm} = \left[c_1 J_n(k_{nm}\rho) + c_2 Y_n(k_{nm}\rho)\right]\cos n(\varphi - \varphi_0), \qquad \text{with } n = 0, 1, 2, \ldots,$$

where the eigenvalues k_{nm} of the transverse wave number k_t have to be chosen to satisfy the boundary conditions at both $\rho = a$ and $\rho = b$.

For the E-modes, the boundary condition Eq. (7.124) yields the following system of two equations for the constants $c_{1,2}$:

$$c_1 J_n(k_{nm}a) + c_2 Y_n(k_{nm}a) = 0,$$
$$c_1 J_n(k_{nm}b) + c_2 Y_n(k_{nm}b) = 0.$$

These two linear, homogeneous equations are compatible if the denominator of the system equals zero. This requirement yields

$$J_n(k_{nm}a)Y_n(k_{nm}b) = J_n(k_{nm}b)Y_n(k_{nm}a),$$

where the integer index m (taking the values 1, 2, 3,…) numbers the roots of this characteristic equation, for each fixed angular index n (taking the values 0, 1, 2,…). Introducing the dimensionless variable $\xi_{nm} \equiv k_{nm}a$, we may rewrite this characteristic equation as

$$J_n(\xi_{nm})Y_n\left(\xi_{nm}\frac{b}{a}\right) = J_n\left(\xi_{nm}\frac{b}{a}\right)Y_n(\xi_{nm}). \qquad (*)$$

b/a	$n=0$, $m=1$	$n=1$, $m=1$	$n=0$, $m=2$	$n=1$, $m=2$
4	3.073	3.336	6.243	6.403
2	3.123	3.197	6.273	6.312
$\to 1$	$\to \pi$	$\to \pi$	$\to 2\pi$	$\to 2\pi$

The table on the right shows the results for the product $(b/a - 1)\xi_{nm} \equiv k_{nm}(b - a)$, obtained by numerical solution of this equation for a few values of the ratio b/a, and a few lowest numbers n and m.[48] It shows that the product $k_{nm}(b - a)$ changes very slowly with that ratio, staying close to the asymptotic value $m\pi$ (reached at $b/a \to 1$), for all reasonable values of b/a. This result may be readily understood physically: at the cutoff frequency $\omega_c = k_{nm}v$ (when $k_t = k = 2\pi/\lambda$), nearly m TEM half-wavelengths fit the distance between the external and internal conductors – almost independently of n, provided that the latter number is not too high. (Still, as could be expected, the axially-symmetric distribution with $n = 0$ gives the lowest k_t, and hence the lowest cutoff frequency.)

For the H-modes, we need to use another boundary condition, Eq. (7.126), that gives, instead of Eq. (*), a different characteristic equation:

$$J_n'(\xi_{nm})Y_n'\left(\xi_{nm}\frac{b}{a}\right) = J_n'\left(\xi_{nm}\frac{b}{a}\right)Y_n'(\xi_{nm}), \qquad (**)$$

b/a	$n=1$, $m=1$	$n=1$, $m=2$
4	2.055	3.760
2	2.031	4.023
$\to 1$	$\to 2$	$\to 4$

where the prime sign means the derivative of the Bessel function over its whole argument. The table on the right gives results for a different dimensionless combination, $(1 + b/a)\xi_{nm} \equiv k_{nm}(a + b)$, which is more relevant (virtually parameter-independent) in this case, for the two lowest modes with $n = 1$.[49] It shows that for any realistic b/a ratio, this combination is close to $2m$. Physically, this means that at the cutoff frequency, m TEM wavelengths $\lambda = 2\pi/k_t$ fit the average circumference $p = \pi(a + b)$ of the cable's cross-section.

Now we can compare the approximate values of the lowest k_{nm} for the E- and H-modes:

$$\left.\frac{\pi}{b-a}\right|_{\text{for } E_{01}} \leftrightarrow \left.\frac{2}{b+a}\right|_{\text{for } H_{11}},$$

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[48] H. Dwight, *J. Math. Phys*. **27**, 84 (1949).

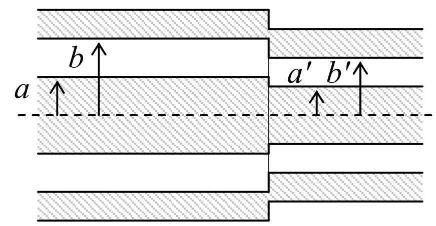
[49] For $n = 0$, we can use the recurrence relations (2.143), which in particular yield $J_0' = -J_1$, $Y_0' = -Y_1$, to rewrite Eq. (**) in a form similar to Eq. (*) with $n = 1$:

$$J_1(\xi_{0m})Y_1\left(\xi_{0m}\frac{b}{a}\right) = J_1\left(\xi_{0m}\frac{b}{a}\right)Y_1(\xi_{0m}).$$

This comparison shows that values k_{0m} of the H-modes are equal to those of k_{1m} of the E-modes (for the same m), and hence are higher than the values k_{0m} of the latter modes, so an H-mode with $n = 0$ cannot be the lowest one.

so (just as in the circular waveguide analyzed in Sec. 7.7 of the lecture notes), the lowest non-TEM mode is H_{11}, with the cutoff frequency $\omega_c \approx 2v/(a + b)$, i.e. the TEM wavelength $\lambda_{max} \approx \pi(a + b)$. This result (which, at $a/b \to 0$, is close to Eq. (7.145) for the single-hole circular waveguide) is important because it imposes a practical limit, $\lambda > \pi(a + b)$ for using coaxial cables as TEM transmission lines, in order to avoid unintentional excitation of non-TEM modes on unavoidable small inhomogeneities.

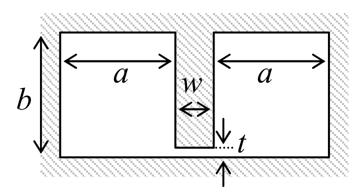
Note, however, that in some practical systems with long cables, the wavelength may be restricted even more severely by the wave attenuation effects – see Sec. 7.9 of the lecture notes.



Problem 7.27. Two coaxial cable sections are connected coaxially – see the figure on the right, which shows the system's cut along its symmetry axis. Eqs. (7.118) and (7.120) of the lecture notes seem to imply that if the ratios b/a of these sections are equal, their impedance matching is perfect, i.e. a TEM wave incident from one side on the connection would pass it without any reflection at all: $R = 0$. Is this statement correct?

Solution: The derivations of both formulas mentioned in the assignment were based on the assumption that the wave fields in both sections are strictly of the TEM type, with the field vectors **E** and **H** exactly normal to the wave vector **k**, i.e. to the system's axis. This cannot be true close to the connection point; for example, in the connection plane, this would lead to the vector **E** being parallel to the "quasi-lid" surfaces of both conductors, thus contradicting the first of the boundary conditions (7.104). Thus there is always some field perturbation, and as a result, some wave reflection from such a sharp jump of the cable's parameters. Moreover, according to the solution of the previous problem, if the wavelength is smaller than $\lambda_c \approx \pi(a + b)$, waves of other types may propagate at least in the wider section and may be effectively excited by the TEM wave incident on the step.[50]

However, if the wavelength λ is much longer than this λ_c (as it is in most practical applications of coaxial cables), such propagation is impossible, and the field perturbations may be only local, spreading at distances of the order of a, b from the connection plane. Since the telegrapher's equations (7.110)-(7.111), whose corollary Eq. (7.118) is, are derived by averaging the field effects at distances much larger than the cross-section's dimensions (in our case, a and b), the effect of these perturbations scales as $\lambda_c/\lambda << 1$. Thus if the matching condition $a'/b' = a/b$ is fulfilled, the reflection coefficient R is not exactly zero but scales as the (small) λ_c/λ ratio.



Problem 7.28. Calculate the cutoff frequencies ω_c of the fundamental mode and the next lowest mode in the so-called *ridge waveguide* with the cross-section shown in the figure on the right, in the limit $t << a$, b, w. Briefly discuss possible advantages and drawbacks of

[50] Note that some (even very popular) textbooks describe over-simplified calculations of the higher mode excitation at waveguide connections, based on expansions of the perturbed fields in series over all possible *traveling-wave* modes. Such calculations, neglecting the field perturbations *localized* at the connection, are not strictly valid at frequencies below the highest ω_c of the connected parts; in this case, their results may be used as estimates at best.

such waveguides for signal transfer and physical experiment.

Solution: As was discussed in Sec. 7.5 of the lecture notes, at the cutoff frequency, $k_z = 0$, i.e. the field distribution is two-dimensional: both **E** and **H** at a point are uniquely defined by its position at the waveguide's cross-section. Hence our task is reduced to finding the oscillation frequency of such a 2D standing wave.

Next, looking at Fig. 7.32 of the lecture notes for the rectangular waveguide, it is easy to understand how its fundamental mode is deformed by an increasing ridge: the electric field becomes more and more concentrated in the gap (of thickness t) between the ridge and the bottom wall. In our limit $t << a, b, w$, this gap is essentially a plane capacitor with a very large capacitance given by Eq. (2.28):

$$C = \frac{\varepsilon_0 wl}{t},$$

where $l >> a, b, w$ is the length of the waveguide.

In this mode, the surface charges of all other surfaces and the electric field outside the gap are negligibly small. Hence, the linear density J of the currents flowing on the waveguide's wall surfaces as a result of the periodic recharging of the capacitance C is also constant outside the gap. As a result, each of the two long, rectangular side cylinders with the cross-section's area $A = a\times b$, separated by the ridge, work as lumped single-wire-turn solenoids. According to Eq. (5.75) with $N = 1$, the inductance of each of them is

$$L_1 = \frac{\mu_0 A}{l} = \frac{\mu_0 ab}{l}.$$

Since these two inductances are connected to the capacitance C in parallel, the effective inductance L equals $L_1/2$, and the oscillation frequency of the resulting LC circuit (and hence the cutoff frequency of the fundamental mode of the ridge waveguide) is

$$(\omega_c)_{LC} = \frac{1}{(LC)^{1/2}} = \left(\frac{2}{L_1 C}\right)^{1/2} = \left(\frac{2t}{\varepsilon_0\mu_0 wab}\right)^{1/2} \equiv c\left(\frac{2t}{wab}\right)^{1/2}.$$

(As the easiest sanity check, the frequency does not depend on l.) The corresponding transverse wave number (7.123) is

$$(k_t)_{LC} \equiv \frac{1}{c}(\omega_c)_{LC} = \left(\frac{2t}{wab}\right)^{1/2}.$$

Since we have assumed that $t << a, b, w$, this wave number is much smaller than any reciprocal dimension of the cross-section. On the other hand, the next lowest mode is evidently the H_{10} wave inside each of the side $a\times b$-cylinders;[51] according to Eq. (7.133), its transverse wave number is

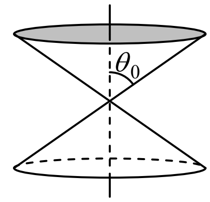
$$(k_t)_{10} \equiv \frac{\pi}{\max[a,b]}, \qquad \text{so that } (\omega_c)_{10} \equiv c(k_t)_{10} = \frac{\pi c}{\max[a,b]} >> (\omega_c)_{10}. \qquad (*)$$

Hence, the ridge waveguide has a very large gap between the cutoff frequencies of the fundamental LC mode and all higher modes – the property to some extent similar to that of the coaxial

[51] Actually, the gap with $t \neq 0$ couples these modes in two side volumes, splitting their cutoff frequency into two different ones, one of them lower than the value given by Eq. (*). However, at $t << a, b, w$, this shift is very small.

cable.[52] According to the general Eq. (7.122), within most of this frequency gap, i.e. at frequencies $(\omega_c)_{LC} << \omega < (\omega_c)_{10}$, such a waveguide has only one mode, with low dispersion, making it convenient for signal transfer. In addition, the uniformity of the electric field inside the t-gap and of the magnetic field outside it makes such waveguides very convenient for some experiments. On the other hand, the wave attenuation of the LC mode is somewhat larger than that of the fundamental mode of a simple rectangular waveguide – with the same external dimensions, at the same frequency.[53]

Finally, note a close similarity between this problem and Problem 7.41 below.



Problem 7.29. Prove that TEM-like waves may propagate, in the radial direction, in the free space between two coaxial, round, well-conducting cones – see the figure on the right. Can this system be characterized by a certain transmission line impedance Z_W, as defined by Eq. (7.115) of the lecture notes?

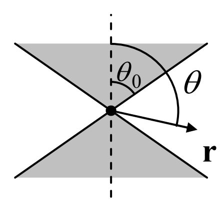
Solution: The term "radial direction" means that the wave vector **k** has to be directed along the radius **r** (with the origin at the point where the cones meet). Any "TEM-like" wave should have its vectors **E** and **H** directed normally to the vector **k**, and to each other. Due to the axial symmetry of the system, this means that the complex amplitudes of the (monochromatic) wave fields, similar to those defined by Eq. (7.98), may have only one of the following two spatial structures: either

$$\mathbf{E}_\omega(\mathbf{r}) = \mathbf{n}_\theta E(r,\theta), \quad \text{and} \quad \mathbf{H}_\omega(\mathbf{r}) = \mathbf{n}_\varphi H(r,\theta), \tag{*}$$

or, vice versa,

$$\mathbf{E}_\omega(\mathbf{r}) = \mathbf{n}_\varphi E(r,\theta), \quad \text{and} \quad \mathbf{H}_\omega(\mathbf{r}) = \mathbf{n}_\theta H(r,\theta), \tag{**}$$

where r, θ, and φ are the usual spherical coordinates – see the figure on the right, showing the axial cross-section of the structure.



However, only the wave (*) can satisfy the coarse-grain boundary conditions (7.104) at the cone surfaces. For its complex amplitudes, the four homogeneous Maxwell equations (7.2) written in the spherical coordinates,[54] with $\varepsilon = \varepsilon_0$, and $\mu = \mu_0$, are reduced to

$$\frac{1}{r}\frac{\partial(rE)}{\partial r} - i\omega\mu_0 H = 0, \qquad -\frac{1}{r}\frac{\partial(rH)}{\partial r} + i\omega\varepsilon_0 E = 0, \tag{***}$$

$$\frac{1}{r\sin\theta}\frac{\partial(E\sin\theta)}{\partial\theta} = 0, \qquad 0 = 0, \tag{****}$$

the last (trivial) relation meaning that the Maxwell equation $\nabla\cdot\mathbf{H} = 0$ is automatically satisfied for any vector function $\mathbf{H}(\mathbf{r}) = \mathbf{n}_\varphi H(r, \theta)$. The nontrivial relation of Eqs. (****) immediately says that $e \equiv E\sin\theta$ may be a function of r alone. Since Eqs. (***) do not involve the polar angle θ, they may be identically satisfied for all θ only if H follows the same dependence on the polar angle, i.e. if

[52] Note, however, that in contrast with the coaxial cable, $(\omega_c)_{LC} \neq 0$, and the corresponding fundamental mode is *not* a TEM one – just as it should be for any waveguide with singly-connected walls.
[53] The calculation of this attenuation is a simple additional exercise, highly recommended to the reader.
[54] See, e.g., MA Eqs. (10.10) and (10.11) with $\partial/\partial\varphi = 0$.

$$E(r,\theta)=\frac{e(r)}{\sin\theta}, \qquad H(r,\theta)=\frac{h(r)}{\sin\theta}.$$

Now Eqs. (***), multiplied by r, may be rewritten as

$$\frac{d(re)}{dr}-i\omega\mu_0(rh)=0, \qquad -\frac{d(rh)}{dr}+i\omega\varepsilon_0(re)=0 .$$

But this means that the products $re \equiv E(r,\theta)\, r\sin\theta$ and $rh \equiv H(r,\theta)\, r\sin\theta$ are related exactly as the complex wave amplitudes in the usual plane waves in free space, i.e. have r-independent amplitudes, may propagate along the radius $\mathbf{r}$ (in any direction) with velocity $c \equiv 1/(\varepsilon_0\mu_0)^{1/2}$ and have an r- and θ-independent ratio (7.8):

$$\frac{re}{rh}\equiv\frac{E(r,\theta)}{H(r,\theta)}=Z_0\equiv\left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2}=\text{const} .$$

Let us check whether the wave impedance Z_W of such a "transmission line", defined by the first of Eqs. (7.115), depends on r. The complex amplitude V_ω of the voltage between two points of the opposite cones, located as the same distance r from the center of the system, may be calculated as

$$V_\omega = \int_{\theta=\theta_0}^{\theta=\pi-\theta_0}\mathbf{E}_\omega\cdot d\mathbf{r} = \int_{\theta_0}^{\pi-\theta_0} Erd\theta = 2(re)\int_{\theta_0}^{\pi/2}\frac{d\theta}{\sin\theta} = -2(re)\int_{\theta=\theta_0}^{\theta=\pi/2}\frac{d(\cos\theta)}{1-\cos^2\theta} = -2(re)\int_{\theta=\theta_0}^{\theta=\pi/2}\frac{d\xi}{1-\xi^2}$$

$$= -(re)\int_{\theta=\theta_0}^{\theta=\pi/2}\left(\frac{1}{1+\xi}+\frac{1}{1-\xi}\right)d\xi = (re)\ln\frac{1+\cos\theta_0}{1-\cos\theta_0}.$$

Similarly, let us calculate the complex amplitude I_ω of the current flowing in each cone at a distance r from the center, where the cone's cross-section radius is $\rho(r) = r\sin\theta_0$. Applying the general Eq. (6.38) to the relation between the linear density of the cone's surface current (flowing in the radial direction) and the azimuthal magnetic field (*) near the surface, we get

$$I_\omega = 2\pi\rho J_\omega = 2\pi r\sin\theta_0 H(r,\theta_0) = 2\pi(rh),$$

so the impedance

$$Z_W \equiv \frac{V_\omega}{I_\omega}=\frac{1}{2\pi}\frac{(re)}{(rh)}\ln\frac{1+\cos\theta_0}{1-\cos\theta_0}\equiv\frac{Z_0}{2\pi}\ln\frac{1+\cos\theta_0}{1-\cos\theta_0}$$

is also independent of r, i.e. is a constant parameter of the conical structure. (Moreover, the expression for Z_W is close in structure to Eq. (7.120) of the lecture notes for the usual coaxial cable, in particular featuring a similar weak (logarithmic) divergence at $\theta_0 \to 0$, i.e. at the vanishing cone's "thickness" $\rho_c(r)/r = \sin\theta_0$.)

These relations are the basis for the design of the so-called *conical antennas*, which may provide wideband coupling of free-space waves to small-size (lumped) sources and detectors of electromagnetic radiation.

Problem 7.30. Use the recipe outlined in Sec. 7.7 of the lecture notes to prove the characteristic equation (7.161) for the *HE* and *EH* waves in step-index optical fibers with a round cross-section.

Solution: Let us start with expressing the constant coefficient in Eq. (7.160) via the amplitude f_l defined by Eq. (7.156), from the requirement that at the core-to-cladding interface ($\rho = R$), both E_z and H_z have to be continuous (as the components tangential to the interface), i.e. $f_+ = f_-$. This relation, combined with Eq. (7.156) for f_-, enables us to make Eq. (7.160) more specific:

$$f_+ = f_l \frac{J_n(k_t R)}{K_n(\kappa_t R)} K_n(\kappa_t \rho) \cos n(\varphi - \varphi_0).$$

Since we are now dealing with a linear superposition of two longitudinal fields, it is easier to operate with the complex-exponential form of this expression and Eq. (7.156):

$$f_+ = f_l \frac{J_n(k_t R)}{K_n(\kappa_t R)} K_n(\kappa_t \rho) e^{in\varphi}, \qquad f_- = f_l J_n(\kappa_t \rho) e^{in\varphi}, \qquad \text{where } f_l = \begin{cases} E_l, & \text{for } f_\pm = E_\pm, \\ H_l, & \text{for } f_\pm = H_\pm, \end{cases}$$

where the constant phases φ_0 (which may be different for E_z and H_z) are incorporated into the complex amplitudes f_l.[55] Now plugging these expressions into the first of Eqs. (7.121), and using the general vector-algebra expressions for the cylindrical coordinates,[56]

$$(\nabla_t f)_\rho = \frac{\partial f}{\partial \rho}, \qquad (\nabla_t f)_\varphi = \frac{1}{\rho}\frac{\partial f}{\partial \varphi}, \qquad (\mathbf{n}_z \times \nabla_t f)_\rho = \frac{1}{\rho}\frac{\partial f}{\partial \varphi}, \qquad (\mathbf{n}_z \times \nabla_t f)_\varphi = -\frac{\partial f}{\partial \rho},$$

we get the following formulas for the transverse components of the electric field:

$$E_\rho^- = \frac{i}{k_t^2}\left[k_z k_t J_n'(k_t\rho)E_l - k_- Z_- \frac{in}{\rho} J_n(k_t\rho) H_l\right] e^{in\varphi},$$

$$E_\varphi^- = \frac{i}{k_t^2}\left[k_z \frac{in}{\rho} J_n(k_t\rho)E_l + k_- Z_- k_t J_n'(k_t\rho) H_l\right] e^{in\varphi},$$

$$E_\rho^+ = -\frac{i}{\kappa_t^2}\frac{J_n(k_t R)}{K_n(\kappa_t R)}\left[k_z \kappa_t K_n'(\kappa_t\rho)E_l - k_+ Z_+ \frac{in}{\rho} K_n(\kappa_t\rho) H_l\right] e^{in\varphi},$$

$$E_\varphi^+ = -\frac{i}{\kappa_t^2}\frac{J_n(k_t R)}{K_n(\kappa_t R)}\left[k_z \frac{in}{\rho} K_n(\kappa_t\rho)E_l + k_+ Z_+ \kappa_t K_n'(\kappa_t\rho) H_l\right] e^{in\varphi}.$$

where the prime signs denote the differentiation of each Bessel function over its whole argument.

Now we should write the boundary conditions for the transverse components of the electric field:

$$\varepsilon_- E_\rho^- = \varepsilon_+ E_\rho^+, \quad E_\varphi^- = E_\varphi^+, \qquad \text{at } \rho = R.$$

Requiring these conditions to be satisfied at all angles φ, we get two linear equations for the complex constants E_l and H_l. At $\mu_- = \mu_+ = \mu_0$, the boundary conditions for the transverse components of the magnetic field give an equivalent system. The same equality for μ ensures that the ratio $\varepsilon_+/\varepsilon_-$ may be represented as k_+^2/k_-^2, and that $k_+Z_+ = k_-Z_- = \omega\mu_0$, so the system of two equations reduces to

[55] This means that here we are working with the complex amplitudes of the fields, defined as in Eq. (7.98), for brevity implying the index ω.

[56] See, e.g., MA Eq. (10.2) with $\partial/\partial z = 0$, so the 3D operator ∇ reduces to the 2D operator ∇_t.

$$k_z\left(\frac{k_-^2}{k_t}\frac{J_n{}'}{J_n}+\frac{k_+^2}{\kappa_t}\frac{K_n{}'}{K_n}\right)E_l-\frac{in}{R}\omega\mu_0\left(\frac{k_-^2}{k_t^2}+\frac{k_+^2}{\kappa_t^2}\right)H_l=0,$$

$$k_z\frac{in}{R}\left(\frac{1}{k_t^2}+\frac{1}{\kappa_t^2}\right)E_l+\omega\mu_0\left(\frac{1}{k_t}\frac{J_n{}'}{J_n}+\frac{1}{\kappa_t}\frac{K_n{}'}{K_n}\right)H_l=0,$$

where the Bessel functions and their derivatives are taken at $\rho = R$. Now requiring the determinant of this system to be zero, so the equations would be consistent, we indeed arrive at Eq. (7.161):

$$\left(\frac{k_-^2}{k_t}\frac{J_n{}'}{J_n}+\frac{k_+^2}{\kappa_t}\frac{K_n{}'}{K_n}\right)\left(\frac{1}{k_t}\frac{J_n{}'}{J_n}+\frac{1}{\kappa_t}\frac{K_n{}'}{K_n}\right)=\frac{n^2}{R^2}\left(\frac{k_-^2}{k_t^2}+\frac{k_+^2}{\kappa_t^2}\right)\left(\frac{1}{k_t^2}+\frac{1}{\kappa_t^2}\right).$$

Problem 7.31. Derive an approximate equation describing spatial variations of the complex amplitude of a general monochromatic paraxial beam propagating in a uniform medium, for the case when these variations are sufficiently slow. Is the Gaussian beam described by Eq. (7.181) one of the possible solutions of this equation? Give your interpretation of the last result.

Solution: As was discussed in 7.8 of the lecture notes (in the context of resonant cavities), for a monochromatic wave of frequency ω, Eqs. (7.3) (similar for both fields, **E** and **H**) are reduced to the same 3D Helmholtz equation (7.204) for their complex amplitudes $\mathcal{R}_\omega(\mathbf{r})$:[57]

$$\left(\nabla^2+k^2\right)\mathcal{R}_\omega=0,\qquad \text{with } k^2\equiv\frac{\omega^2}{v^2}=\varepsilon\mu\omega^2.$$

Looking for its solution in the form $\mathcal{R}_\omega(\mathbf{r}) = f(\mathbf{r})\exp\{ikz\}$, and assuming that the medium is uniform, so v (and hence k) are spatially-independent, we get

$$\left(\nabla_t^2+2ik\frac{\partial}{\partial z}+\frac{\partial^2}{\partial z^2}\right)f=0, \qquad (*)$$

where, as in Secs. 7.5-7.6, ∇_t is the del operator acting only in the directions normal to the z-axis.

So far, this is an exact equation. However, in many practical situations, especially in optics, the function f changes slowly on the wavelength scale:

$$|\nabla f| << |kf|.$$

In a less formal language, this means that our wave propagates predominantly along the z-axis, with its complex amplitude changing slowly on the wavelength's scale. (The paraxial beams discussed in Sec. 7.7 is a typical but not the only possible example of such a situation.) In this case, the second derivative $\partial^2 f/\partial z^2$ in Eq. (*) is negligible in comparison with $k\partial f\partial z$, and this equation reduces to the so-called *paraxial* (or "parabolic") *wave equation*

$$\left(\nabla_t^2+2ik\frac{\partial}{\partial z}\right)f=0, \qquad (**)$$

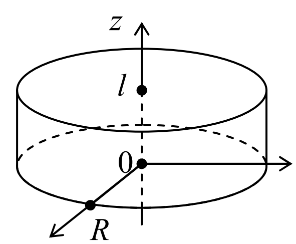
which is, for many problems, more convenient than the initial Helmholtz equation.[58]

[57] In the quasi-plane-wave situation we are going to discuss, the amplitudes of both fields are proportional to each other, and hence to the same scalar function $\mathcal{R}(\mathbf{r})$.

Now, plugging Eq. (7.181), which describes the Gaussian beam of width a,

$$f(\mathbf{r}) = f_0 \exp\left\{-\frac{\rho^2}{2a^2}\right\}, \qquad \text{where } \rho^2 \equiv x^2 + y^2, \qquad (***)$$

into the paraxial wave equation, we may readily see that if a is a z-independent parameter, then Eq. (***) is *not* a solution of Eq. (**). As will be discussed in Chapter 8, this is a result of the beam's *diffraction*, leading to its gradual broadening at distances $\Delta z \sim ka^2$.[59] The reason why we have received Eq. (***) with a = const in Sec. 7.7 is that the radial ρ-dependence of the dielectric constant in a graded fiber, which is *not* described by the paraxial equation in its simplest form (**), may provide effective beam focusing that continuously compensates for its diffraction.



Problem 7.32. Calculate the lowest resonance frequencies and the corresponding field distributions of standing electromagnetic waves inside a round cylindrical cavity with well-conducting walls (see the figure on the right), neglecting the skin depth δ_s in comparison with l and R.

Solution: Due to the simple geometry of the system, and hence the simple structure of the fields in it, we can use the first approach to the eigenfrequency calculation, described in Sec. 7.8 of the lecture notes, by employing the analysis of circular metallic waveguides in Sec. 7.6. In particular, for the H-modes we may use Eqs. (7.121), and an evident generalization of Eq. (7.144):

$$H_z = H_l J_n\left(\xi'_{nm}\frac{\rho}{R}\right)\cos n(\varphi - \varphi_0).$$

Without calculating the transverse fields $\mathbf{E}_t$ and $\mathbf{H}_t$ explicitly from Eq. (7.121) with $E_z = 0$, we may use that formula to notice that $\mathbf{E}_t$ has the same phase as H_z, which does not depend on the point's position on the cross-section z = const. Hence if a couple of well-conducting walls, normal to the z-axis, are inserted into the waveguide at any of the distances $\Delta z = p(\lambda_z/2) = \pi p/k_z$ (where p = 1, 2, ..) between them, they do not disturb the field distribution – besides turning the traveling wave into a standing one.

Now by requiring this Δz to be equal l, and using the general relation (7.122) in the form

$$k^2 \equiv \frac{\omega^2}{v^2} = k_t^2 + k_z^2,$$

and Eq. (7.143) for the eigenvalues of the transverse component k_t of the wave vector, we get the following eigenfrequency spectrum of the H-modes:

$$\omega_{nmp}^2 = v^2 k^2 = v^2\left[\left(\frac{\xi'_{nm}}{R}\right)^2 + \left(\frac{\pi p}{l}\right)^2\right]. \qquad (*)$$

---

[58] Conceptually, the parabolic equation is very close to the van der Pol approximation in the classical theory of oscillations (called RWA in quantum mechanics) – see, e.g., CM Secs. 5.3-5.5 and QM Secs. 6.5 and 9.4.

[59] Just for the reader's reference: a gradually broadening Gaussian beam with the following z-dependent width: $a = [a_0^2 + (z - z_0)^2/k^2a_0^2]^{1/2}$ (where a_0 and z_0 are arbitrary constants), and certain z-dependences of its amplitude and phase (see Chapter 8 for details) *does* satisfy the paraxial equation (**).

The sets of possible integer numbers m and p start with 1, so according to Table 7.1, the lowest of these frequencies is

$$\omega_{011}^2 = v^2\left[\left(\frac{\xi'_{11}}{R}\right)^2 + \left(\frac{\pi}{l}\right)^2\right] \approx v^2\left[\left(\frac{1.841}{R}\right)^2 + \left(\frac{\pi}{l}\right)^2\right]. \qquad (**)$$

(As a reminder, v is the plane wave's velocity in the dielectric filling of the resonator: $v^2 = 1/\varepsilon\mu$.)

However, this is not necessarily the lowest frequency of the resonator. Indeed, using an absolutely similar analysis of the E-waves, whose longitudinal electric field is described by the natural modification of Eq. (7.144) of the lecture notes:[60]

$$E_z = E_l J_n\left(\xi_{nm}\frac{\rho}{R}\right)\cos n(\varphi - \varphi_0),$$

for the corresponding modes of the resonator, we get an expression very similar to Eq. (*):

$$\omega_{nmp}^2 = v^2\left[\left(\frac{\xi_{nm}}{R}\right)^2 + \left(\frac{\pi p}{l}\right)^2\right].$$

However, for the E modes, the boundary conditions at the lid surfaces ($\mathbf{E}_t = 0$, $\partial H_z/\partial z = 0$) allow non-zero fields even if $p = 0$. (In this case, all components of the fields $\mathbf{E}$ or $\mathbf{H}$ are z-independent.) Hence the lowest frequency of these modes is independent of l:

$$\omega_{010}^2 = v^2\left(\frac{\xi_{01}}{R}\right)^2 \approx v^2\left(\frac{2.405}{R}\right)^2. \qquad (***)$$

Note that the field distribution in this E_{010} mode is very similar to that in the fundamental mode of the rectangular resonator – see Fig. 7.30, where b should be replaced with l, and a with $2R$.

A direct comparison of Eqs. (**) and (***) shows that the latter frequency is lower, and hence E_{010} with the eigenfrequency (***) is the fundamental mode of the resonator if $l < \pi R/(\xi^2_{01} - \xi'^2_{11})^{1/2} \approx 2.03R$. In the opposite case, H_{011}, with the eigenfrequency given by Eq. (**), is the fundamental mode.[61]

Just for the reader's reference, let me note that the E-modes with $p = 0$ (i.e. with the fields independent of the z-coordinate directed along the cylinder's axis), and high indices $n >> 1$, and low indices $m \sim 1$, are called the *whispering gallery* modes. This name[62] is due to the fact that (as, for example, Fig. 2.18 implies) the fields of such modes are localized mostly at the cavity's walls (at $\rho \approx R$) – just as acoustic waves may be virtually localized at the walls of a round hall or gallery.

Problem 7.33. Analyze electromagnetic waves that may propagate inside a relatively narrow gap between two well-conducting concentric spherical shells of radii R and $R + d$, in the limit $d << R$.

---

[60] Note the change from the roots ξ'_{nm} to the roots ξ_{nm}, due to the different relevant boundary conditions on the side wall of the resonator (at $\rho = R$): for the E waves, $E_z = 0$, instead of the $dH_z/d\rho = 0$ for the H waves.

[61] A similar crossover in the rectangular resonator, disguised by the similarity of field distribution along its sides, takes place at l = min [a, b], i.e. essentially at the same condition as in the circular resonator, if we parallel its diameter $2R$ with the smallest lateral size of the rectangular resonator.

[62] It is due to Lord Rayleigh who was first (in 1878) to notice this effect – for human whispers in the circular gallery of London's St. Paul Cathedral.

(i) Within the coarse-grain approximation, derive the 2D equation describing such waves with relatively large wavelengths $\lambda \sim R >> d$.

(ii) Calculate the lowest resonance frequencies of the system.

Solutions:

(i) On the scale of the distance $d << R$, both spherical surfaces are locally flat, so the differential equation describing the wave propagation between them is the same as for the gap between two parallel plane surfaces of good conductors. In the coarse-grain approximation, the wave's fields at the exterior sides of these conductors have to satisfy the boundary conditions (7.104):

$$\mathbf{E}_\tau = 0, \qquad H_n = 0 .$$

In such a system, the requirement $\lambda >> d$ may be satisfied only by TEM waves with their electric field normal to these surfaces and the magnetic field tangential to them, both independent of the coordinate normal to the surfaces. For these fields, the derivatives $\partial^2/\partial z^2$ vanish, so the general 3D Helmholtz equation (7.204) for their complex amplitude $\mathcal{F}(\mathbf{r})$ is reduced to a 2D form:

$$\left(\nabla_t^2 + k^2\right)\mathcal{F}(\boldsymbol{\rho}) = 0,$$

where $\boldsymbol{\rho}$ is the 2D radius vector in the plane of the conductors' surfaces, and ∇_t is the del operator acting only in this plane.

Now returning to the "global" (spherical) geometry of our system, in the usual spherical coordinates, we may spell out this equation as[63]

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{(\sin\theta)^2}\frac{\partial^2}{\partial\varphi^2} + (kR)^2\right]\mathcal{F}(\theta,\varphi) = 0 . \qquad (*)$$

(ii) We may apply, to this equation, the variable separation method as this was done with the Laplace equation in Sec. 2.8, besides that now the solution has to be r-independent:

$$\mathcal{F}(\theta,\varphi) = \sum_k c_k \mathcal{P}_k(\cos\theta)\mathcal{F}_k(\varphi), \qquad (**)$$

where, so far, k is just a symbol standing for a certain set of eigenvalues. Now plugging this expansion into Eq. (*), we see that the functions $\mathcal{P}$ and $\mathcal{F}$ satisfy, respectively, Eqs. (2.164) and (2.165). As was discussed in Sec. 2.8, their eigenfunctions are, respectively, the associated Legendre functions $\mathcal{P}_l^n(\cos\theta)$ and sinusoidal functions $\mathcal{F}_\nu(\varphi)$. Obviously, all solutions of our current problem, and hence all functions $\mathcal{F}_\nu(\varphi)$, have to be 2π-periodic in the azimuthal angle φ, so the index ν has to be an integer. As a result, taking into account Eq. (2.178), Eq. (**) may be spelled out as

$$\mathcal{F}(\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{n=0}^{l} c_{l,n}\mathcal{P}_l^n(\cos\theta)\mathcal{F}_n(\varphi).$$

The partial products in its right-hand part are the real spherical harmonics $Y_{ln}(\theta, \varphi)$. The only property of these harmonics needed for this solution is that the eigenvalues of Eq. (*) do not depend on the

[63] See, e.g., MA Eq. (10.9) with $r = R = \text{const}$.

azimuthal index n.[64] So, calculating them by plugging the partial solutions $\mathcal{P}_l^n \mathcal{F}_n$ into this equation, we may take $n = 0$, and use the Legendre equation (2.168) for the functions $\mathcal{P}_l^0 \equiv \mathcal{P}_l$. The result is very simple:

$$\left[-l(l+1)+(kR)^2\right]\mathcal{R}=0\,,$$

so we may label the eigenvalues of the wave number k with the index l alone:

$$k_l = \frac{l(l+1)}{R}, \qquad \text{with } l = 0,1,2,...$$

From here and Eq. (7.28), the lowest resonant frequencies of the system are

$$\omega_l \equiv v_l k_l = v_l \frac{l(l+1)}{R}, \qquad \text{where } v_l = \left[\varepsilon(\omega_l)\mu(\omega_l)\right]^{1/2}.$$

Note that the (mathematically acceptable) eigenvalue $l = 0$ gives $\omega_0 = 0$ and describes not a wave but just a possible stationary electric field between the spherical surfaces.

This system, with $v_l = c \approx 3.0\times10^8$ m/s and $R \approx 6.4\times10^6$ m, may be used as a model for the so-called *Schumann resonances* in the layer between the Earth's surface and the lower boundary of its ionosphere. Its frequency spectrum ω_l may be experimentally measured, for example, by observing resonant peaks in the spectral density of the random electromagnetic noise generated by lightning strikes. Very surprisingly for such a crude model, which completely ignores the Earth's surface terrain and (even more importantly) local variations of the ionosphere, it overestimates the lowest resonance frequencies by only ~20%. The lowest and the strongest of the observed resonances, for $l = 1$, is at the cyclic frequency $\omega_1/2\pi \approx 8$ Hz.

Finally, note a very similar CM Problem 8.14 on water waves on the Earth's ocean surface.

Problem 7.34. A molecule with an electric polarizability α is placed inside an otherwise empty macroscopic cavity with well-conducting walls. Express the resulting shifts of its resonance frequencies via the unperturbed field distribution in the corresponding mode.

Hint: You may like to use the adiabatic theorem of classical mechanics in application to a harmonic oscillator.[65]

Solution: Since the effect of a single molecule on a macroscopic cavity is very small, and its size is much smaller than that of the cavity, we may calculate the average energy of its interaction with the electric field from Eqs. (3.15b) and (3.48) of the lecture notes,

$$\overline{U}_{\text{int}} = -\frac{1}{2}\overline{\mathbf{p}\cdot\mathbf{E}(\mathbf{r},t)} = -\frac{\alpha}{2}\overline{E^2(\mathbf{r},t)},$$

by using the unperturbed value of the field at the point $\mathbf{r}$ of the molecule's location. At free oscillations of a j^{th} resonant mode, the field changes sinusoidally, so the time averaging yields

$$\overline{U}_{\text{int}} = -\frac{\alpha}{4}E_j^2(\mathbf{r}),$$

[64] This fact follows from the discussion in Sec. 2.8 of the lecture notes but, admittedly, was not emphasized there.
[65] See, e.g., CM Sec. 10.2.

where E_j is the real amplitude of the field.

On the other hand, the unperturbed energy of the field in the cavity may be calculated from Eq. (6.113) with $E = E_j$ and $B = 0$, because the magnetic field of any standing wave vanishes at the moments when the electric field takes its maximum (amplitude) value:

$$U_{\text{fielsd}} = \frac{\varepsilon_0}{2}\int_V E_j^2(\mathbf{r})d^3r\,.$$

(In the course-grain approximation, valid when the skin depth δ_s is negligible in comparison with the cavity's linear dimensions, V may be taken equal to just its internal volume.) Hence, the relative change of the total field's energy due to the molecule's insertion may be calculated as

$$\frac{\delta U_j}{U_j} = \frac{\overline{U}_{\text{int}}}{U_{\text{fielsd}}} = -\frac{\alpha}{2\varepsilon_0}\frac{E_j^2(\mathbf{r})}{\int_V E_j^2(\mathbf{r})d^3r}\,.$$

However, according to the adiabatic theorem of classical mechanics (where the word "mechanics" is understood in its broad sense),[66] for any harmonic oscillator, this relative change has to be equal to that of the oscillation frequency, so

$$\frac{\delta\omega_j}{\omega_j} = \frac{\delta U_j}{U_j} = -\frac{\alpha}{2\varepsilon_0}\frac{E_j^2(\mathbf{r})}{\int_V E_j^2(\mathbf{r})d^3r}\,.$$

Problem 7.35. A plane monochromatic wave propagates through a medium with an Ohmic conductivity σ and negligible electric and magnetic polarization effects. Calculate the wave's attenuation and relate the result to a certain calculation carried out in Chapter 6 of the lecture notes.

Solution: In a plane wave, the only energy losses are those in the propagation medium, so the attenuation coefficient is completely described by Eq. (7.218) of the lecture notes:

$$\alpha = 2\omega\,\text{Im}\left[\varepsilon^{1/2}(\omega)\mu^{1/2}(\omega)\right]. \qquad (*)$$

For the medium described in the problem's assignment, Eq. (7.46) takes the form

$$\varepsilon(\omega) = \varepsilon_0 + i\frac{\sigma}{\omega},$$

so Eq. (*), with $\mu(\omega) = \mu_0$, yields

$$\alpha = 2\omega\,\text{Im}\left[\left(\varepsilon_0 + i\frac{\sigma}{\omega}\right)^{1/2}\mu_0^{1/2}\right] \equiv 2\frac{\omega}{c}\text{Im}\left[\left(1+\frac{i}{\omega\tau_r}\right)^{1/2}\right] \equiv \sqrt{2}\frac{\omega}{c}\left[\left(1+\frac{1}{\omega^2\tau_r^2}\right)^{1/2}-1\right]^{1/2}, \qquad (**)$$

where $\tau_r \equiv \varepsilon_0/\sigma$ is the charge relaxation time defined by Eq. (4.10), for the particular case $\kappa = 1$.

This result may be simplified in two limiting cases. At relatively high frequencies (or at relatively small conductivity), the formula is reduced to

[66] Note that according to the Bohr-Wilson-Sommerfeld quantization rules (see, e.g., QM Sec. 2.4), the same result is valid for a broad range of quantum systems as well.

$$\alpha \approx \sqrt{2}\,\frac{\omega}{c}\frac{1}{\left(2\omega^2\tau_r^2\right)^{1/2}} \equiv \frac{1}{c\tau_r} \equiv \frac{\sigma}{c\varepsilon_0} \equiv \sigma Z_0 << k_0, \qquad \text{for } \omega\tau_r >> 1, \quad \text{i.e. for } \alpha << k_0,$$

where $Z_0 \equiv (\mu_0/\varepsilon_0)^{1/2}$ and $k_0 = \omega/c$ are, respectively, the wave impedance and the wave number in free space. In this case, the wave decays at a distance ($l_d \equiv 1/\alpha$) much longer than its wavelength $\lambda_0 \equiv 2\pi/k_0$.

In the opposite limit of low frequencies (or high conductivity), the result yields

$$\alpha \approx \sqrt{2}\,\frac{\omega}{c}\frac{1}{\left(\omega\tau_r\right)^{1/2}} \equiv 2\left(\frac{\mu_0\omega\sigma}{2}\right)^{1/2} >> k_0, \qquad \text{for } \omega\tau_r << 1, \quad \text{i.e. } \alpha >> k_0.$$

In its reciprocal $l_d \equiv 2/\alpha$, i.e. the wave decay length, we may readily recognize the skin depth δ_s (6.33), which was derived in Sec. 6.3 without the account of the displacement currents. Indeed, in this limit, the decay length l_d is much smaller than the free-space wavelength λ_0 at this frequency, so the quasistatic approximation is fully justified – see Sec. 6.8. Note that in this limit, it is hardly possible to speak about the wave *propagation*: upon entering the medium, the wave decays almost immediately – physically, because the high Ohmic conductivity results in very high Joule losses (4.39).

Problem 7.36. Generalize the telegrapher's equations (7.110)-(7.111) by accounting for small energy losses:

(i) in the transmission line's conductors, and
(ii) in the medium separating the conductors,

using their simplest (Ohmic) models. Formulate the conditions of validity of the resulting equations.

Solutions:

(i) Physically, Eq. (7.111) is just a balance between Faraday's e.m.f. induced on a unit-length segment of the transmission line (described by the right-hand side of that equation) and its drop on the inductance of that segment. If the line's conductors have the (total) resistance R_0 per unit length, it makes an additional contribution $R_0 I$ to the balance, so the equation becomes

$$L_0\frac{\partial I}{\partial t} = -\frac{\partial V}{\partial z} - R_0 I\,. \qquad (*)$$

(ii) Similarly, Eq. (7.110) has a clear physical sense of the charge conservation law: its left-hand side is the rate of change of the charge at the capacitance of a unit length of the transmission line, while the right-hand side is the balance of the currents flowing in and out of such a unit-length segment. A nonvanishing Ohmic conductance G_0 (per unit length) of the media separating the wires evidently brings into this balance an additional leakage current $G_0 V$, so the equation takes the following form:

$$C_0\frac{\partial V}{\partial t} = -\frac{\partial I}{\partial z} - G_0 V\,. \qquad (**)$$

The correctness of the signs of the additional terms in these generalized telegrapher's equations may be double-checked from the natural requirement that if R_0 and G_0 are positive (dissipative), they lead to the wave's attenuation. In order to calculate the attenuation, we may plug into Eqs. (*) and (**) the standard monochromatic traveling-wave solution $I = \text{Re}[I_\omega e^{i(kz-\omega t)}]$, $V = \text{Re}[V_\omega e^{i(kz-\omega t)}]$, getting the following system of two linear equations for their complex amplitudes:

$$-i\omega L_0 I_\omega = -ikV_\omega - R_0 I_\omega, \qquad -i\omega C_0 V_\omega = -ikI_\omega - G_0 V_\omega$$

– cf. Eqs. (7.113) of the lecture notes. The condition of their consistency,

$$\begin{vmatrix} -i\omega L_0 + R_0 & ik \\ ik & -i\omega C_0 + G_0 \end{vmatrix} = 0,$$

immediately yields the dispersion relation

$$k = \left[(\omega L_0 + iR_0)(\omega C_0 + iG_0)\right]^{1/2},$$

which is a generalization of Eq. (7.114) to the case of nonvanishing Ohmic losses. In particular, in the most important case when the losses are relatively low (Re $k \equiv k' >> k'' \equiv$ Im k),[67] the relation reduces to

$$k = k' + ik'', \qquad \text{with } k' = \omega(L_0 C_0)^{1/2}, \quad k'' = \frac{1}{2}\left(\frac{R_0}{Z_W} + G_0 Z_W\right),$$

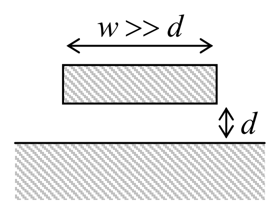
where $Z_W \equiv (L_0/C_0)^{1/2}$ is the lossless transmission line's impedance (7.115). This result shows that, indeed, positive R_0 and G_0 do give positive contributions to the attenuation constant (7.216), $\alpha \equiv 2k''$.

Since the above derivation of Eqs. (*)-(**) was based on such global (or "instant") characteristics of the transmission line as R_0 and G_0, the necessary condition of validity of these equations is the smallness of the linear scale a of the transmission line's cross-section in comparison with wave's decay length:

$$a << l_\mathrm{d} \equiv \frac{1}{2k''} \approx \min\left[\frac{Z_W}{R_0}, \frac{1}{G_0 Z_W}\right].$$

Problem 7.37. Calculate the skin-effect contribution to the attenuation constant α of a TEM wave in the microstrip line discussed in Problem 21 (ii).

Solution: As was discussed in the model solution of Problem 21 (ii), at d, $\delta_\mathrm{s} << w$, both the electric and the magnetic field within the gap between the conductors of the microstrip line (see the figure on the right) are uniform while being negligibly small everywhere outside the gap, besides small fringe regions of width $\sim d$, δ_s near the strip's edges. In addition, if the skin depth is much smaller than d as well,[68] the fields are localized within the area wd of the gap's cross-section, so the transmitted power may be calculated directly from Eq. (7.9) of the lecture notes, as



$$\mathscr{P} = Swd = \left(\frac{\mu}{\varepsilon}\right)^{1/2} H^2 wd.$$

On the other hand, the wave power loss at an elementary length dz of the line, due to the skin effect, may be calculated by multiplying the result given by Eq. (6.36) by $2wdz$ – the area of contact of the magnetic field and the two conductor surfaces:

[67] If they are not, the wave decays so fast that we cannot really speak about its *propagation* in the line.

[68] According to Eq. (7.78), which may be used for crude estimates of the skin effect losses in any skin-effect system, this relation is necessary to keep the attenuation low, $\alpha << k$, which is, in turn, the necessary condition of neglecting the losses at the forthcoming calculation of H and $\mathscr{P}$.

$$d\mathscr{P}_{\text{loss}} = H^2 \frac{\mu\omega\delta_s}{4} 2wdz \, .$$

Plugging these expressions into the definition (7.214) of the attenuation constant, we get a result independent of the microstrip's width w:

$$\alpha_{\text{skin}} \equiv \frac{(d\mathscr{P}_{\text{loss}}/dz)_{\text{skin}}}{\mathscr{P}} = (\varepsilon\mu)^{1/2} \frac{\omega\delta_s}{2d} \equiv \frac{k\delta_s}{2d} .$$

This result may be compared with Eq. (7.225) for another TEM transmission line, the coaxial cable. It also confirms that in order to keep the attenuation low ($\alpha << k$), the skip depth δ_s has to be much smaller than the gap width d.

Problem 7.38. Calculate the skin-effect contribution to the attenuation coefficient α defined by Eq. (7.214) of the lecture notes, for the fundamental (H_{10}) mode propagating in a waveguide well-conducting walls, of a rectangular cross-section – see Fig. 7.22. Use the results to evaluate the wave decay length $l_d \equiv 1/\alpha$ of a 10 GHz wave in the standard X-band waveguide WR-90 (with copper walls, a = 23 mm, b = 10 mm, and no dielectric filling), at room temperature. Compare the result with that (obtained in Sec. 7.9 of the lecture notes) for the standard TV coaxial cable, at the same frequency.

Solution: As was discussed in Sec. 7.6 of the lecture notes, in the H_{10} wave, the electric field has just one Cartesian component, with the complex amplitude

$$E_y(x) = i\frac{ka}{\pi} ZH_l \sin\frac{\pi x}{a} ,$$

while its magnetic field has two components, with the following complex amplitudes:

$$H_z(x) = H_l \cos\frac{\pi x}{a}, \qquad H_x(x) = -i\frac{k_z a}{\pi} H_l \sin\frac{\pi x}{a} , \tag{*}$$

all in the notations of Fig. 7.22. Of those two components, only H_x contributes to the longitudinal (z-) component of the time-averaged Poynting vector

$$\overline{S_z} = \frac{E_x H_y^* - E_y H_x^*}{2} = \frac{kk_z a^2}{2\pi^2} Z|H_l|^2 \sin^2\frac{\pi x}{a} ,$$

which gives the following total power flow along the waveguide:

$$\mathscr{P} = \int_0^a dx \int_0^b dy \, \overline{S}_z = \frac{kk_z a^3 b}{4\pi^2} Z|H_l|^2 , \tag{**}$$

giving us the denominator on the right-hand side of Eq. (7.214).

In order to calculate the numerator of that fraction, i.e. the power dissipation per unit length, we may use Eqs. (*) to integrate the skin-effect losses per unit area, given by Eq. (7.219), over the cross-section's perimeter:

$$\frac{d\mathscr{P}_{\text{loss}}}{dz} = \frac{\mu_0\omega\delta_s}{4} \oint_C |H_0(x)|^2 dl = \frac{\mu_0\omega\delta_s}{4} \left[2\int_0^a \left[|H_x(x)|^2 + |H_z(x)|^2 \right] dx + \int_0^b |H_z(0)|^2 dy + \int_0^b |H_z(b)|^2 dy \right]$$

$$= \frac{\mu_0 \omega \delta_s}{4} |H_l|^2 \left\{ 2\int_0^a \left[\left(\frac{k_z a}{\pi} \right)^2 \sin^2 \frac{\pi x}{a} + \cos^2 \frac{\pi x}{a} \right] dx + 2b \right\} = \frac{\mu_0 \omega \delta_s}{4} |H_l|^2 \left\{ \left[\left(\frac{k_z a}{\pi} \right)^2 + 1 \right] a + 2b \right\}.$$

Per the wave's dispersion relation given by Eqs. (7.122) and (7.133),

$$k_z^2 + k_t^2 = k^2 = \omega^2 \varepsilon \mu, \qquad \text{with } (k_t)_{10} = \frac{\pi}{a},$$

the expression in the last square brackets is just $(ka/\pi)^2$, so using Eq. (**) for $\mathscr{P}$, we finally get

$$\alpha \equiv \frac{1}{\mathscr{P}} \frac{d\mathscr{P}_{\text{loss}}}{dz} = \pi^2 \frac{\mu \omega \delta_s}{Zkk_z a^2 b} \left[\left(\frac{ka}{\pi} \right)^2 + \frac{2b}{a} \right] \equiv \pi \frac{\delta_s}{(k_z a/\pi)ab} \left[\left(\frac{ka}{\pi} \right)^2 + \frac{2b}{a} \right].$$

Note that α scales approximately as δ_s/A, where $A \equiv ab$ is the waveguide's cross-section area. More particularly, α diverges at $b \to 0$, because the transmitted power vanishes while the losses in the broader walls of the waveguide remain constant (at a fixed field amplitude). Another interesting fact is that the dependence of the attenuation on frequency is non-monotonic: α diverges at $\omega \to \omega_c$ where $k_z \to 0$, but also grows (as $k\delta_s \propto \omega^{1/2}$) at $\omega \to \infty$ where $k_z \to k >> a^{-1}, b^{-1}$. As a result, the lowest attenuation is reached at a frequency ~30% above the threshold, i.e. when the higher modes (H_{11}, H_{20}, and E_{11}) still cannot propagate in the waveguide.

For a 10 GHz wave in the WR-90 waveguide we get $\lambda = \lambda_0 \approx 30$ mm, $ka/\pi \equiv 2a/\lambda \approx 1.53$, $k_z a/\pi = [(ka/\pi)^2 - 1]^{1/2} \approx 1.16$, while for copper at room temperature and that frequency, $\delta_s \approx 6.5\times10^{-7}$ m; as a result, our final formula yields $\alpha \approx 0.025$ m^{-1} (i.e. ~0.1 dB/m), i.e. $l_d \equiv 1/\alpha \approx 40$ m. Hence the waveguide may provide an attenuation well below that ($\alpha \approx 0.16$ m^{-1}, see Sec. 7.9) of the standard TV coaxial cable RG-6/U at the same frequency. (Admittedly, the difference is mostly due to the waveguide's larger cross-section rather than to its different wave mode.)

Problem 7.39. Calculate the skin-effect contribution to the attenuation coefficient α of

(i) the fundamental (H_{11}) wave, and
(ii) the H_{01} wave,

in a conductor-wall waveguide with the circular cross-section (see Fig. 7.23a of the lecture notes), and analyze the low-frequency ($\omega \to \omega_c$) and high-frequency ($\omega >> \omega_c$) behaviors of α for each of these modes.

Solutions:

(i) For the H_{11} mode, the longitudinal component H_z of the magnetic field (or rather its complex amplitude) is described by Eq. (7.144) of the lecture notes:

$$H_z(\rho, \varphi) = H_l J_1\left(\xi'_{11} \frac{\rho}{R} \right) \cos\varphi, \quad \text{with } \xi'_{11} \approx 1.841,$$

where the constant φ_0, which does not affect α due to the axial symmetry of the system, was taken for 0. Plugging this result into the second of Eqs. (7.121) with $E_z = 0$, and using the general vector-algebra equalities[69]

[69] See, e.g., MA Eq. (10.2) with $\partial/\partial z = 0$ and/or the model solution of Problem 30.

$$\left(\nabla_t f\right)_\rho = \frac{\partial f}{\partial \rho}, \qquad \left(\nabla_t f\right)_\varphi = \frac{1}{\rho}\frac{\partial f}{\partial \varphi}, \qquad \left(\mathbf{n}_z \times \nabla_t f\right)_\rho = \frac{1}{\rho}\frac{\partial f}{\partial \varphi}, \qquad \left(\mathbf{n}_z \times \nabla_t f\right)_\varphi = -\frac{\partial f}{\partial \rho},$$

we get the following formulas for transverse components of the electric and magnetic fields:

$$E_\rho(\rho,\varphi) = \frac{ikZ}{k_t^2} H_l \frac{1}{\rho} J_1\left(\xi'_{11}\frac{\rho}{R}\right)\sin\varphi, \qquad E_\varphi(\rho,\varphi) = \frac{ikZ}{k_t} H_l J_1'\left(\xi'_{11}\frac{\rho}{R}\right)\cos\varphi,$$
$$H_\rho(\rho,\varphi) = i\frac{k_z}{k_t} H_l J_1'\left(\xi'_{11}\frac{\rho}{R}\right)\cos\varphi, \qquad H_\varphi(\rho,\varphi) = -\frac{ik_z}{k_t^2} H_l \frac{1}{\rho} J_1\left(\xi'_{11}\frac{\rho}{R}\right)\sin\varphi,$$

where $k \equiv \omega(\varepsilon\mu)^{1/2}$, $Z \equiv (\mu/\varepsilon)^{1/2}$, and $k_t = \xi'_{11}/R$, while k_z should be found from the general Eq. (7.102):

$$k_z = \left(k^2 - k_t^2\right)^{1/2}.$$

From here and Eq. (7.219), the time-averaged areal density of the energy loss rate due to the skin effect is

$$\frac{d\mathscr{P}_{\text{loss}}}{dA} = \frac{\mu\omega\delta_s}{4}\left[\left|H_\varphi(R,\varphi)\right|^2 + \left|H_z(R,\varphi)\right|^2\right] = \frac{\mu\omega\delta_s}{4}\left|H_l\right|^2 J_1^2\left(\xi'_{11}\right)\left(\frac{k_z^2}{k_t^4 R^2}\sin^2\varphi + \cos^2\varphi\right).$$

Integrating this expression over the cross-section's perimeter, we can find the full power loss per unit length of the waveguide:

$$\frac{d\mathscr{P}_{\text{loss}}}{dz} = \int_0^{2\pi}\frac{d\mathscr{P}_{\text{loss}}}{dA} R\, d\varphi = \frac{\pi}{4} J_1^2\left(\xi'_{11}\right)\mu\omega\delta_s R\left|H_l\right|^2\left[\frac{1}{(k_t R)^2}\frac{k_z^2}{k_t^2} + 1\right].$$

By the definition of ξ'_{11}, $J_1(\xi'_{11})$ is just the first maximum of Bessel function $J_1(\xi)$, which is close to 0.5819 (see, e.g., Fig. 2.18), while $(k_tR)^2 = (\xi'_{11})^2 \approx 3.389$.

The average longitudinal component of the Poynting vector is

$$\overline{S_z} = \frac{E_\rho H_\varphi^* - E_\varphi H_\rho^*}{2} = \frac{Z}{2}\left|H_l^2\right|\frac{kk_z}{k_t^2}\left[\frac{1}{k_t^2\rho^2} J_1^2\left(\xi'_{11}\frac{\rho}{R}\right)\sin^2\varphi + J_1'^2\left(\xi'_{11}\frac{\rho}{R}\right)\cos^2\varphi\right].$$

Integrating it over the waveguide's cross-section, for the average propagating power we get

$$\mathscr{P} = \int_0^{2\pi} d\varphi \int_0^R \rho\, d\rho\, \overline{S}_z = \frac{\pi}{2} Z\left|H_l^2\right|\frac{kk_z}{k_t^2}\int_0^R\left[\frac{1}{k_t^2\rho^2}J_1^2\left(\xi'_{11}\frac{\rho}{R}\right) + J_1'^2\left(\xi'_{11}\frac{\rho}{R}\right)\right]\rho\, d\rho = \frac{\pi I}{\xi'^4_{11}} Z\left|H_l^2\right| kk_z R^4,$$

where I is a numerical constant:

$$I \equiv \int_0^{\xi'_{11}}\left[\frac{J_1^2(\xi)}{\xi^2} + J_1'^2(\xi)\right]\xi\, d\xi = \frac{1}{2}\int_0^{\xi'_{11}}\left[J_0^2(\xi) + J_2^2(\xi)\right]\xi\, d\xi \approx 0.5822.$$

From here, taking into account that $\mu\omega/kZ = 1$, the attenuation constant is

$$\alpha \equiv \frac{1}{\mathscr{P}}\frac{d\mathscr{P}_{\text{loss}}}{dz} = \frac{J_1^2\left(\xi'_{11}\right)\xi'^2_{11}}{4I}\frac{\delta_s}{k_z R^3}\left(\frac{k_z^2}{k_t^2} + \xi'^2_{11}\right) \approx 0.4929\frac{\delta_s}{k_z R^3}\left(\frac{k_z^2}{k_t^2} + 3.389\right).$$

The attenuation's dependence on the wave frequency ω, following from this formula, is very similar to that for the rectangular waveguide (see the solution of the previous problem): α diverges when ω is reduced to the cutoff frequency $\omega_c = k_t/(\varepsilon\mu)^{1/2}$ because at that point $k_z \to 0$, and also increases as $k\delta_s \propto \omega^{1/2}$ at high frequencies $\omega >> \omega_c$, where $k_z \approx k >> k_t$. (The minimum of α is reached between these two extremes, at $\omega \approx 4\omega_c$.)

(ii) For the H_{01} mode, the longitudinal field is described by Eqs. (7.141) and (7.143) with $n = 0$ and $m = 1$:

$$H_z(\rho,\varphi) = H_l J_0\left(\xi'_{01}\frac{\rho}{R}\right), \qquad \xi'_{01} \approx 3.831,$$

i.e. is independent of the azimuthal angle. Because of that independence, the field calculations, absolutely similar to those carried out in Task (i), yield simpler results:

$$E_\rho(\rho,\varphi) = 0, \qquad E_\varphi(\rho,\varphi) = -i\frac{k}{k^2}ZH_l J_0'\left(\xi'_{01}\frac{\rho}{R}\right) \equiv i\frac{k}{k_t}ZH_l J_1\left(\xi'_{01}\frac{\rho}{R}\right),$$

$$H_\rho(\rho,\varphi) = i\frac{k_z}{k_t}H_l J_0'\left(\xi'_{01}\frac{\rho}{R}\right) \equiv -i\frac{k_z}{k_t}H_l J_1\left(\xi'_{01}\frac{\rho}{R}\right), \qquad H_\varphi(\rho,\varphi) = 0.$$

where $k_t = \xi'_{01}/R$. As a result of the vanishing azimuthal magnetic field, the skin-effect losses are determined only by the longitudinal component of $\mathbf{H}$:

$$\frac{d\mathscr{P}_{\text{loss}}}{dA} = \frac{\mu\omega\delta_s}{4}\left|H_z(R,\varphi)\right|^2 = \frac{\mu\omega\delta_s}{4}\left|H_l\right|^2 J_0^2(\xi'_{01}), \quad \text{so} \quad \frac{d\mathscr{P}_{\text{loss}}}{dz} = R\int_0^{2\pi}\frac{\mathscr{P}_{\text{loss}}}{A}d\varphi = \frac{\pi}{2}J_0^2(\xi'_{01})\mu\omega\delta_s R\left|H_l\right|^2.$$

The longitudinal component of the Poynting vector is also contributed by only one component product:

$$\overline{S_z} = -\frac{E_\varphi H_\rho^*}{2} = \frac{Z}{2}\left|H_l^2\right|\frac{kk_z}{k_t^2}J_1^2\left(\xi'_{01}\frac{\rho}{R}\right),$$

so the total average power flow is

$$\mathscr{P} = \int_0^{2\pi}d\varphi\int_0^R \rho\, d\rho\, \overline{S}_z = \pi Z\left|H_l^2\right|\frac{kk_z}{k_t^2}\int_0^R J_1^2\left(\xi'_{01}\frac{\rho}{R}\right)\rho\, d\rho = \pi I'Z\left|H_l^2\right|\frac{kk_z}{k_t^2}R^2, \quad \text{with } I' \equiv \int_0^{\xi'_{01}} J_1^2(\xi)\xi\, d\xi \approx 0.6384.$$

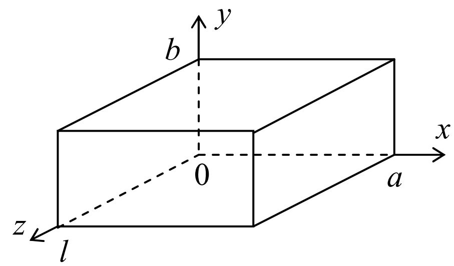
As a result, the expression for the attenuation constant,

$$\alpha \equiv \frac{1}{\mathscr{P}}\frac{\mathscr{P}_{\text{loss}}}{dz} = \frac{J_0^2(\xi'_{01})}{2I'}\frac{\delta_s k_t^2 R}{k_z} \approx 1.865\frac{\delta_s}{k_z R^3},$$

has a rather different frequency dependence: while still diverging at $\omega \to \omega_c$ (where $k_z \to 0$), it is proportional to $\delta_s/k \propto \omega^{-1/2} \to 0$ at $\omega/\omega_c \to \infty$, where $k_z \to k >> k_t$.

This property, which makes the H_{01} mode (as well as higher H_{0m} modes) attractive for long-distance microwave energy transfer, is due to the fact that the transverse component of the magnetic field vanishes at the wall surface ($\rho = R$). As a result, the skin-effect losses are due only to the longitudinal component of the magnetic field, which decreases (at a fixed power of the wave) as its frequency is increased.

Problem 7.40. For a rectangular cavity of dimensions $a\times b\times l$, with $b \leq a$, l, calculate the Q-factor of the fundamental oscillation mode, due to the skin-effect losses in its conducting walls. Evaluate the factor for a $23\times23\times10\ \text{mm}^3$ cavity with copper walls, at room temperature.



Solution: Selecting the Cartesian coordinates as in Fig. 7.29 of the lecture notes (partly reproduced on the right) and using Eq. (7.137) with $k_z = \pm l/\pi$, for the fundamental (H_{101}) mode discussed in Sec. 7.8, we may write the magnetic field distribution as follows:

$$H_x = \frac{a}{l}H_l \sin\frac{\pi x}{a}\cos\frac{\pi z}{l}, \quad H_y = 0, \quad H_z = H_l \cos\frac{\pi x}{a}\sin\frac{\pi z}{l}.$$

The electromagnetic field energy in the cavity may be calculated, for example, as the largest energy of the magnetic field:

$$U = \frac{\mu_0}{2}\int_0^a dx\int_0^b dy\int_0^l dz\left(H_x^2 + H_z^2\right) = \frac{\mu_0}{2}|H_l|^2 b\int_0^a dx\int_0^l dz\left[\frac{a^2}{l^2}\sin^2\frac{\pi x}{a}\cos^2\frac{\pi z}{l} + \cos^2\frac{\pi x}{a}\sin^2\frac{\pi z}{l}\right]$$
$$= \frac{\mu_0}{8}|H_l|^2 abl\left(\frac{a^2}{l^2}+1\right),$$

where it assumed that the wall's conductivity is sufficiently high to have $\delta_s << a, b, l$, In this case, the time-averaged power losses due to the skin effect may be calculated just as it was for a waveguide in the model solution of Problem 38, i.e. by integrating Eq. (7.219) over the area of all the walls:

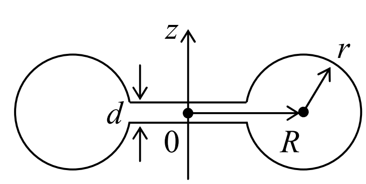
$$\mathscr{P}_{\text{loss}} = \frac{\mu_0\omega\delta_s}{4}\left\{2b\left[\int_0^a |H_x(x,0)|^2 dx + \int_0^l |H_z(0,z)|^2 dz\right] + 2\int_0^a dx\int_0^l dz\left[|H_x(x,z)|^2 + |H_z(x,z)|^2\right]\right\}$$
$$= \frac{\mu_0\omega\delta_s}{4}|H_l|^2 l\left[b\left(\frac{a^3}{l^3}+1\right) + \frac{a}{2}\left(\frac{a^2}{l^2}+1\right)\right].$$

As a result, the Q-factor defined by Eq. (7.227) is

$$Q \equiv \omega\frac{U}{\mathscr{P}_{\text{loss}}} = \frac{1}{2\delta_s}\left[ab\left(\frac{a^2}{l^2}+1\right)\right]\Bigg/\left[b\left(\frac{a^3}{l^3}+1\right)+\frac{a}{2}\left(\frac{a^2}{l^2}+1\right)\right].$$

As could be expected from Eq. (7.78), Q scales as the ratio of some effective size of the cavity (at $l << a$, $b \sim \lambda$, tending to l), to the skin depth δ_s. According to Eq. (7.206) of the lecture notes, the resonance frequency of the cavity specified in the assignment is $\omega \approx 5.7\times10^{10}\ \text{s}^{-1}$ ($f \approx 9.2$ GHz), so the skin depth δ_s for copper walls is close to 6.8×10^{-7} m, and the above formula yields $Q \approx 7.9\times10^3$. For a microwave metallic cavity of a practicable size, operating at its fundamental mode, this is almost as high a quality factor as you can get at room temperature. Note that for these numbers, the condition $Q >> 1$ is well satisfied, so our approximate method of calculation of the Q-factor is indeed legitimate.

Problem 7.41.* Calculate the lowest eigenfrequency and the Q-factor (due to the skin effect) of the axially symmetric toroidal cavity with



well-conducting walls and the interior's cross-section shown in the figure on the right, for the case $d \ll r, R$.[70]

Solution: Looking at the field distribution in the fundamental, H_{101} mode of a rectangular cavity (Fig. 7.30 of the lecture notes), and the virtually similar distribution in the E_{010} mode of the circular cylindrical cavity (see the model solution of Problem 32), it is easy to understand that the lowest mode of the toroidal cavity has the electric field concentrated almost exclusively in its central part, with the distance d between the parallel planar walls, while the magnetic field is virtually limited to the toroidal ("doughnut") part of radii r and R. Due to this field separation, the resonance frequency may be calculated just as for the lumped LC circuit,

$$\omega = \frac{1}{(LC)^{1/2}},$$

with the capacitance equal to that of the plane capacitor of thickness d and area $A = \pi(R-r)^2$,

$$C = \varepsilon \frac{\pi(R-r)^2}{d},$$

and the inductance calculated as in Problem 5.20, but with $N = 1$:

$$L = \mu\left[R - \left(R^2 - r^2\right)^{1/2}\right],$$

so, finally,

$$\omega^2 = \frac{d}{\pi\varepsilon\mu\,(R-r)^2\left[R - \left(R^2 - r^2\right)^{1/2}\right]}. \qquad (*)$$

The TEM wavelength corresponding to the frequency is

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{(\varepsilon\mu)^{1/2}\omega} = 2\pi^{3/2}\frac{(R-r)}{d^{1/2}}\left[R - \left(R^2 - r^2\right)^{1/2}\right]^{1/2}.$$

This formula shows that at $r \sim R$, the wavelength scales as $(R^3/d)^{1/2} \gg R, r \gg d$, so our quasistationary treatment of the electric and magnetic fields in indeed valid.

To calculate the Q factor, we need to evaluate both the numerator and denominator of the right-hand side of Eq. (7.227) of the lecture notes. Assuming that $Q \gg 1$, this may be done neglecting the effect of losses on the field distribution. In this approximation, the average energy of oscillations may be found as the maximum energy of the electric field:

$$U = \frac{|Q_\omega|^2}{2C} = \frac{|Q_\omega|^2 d}{2\pi\varepsilon(R-r)^2}, \qquad (**)$$

where Q_ω is the complex amplitude of the total electric charge of each planar "lid" of the cavity.

In order to calculate the energy loss rate due to the skin effect, we can use Eq. (7.219). With the account of the fundamental Eq. (6.38), it may be rewritten as

[70] Such resonators are broadly used in particle accelerators and also in vacuum electron devices for high-power microwave amplification and generation (e.g., the so-called *klystrons*), where the electric field has to be concentrated in the region of charged particle passage – typically, along the symmetry axis (in the figure above, the z-axis), through a pair of small holes in the cavity's walls, which do not affect the field distribution substantially.

$$\frac{d\mathscr{P}_{\text{loss}}}{dA} = \frac{\mu\omega\delta_s}{4}\left|J_\omega(\rho)\right|^2, \qquad (***)$$

where $J_\omega(\rho)$ is the complex amplitude of the linear density of the surface currents at the point of the cavity wall surface, that is separated from the z-axis by distance ρ. Let us express this density via its value $J_0 \equiv J_\omega(R-r)$ in the cavity's "neck", i.e. at the connection of its planar and toroidal parts, because J_0 may be readily related to Q_ω by using the charge conservation law. With the account of the sinusoidal law of time evolution of all the variables ($\partial/\partial t \to -i\omega$), this relation yields

$$-i\omega Q_\omega = 2\pi(R-r)J_0, \qquad \text{i.e. } J_0 = i\omega\frac{1}{2\pi(R-r)}Q_\omega. \qquad (****)$$

The current density in the toroidal part of the cavity, with negligible displacement currents, may be calculated from the conservation of the total surface current, $2\pi\rho J_\omega(\rho) = 2\pi(R-r)J_0$, giving

$$J_\omega(\rho) = J_0\frac{R-r}{\rho}, \qquad \text{for } \rho \geq R-r.$$

However, in the planar part of the device (with $\rho < R-r$), the displacement currents are substantial, because of the smallness of d. The easiest way to calculate J_ω here is to write the charge conservation law similar to that written above for J_0, $-i\omega dQ_\omega(\rho)/dt = -2\pi\rho J_\omega(\rho)$, where $Q_\omega(\rho) = Q_\omega[\rho/(R-r)]^2$ is the part of the full charge Q_ω, that is located within a circle of radius ρ.[71] As a result, we get

$$J_\omega(\rho) = J_0\left(\frac{\rho}{R-r}\right)^2, \qquad \text{for } \rho \leq R-r.$$

As the sanity check, $J_\omega(R-r) = J_0$, and $J_\omega(0) = 0$.

Now we have everything ready for the energy loss calculation using Eq. (***):

$$\mathscr{P}_{\text{loss}} = \frac{\mu\omega\delta_s}{4}\int\limits_{\substack{\text{over all}\\ \text{surfaces}}}\left|J_\omega(\rho)\right|^2 d^2r = \frac{\mu\omega\delta_s}{4}\left|J_0\right|^2\left[2\int_0^{R-r}\left(\frac{\rho}{R-r}\right)^4 2\pi\rho\, d\rho + 2\int_0^{\pi}\left(\frac{R-r}{\rho}\right)^2 2\pi\rho\, r d\theta\right],$$

where θ is the angle pointing to the considered point of the torus from the center of its cross-section, so $\rho = R + r\cos\theta$. An easy integration[72] yields

$$\mathscr{P}_{\text{loss}} = \frac{\mu\omega\delta_s}{4}\left|J_0\right|^2\left[\frac{2\pi}{3}(R-r)^2 + 4\pi^2 r\frac{(R-r)^2}{\left(R^2-r^2\right)^{1/2}}\right].$$

Finally, combining this expression with Eqs. (*), (**), and (****), we get

$$Q \equiv \omega\frac{U}{\mathscr{P}_{\text{loss}}} = \frac{R-\left(R^2-r^2\right)^{1/2}}{\delta_s}\Bigg/\left[\frac{1}{12\pi} + \frac{r}{2(R^2-r^2)^{1/2}}\right].$$

This formula shows that in the quasistatic approximation, the Q-factor does not depend on d and that at $r \sim R$,[73] it scales as $\sim R/\delta_s$, i.e. is much lower than the value $\sim\lambda/\delta_s$ that we could get from the

[71] This particular calculation is similar to the one carried out in the model solution of Problem 6.30.

[72] In the second case, by using MA Eq. (6.3c).

"usual" resonance cavity, with all three linear dimensions comparable. This is the price we have to pay for the system's convenience for particle-beam applications. This result explains why in such unique applications as particle accelerators where refrigeration costs are not the primary concern, accelerating rf cavities are fabricated from a superconducting material (usually, niobium with the critical temperature of 9.2 K) and are kept at "helium" temperatures of a few K, to keep Q sufficiently high.

Problem 7.42. Express the contribution to the damping coefficient (the reciprocal Q-factor) of a resonant cavity by small energy losses in the dielectric that fills it, via the complex functions $\varepsilon(\omega)$ and $\mu(\omega)$ of the material.

Solution: For the dispersion-free and loss-free case, Eq. (7.202) of the lecture notes has the partial variable-separated solutions (7.203) whose temporal factors obey the ordinary differential equation similar to the second of Eqs. (7.201):

$$\ddot{\mathcal{T}} + \omega^2 \mathcal{T} = 0, \qquad \text{where } \omega^2 = k^2 v^2 \equiv \frac{k^2}{\varepsilon\mu}, \tag{*}$$

and k is the corresponding eigenvalue of the Helmholtz equation (7.204). For a lossless medium, ε and μ are real, so $\mathcal{T}$ describes persisting sinusoidal oscillations of frequency ω:

$$\mathcal{T}(t) = \mathrm{Re}\left[\mathcal{T}_\omega e^{-i\omega t}\right] = \mathrm{Re}\,\mathcal{T}_\omega \cos\omega t + \mathrm{Im}\,\mathcal{T}_\omega \sin\omega t\,.$$

Acting in analogy with the derivation of Eq. (7.28), we may generalize Eq. (*) to the case of a dispersive medium as

$$\omega^2 \equiv \frac{k^2}{\varepsilon(\omega)\mu(\omega)}. \tag{**}$$

If the dielectric is lossy, then $\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$, and $\mu(\omega) = \mu'(\omega) + i\mu''(\omega)$,[74] and the solution of this equation is complex, $\omega = \omega' + i\omega''$. This means that the solution of Eq. (**) is not exactly sinusoidal, but decays with time:

$$\mathcal{T}(t) = \mathrm{Re}\left[\mathcal{T}_\omega e^{-i\omega t}\right] = \mathrm{Re}\left[\mathcal{T}_\omega e^{-i(\omega' + i\omega'')t}\right] = e^{\omega'' t}\,\mathrm{Re}\left[\mathcal{T}_\omega e^{-i\omega' t}\right],$$

so the oscillations' energy decays as

$$U(t) = U(0)e^{2\omega'' t}\,.$$

Comparing this expression with Eq. (7.228), we see that the contribution of the complex ω to the damping coefficient $1/Q$ is

$$\left(\frac{1}{Q}\right)_{\text{dielectric}} = -\frac{2\omega''}{\omega'}\,.$$

[73] For fixed R and δ_s, Q reaches its maximum at $r \approx 0.8R$, while vanishing at both $r \to 0$ and $r \to R$. This fact justifies our efforts to carry out the calculations for arbitrary ratio r/R.

[74] For most dielectrics, $\mu(\omega)$ is very close to μ_0, so $\mu''(\omega)$ is negligible, and in this solution, that component is kept mostly for the sake of generality.

In the most important case when the losses are small, $|\varepsilon''(\omega)| << \varepsilon'(\omega)$ and $|\mu''(\omega)| << \mu'(\omega)$, so $|\omega''| << \omega'$, and the characteristic equation (**) rewritten as

$$(\omega' + i\omega'')^2(\varepsilon' + i\varepsilon'')(\mu' + i\mu'') = k^2, \qquad (***)$$

may be solved by successive approximations. In the 0th approximation, $\omega'' = 0$, while ω' is just the unperturbed real frequency ω_0 of the cavity, which has to be found self-consistently from the equation $\omega_0^2 = k^2/\varepsilon'(\omega_0)\mu'(\omega_0)$. In the 1st approximation, we may plug this result, $\omega' = \omega_0$, into Eq. (***) linearized with respect to small ω'', ε'', and μ'':

$$2\omega'\omega''\varepsilon'\mu' + \omega'^2(\varepsilon'\mu'' + \varepsilon''\mu') \approx 0.$$

Its solution is

$$\omega'' \approx -\frac{\omega'(\varepsilon'\mu'' + \varepsilon''\mu')}{2\varepsilon'\mu'} \equiv -\frac{\omega'}{2}\left(\frac{\varepsilon''}{\varepsilon'} + \frac{\mu''}{\mu'}\right),$$

so, finally, we get a very simple and natural expression:

$$\left(\frac{1}{Q}\right)_{\text{dielectric}} \approx \frac{\varepsilon''}{\varepsilon'} + \frac{\mu''}{\mu'} \equiv \frac{\operatorname{Im}\varepsilon(\omega_0)}{\operatorname{Re}\varepsilon(\omega_0)} + \frac{\operatorname{Im}\mu(\omega_0)}{\operatorname{Re}\mu(\omega_0)}.$$

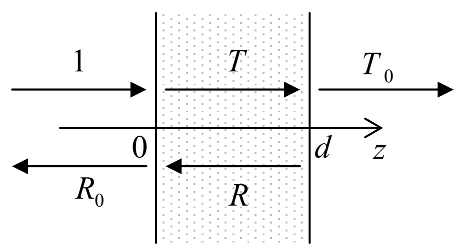
Problem 7.43. For the dielectric Fabry-Pérot resonator (Fig. 7.31 of the lecture notes) with the normal wave incidence, calculate the Q-factor due to radiation losses, in the limit of a strong impedance mismatch ($Z >> Z_0$), by using two approaches:

(i) from the energy balance, by using Eq. (7.227), and

(ii) from the frequency dependence of the power transmission coefficient, by using Eq. (7.229).

Compare the results.

Solutions:



(i) The calculation of the field distribution in such a system, with a unit-amplitude external wave incident from one side, was the subject of Problem 13. The last two equations of the full system for the traveling wave amplitudes T, T_0, R, and R_0, normalized to the incident wave amplitude E_ω (for notation, see the figure on the right), may be readily solved to give

$$Te^{ikd} = \frac{Z_0 + Z}{2Z_0}T_0, \qquad Re^{-ikd} = \frac{Z_0 - Z}{2Z_0}T_0.$$

At the exact resonance ($kd = k_m d = \pi m$, where $m = 1, 2, 3, \ldots$), both exponential factors are equal to $(-1)^m$, and from the solution of Problem 13, we know that $T_0 = 1$, so at $Z >> Z_0$, $R \approx -T$, with

$$|T|^2 \approx |R|^2 \approx \left(\frac{Z}{2Z_0}\right)^2 >> 1.$$

Thus the field inside the resonator ($0 \le z \le d$) may be approximated with a standing wave:

$$E(z) = E_\omega\left(Te^{ikz} + R\,e^{-ikz}\right) \approx -E_\omega \frac{Z}{2Z_0}\left(e^{ikz} - e^{-ikz}\right) \equiv -iE_\omega \frac{Z}{Z_0}\sin\frac{\pi mz}{d}.$$

Note that the standing wave's amplitude is much higher than that (E_ω) of the incident wave. (This is the well-known effect of energy accumulation at resonance.). Also notable is the $\pi/2$ phase shift between the standing wave and the incident wave (represented by the coefficient $-i \equiv \exp\{-i\pi/2\}$), which is also typical for the exact resonance.[75]

Now we can use Eq. (3.79) to calculate the full (time-independent) oscillation energy as the time-maximum electric field energy in the resonator (per unit area):

$$\frac{U}{A} = \frac{\varepsilon}{2}\int_0^d \max_t\left[E^2(z,t)\right]dz = \frac{\varepsilon}{2}\left|E_\omega\right|^2\left(\frac{Z}{Z_0}\right)^2\int_0^d \sin^2\frac{\pi mz}{d}dz = \frac{\varepsilon}{2}\left|E_\omega\right|^2\left(\frac{Z}{Z_0}\right)^2\frac{d}{2}.$$

The resonator's energy loss (per unit time and unit area), due to the radiation through its interfaces, is the sum of the time-averaged Poynting vectors of the two outcoming waves, whose amplitudes T_0 and R_0 are (in our approximation) equal to each other:

$$\frac{\mathscr{P}_{\text{loss}}}{A} = 2\overline{S}_T = \frac{1}{Z_0}\left|T_0E_\omega\right|^2,$$

so at the resonance ($T_0 = 1$), Eq. (7.227) yields

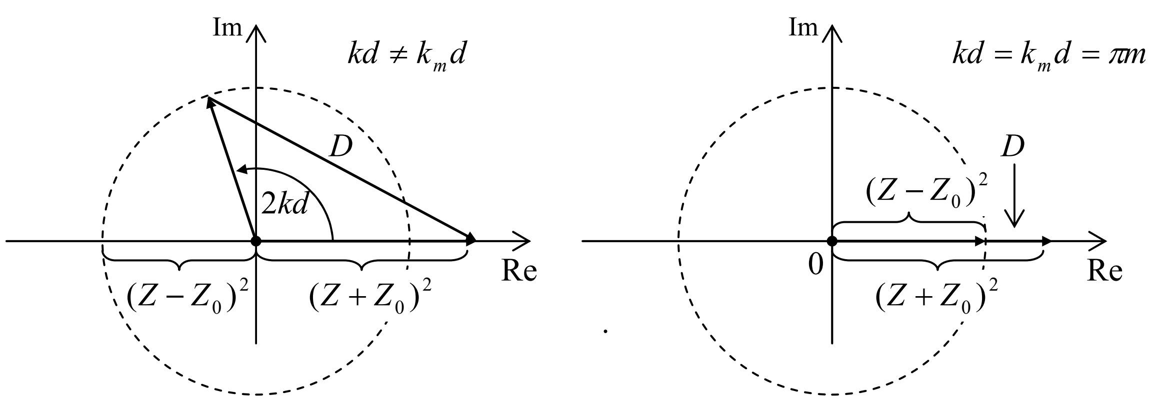
$$Q \equiv \omega_m\frac{U}{\mathscr{P}_{\text{loss}}} = \omega_m\frac{\varepsilon d}{4}Z_0\left(\frac{Z}{Z_0}\right)^2 = \frac{k_m d}{4Z}Z_0\left(\frac{Z}{Z_0}\right)^2 = \frac{\pi m}{4}\frac{Z}{Z_0}. \qquad (*)$$

This formula is only valid if $Q >> 1$, as it is in our case $Z >> Z_0$.

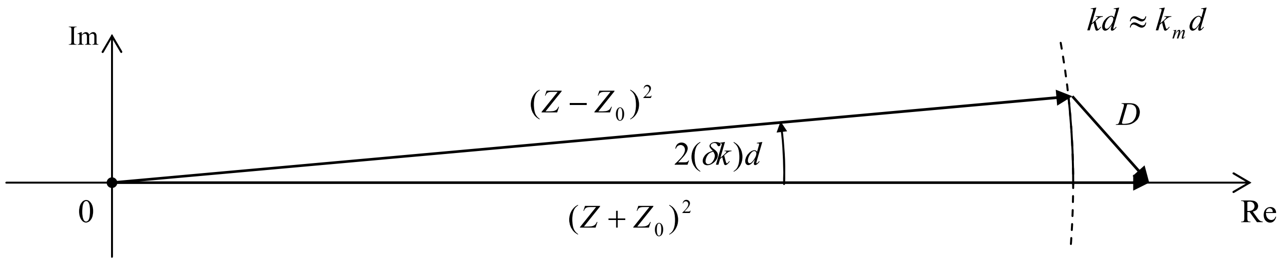
(ii) The final result of the solution of Problem 13 was the following expression for the (amplitude) transmission coefficient:

$$T_0 = \frac{4ZZ_0}{(Z+Z_0)^2 - (Z-Z_0)^2\exp\{2ikd\}}\exp\{i(k-k_0)d\} \equiv \frac{N}{D},$$

whose denominator D (a complex number) may be conveniently represented by the 2D vector diagrams reproduced in the figure below. (The top-right panel of that figure confirms, in particular, that the resonance transmission ($T_0 = 1$) is reached at $k_m d = \pi m$.)



[75] See, e.g., CM Sec. 5.1, in particular, the right panel of Fig. 5.1.



The diagrams indicate that at a strong impedance mismatch ($Z >> Z_0$), even a small deviation $\delta k \equiv k - k_m$ from the exact resonance leads to a fast growth of the denominator D and hence to a fast suppression of $|T|$. As a result, in order to find the power transmission coefficient $|T|^2$ near the resonance, we may use the following approximation (see the bottom panel of the figure above):

$$|D|^2 \approx \left[(Z+Z_0)^2 - (Z-Z_0)^2\right]^2 + \left[(Z-Z_0)^2 2(\delta k)d\right]^2 \approx \left[4ZZ_0\right]^2 + \left[Z^2 2(\delta k)d\right]^2.$$

The denominator increases to twice its resonance value (and hence the transmitted power drops to one-half of its resonance value) when the second term in this expression becomes equal to the first one, i.e. when

$$(\delta k)\big|_{1/2} d = \pm \frac{2Z_0}{Z}.$$

According to Eq. (7.229),[76] the Q factor of the resonator may be determined from the "full-width half-maximum" (FWHM) bandwidth:

$$Q = \frac{\omega_m}{\Delta\omega} = \frac{k_m d}{(\Delta k)d} = \frac{\pi m}{2\left|\delta k\right|_{1/2} d} = \frac{\pi m Z}{4Z_0},$$

i.e. exactly the same result as given by Eq. (*).

[76] Again, the (quite simple) derivation of that formula may be found, for example, in CM Sec. 5.1.

Chapter 8. Radiation, Scattering, Interference, and Diffraction

Problem 8.1. Equation (8.8) of the lecture notes obviously has standing-wave solutions $\chi(r, t) = \text{Re}\,[C\sin kr \exp\{-i\omega t\}]$, turning the scalar potential $\phi = \chi/r$ into a finite constant at $r = 0$ and into zero at $kr = \pi n$, with $n = 1, 2, 3,\ldots$ This fact seems to imply that a cavity of radius R, carved inside a good conductor, has resonant modes with a purely radial electric field $\mathbf{E}(\mathbf{r}) = \mathbf{n}_r E(r)$ and that the lowest nonvanishing of them, with $k = \pi/R$, gives the lowest (fundamental) frequency $\omega \equiv vk = \pi(v/R)$ of the cavity. Is this conclusion correct?

Solution: The described electric field distributions would indeed satisfy the wave equation (7.3) inside the cavity ($0 \leq r \leq R$) and the coarse-grain boundary conditions (7.104)

$$E_\theta = E_\varphi = 0,$$

on its surface. However, as it follows from simple symmetry arguments[77] (and as MA Eq. (10.11) confirms), the curl of such a purely radial field equals zero at all points, so according to the first of the Maxwell equations (6.100a), the corresponding $\partial\mathbf{B}/\partial t$ vanishes everywhere in the cavity, making the standing electromagnetic wave impossible.

Just for the reader's reference, the actual fundamental frequency of the cavity is even a bit lower: $\omega_{11} = \zeta_{11}(v/R)$, where $\zeta_{11} \approx 2.744 < \pi$ is the first root of the following transcendental equation:

$$\frac{d}{d\zeta}\left[\zeta\, j_1(\zeta)\right] = 0, \qquad \text{where } j_1(\zeta) \equiv \frac{\sin\zeta}{\zeta^2} - \frac{\cos\zeta}{\zeta}.$$

This $j_1(\zeta)$ is one of the so-called *spherical Bessel functions* $j_l(\zeta)$.[78] (The scalar potential's distribution suggested in the assignment is proportional to another function of the same set: $\phi \propto j_0(\zeta) = \sin\zeta/\zeta$, with $\zeta = kr$.)

Problem 8.2. Simplify the Lorentz reciprocity theorem (6.121) for space-localized field sources. Then find out what it says about the fields of two compact, well-separated sources of electric-dipole radiation.

Solution: Let both field sources $\mathbf{j}_1(\mathbf{r})$ and $\mathbf{j}_2(\mathbf{r})$, participating in the Lorentz theorem,

$$\oint_S \left(\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1\right)_n d^2r = \int_V \left(\mathbf{E}_2 \cdot \mathbf{j}_1 - \mathbf{E}_1 \cdot \mathbf{j}_2\right) d^3r, \qquad (*)$$

be localized in space, and select the surface S in the form of a much larger sphere, with the center at these sources. As we know from Sec. 8.1 of the lecture notes, the fields induced by any localized source are generally a combination of a radiated wave decreasing with distance as $1/r$ and near-zone fields decreasing even faster. Since the sphere's surface grows with its radius as r^2, the contribution of the

[77] Indeed, if the vector $\nabla\times\mathbf{E}$ was not equal to zero, where would it be directed without violating the spherical symmetry? (By the curl's definition, it cannot be radial.)

[78] These functions, already mentioned in Sec. 2.7, will be discussed in the QM part of this series.

latter fields to the integral on the left-hand side of Eq. (*) tends to zero at $r \to 0$. On the other hand, the wave fields are, on the sphere's surface, quasi-planar, i.e. obey Eq. (7.6),

$$\mathbf{E}_1 = Z\mathbf{H}_1 \times \mathbf{n}, \qquad \mathbf{E}_2 = Z\mathbf{H}_2 \times \mathbf{n},$$

where in this case, $\mathbf{n} \equiv \mathbf{r}/r$, so the expression under the surface integral is proportional to

$$(\mathbf{H}_1 \times \mathbf{n}) \times \mathbf{H}_2 - (\mathbf{H}_2 \times \mathbf{n}) \times \mathbf{H}_1 \equiv \mathbf{H}_1 \times (\mathbf{H}_2 \times \mathbf{n}) - \mathbf{H}_1 \times (\mathbf{H}_2 \times \mathbf{n}).$$

Applying to each of these double vector products the *bac minus cab* rule,[79] we get

$$[\mathbf{H}_2(\mathbf{H}_1 \cdot \mathbf{n}) - \mathbf{n}(\mathbf{H}_1 \cdot \mathbf{H}_2)] - [\mathbf{H}_1(\mathbf{H}_2 \cdot \mathbf{n}) - \mathbf{n}(\mathbf{H}_2 \cdot \mathbf{H}_1)] \equiv \mathbf{H}_2(\mathbf{H}_1 \cdot \mathbf{n}) - \mathbf{H}_1(\mathbf{H}_2 \cdot \mathbf{n}).$$

But the fields of these transverse waves are normal to the direction **n** of their propagation, so both terms in the last form of this expression, and hence the whole left-hand side of Eq. (*) vanish, and the Lorentz reciprocity relation (as a reminder, valid for complex amplitudes of ac fields) is reduced to simply

$$\int_V \mathbf{E}_1(\mathbf{r}) \cdot \mathbf{j}_2(\mathbf{r}) d^3r = \int_V \mathbf{E}_2(\mathbf{r}) \cdot \mathbf{j}_1(\mathbf{r}) d^3r, \qquad (**)$$

i.e. becomes very similar to the reciprocal relation for dc fields in electrostatics,

$$\int_V \phi_1(\mathbf{r}) \rho_2(\mathbf{r}) d^3r = \int_V \phi_2(\mathbf{r}) \rho_1(\mathbf{r}) d^3r, \qquad (***)$$

which was proved in Chapter 1.

Formula (**) is an important tool for electromagnetic system design. In particular, it means that the angular distribution of the radiation emitted by an antenna is similar to its receiving pattern. Similarly, applied to a system that may either emit or absorb radiation (e.g., an atom), it implies that if the system is subjected to the external radiation handcrafted to reproduce its own emission pattern, the radiation may be fully absorbed, without any reflection – the effect sometimes called *coherent perfect absorption*.[80]

Now note that in Eq. (**), just as in Eq. (***), each integration may be restricted to the volume where the corresponding source density, either $\mathbf{j}(\mathbf{r})$ or $\rho(\mathbf{r})$, is different from zero. Hence if we consider two sources of dipole radiation, each of a size small in comparison not only with the distance between them but also with the emitted wavelength, then both electric field vectors in Eq. (**) may be taken out of the integrals:

$$\mathbf{E}_1(\mathbf{r}_2) \cdot \int_V \mathbf{j}_2(\mathbf{r}) d^3r = \mathbf{E}_2(\mathbf{r}_1) \cdot \int_V \mathbf{j}_1(\mathbf{r}) d^3r.$$

But according to Eq. (8.22) of the lecture notes, such integrals of the instant current densities $\mathbf{j}(\mathbf{r}, t)$ are just the time derivatives of the corresponding dipole moments. Since for the monochromatic fields we are discussing, the time derivative is equivalent to the multiplication by a constant coefficient ($-i\omega$), these coefficients on both sides of the reciprocity relation cancel, and it becomes simply

$$\mathbf{E}_1(\mathbf{r}_2) \cdot \mathbf{p}_2 = \mathbf{E}_2(\mathbf{r}_1) \cdot \mathbf{p}_1.$$

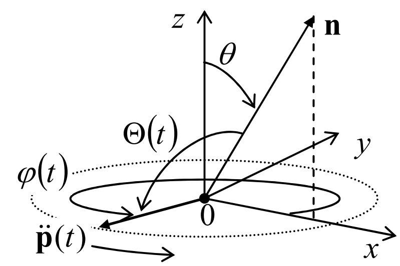
[79] See, e.g., MA Eq. (7.5).

[80] See, e.g., the experiments by W. Wan *et al*., *Science* **331**, 889 (2011).

Problem 8.3. In the electric-dipole approximation, calculate the angular distribution and the total power of electromagnetic radiation by the hydrogen atom within the following classical model: an electron rotates, at a constant distance R, about a much heavier proton. Use the result to calculate the law of a gradual reduction of R in time. Finally, evaluate the classical lifetime of the atom by borrowing the initial value of R from quantum mechanics: $R(0) = r_B \approx 0.53\times10^{-10}$ m.

Solution: For the radial component of the instantaneous Poynting vector, we can use Eq. (8.26) of the lecture notes:

$$S_r = \frac{Z}{(4\pi vr)^2}\ddot{p}^2 \sin^2\Theta.$$



In our current problem, with the dipole moment $\mathbf{p} = q\mathbf{R} = -e\mathbf{R}$ (where $\mathbf{R}$ is the electron's radius vector, with $R << r$) rotating, the angle Θ is a function of time – see the figure on the right, where the z-axis is taken to be normal to the rotation plane. The time average of $\sin^2\Theta$ may be calculated in several equivalent ways, for example by returning to Eq. (8.24) to write

$$\ddot{p}^2 \sin^2\Theta = \left|\mathbf{n}\times\ddot{\mathbf{p}}\right|^2,$$

and calculating this vector product in some immobile reference frame – for example, that shown in the figure above, with the x- and y-axes, both within the plane of particle's rotation, chosen so that the observation point $\mathbf{r}$ is within the plane $[x, z]$, i.e. for that point, $y = 0$. As the figure shows, in this frame,

$$\mathbf{n} = \mathbf{n}_x \sin\theta + \mathbf{n}_z \cos\theta, \qquad \ddot{\mathbf{p}} = eR\omega^2\left(\mathbf{n}_x \cos\varphi + \mathbf{n}_y \sin\varphi\right), \qquad \text{with } \varphi \equiv \omega t + \text{const},$$

where ω is the angular velocity of the electron's rotation. Now the vector multiplication yields

$$\mathbf{n}\times\ddot{\mathbf{p}}(t) = eR\omega^2\begin{vmatrix}\mathbf{n}_x & \mathbf{n}_y & \mathbf{n}_z\\ \sin\theta & 0 & \cos\theta\\ \cos\varphi & \sin\varphi & 0\end{vmatrix} = eR\omega^2\left(-\mathbf{n}_x\cos\theta\sin\varphi + \mathbf{n}_y\cos\theta\cos\varphi + \mathbf{n}_z\sin\theta\sin\varphi\right).$$

Averaging the square of this vector over the period of rotation, i.e. over the interval $\Delta\varphi = 2\pi$, we get

$$\overline{(\ddot{p}\sin\Theta)^2} = q^2R^2\omega^4\overline{\left(\cos^2\theta\sin^2\varphi + \cos^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi\right)} = e^2R^2\omega^4\left(\cos^2\theta + \frac{1}{2}\sin^2\theta\right).$$

This formula shows that the angular distribution of the average radiated power is indeed different from that produced by an oscillating dipole of a fixed orientation: the radiation is strongest at $\theta = 0$ and $\theta = \pi$, i.e. along the direction normal to the charge rotation plane (in our notation, axis z), but is also nonvanishing along any other direction. The total average power of the radiation is

$$\mathscr{P} = \int_{4\pi}\overline{S_r}r^2 d\Omega = Z_0\left(\frac{\omega^2 eR}{4\pi c}\right)^2 2\pi\int_0^{\pi}\left(\cos^2\theta + \frac{1}{2}\sin^2\theta\right)\sin\theta d\theta = Z_0\left(\frac{\omega^2 eR}{4\pi c}\right)^2 2\pi\frac{4}{3} = \frac{Z_0\omega^4 e^2R^2}{6\pi c^2}.$$

Comparing this result with Eq. (8.29) for a fixed-direction dipole, we see that the radiation power by a rotating charge equals a sum of those from its two oscillating components $p_x(t)$ and $p_y(t)$, calculated independently. The physical reason for this independence (which is not immediately apparent and requires a proof, for example, the one given above) is that the polarization directions of the partial waves radiated by the dipole components are mutually perpendicular.

Proceeding to the calculation of the classical atom's lifetime, elementary classical mechanics says that at a circular motion of a non-relativistic particle in the Coulomb attractive field (with the potential energy $U = -e^2/4\pi\varepsilon_0 R < 0$), its kinetic energy $T \equiv mv^2/2 = -U/2 > 0$, so the full energy $\mathscr{E} \equiv T + U$ equals $U/2$. In our case, this means that

$$U = -\frac{e^2}{4\pi\varepsilon_0 R}, \qquad \mathscr{E} = -\frac{e^2}{8\pi\varepsilon_0 R}, \qquad T = \frac{e^2}{8\pi\varepsilon_0 R}, \quad \text{so } \omega^2 = \frac{v^2}{R^2} = \frac{2T/m_\text{e}}{R^2} = \frac{e^2}{4\pi\varepsilon_0 m_\text{e} R^3}.$$

Weak radiation of power $\mathscr{P} << \omega\mathscr{E}$ causes a relatively slow energy reduction: $d\mathscr{E}/dt = -\mathscr{P}$, where $\mathscr{P}$ may be calculated (as this was done above) for a circular orbit. In our particular case, this equation is

$$\frac{d\mathscr{E}}{dt} \equiv -\frac{e^2}{8\pi\varepsilon_0}\frac{d}{dt}\left(\frac{1}{R}\right) = -\frac{Z_0\omega^4 e^2 R^2}{6\pi c^2} \equiv -\frac{Z_0 e^6}{6\pi c^2 (4\pi\varepsilon_0)^2 m_\text{e}^2}\frac{1}{R^4} < 0.$$

This equation, describing a monotonic increase of $(1/R)$, i.e., the electron's fall on the point-like nucleus (proton), may be rewritten in the dimensionless form

$$\tau\dot{\xi} = -\frac{1}{3\xi^2}, \qquad \text{i.e. } 3\xi^2 d\xi = -\frac{dt}{\tau},$$

where $\xi \equiv R(t)/R(0)$, and τ is the time scale of the decay process:

$$\tau \equiv \frac{4\pi^2 m_\text{e}^2 R^3(0)}{e^4 Z_0 \mu_0}.$$

This dimensionless differential equation may be readily integrated to give

$$\xi(t) = \left(1 - \frac{t}{\tau}\right)^{1/3}, \qquad \text{i.e. } R(t) = R(0)\left(1 - \frac{t}{\tau}\right)^{1/3},$$

showing that $R(t)$ vanishes (i.e. the electron drops on the proton) exactly at $t = \tau$. For an electron ($e \approx -1.60\times10^{-19}$ C, $m_\text{e} \approx 0.91\times10^{-30}$ kg), initially rotating at the Bohr radius $R(0) = r_\text{B} \approx 0.53\times10^{-10}$ m, the above formula gives $\tau \approx 1.56\times10^{-11}$ s.

Such atomic collapse is very fast on the human scale of events (so let us thank quantum mechanics for preventing this disaster!), but since for our parameters $\omega \sim 4\times10^{16}$ s^{-1}, this process is slow on the rotation period scale: $\omega\tau \sim 10^6 >> 1$. This strong relation justifies this gradual approach to the problem, which neglects the loss of energy (and hence of R) during one rotation period, at the radiated power calculation stage. Another necessary sanity check is that the radiation wavelength scale, $\lambda = 2\pi c/\omega \sim 5\times10^{-8}$ m, is much larger than $R \sim 10^{-10}$ m, so the radiating charge could indeed be treated as an electric dipole – as it was.

Problem 8.4. A non-relativistic particle of mass m, with electric charge q, is placed into a time-independent uniform magnetic field **B**. Derive the law of decrease of the particle's kinetic energy due to its electromagnetic radiation at the *cyclotron frequency* $\omega_\text{c} = qB/m$. Evaluate the rate of such radiation cooling for electrons in a magnetic field of 1 T, and estimate the electron energy interval in which this result is quantitatively correct.

Hint: The cyclotron motion will be discussed in detail (for arbitrary particle velocities) in Sec. 9.6 of the lecture notes, but I hope that the reader already knows that in the non-relativistic case ($v \ll c$), the above formula for ω_c may be readily obtained by combining the 2nd Newton law $mv_\perp^2/R = qv_\perp B$ for the particle's circular rotation under the effect of the magnetic component of the Lorentz force (5.10), and the geometric relation $v_\perp = R\omega_c$. (Here $\mathbf{v}_\perp$ is the particle's velocity in the plane normal to the vector $\mathbf{B}$.)

Solution: Let us assume that the particle's energy loss because of the radiation during one revolution is much smaller than the energy itself. (This condition has to be verified *a posteriori* for the application of the calculation result to any particular system.) Then its orbit in the plane normal to the field is approximately circular, and we may reuse the following intermediate result of the solution of the previous problem: the power of the electric-dipole radiation into the free space by a particle of charge q, moving on a circular orbit of radius R with angular velocity ω, averaged over the rotation period, is

$$\mathscr{P} = \frac{Z_0 q^2 \omega^4 R^2}{6\pi c^2} \equiv \frac{Z_0 q^2}{6\pi c^2}\omega^2 \left(\omega R\right)^2 .$$

For our current case, $\omega = \omega_c = qB/m$, and $\omega R = v_\perp$, so[81]

$$\mathscr{P} = \frac{Z_0 q^2}{6\pi c^2}\left(\frac{qB}{m}\right)^2 v_\perp^2 \equiv \frac{Z_0 q^4 B^2}{3\pi c^2 m^3} T , \qquad (*)$$

where $T \equiv mv_\perp^2/2$ is the kinetic energy of the particle's motion in the plane normal to the magnetic field's direction. The magnetic field's force $q\mathbf{v}\times\mathbf{B}$ cannot do any work on the particle, i.e. does not change its kinetic energy, so we may represent the left-hand side of Eq. (*) as the kinetic energy's decrease rate, $-dT/dt$. As a result, we get a very simple equation for the energy's evolution:

$$\frac{dT}{dt} = -\frac{T}{\tau}, \qquad \text{where } \tau \equiv \frac{3\pi c^2 m^3}{Z_0 q^4 B^2},$$

with the evident solution $T(t) = T(0)\exp\{-t/\tau\}$, so the constant τ defined above gives a fair scale of the energy decay. (Note that in contrast with the previous problem, the decay time is independent of the initial energy of the particle.)

For an electron ($q = -e \approx 1.60\times10^{-19}$ C, $m = m_e \approx 0.91\times10^{-30}$ kg) in the field B = 1 T, the above expression yields $\tau \approx 2.5$ s, while $\omega_c \approx 1.8\times10^{11}\text{s}^{-1}$. This means that the relative energy loss $\Delta T/T \approx 2\pi/\omega_c\tau$ during one rotation is indeed very small ($\sim10^{-11}$), so the applied approximation of constant energy at each revolution is indeed valid.

Another condition of validity of our result is imposed by the applicability of the electric dipole approximation: $k_c R \ll 1$, where $k_c \equiv \omega_c/c$ is the radiation's wave vector. Since $R = v_\perp/\omega_c$, this condition is reduced to $v_\perp \ll c$, i.e. that the particle is non-relativistic, meaning that energy T has to be much *lower* than mc^2. (For an electron, $mc^2 \approx 0.5$ MeV.)

[81] The generalization of this relation to the relativistic case (when this effect is referred to as *synchrotron radiation*) will be discussed in Sec. 10.3 of the lecture notes.

Finally, we have treated the particle as a classical one. In the magnetic field, this treatment is only valid if T is much *higher* than the distance $\hbar\omega_c$ between the so-called *Landau levels*,[82] where $\hbar \approx 1.055\times10^{-34}$ J·s is the Planck constant. For the above example, $\hbar\omega_c \approx 0.12$ meV. Thus, the above evaluation is valid within at least 9 orders of the electron's energy.

<u>Problem 8.5</u>. A particle with mass m, electric charge q, and an initial kinetic energy $T << mc^2$ collides head-on with a much more massive particle of charge $\mathcal{Z}q$, in free space. Calculate the total energy of electromagnetic radiation during this collision, assuming it to be much lower than T.

Solution: The last condition in the assignment means that we may calculate the particle's velocity and acceleration neglecting the energy losses to radiation. Also, due to the large mass of the heavier particle, we may consider it as an immobile center providing a repulsive Coulomb force described by Eq. (1.3). In our case of a head-on collision, the force has just one Cartesian component:

$$F = \frac{\mathcal{Z}q^2}{4\pi\varepsilon_0 r^2},$$

where r is the distance between the particles, so according to the 2nd Newton equation of the 1D motion of the lighter particle, its acceleration is

$$\ddot{r} = \frac{F}{m} = \frac{\mathcal{Z}q^2}{4\pi\varepsilon_0 m r^2}. \qquad (*)$$

Hence, according to the Larmor formula (8.26) with $p = qr$, the instantaneous power of the electric-dipole radiation (which dominates the electromagnetic radiation of this non-relativistic system) is

$$\mathscr{P} = \frac{Z_0}{6\pi c^2}\ddot{p}^2 = \frac{Z_0}{6\pi c^2}\left(\frac{\mathcal{Z}q^3}{4\pi\varepsilon_0 m r^2}\right)^2.$$

Due to the time symmetry of the collision process with respect to the nearest-approach point $r_0 = r(t_0)$, the total radiated energy may be calculated as

$$\mathscr{E}_{\text{rad}} = 2\int_{t_0}^{\infty}\mathscr{P}dt = 2\frac{Z_0}{6\pi c^2}\left(\frac{\mathcal{Z}q^3}{4\pi\varepsilon_0 m}\right)^2\int_{t_0}^{\infty}\frac{dt}{r^4} = 2\frac{Z_0}{6\pi c^2}\left(\frac{\mathcal{Z}q^3}{4\pi\varepsilon_0 m}\right)^2\int_{r_0}^{\infty}\frac{dr}{u(r)r^4}, \quad \text{where } u \equiv \frac{dr}{dt}. \qquad (**)$$

The function $u(r)$, needed for working out this integral, may be obtained from the energy conservation law – essentially the first integral of Eq. (*):

$$\frac{mu^2}{2} + \frac{\mathcal{Z}q^2}{4\pi\varepsilon_0 r} = T, \qquad \text{giving } u = \left[\frac{2T}{m}\left(1-\frac{r_0}{r}\right)\right]^{1/2}, \qquad \text{where } r_0 \equiv \frac{\mathcal{Z}q^2}{4\pi\varepsilon_0 T}.$$

so Eq. (**) becomes

$$\mathscr{E}_{\text{rad}} = 2\frac{Z_0}{6\pi c^2}\left(\frac{\mathcal{Z}q^3}{4\pi\varepsilon_0 m}\right)^2\left(\frac{m}{2T}\right)^{1/2}\frac{1}{r_0^3}I, \qquad \text{where } I \equiv \int_1^{\infty}\frac{d\xi}{(1-1/\xi)^{1/2}\xi^4}.$$

[82] See, e.g., QM Sec. 3.2.

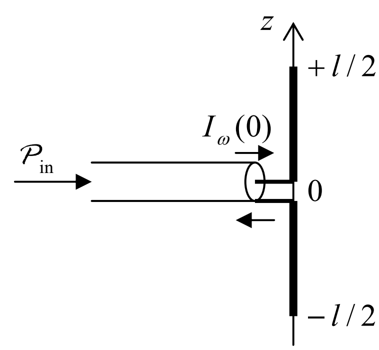
I may be reduced to a table integral[83] by using the variable substitution $\xi \equiv 1 + \zeta^2$:

$$I \equiv \int_1^{\infty} \frac{d\xi}{\left(1-1/\xi\right)^{1/2} \xi^4} = \int_{-\infty}^{+\infty} \frac{d\zeta}{\left(1+\zeta^2\right)^{7/2}} = \frac{16}{15},$$

so we finally get

$$\mathscr{E}_{\text{rad}} = 2\frac{Z_0}{6\pi c^2}\left(\frac{\mathcal{Z}q^3}{4\pi\varepsilon_0 m}\right)^2 \left(\frac{m}{2T}\right)^{1/2} \left(\frac{4\pi\varepsilon_0 T}{\mathcal{Z}q^2}\right)^3 \frac{16}{15} \equiv \frac{64}{45\sqrt{2}}\frac{1}{\mathcal{Z}}\left(\frac{T}{mc^2}\right)^{3/2} T.$$

This result (curiously, independent of q!) is important because it gives the scale (and, moreover, the largest value) of the radiation energy at the Coulomb scattering with an arbitrary impact parameter b. In particular, it shows that for non-relativistic particles, with $T << mc^2$, the initial assumption that the particle's energy loss to radiation is always relatively small, $\mathscr{E}_{\text{rad}} << T$, is indeed justified.



Problem 8.6. Solve the dipole antenna radiation problem discussed in Sec. 8.2 of the lecture notes (see Fig. 8.3, partly reproduced on the right) for the optimal value $l = \lambda/2$ of its length, assuming[84] that the current distribution in each of its arms is sinusoidal: $I(z, t) = I_0\cos(\pi z/l) \cos\omega t$.

Solution: Since the main condition of the dipole approximation condition, $kl << 1$, is not satisfied for the antenna that long ($kl \equiv (2\pi/\lambda)l = \pi$), we cannot use the formulas derived in Sec. 8.2, and need to return to the general expressions (8.17) for the retarded potentials. However, we may use the experience of Sec. 8.2, indicating that the fields in the far-field zone ($kr >> 1$) may be more readily calculated from the vector potential than from the scalar potential. Integrating Eq. (8.17b) over the antenna wire's cross-section, we get the following expression for the vector potential in the free space around the antenna: [85]

$$\mathbf{A}(\mathbf{r},t) = \mathbf{n}_z \frac{\mu_0 I_0}{4\pi}\int_{-l/2}^{+l/2} \cos\frac{\pi z'}{l}\cos\left[\omega\left(t-\frac{R}{c}\right)\right]\frac{dz'}{R} \equiv \mathbf{n}_z \frac{\mu_0 I_0}{4\pi}\text{Re}\left[\int_{-l/2}^{+l/2}\cos\frac{\pi z'}{l}\exp\left\{-i\omega\left(t-\frac{R}{c}\right)\right\}\frac{dz'}{R}\right], \quad (*)$$

where $R = [\rho^2 + (z - z')^2]^{1/2}$, with ρ and z being the cylindrical coordinates of the observation point $\mathbf{r}$ in the reference frame shown in the figure above. At large distances, $r \equiv (\rho^2 + z^2)^{1/2} >> l$, the product in the complex exponent may be approximated as

$$\omega\left(t-\frac{R}{c}\right) \equiv \omega t - k\left[\rho^2 + (z-z')^2\right]^{1/2} \approx \omega t - k\left(\rho^2 + z^2 - 2zz'\right)^{1/2} \approx \omega t - kr\left(1-\frac{zz'}{r^2}\right) \equiv (\omega t - kr) + kz'\cos\Theta,$$

where $k = \omega/c$ is the wave number and $\Theta \equiv \cos^{-1}(z/r)$ is the angle between the direction toward the observation point and the z-axis. In the same limit, R in the denominator of Eq. (*) may be approximated

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[83] See, e.g., MA Eq. (6.5d) with $n = 3$.

[84] As was emphasized in the lecture notes, this is a reasonable guess rather than a controllable approximation. The exact (rather involved!) theory shows that this assumption gives errors ~5%, depending on the wire's diameter.

[85] Just as in Sec. 8.2, we ignore the possible distortions of the field by the generator and/or transmission line used to create the current $I(0, t)$ in the antenna. Since the TEM line's cross-section may be much less than λ, in many practical cases this is a very good approximation.

with r, and moved out of the integral over z'.[86] These simplifications enable us to reduce Eq. (*), in the far-field zone, to

$$\mathbf{A}(\mathbf{r},t) \approx \mathbf{n}_z \frac{\mu_0 I_0}{4\pi r} \operatorname{Re}\left[\exp\{i(kr-\omega t)\}\int_{-l/2}^{+l/2}\cos\frac{\pi z'}{l}\exp\{-ikz'\cos\Theta\}dz'\right], \qquad \text{at } kr, kl >> 1, \qquad (**)$$

and to work out the remaining integral analytically:

$$\int_{-l/2}^{+l/2}\cos\frac{\pi z'}{l}\exp\{-ikz'\cos\Theta\}dz' \equiv \int_{-\pi/2k}^{+\pi/2k}\cos kz'\exp\{-ikz'\cos\Theta\}dz' \equiv \frac{1}{2}\sum_{\pm}\int_{-\pi/2k}^{+\pi/2k}\exp\{ikz'(\pm 1-\cos\Theta)\}dz'$$

$$= \frac{1}{2}\sum_{\pm}\frac{\exp\{ikz'(\pm 1-\cos\Theta)\}}{ik(\pm 1-\cos\Theta)}\Bigg|_{kz'=-\pi/2}^{kz'=+\pi/2} \equiv \sum_{\pm}\frac{\sin\left(\pm\frac{\pi}{2}-\frac{\pi}{2}\cos\Theta\right)}{k(\pm 1-\cos\Theta)} \equiv 2\frac{\cos\left(\frac{\pi}{2}\cos\Theta\right)}{k\sin^2\Theta}.$$

Since this result is purely real, the operator in Eq. (**) is redundant and the vector potential is

$$\mathbf{A}(\mathbf{r},t) = \mathbf{n}_z\frac{\mu_0 I_0}{2\pi}\frac{\cos\left(\frac{\pi}{2}\cos\Theta\right)}{\sin^2\Theta}\frac{\cos(\omega t-kr)}{kr} \equiv \mathbf{n}_z\frac{\mu_0 I(0,t-r/c)}{2\pi kr}\frac{\cos\left(\frac{\pi}{2}\cos\Theta\right)}{\sin^2\Theta}.$$

The structure of this expression is exactly the same as that of Eq. (8.23), with the following replacement:

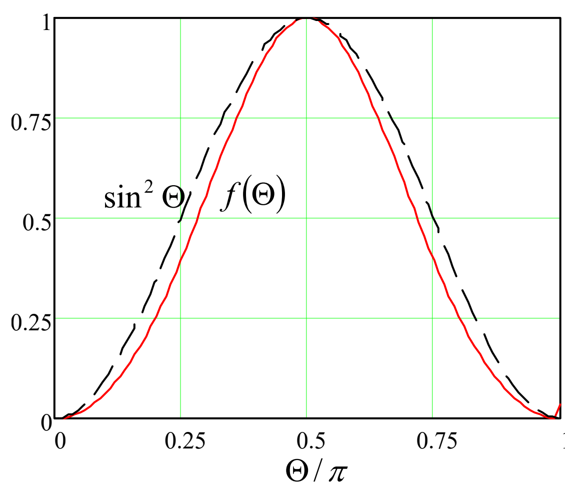
$$\dot{\mathbf{p}}(t) \to \mathbf{n}_z\frac{2I(0,t)}{k}\frac{\cos\left(\frac{\pi}{2}\cos\Theta\right)}{\sin^2\Theta},$$

and we may repeat all the steps from that formula to Eq. (8.26) to get the following radiation power density, i.e. the Poynting vector' radial component:

$$S_r = \frac{Z_0}{(4\pi cr)^2}\left(\frac{2\dot{I}(0,t)}{k}\right)^2 f(\Theta) = \frac{Z_0 I_0^2\cos^2(\omega t-kr)}{4\pi^2 r^2}f(\Theta),$$

where

$$f(\Theta) \equiv \frac{\cos^2[(\pi/2)\cos\Theta]}{\sin^2\Theta}.$$



This expression shows that the angular distribution of the radiation, shown in the figure on the right, is somewhat (though not too much) different from that ($\sin^2\Theta$) of a radiating dipole – in particular, of a short dipole antenna analyzed in Sec. 8.2. Because of that, its solid-angle integral,

$$J \equiv \oint_{4\pi} f(\Theta)d\Omega = 2\pi\int_0^{\pi} f(\Theta)\sin\Theta d\Theta,$$

[86] Note that these are essentially the same approximations as used at the derivation of Eq. (8.78), which expresses the Huygens principle, so the reader who needs more detailed explanations may be referred to that part of the lecture notes.

is also slightly different – it equals approximately 7.658 instead of $8\pi/3 \approx 8.378$ for $\sin^2\Theta$. As a result, the total average radiation power is

$$\mathscr{P} \equiv r^2 \oint_{4\pi} \overline{S}_r d\Omega \approx Z_0 \frac{J}{4\pi^2} \frac{I_0^2}{2}.$$

Now, just as was done in Sec. 8.2 for the short antenna, we may recalculate this result into the antenna's impedance as "seen" by the generator (or transmission line) feeding it:

$$\operatorname{Re} Z_A = \frac{J}{4\pi^2} Z_0 \approx \frac{7.658}{4\pi^2} Z_0 \approx 73.1\,\Omega.$$

This impedance is very close to one of the coaxial cable standard values (75 Ω), making its matching with the antenna straightforward. (If this statement is not clear, please revisit the discussion at the end of Sec. 7.5 of the lecture notes.) Again, all these results are only valid with an accuracy of a few percent, because they are based on the exactly sinusoidal (rather than self-consistently calculated) distribution of the current along the antenna's length.

Problem 8.7. A plane wave is scattered by a localized object in free space. Relate the differential cross-section of the wave's scattering to the average force it exerts on the object. Use this general relation to calculate the force exerted by a plane monochromatic wave on a free non-relativistic particle and compare the result with those obtained in Problems 7.4 and 7.5.

Solution: According to Eq. (6.115) of the lecture notes, a plane wave with a Poynting vector **S** carries linear momentum equal to $c\mathbf{g} = \mathbf{S}/c$ per unit time per unit front area. In the problem we are considering, this expression is applicable not only to the incident wave but also to the scattered wave if its intensity **S** is measured (as it is accepted in virtually all theories of scattering) at a sufficiently large distance from the scatterer where the wave is locally-planar.

Per the 2nd Newton law, the force **F** exerted on an object has to be equal to the change of its linear moment per unit time, and according to the 3rd Newton law, this change has to be equal and opposite to the change of the net momentum (also per unit time) of the agents exerting the force – in our current case, of the electromagnetic waves. By using the definitions of the full and differential cross-sections, for the values averaged over the wave's period,[87] this balance may be represented as

$$\overline{\mathbf{F}} = \frac{\overline{\mathbf{S}}_{\text{incident}}}{c}\sigma - \oint_{4\pi} \frac{\overline{\mathbf{S}}_{\text{scattered}}}{c} r^2 d\Omega \equiv \frac{\overline{S}_{\text{incident}}}{c}\left(\mathbf{n}_0\sigma - \oint_{4\pi} \mathbf{n}\frac{d\sigma}{d\Omega}d\Omega\right), \qquad (*)$$

where $\mathbf{n}_0$ is the direction of the incident wave and **n** is that of the scattered one – over which the integration has to be carried out.[88]

Superficially, Eq. (*) contradicts the results obtained in the solution of Problems 7.4 and 7.5. Indeed, in the first of these solutions, we have concluded that a traveling wave does *not* exert any average force on a free point charge. This is not what Eq. (*) predicts. Indeed, according to Eqs. (8.27) and (8.36), the charge's differential cross-section is proportional to $\sin^2\Theta$, where Θ is the angle between

[87] Such averaging allows us not to worry about the (possibly, different) time delays between the waves' interaction with the object and the measurement of their intensity.

[88] Alternatively, this relation may be derived from the Maxwell stress tensor, to be discussed in Sec. 9.8.

the electric field vector **E** of the incident wave and the direction **n** toward the observer. Taking the direction of **E** for the z-axis (so that Θ would coincide with the usual polar angle θ), we see that all Cartesian components of the second term in Eq. (*) vanish:

$$\oint_{4\pi} \mathbf{n}\frac{d\sigma}{d\Omega}d\Omega \propto \oint_{4\pi} \mathbf{n}\sin^2\theta d\Omega = \int_0^{\pi}\sin\theta d\theta \int_0^{2\pi} d\varphi \left(\mathbf{n}_x \sin\theta\cos\varphi + \mathbf{n}_y \sin\theta\sin\varphi + \mathbf{n}_z \cos\theta\right)\sin^2\theta$$
$$= 2\pi\mathbf{n}_z \int_0^{\pi}\sin\theta d\theta \cos\theta \sin^2\theta = 2\pi\mathbf{n}_z\int_{-1}^{+1}\left(1-\cos^2\theta\right)\cos\theta\, d(\cos\theta) = 0.$$

Hence we are left with the first term; according to Eqs. (7.9b) and (8.40), it gives a non-zero force directed along the propagation of the incident wave:

$$\overline{\mathbf{F}} = \frac{S_{\text{incident}}}{c}\sigma\,\mathbf{n}_0 = \frac{\overline{E^2}}{Z_0 c}\frac{Z_0^2 q^4}{6\pi c^2 m^2}\mathbf{n}_0 \neq 0, \qquad (**)$$

where q is the particle's charge and m is its mass.

In order to reveal the nature of this contradiction, let us compare the magnitude of the force given by Eq. (**) with the maximum value of the force exerted by a *standing* plane wave of frequency ω, which was calculated in Problem 7.5 using the same approach as in Problem 7.4:

$$\left|\overline{F}'\right|_{\max} = \frac{q^2}{m\omega c}E_\omega E_\omega^* = \frac{q^2}{m\omega c}2\overline{E^2},$$

where $k = \omega/c$ is the wave number. The comparison yields a ratio independent of the wave's amplitude:

$$\frac{\left|\overline{F}\right|}{\left|\overline{F}'\right|} = \frac{Z_0 q^4}{6\pi c^3 m^2}\Bigg/\frac{2q^2}{m\omega c} \equiv \frac{kr_{\text{c}}}{3} = \frac{2r_{\text{c}}}{\lambda},$$

where $\lambda = 2\pi/k$ is the wavelength and $r_{\text{c}} \equiv q^2/4\pi\varepsilon_0 mc^2$ is the classical radius of the particle (for an electron, $\sim 3\times10^{-15}$ m – see Eq. (8.41) and its discussion), so the above ratio is extremely small for all frequencies where classical electrodynamics makes sense. As a result, the expressions obtained in the solutions of Problems 7.4 and 7.5 are sufficient for virtually all practical purposes. (The reason why they are not exact is that at their derivation, the contribution of the scattered wave into the ac electric field applied to the particle has been neglected.[89])

One more remark: as it follows from the above derivation of Eq. (*), if the object not only scatters but also partly absorbs the incident wave, the corresponding total cross-section σ_{a} (see Problem 15 below) has to be added to its first term, while the second term remains unaltered.

Problem 8.8. Use the Lorentz oscillator model of a bound charge, described by Eq. (7.30) of the lecture notes, to explore the transition between the two scattering limits discussed in Sec. 8.3 of the lecture notes and, in particular, the *resonant scattering* taking place at $\omega \approx \omega_0$. In the last context, discuss the contribution of the scattering to the oscillator's damping.

Solution: The differential equation Eq. (7.30) describing the charge dynamics is linear; hence the forced oscillations $x(t)$ of the charge's displacement and its dipole moment $p(t) = qx(t)$, induced by a

[89] Such problems are common for classical electrodynamics and will be further discussed in Sec. 10.6.

monochromatic incident wave with frequency ω, are sinusoidal, with the same frequency. The average power of the wave radiated (in the case of scattering, re-radiated, i.e. scattered) by such a sinusoidally oscillating dipole into free space is given by Eq. (8.28) with $Z = Z_0$ and $v = c$:

$$\overline{\mathscr{P}} = \frac{Z_0\omega^4}{12\pi c^2}\left| p_\omega \right|^2 = \frac{Z_0\omega^4}{12\pi c^2} q^2 \left| x_\omega \right|^2 , \tag{*}$$

where x_ω is the complex amplitude of the charge's oscillations. Using Eq. (7.31) to express this amplitude via that of the electric field of the incident wave, we get

$$\overline{\mathscr{P}} = \frac{Z_0\omega^4 q^4}{12\pi m^2 c^2} \frac{\left| E_\omega \right|^2}{\left(\omega_0^2 - \omega^2\right)^2 + \left(2\omega\delta_0\right)^2},$$

so Eq. (8.39) yields the following total cross-section of scattering:

$$\sigma = \frac{Z_0^2 q^4}{6\pi\ m^2 c^2} \frac{\omega^4}{\left(\omega_0^2 - \omega^2\right)^2 + \left(2\omega\delta_0\right)^2}. \tag{**}$$

In the high-frequency limit, $\omega/\omega_0 \to \infty$, this expression reduces to the Thomson scattering formula (8.40)-(8.41), while in the opposite limit, $\omega/\omega_0 \to 0$, it leads to Eq. (8.45) with

$$\alpha \equiv \frac{q x_\omega}{E_\omega}\Big|_{\omega \ll \omega_0} = \frac{q^2}{m\omega_0^2},$$

and hence to the Rayleigh scattering. However, Eq. (**) shows that on the way between these two limits, the scattering cross-section may have a very high peak at $\omega \approx \omega_0$, with a height limited only by the oscillator's damping:

$$\sigma\Big|_{\omega=\omega_0} = \frac{Z_0^2 q^4}{6\pi m^2 c^2}\left(\frac{\omega_0}{2\delta_0}\right)^2 \equiv Q^2 \sigma\Big|_{\omega\to\infty},$$

where Q is the Q-factor of the resonance, which may be much larger than 1. This is *resonant scattering*.

The damping may be contributed not only by some internal energy losses inside the bound charge system but also by the scattering itself. Indeed, the average rate of the energy loss (i.e. the dissipation power) at forced oscillations of a harmonic oscillator is[90]

$$\overline{\mathscr{P}}_{\text{loss}} = 2m\overline{\dot{x}^2}\delta_0 = m\omega^2 \left| x_\omega \right|^2 \delta_0 .$$

If the electromagnetic wave's (re-) radiation is the only energy loss mechanism of the oscillator, this expression has to be equal to the one given at $\omega \approx \omega_0$ by Eq. (*). As a result,

$$Q^{-1} \equiv \frac{2\delta_0}{\omega} = \frac{Z_0\omega q^2}{6\pi c^2 m} \equiv \frac{2}{3}\left(Z_0 c\varepsilon_0\right)\left(\frac{\omega}{c}\right)\left(\frac{q^2}{4\pi\varepsilon_0 mc^2}\right) \equiv \frac{2}{3} k r_{\text{c}},$$

[90] This expression may be readily obtained by interpreting the second term on the left-hand side of Eq. (7.30) as a viscous friction force proportional to the charge's velocity: $F_v = -2mv\delta_0$, giving the instant dissipation power $\mathscr{P}_{\text{loss}} = -F_v v = 2mv^2\delta_0$.

where $k = \omega/c$ is the wave number, and r_c is the classical radius defined by Eq. (8.41) of the lecture notes. For an electron, $r_c \sim 3\times10^{-15}$ m, so the product kr_c is very small at all frequencies up to hard γ rays. Hence, the radiation-defined Q is very large, so the resonant scattering may be much stronger than the Thomson and Rayleigh scattering.

Quantum mechanics[91] not only confirms the above formulas for the harmonic oscillator, but also shows that the resonant scattering takes place for *any* system with discrete energy levels E_n, as soon as the wave frequency ω approaches *any* of the quantum transition frequencies $\omega_{nn'} = (E_n - E_{n'})/\hbar$, with $n \neq n'$, provided that the corresponding matrix element of the electric dipole moment's operator is different from zero and that at least one of the involved energy levels is occupied.

Problem 8.9.* A sphere of radius R, made of a material with a uniform permanent electric polarization $\mathbf{P}_0$ and a constant mass density ρ, is free to rotate about its center. Calculate its average total cross-section for scattering of a linearly polarized plane electromagnetic wave with frequency $\omega << c/R$, incident from free space, in the weak-wave limit, assuming that the initial orientation of the polarization vector is random.

Solution: As we know from the solution of Problem 3.13, the electric field induced by a polarized sphere is identical to that of a point electric dipole, placed in the sphere's center, with the moment given by a very simple and natural formula

$$\mathbf{p} = V\mathbf{P}_0 = \frac{4\pi}{3}R^3\mathbf{P}_0 .$$

At the given condition $\omega << c/R$, the wave's electric field $\mathbf{E}$ is virtually uniform on the distances of the order of R. Per Eq. (3.17) of the lecture notes, such a uniform electric field $\mathbf{E}$ exerts on such a dipole (i.e. on our sphere) the following mechanical torque:

$$\boldsymbol{\tau} = \mathbf{p}\times\mathbf{E} .$$

As we know from the basic classical mechanics,[92] under the effect of such a torque, the mechanical angular momentum $\mathbf{L}$ of the sphere evolves as

$$\dot{\mathbf{L}} = \boldsymbol{\tau} .$$

Classical mechanics also says[93] that the angular momentum $\mathbf{L}$ of a spherically-symmetric body (frequently called the *spherical top*) equals $I\boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is the instantaneous angular velocity vector, and I is the (only) principal moment of inertia of such a body (with the mass $M = \rho V = \rho(4\pi/3)R^3$): [94]

$$I = \frac{2}{5}MR^2 = \frac{8\pi}{15}\rho R^5 .$$

Combining the above relations, we get the following equation of time evolution of the vector $\boldsymbol{\Omega}$:

$$I\dot{\boldsymbol{\Omega}} = \mathbf{p}\times\mathbf{E} . \qquad (*)$$

[91] See, e.g., QM Chapters 6 and 9.
[92] See, e.g., CM Eq. (1.33).
[93] See, e.g., CM Secs. 4.1-4.2, in particular, Eqs. (4.31)-(4.32).
[94] See, e.g., the model solution of CM Problem 4.1.

One more relation between the vectors **Ω** and **p** follows from the kinematics of rotation of any vector of constant length, in particular, **p**:[95]

$$\dot{\mathbf{p}} = \mathbf{\Omega} \times \mathbf{p}. \qquad (**)$$

A joint solution of the two vector equations (*) and (**), with a time-dependent (oscillating) vector **E**, is not a trivial task but it may be simplified if the incident wave's amplitude is relatively weak. To see this, let us differentiate Eq. (**) over time, and then plug Eq. (*) and (again) Eq. (**) into the resulting right-hand side:

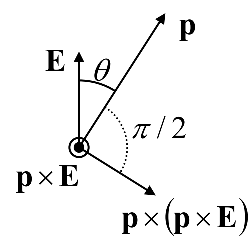
$$\ddot{\mathbf{p}} = \dot{\mathbf{\Omega}} \times \mathbf{p} + \mathbf{\Omega} \times \dot{\mathbf{p}} = \frac{1}{I}(\mathbf{p} \times \mathbf{E}) \times \mathbf{p} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{p}). \qquad (***)$$

Since p = const, the first term in the last expression contains a component oscillating with the wave's frequency ω, and proportional to its amplitude E_ω. On the other hand, Eq. (*) shows that the amplitude of such oscillations of the vector **Ω** is also proportional to E_ω, so at $E_\omega \rightarrow 0$, the amplitude of the fast oscillations of the term **Ω**×(**Ω**×**p**) is proportional to $|E_\omega|^2$, i.e. is much smaller than the first term. (This reasoning is strict in the absence of the initial rotation of the sphere, but even if such a rotation is present, its angular velocity Ω_0 has to be as high as ω to affect the approximation used below.)

Dropping, on this grounds, the second term on the right-hand side of Eq. (***) (but only for a while – see below), we get

$$\ddot{\mathbf{p}} = -\frac{1}{I}\mathbf{p} \times (\mathbf{p} \times \mathbf{E}).$$

According to this expression, if the incident wave is linearly polarized, i.e. the vector **E** is rapidly oscillating along a fixed direction, the resulting vector $\ddot{\mathbf{p}}$ also oscillates, with the same frequency, along a line normal to the vector **p** (see the figure on the right), with the following magnitude:



$$\ddot{p} = -\frac{p^2 E}{I}\sin\theta,$$

where θ is the angle between the vectors **E** and **p**. Plugging this expression into the general Eq. (8.27), for the average power of radiation into free space (with $Z = Z_0$ and $v = c$), we get a frequency-independent result:

$$\overline{\mathscr{P}} = \frac{Z_0}{6\pi c^2}\left(\frac{p^2}{I}\right)^2 \overline{E^2(t)}\sin^2\theta = \frac{Z_0}{12\pi c^2}\left(\frac{p^2}{I}\right)^2 |E_\omega|^2 \sin^2\theta.$$

Here the power has been averaged over the period of the wave frequency ω. Now returning to Eq. (***), we have to notice that the second term on its right-hand side, neglected in the above calculation, also has a certain small (proportional to $|E_\omega|^2$) but nonvanishing average over the period. This average component of the second derivative of the vector **p** results in a rather complex slow evolution of the direction of the vector, and hence to a change of the angle θ. Though slow on the scale of the wave frequency ω, this evolution may be noticeable on the scale of the scattering measurement experiment and, generally speaking, should be accounted for in theory. However, if the initial direction

[95] See, e.g., CM Sec. 4.1, in particular Eq. (4.9).

of the sphere's polarization vector $\mathbf{P}_0$ is random, the evolution details are not important for the statistical average[96] of the radiation power, which may be calculated as

$$\left\langle \overline{\mathscr{P}} \right\rangle = \frac{1}{4\pi}\oint_{4\pi} \overline{\mathscr{P}} d\Omega = \frac{1}{4\pi} 2\pi \int_0^\pi \overline{\mathscr{P}} \sin\theta d\theta = \frac{1}{2}\frac{Z_0}{12\pi c^2}\left(\frac{p^2}{I}\right)^2 \left|E_\omega\right|^2 \int_0^\pi \sin^2\theta \sin\theta d\theta .$$

By using the standard variable substitution $\xi \equiv \cos\theta$, it is easy to see that the last integral equals 4/3, so, finally,

$$\left\langle \overline{\mathscr{P}} \right\rangle = \frac{1}{2}\frac{Z_0}{12\pi c^2}\left(\frac{p^2}{I}\right)^2 \left|E_\omega\right|^2 \frac{4}{3} \equiv \frac{Z_0 p^4}{18\pi c^2 I^2}\left|E_\omega\right|^2 .$$

For the similarly averaged cross-section (8.39), this result yields

$$\langle \sigma \rangle = \frac{\left\langle \overline{\mathscr{P}} \right\rangle}{\left|E_\omega\right|^2 / 2Z_0} = \frac{Z_0^2 p^4}{9\pi c^2 I^2} .$$

Plugging in the above expressions of p and I for a uniform sphere, we get

$$\langle \sigma \rangle = \frac{Z_0^2}{9\pi c^2}\left(\frac{4\pi}{3}R^3 P_0\right)^4 \left(\frac{8\pi}{15}\rho R^5\right)^{-2} \equiv \frac{100}{81}\left(\frac{P_0^2/\varepsilon_0}{\rho c^2}\right)^2 \pi R^2 . \qquad (****)$$

It is curious that the scattering cross-section is proportional to the physical cross-section $\sigma_0 \equiv \pi R^2$ of the sphere; however, for any non-exotic matter, it is much smaller. Indeed, as was mentioned in Sec. 3.3 of the lecture notes, the highest values of the remnant polarization P_0 in ferroelectrics do not exceed ~1 C/m^2, while their density ρ is of the order of 10^4 kg/m^3. Plugging these values (both in the SI units) into our result, we see that the ratio $\langle\sigma\rangle/\sigma_0$ can hardly be higher than $\sim 10^{-20}$.[97]

This smallness is not occasional; indeed, the largest polarization of usual condensed matter corresponds to the displacement of a few outer-shell electrons by the distance of the order of interatomic distances a, which are of the order of the Bohr radius r_B, so the largest dipole moment per atom is $p_0 \sim er_B$, and the largest polarization is $P_0 \sim p_0/r_B^3 \sim e/r_B^2$. Plugging in the quantum-mechanical expression for the Bohr radius,[98] $r_B = \hbar^2/m_e(e^2/4\pi\varepsilon_0)$, and the natural (though crude) estimate of the material's density, $\rho \sim Am_p/r_B^3$, where A is the atomic number, we see that the fraction in the last form of Eq. (****) is of the order of $(m_e/Am_p)\alpha^2 << 1$, where $\alpha \equiv e^2/4\pi\varepsilon_0\hbar c \approx 1/137$ is the fine-structure constant, which characterizes the strength (or rather the weakness :-) of electromagnetic interactions on the quantum-relativistic scale – see also the model solutions of Problems 5.18 and 9.14.

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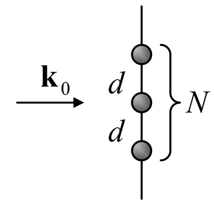
[96] Such an average is all we need if we measure the simultaneous, independent scattering of a wave by many similar spheres with random polarization directions – just as was discussed in Sec. 8.3 for the Rayleigh scattering. (Please note the *statistical* averaging sign ⟨ ⟩, which differs from the *temporal* averaging sign ‾; it will be used very often in the QM and SM parts of this series.)

[97] This extreme smallness of the calculated effect makes it hard to detect on the masking background of other effects, not accounted for in our simple model, such as small but nonvanishing differential ("dynamic") polarizability $\alpha_{dyn} \equiv dp_0/dE$ of the ferroelectric's atoms.

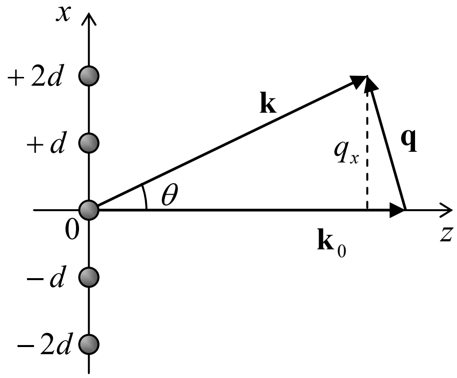
[98] It was already mentioned in Sec. 2.1 of the lecture notes, and in the model solutions of Problems 5.14 and 5.15, and will be discussed in detail in the QM part of this series.

Problem 8.10. Use Eq. (8.56) of the lecture notes to analyze the interference/diffraction pattern produced by a plane wave's scattering on a set of N similar equidistant small objects located on a straight line normal to the direction of the incident wave's propagation – see the figure on the right. Discuss the trend(s) of the pattern in the limit $N \to \infty$.



Solution: As was discussed in Sec. 8.4 of the lecture notes, in the case of similar small scatterers located at points $\mathbf{r}_j$, the scattered wave intensity is proportional (besides the smooth polarization-related function $\sin^2\Theta$) to the scattering function $F = |I(\mathbf{q})|^2$, where $I(\mathbf{q})$ is the phase sum (8.57):

$$I(\mathbf{q}) \equiv \sum_j \exp\{-i\mathbf{q}\cdot\mathbf{r}_j\},$$



and $\mathbf{q} \equiv \mathbf{k} - \mathbf{k}_0$ is the wave vector's change at scattering. Introducing the Cartesian coordinates just as in Fig. 8.6 (see the figure on the right), we get $\mathbf{r}_j = x_j\mathbf{n}_x$, with $x_j = jd$, so for the observation point within the plane of the drawing (common for the line of the elementary scatterers and the wave vector $\mathbf{k}_0$), where $q_x = k\sin\theta$, we get

$$I(\mathbf{q}) = \sum_{j=1}^{N} \exp\{-ikjd\sin\theta\} \equiv \sum_{j=1}^{N} \exp\{-ij\zeta\}, \qquad (*)$$

where $\zeta \equiv kd\sin\theta$, so

$$F(\zeta) = |I(\mathbf{q})|^2 \equiv I(\mathbf{q})I^*(\mathbf{q}) = \left(\sum_{j=1}^{N} \exp\{-ij\zeta\}\right)\left(\sum_{j'=1}^{N} \exp\{-ij'\zeta\}\right)^* \equiv \sum_{j,j'=1}^{N} \exp\{i(j'-j)\zeta\}.$$

The three most significant features of the function $F(\zeta)$, describing an interplay between the interference of the waves scattered from the adjacent objects and the diffraction of the incident wave on the system as a whole, are immediately apparent from this formula:

(i) the function is even because every term of the last sum is invariant with respect to the simultaneous replacements $\{\zeta \to -\zeta,\ j \leftrightarrow j'\}$ and the sums over j and j' are similar;

(ii) the function is 2π-periodic because each term of the sum is; and

(iii) the function reaches its major maxima,[99] $F_{\max} = N^2$, at $\zeta = 0$ (and hence at all $\zeta = 2\pi n$, with any integer n) because at these points, each term of the sum reaches its maximum equal to 1.

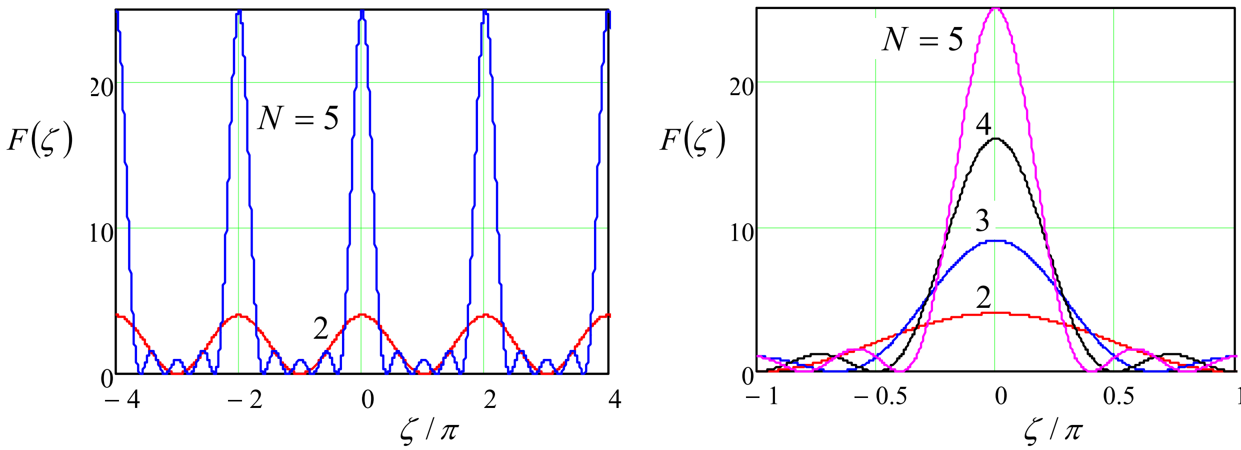
These features are clearly visible on the numerical plots of the function for several values of N, shown in the figure below. (Its right panel is a zoom on any of the major peaks.) The plots show that as the number N of scatterers is increased, the width $\delta\zeta$ of the function's major peals decreases as $\sim \pi/N$,[100] and that at $N \to \infty$, the function rapidly approaches a periodic set of the Fraunhofer diffraction patterns – see Fig. 8.8 of the lecture notes:

$$F_n(\zeta) \approx \left(N\frac{\sin\xi}{\xi}\right)^2 \equiv N^2\operatorname{sinc}^2\xi, \qquad \text{with } \xi \equiv \frac{N(\zeta - 2\pi n)}{2} = \frac{k(Nd)\sin\theta}{2} - \pi N n, \qquad \text{for } N >> 1.$$

[99] As the figure below shows, the function has $(N-1)$ additional, lower maxima at each period.

[100] The exact coefficient in this relation depends on the accepted quantitative definition of the peak's width.

This is natural because, at $N \to \infty$, the sum in Eq. (*) tends to the phase integral defined by Eq. (8.63) and given, for our case, by Eq. (8.64) with the replacement $V \to N$.



Note, however, that in terms of the observation angle $\theta = \sin^{-1}(\zeta/kd)$, the transition $N \to \infty$ looks rather different in the two ultimate cases.

(i) If the total length $a \equiv Nd$ of the scattering system is kept fixed (i.e. the distance d between the adjacent scatterers is decreased as $1/N$), then the width

$$\delta(\sin\theta) \equiv \frac{2}{ka}\delta\xi \equiv \frac{N}{ka}\delta\zeta \approx \frac{\pi}{ka}$$

of each diffraction peak does not depend on N, while the intervals between the peaks,

$$\Delta\xi \equiv \frac{ka}{2}\Delta(\sin\theta) = \frac{N}{2}\Delta\zeta = N\pi\,,$$

grow as N. As a result, since the largest magnitude of $\sin\theta$ is 1, in the case $ka \sim 1$, none of the diffraction peaks but one (with $n = 0$, i.e. $\theta \approx 0$) exist. This is the case of diffraction, which was discussed in Sec. 8.4 of the lecture notes.

(ii) In the opposite case when the period d of the scattering structure is kept fixed, so the increase of N leads to the proportional increase of its total length $a = Nd$, the intervals

$$\Delta(\sin\theta) \equiv \frac{2}{ka}\Delta\xi = \frac{N}{ka}\Delta\zeta = \frac{2\pi}{kd}$$

between the major peaks remain constant, so they show up at the same values of the observation angle θ. In contrast, the width $\delta\theta$ of each peak shrinks:

$$\delta(\sin\theta) \approx \frac{\pi}{kd}\frac{1}{N} \to 0\,,$$

i.e. the diffraction peaks become very sharp, so the difference between their positions for slightly different wavelengths $\lambda = 2\pi/k$ may be used for high-resolution spectroscopy. This is exactly the limit pursued in diffraction gratings – see Sec. 8.8 of the lecture notes for a brief discussion of these important devices.

Finally, note that if the observation point is off the common plane shown in the first figure above, the interference/diffraction pattern is quantitatively different because the vector **q** also has an off-plane component, so the scattering angle θ is now different from $(\pi/2 - \Theta)$ – see, e.g., Fig. 8.5 in the lecture notes. However, in the most interesting case $N >> 1$ and $kd \sim 1$, when the most important features of the pattern pertain to small scattering angles, they remain qualitatively the same.

Problem 8.11. Use the Born approximation to calculate the differential cross-section of a plane wave's scattering by a uniform dielectric sphere of an arbitrary radius R. In the limits $kR << 1$ and $1 << kR$ (where k is the wave number), analyze the angular dependence of the differential cross-section and calculate the total cross-section of scattering.

Solution: According to Eqs. (8.62)-(8.63) of the lecture notes, the differential cross-section may be calculated as

$$\frac{d\sigma}{d\Omega} = \left(\frac{k^2}{4\pi}\right)^2 (\kappa - 1)^2 \left|I(\mathbf{q})\right|^2 \sin^2\Theta,$$

with

$$I(\mathbf{q}) \equiv \int_V \exp\{-i\mathbf{q}\cdot\mathbf{r}\} d^3r, \qquad (*)$$

where V is the scatterer's volume, and $\mathbf{q} \equiv \mathbf{k} - \mathbf{k}_0$ is the wave vector's change due to scattering. Since, at the elastic scattering we are discussing, $k = k_0$, the magnitude of **q** is related to the scattering angle θ (i.e. the angle between the vectors **k** and $\mathbf{k}_0$) as $q = 2k\sin(\theta/2)$.

For a sphere, it is natural to work out the phase integral (*) by taking the direction of the vector **q** for the polar axis, so $\mathbf{q}\cdot\mathbf{r} = qr\cos\alpha$, and

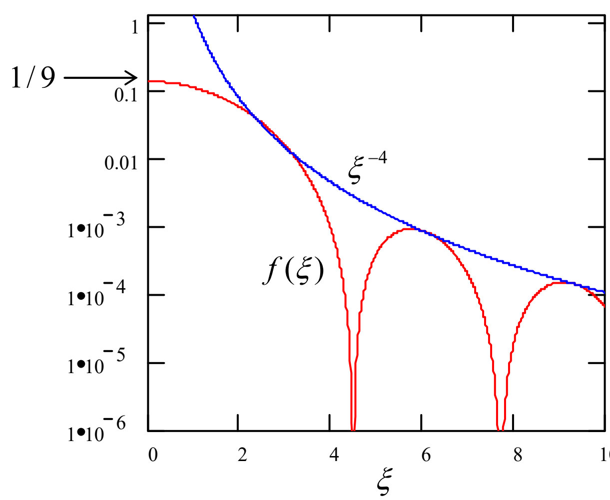
$$I(\mathbf{q}) = 2\pi\int_0^R r^2 dr \int_0^\pi \exp\{-iqr\cos\alpha\}\sin\alpha d\alpha \equiv 2\pi\int_0^R r^2 dr\int_{-1}^{+1}\exp\{-iqr\cos\alpha\}d(\cos\alpha)$$
$$= 2\pi\int_0^R r^2 dr \frac{1}{-iqr}\left(e^{-iqr} - e^{iqr}\right) \equiv \frac{4\pi}{q}\int_0^R rdr\sin qr = \frac{4\pi}{q^3}\left(\sin qR - qR\cos qR\right).$$

As a result, the differential cross-section is

$$\frac{d\sigma}{d\Omega} = \left(k^2R^3\right)^2 (\kappa-1)^2 \sin^2\Theta\, f(qR), \qquad (**)$$

where the function f, defined as

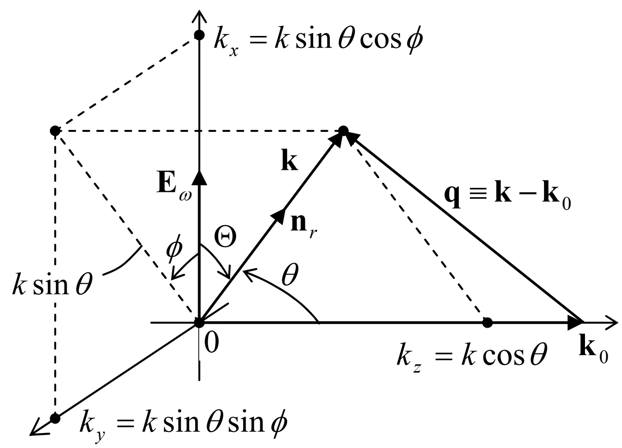
$$f(\xi) \equiv \frac{\left(\sin\xi - \xi\cos\xi\right)^2}{\xi^6},$$



behaves as the red line in the figure on the right shows. As Fig. 8.5 (reproduced, with additions, below) shows, the polarization angle Θ participating in Eq. (**), i.e. the angle between the vectors $\mathbf{E}_\omega$ and **k**, is related to the scattering angle θ and the azimuthal angle ϕ as follows:

$$\sin^2\Theta = \frac{k_y^2 + k_z^2}{k^2} = \sin^2\theta \sin^2\phi + \cos^2\theta . \qquad (***)$$

For a relatively large sphere, $kR >> 1$, the function $f(qR)$ oscillates already at relatively small values of the scattering angle θ. As Eq. (***) shows, at such angles, $\Theta \approx \pi/2$, so $d\sigma/d\Omega$ is virtually independent of the azimuthal angle ϕ, so Eq. (**) describes a round-ring diffraction pattern. Per this formula, as the number n of the intensity maximum is increased, its position rapidly approaches the following asymptotic value:



$$\xi_n = \left(n + \frac{1}{2}\right)\pi,$$

i.e. the following scattering angle:

$$\theta_n \approx \frac{\xi_n}{kR} = \left(n + \frac{1}{2}\right)\frac{\pi}{kR} << 1 .$$

As qR is increased, the envelope of the intensity oscillations decreases very fast, as $(qR)^{-4}$ – see the blue dashed line in the first figure above. As a result, the integral (8.50), which gives the total cross-section of scattering, converges already at small scattering angles $\theta << 1$, where we may take $\sin^2\Theta \approx 1$, $q \approx k\theta$, and $\sin\theta \approx \theta$, so

$$\sigma \approx \int_{4\pi} \frac{d\sigma}{d\Omega}\Big|_{1/k<<\theta<<1} d\Omega \approx 2\pi k^4 R^6 (\kappa - 1)^2 \int_0^{1/k<<\theta<<1} f(kR\theta)\theta d\theta = 2\pi k^2 R^4 (\kappa-1)^2 \int_0^\infty f(\xi)\xi d\xi .$$

The last integral is equal to ¼,[101] so, finally,

$$\sigma = \frac{\pi}{2}(kR^2)^2(\kappa-1)^2 \equiv \frac{1}{2}\sigma_0 (kR)^2 (\kappa-1)^2, \qquad \text{for } kR >> 1,$$

where $\sigma_0 \equiv \pi R^2$ is the geometric cross-section of the sphere, as "viewed" by the incident wave. Since $kR >> 1$, it is tempting to conclude in this limit, that σ may be larger than σ_0. However, a straightforward analysis of the validity of the Born approximation for extended uniform objects[102] yields the condition $\sigma << \sigma_0$, so the above result is only valid if $(\kappa - 1) << 1/kR$.

In the opposite limit, $kR << 1$, we can use Eq. (**) with $qR << 1$ and $f(qR) \approx f(0) = 1/9$ for all possible directions of the vector $\mathbf{k}$ (because $qR \leq 2kR$), so

$$\frac{d\sigma}{d\Omega} = \left(\frac{k^2R^3}{3}\right)^2 (\kappa-1)^2 \sin^2\Theta.$$

This result coincides with the general Eq. (8.53) with $V = (4\pi/3)R^3$ and shows that in this limit, there is no diffraction pattern to speak about. As we have already seen in the derivation of Eq. (8.27), the solid-angle integral of $\sin^2\Theta$ equals $8\pi/3$, so the total cross-section in this limit is

[101] See, e.g., MA Eq. (6.3b).
[102] See the model solution of Problem 14 below.

$$\sigma = \frac{8\pi}{27} k^4 R^6 (\kappa - 1)^2 \equiv \frac{8}{27} \sigma_0 (kR)^4 (\kappa - 1)^2, \qquad \text{for } kR << 1. \tag{****}$$

The last expression clearly shows that in this limit, σ is also much smaller than the geometric cross-section σ_0 of the sphere.

Problem 8.12. A sphere of radius R is made of a uniform dielectric material, with an arbitrary dielectric constant. Calculate its total cross-section of scattering a linearly-polarized low-frequency ($k << 1/R$) wave and compare the result with the solution of the previous problem.

Solution: In the low-frequency limit $kR << 1$, i.e. when the wavelength $\lambda = 2\pi/k$ is much larger than the sphere's radius R, the incident wave's field is essentially uniform at distances of the order of R. As the analysis in Sec. 3.4 has shown, in this case, the electric field created by the sphere's polarization outside it is exactly the same as that of a point electric dipole with the dipole moment (3.64):[103]

$$\mathbf{p}_\omega = 4\pi\varepsilon_0 R^3 \frac{\kappa - 1}{\kappa + 2} \mathbf{E}_\omega . \tag{*}$$

This means, in particular, that the power re-radiated by the sphere is given by Eq. (8.28) of the lecture notes, with $\mathbf{p}_\omega$ given by Eq. (*). Now using Eq. (8.39), we get

$$\sigma = \frac{8\pi}{3} (kR)^4 \left(\frac{\kappa - 1}{\kappa + 2} \right)^2 R^2 .$$

Comparing this expression with Eq. (****) in the solution of the previous problem (which was obtained in the Born approximation) in the same frequency limit, we see that they coincide only in the limit $\kappa \to 1$, when $(\kappa + 2)^2 \to 9$. On the other hand, if the dielectric constant is increased, the cross-section gradually approaches its largest possible value

$$\sigma \approx \sigma_{\max} = \frac{8\pi}{3} (kR)^4 R^2, \qquad \text{so that } \frac{\sigma_{\max}}{\pi R^2} = \frac{8}{3} (kR)^4, \qquad \text{for } \kappa >> 1 .$$

The last expression clearly shows that in the low-frequency limit ($kR << 1$) even this largest value is much smaller than the geometric cross-section $\sigma_0 \equiv \pi R^2$ of the sphere.

Problem 8.13. Use the Born approximation to calculate the differential cross-section of a plane wave's scattering on a right circular cylinder of length l and radius R, for an arbitrary angle of incidence.

Solution: In order to spell out Eq. (8.62) of the lecture notes, we have to calculate the phase integral (8.63):

$$I(\mathbf{q}) \equiv \int_V \exp\{-i\mathbf{q}\cdot\mathbf{r}'\} d^3 r' = \int_{-l/2}^{+l/2} dz' \exp\{-iq_z z'\} \int_0^R \rho' d\rho' \int_0^{2\pi} d\varphi' \, \exp\{-iq_\perp \rho' \cos\varphi'\},$$

where q_z is the component of the scattering vector $\mathbf{q} = \mathbf{k} - \mathbf{k}_0$ along the cylinder's axis (taken for the z-axis), $q_\perp$ is the magnitude of its component in the plane of the cylinder's cross-section, and φ' is the

[103] Here and below, for any material with frequency-dependent complex electric permittivity $\varepsilon(\omega)$, the dielectric constant κ has to be understood as the ratio $\varepsilon(\omega)/\varepsilon_0$.

angle between that component and the vector $\boldsymbol{\rho}' \equiv \mathbf{r}' - \mathbf{n}_z z'$. The first integral is simple – see, e.g., Eqs.(8.64)-(8.66) and Fig. 8.7:

$$\int_{-l/2}^{+l/2} e^{-iq_z z'} dz' = l \operatorname{sinc}\frac{q_z l}{2},$$

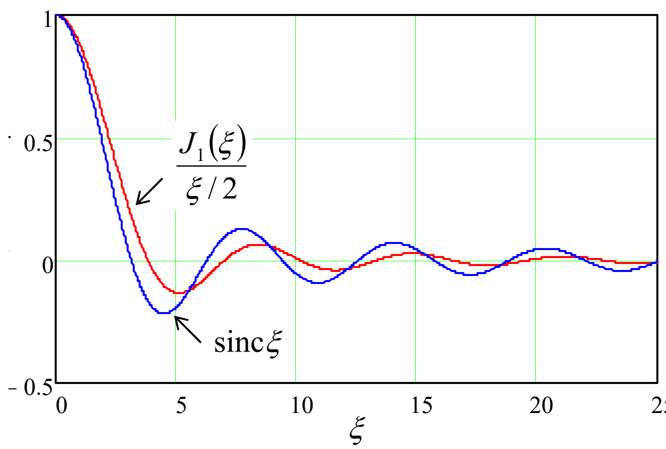
while the second (double) integral may be expressed via a Bessel function of the first kind, by using its integral representation[104] and then the recurrence relation (2.143) for $n = 1$:

$$\int_0^R \rho' d\rho' \int_0^{2\pi} d\varphi' \, e^{-iq_\perp \rho' \cos\varphi'} = 2\pi \int_0^R J_0(q_\perp \rho')\rho' d\rho' \equiv \frac{2\pi}{q_\perp^2}\int_0^{q_\perp R} J_0(\xi)\xi d\xi = \frac{2\pi}{q_\perp^2}\left[\xi J_1(\xi)\right]_0^{q_\perp R} \equiv \frac{2\pi R}{q_\perp} J_1(q_\perp R).$$

With these substitutions, Eqs. (8.62)-(8.63) yield

$$\frac{d\sigma}{d\Omega} = \left(\frac{k^2 l R^2}{4}\right)^2 (\kappa - 1)^2 \left[\frac{J_1(q_\perp R)}{q_\perp R/2}\right]^2 \sin^2\Theta \operatorname{sinc}^2\frac{q_z l}{2}.$$

Here the factors inside the square brackets have been grouped to emphasize that this fraction is very much similar to the sinc function, and in particular tends to 1 when its argument (in our case, $q_\perp R$) tends to zero – see the figure on the right.



In this form, it is evident that for a very small cylinder with kl, kR << 1, our result reduces to Eq. (8.53) with $V \equiv \pi R^2 l$. However, for relatively large cylinders, our apparently simple analytical result describes a relatively complex angular pattern, because the relation between q_z, $q_\perp$, and Θ depends on the direction of the vectors **E**, **k**, and $\mathbf{k}_0$ relative to the cylinder's axis – see Fig. 8.5 of the lecture notes.

Problem 8.14. Formulate the quantitative condition of the Born approximation's validity for a uniform dielectric scatterer, with all linear dimensions of the order of the same scale *a*.

Solution: As was mentioned at the beginning of Sec. 8.3 of the lecture notes, the general condition of the Born approximation's validity is the weakness of the scattered wave in comparison with the incident wave $\mathbf{E}_{\text{in}}$ inside the scatterer. For a linear-dielectric, non-magnetic object that scatters a plane, monochromatic incident wave with a wave vector $\mathbf{k}_0$, this means that its electric polarization **P** at point $\mathbf{r}'$ is given by Eq. (3.45) with $\mathbf{E}(\mathbf{r}') = \mathbf{E}_{\text{in}}(\mathbf{r}') = \text{Re}[(\mathbf{E}_\omega)_{\text{in}}\exp\{i\mathbf{k}_0\cdot\mathbf{r}'\}]$:

$$\mathbf{P}(\mathbf{r}') = (\kappa - 1)\varepsilon_0 \mathbf{E}_{\text{in}}(\mathbf{r}') = (\kappa - 1)\varepsilon_0 \,\text{Re}\left[(\mathbf{E}_\omega)_{\text{in}} e^{i\mathbf{k}_0\cdot\mathbf{r}'}\right].$$

[104] See, e.g., MA Eq. (6.15a) with $n = 0$.

Using this expression to calculate the dipole moment $d\mathbf{p}(\mathbf{r}') = \mathbf{P}(\mathbf{r}')\, d^3r'$ of an elementary volume of the scatterer, we may use Eq. (8.17b) to find the contribution of this volume to the complex amplitude of the vector potential of the scattered wave at a certain observation point $\mathbf{r}$:

$$d\mathbf{A}_\omega(\mathbf{r}) = \frac{\mu_0}{2\pi}\frac{j_\omega(\mathbf{r}')}{R} e^{i\mathbf{k}\cdot\mathbf{R}} d^3r' = \frac{\mu_0}{2\pi}\frac{(-i\omega d\mathbf{p}_\omega)}{R} e^{i\mathbf{k}\cdot\mathbf{R}} = -i\frac{\mu_0\omega}{2\pi}(\kappa-1)\varepsilon_0\mathbf{E}_\omega \frac{1}{R} e^{i\mathbf{k}\cdot\mathbf{R}} e^{i\mathbf{k}_0\cdot\mathbf{r}'} d^3r',$$

where $\mathbf{R} \equiv \mathbf{r} - \mathbf{r}'$ is the vector connecting the source and observation points. Summing up these contributions over all the scatterer's volume V, and taking into account that according to Eq. (6.7), the vector potential's amplitude in the (plane and monochromatic) incident wave is $(\mathbf{A}_\omega)_{\text{in}} = (\mathbf{E}_\omega)_{\text{in}}\exp\{i\mathbf{k}_0\cdot\mathbf{r}\}/(-i\omega)$, we get[105]

$$\begin{aligned}\left|\frac{(\mathbf{A}_\omega)_{\text{scat}}}{(\mathbf{A}_\omega)_{\text{in}}}\right| &= \frac{\varepsilon_0\mu_0\omega^2}{2\pi}(\kappa-1)\left|e^{-i\mathbf{k}_0\cdot\mathbf{r}}\int_V e^{i\mathbf{k}\cdot\mathbf{R}} e^{i\mathbf{k}_0\cdot\mathbf{r}'}\frac{d^3r'}{R}\right| \equiv \frac{k^2}{2\pi}(\kappa-1)\left|e^{-i\mathbf{k}_0\cdot\mathbf{r}}\int_V e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} e^{i\mathbf{k}_0\cdot\mathbf{r}'}\frac{d^3r'}{R}\right| \\ &\equiv \frac{k^2}{2\pi}(\kappa-1)\left|e^{i(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{r}}\right|\left|\int_V e^{i\mathbf{k}\cdot\mathbf{r}'} e^{i\mathbf{k}_0\cdot\mathbf{r}'}\frac{d^3r'}{R}\right| = \frac{k^2}{2\pi}(\kappa-1)\left|\int_V e^{-i\mathbf{q}\cdot\mathbf{r}'}\frac{d^3r'}{|\mathbf{r}-\mathbf{r}'|}\right|, \quad \text{with } \mathbf{q} \equiv \mathbf{k}-\mathbf{k}_0.\end{aligned} \qquad (*)$$

The Born approximation is valid if the right-hand side of this equality is much less than 1 at any point of the scatterer. For a sample much smaller than the wavelength, we can take $qr' << 1$ at any point, so the integral in the last form of Eq. (*) is of the order of a^2,[106] and the approximation's validity condition is

$$\frac{(ka)^2}{2\pi}(\kappa-1) << 1, \qquad \text{for } ka << 1. \qquad (**)$$

Comparing this expression with Eq. (8.53) of the lecture notes, which gives the total cross-section

$$\sigma \equiv \oint_{4\pi}\frac{d\sigma}{d\Omega}d\Omega = \left[\frac{k^2V}{4\pi}(\kappa-1)\right]^2 \oint_{4\pi}\sin^2\Theta d\Omega = \frac{4\pi}{3}\left[\frac{k^2V}{4\pi}(\kappa-1)\right]^2,$$

with the obvious estimates $V \sim a^3$ for the scatterer's volume and $\sigma_0 \sim a^2$ for its physical cross-section, we see that Eq. (**) may be rewritten just as

$$\sigma << \sigma_0.$$

On the other hand, for an extended object, with $a >> \lambda$, the function under the integral in Eq. (*) rapidly oscillates in the direction of the vector $\mathbf{q}$, so the contribution of this direction to the integral is accumulated only on distances $\sim\lambda \sim 1/k$ rather than a. As a result, a fairer estimate of the integral is ka, so the Born approximation is valid if[107]

$$\frac{ka}{2\pi}(\kappa-1) << 1, \qquad \text{for } 1 << ka. \qquad (***)$$

[105] For distant observation points (R, $r >> a$), this integral is just $I(\mathbf{q})/r$, where $I(\mathbf{q})$ is the phase integral (8.63), but for our current task, we need to consider points inside the scatterer: $r \sim a$.

[106] Let me remind the reader that the 3D singularity $1/r$ is integrable: $\int d^3r/r = 2\pi r^2 \to 0$ at $r \to 0$.

[107] For the analog of the conditions (**) and (***) for the Born approximation in quantum mechanics, see QM Eq. (3.77).

However, we should remember that the estimates (**) and (***) are only valid if all linear dimensions of the scatterer are comparable, and should be modified, for example, for strongly elongated objects – say, long/thin rods. In such objects, for certain parameters, the Born approximation may work well for some scattering directions, and give wrong results for others.

Problem 8.15. If a scatterer absorbs some part of the incident wave's power, it may be characterized by an *absorption cross-section* σ_a defined similarly to Eq. (8.39) for the scattering cross-section:

$$\sigma_a \equiv \frac{\overline{\mathscr{P}_a}}{\left|E_\omega\right|^2 / 2Z_0},$$

where the numerator is the time-averaged absorbed power. Use two different approaches to calculate σ_a for a small sphere of radius $R << k^{-1}$, δ_s, made of a nonmagnetic material with an Ohmic conductivity σ and the high-frequency permittivity $\varepsilon_{opt} = \varepsilon_0$. Can σ_a of such a sphere be larger than its geometric cross-section πR^2?

Solution:

Approach 1. Due to the condition $R << k^{-1}$, the displacement currents are negligible, and the external field is virtually uniform on the spatial scale of R, while due to the condition $R << \delta_s$, the magnetic field of the eddy currents is also negligible. In this case, we may calculate the electric field inside the sphere from the first of Eqs. (3.65) of the lecture notes, which may be rewritten as

$$\mathbf{E}_{int} = \frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0}\mathbf{E}_0,$$

with the replacements discussed in Sec. 7.2: $\mathbf{E}_{int} \to (\mathbf{E}_{int})_\omega$, $\mathbf{E}_0 \to \mathbf{E}_\omega$, $\varepsilon \to \varepsilon(\omega)$. For an Ohmic conductor in question, the complex electric permittivity may be calculated using Eq. (7.46), with $\varepsilon_{opt} = \varepsilon_0$. Combining these replacements, we get

$$\left(\mathbf{E}_{int}\right)_\omega = \frac{3\varepsilon_0}{\varepsilon(\omega) + 2\varepsilon_0}\mathbf{E}_\omega = \frac{3\varepsilon_0}{\left(\varepsilon_0 + i\sigma/\omega\right) + 2\varepsilon_0}\mathbf{E}_\omega \equiv \frac{1}{1 + i/3\omega\tau_r}\mathbf{E}_\omega,$$

where $\tau_r \equiv \varepsilon_0/\sigma$ is the charge relaxation time of the sphere's material,[108] which was discussed in Sec. 4.2 – see the second of Eqs. (4.10) with $\kappa = 1$.

Now calculating the temporal average of the Joule power loss (4.40) exactly as this was done for the Poynting vector in Eqs. (7.41)-(7.42) of the lecture notes, we get

$$\overline{\mathscr{P}_a} = \sigma\int_V \overline{E_{int}^2}\, d^3r = \sigma\overline{\left\{\mathrm{Re}\left[\left(E_{int}\right)_\omega e^{-i\omega t}\right]\right\}^2}\, V = \sigma\frac{1}{2}\left|\left(E_{int}\right)_\omega\right|^2 \frac{4\pi}{3}R^3 = \sigma\frac{1}{2}\left|\frac{1}{1 + i/3\omega\tau_r}E_\omega\right|^2 \frac{4\pi}{3}R^3$$
$$= \sigma\frac{1}{2}\frac{1}{1 + \left(1/3\omega\tau_r\right)^2}\left|E_\omega\right|^2 \frac{4\pi}{3}R^3,$$

[108] Let me hope that my usage, in this solution, of the Greek letter σ in two different meanings would not result in confusion. (It denotes the cross-section only when used with the index "a".)

so the absorption cross-section is

$$\sigma_{\mathrm{a}} \equiv \frac{\overline{\mathscr{P}_a}}{\left|E_\omega\right|^2/2Z_0} = \frac{4\pi}{3}\frac{Z_0\sigma R^3}{1+\left(1/3\omega\tau_{\mathrm{r}}\right)^2} \quad . \qquad (*)$$

Approach 2. The initial step of another way to express the power absorbed in the sphere may be used for any object of a size well below the wavelength. Integrating Eq. (4.38) over its volume, we get

$$\mathscr{P}_a = \int_V \mathbf{E}\cdot\mathbf{j}\,d^3r\,.$$

If at all points, we take $\mathbf{E}$ equal to the unperturbed field $\mathbf{E}_0$,[109] which may only change on the spatial scale much larger than the object's size, we may take the electric field out of the integral. Representing the remaining integral of $\mathbf{j}$ in the form of a sum over all charges of the object, just as it was done at the derivation of Eq. (8.133), we may express it as the time derivative of its dipole moment $\mathbf{p}$:

$$\int_V \mathbf{j}d^3r = \sum_k q_k\mathbf{v}_k \equiv \sum_k q_k\frac{d\mathbf{r}_k}{dt} = \frac{d}{dt}\sum_k q_k\mathbf{r}_k \equiv \frac{d\mathbf{p}}{dt}.$$

If $\mathbf{p}$ is a linear function of $\mathbf{E}_0$, it changes with the frequency ω of the wave, so the time-averaged absorbed power becomes

$$\overline{\mathscr{P}_{\mathrm{a}}} = \overline{\mathrm{Re}\left[\mathbf{E}_\omega e^{-i\omega t}\right]\cdot\frac{d}{dt}\mathrm{Re}\left[\mathbf{p}_\omega e^{-i\omega t}\right]} = \frac{i\omega}{4}\left(\mathbf{E}_\omega\cdot\mathbf{p}_\omega^* - \mathbf{E}_\omega^*\cdot\mathbf{p}_\omega\right) \equiv -\frac{\omega}{2}\mathrm{Im}\left[\mathbf{E}_\omega\mathbf{p}_\omega^*\right].$$

Assuming that the polarization is isotopic: $\mathbf{p}_\omega \parallel \mathbf{E}_\omega$, and defining the complex polarizability $\alpha(\omega) \equiv \alpha'(\omega) + i\alpha''(\omega)$ similarly to Eq. (3.48), i.e. by the relation $\mathbf{p}_\omega = \alpha(\omega)\mathbf{E}_\omega$, we get

$$\overline{\mathscr{P}_{\mathrm{a}}} = \frac{\omega}{2}\alpha''(\omega)\left|E_\omega^2\right|,.$$

giving a very useful general relation

$$\sigma_{\mathrm{a}} = Z_0\omega\alpha''(\omega).$$

For our particular case of a sphere, we may take $\mathbf{p}$ from another result of the same electrostatic solution, namely Eq. (3.64), giving

$$\alpha = 4\pi R^3\frac{\kappa-1}{\kappa+2}\varepsilon_0\,.$$

Replacing κ in this formula with $\varepsilon(\omega)/\varepsilon_0$, with $\varepsilon(\omega)$ from the same Eq. (7.46), we arrive at the same Eq. (*) as in Approach 1.

This result shows that in the limit of low frequencies, now in the sense $\omega << 1/\tau_{\mathrm{r}} = \sigma/\varepsilon_0$, the absorption cross-section is proportional to the material's *resistivity* $1/\sigma$:

$$\sigma_{\mathrm{a}} \approx \frac{4\pi}{3}\frac{Z_0\sigma R^3}{\left(1/3\omega\tau_{\mathrm{r}}\right)^2} \equiv 12\pi Z_0\sigma R^3\omega^2\tau_{\mathrm{r}}^2 \equiv 12\pi\frac{Z_0\varepsilon_0^2R^3\omega^2}{\sigma}, \qquad \text{for } \omega\tau_{\mathrm{r}} << 1\ ,$$

[109] This is a nontrivial step of this approach. It may be strictly justified in the dipole approximation, similarly to the derivation of Eq. (3.15) in Sec. 3.1 of the lecture notes – a simple exercise, highly recommended to the reader.

and vanishes at $\omega \to 0$. This is natural because the low-frequency field is effectively screened out of the conductor ($E_{\text{int}} \propto \omega\tau_{\text{r}} \to 0$).

As the wave's frequency is increased, the σ_{a} given by Eq. (*) grows, but then saturates at a frequency-independent value proportional to the sphere material's *conductivity* σ:

$$\sigma_{\text{a}} = \frac{4\pi}{3} Z_0 \sigma R^3, \qquad \text{so that } \frac{\sigma_{\text{a}}}{\pi R^2} = \frac{4}{3} Z_0 (\sigma R), \qquad \text{for } \omega\tau_{\text{r}} >> 1.$$

(Note that, as the elementary Eq. (4.21) shows, the right-hand side of the last relation, besides a numerical factor of the order of 1, is equal to the ratio of the free-space wave impedance $Z_0 \approx 377\ \Omega$ to sphere's dc resistance.[110]) Plugging into this result the above high-frequency condition in the form $\sigma << \varepsilon_0\omega$, and the first condition accepted at this solution, $R << 1/k \equiv c/\omega$, we get

$$\frac{\sigma_{\text{a}}}{\pi R^2} \sim Z_0 \sigma R << Z_0 (\varepsilon_0 \omega) \frac{c}{\omega} \equiv Z_0 \varepsilon_0 c \equiv 1,$$

so our result is valid only if the ratio is much less than one. The second condition, $R << \delta_{\text{s}} \equiv (2/\mu_0\sigma\omega)^{1/2}$, also together with the high-frequency condition $\sigma << \varepsilon_0\omega$, leads to a similar restriction:

$$\frac{\sigma_{\text{a}}}{\pi R^2} \sim Z_0 \sigma R << Z_0 \left(\frac{\sigma}{\mu_0 \omega} \right)^{1/2} << Z_0 \left(\frac{\varepsilon_0 \omega}{\mu_0 \omega} \right)^{1/2} \equiv 1.$$

This means that σ_{a}, within the specified restrictions $R << k^{-1}, \delta_{\text{s}}$, cannot be larger than the geometric cross-section πR^2 of the sphere.

<u>Problem 8.16</u>. Use the Huygens principle to calculate the wave's intensity at the symmetry plane of the slit diffraction experiment ($x = 0$ in Fig. 8.12 of the lecture notes), for an arbitrary ratio z/ka^2.

Solution: The general solution of the problem is given by Eqs. (8.105)-(8.107) of the lecture notes. For the particular case $x = 0$, Eq. (8.107) becomes

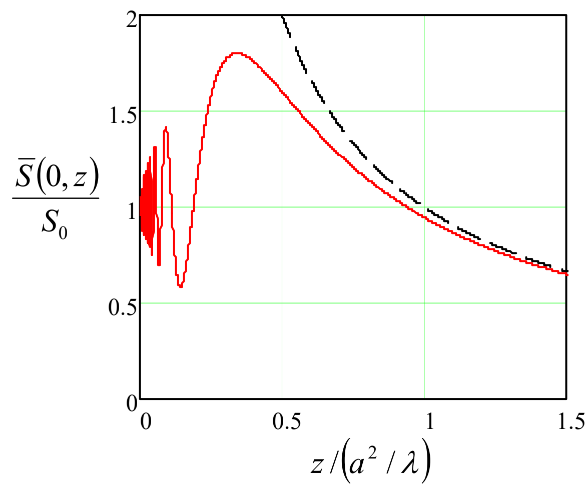
$$\begin{aligned} I_x &= \int_{-a/2}^{+a/2} \exp \frac{ikx'^2}{2z} dx' \equiv 2 \int_0^{+a/2} \left[\cos \frac{kx'^2}{2z} + i \sin \frac{kx'^2}{2z} \right] dx' = 2 \left(\frac{2z}{k} \right)^{1/2} \int_0^{\xi} \left[\cos(\zeta^2) + i \sin(\zeta^2) \right] d\zeta \\ &= 2 \left(\frac{\pi z}{k} \right)^{1/2} \left[\mathscr{C}(\xi) + i \mathscr{S}(\xi) \right], \qquad \text{where } \xi \equiv \left(\frac{k}{8z} \right)^{1/2} a \equiv \left(\frac{\pi}{4} \frac{a^2}{\lambda z} \right)^{1/2}, \end{aligned}$$

and $\mathscr{C}(\xi)$ and $\mathscr{S}(\xi)$ are the Fresnel integrals defined by Eqs. (8.112). Combining this relation with Eqs. (8.105) and (8.106), for the relative wave intensity we get

$$\frac{\bar{S}(0,z)}{S_0} = \left| \frac{f_\omega(0,z)}{f_0} \right|^2 = 2\left[\mathscr{C}^2(\xi) + \mathscr{S}^2(\xi) \right].$$

[110] Assuming that the resistance is measured between two electrodes reasonably distributed on the opposite sides of a sphere's diameter.

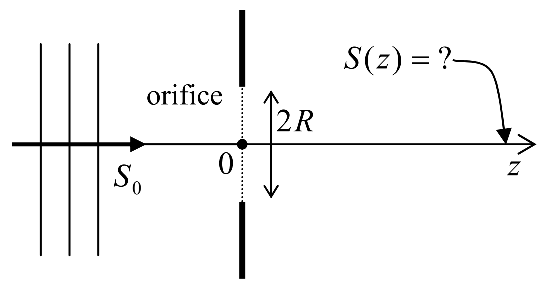
This result is shown with the solid red curve in the figure on the right. As the distance z of the observation point from the slit is increased (i.e. the argument ξ of the Fresnel integrals is *decreased*), the intensity first oscillates, with an increasing amplitude, about the average value equal to S_0, but just beyond the Fresnel-to-Fraunhofer diffraction crossover, the intensity starts a monotonic drop (see the dashed line in the same figure):



$$\frac{\bar{S}(0,z)}{S_0} \approx \frac{4}{\pi}\xi^2 \equiv \frac{a^2}{\lambda z} << 1, \qquad \text{at } z >> \frac{a^2}{\lambda}.$$

Qualitatively, this behavior is similar to that for the diffraction at a circular orifice (see the next problem), but the intensity oscillations with z are less pronounced (with the maximum intensity never reaching $2S_0$, and the minimal intensity not approaching zero), and its drop at large distances is not so fast – all because in our current case, the diffraction takes place is only one dimension.

Problem 8.17. A plane wave with wavelength λ is normally incident on an opaque planar screen with a round orifice of radius $R >> \lambda$. Use the Huygens principle to calculate the passing wave's intensity on the system's symmetry axis, at distances $z >> R$ from the screen (see the figure on the right), and analyze the result.



Solution: According to Eq. (8.91) of the lecture notes, the complex amplitude of the wave is

$$f_\omega(z) = f_0 \frac{k}{2\pi i}\int_0^R \rho' d\rho' \int_0^{2\pi} d\varphi' \frac{\exp\left\{ik\left(\rho'^2 + z^2\right)^{1/2}\right\}}{\left(\rho'^2 + z^2\right)^{1/2}}.$$

Using the axial symmetry of the problem, and the conditions $\lambda << R << z$, this expression may be simplified, just as it has been done in Sec. 8.5 – see, e.g., Eq. (8.86):

$$f_\omega(z) \approx f_0 \frac{k}{iz} e^{ikz} \int_0^R \exp\left\{i\frac{k\rho'^2}{2z}\right\}\rho' d\rho' \equiv f_0 \frac{k}{iz} e^{ikz} \frac{z}{k}\int_0^{kR^2/2z} \exp\{i\xi\} d\xi = -f_0 e^{ikz}\left(\exp\left\{i\frac{kR^2}{2z}\right\} - 1\right),$$

so the wave's intensity is

$$S(z) = S_0\left|\exp\left\{i\frac{kR^2}{2z}\right\} - 1\right|^2 = 2S_0\left(1 - \cos\frac{kR^2}{2z}\right) \equiv 2S_0\left(1 - \cos\frac{\pi R^2}{z\lambda}\right).$$

The result shows that the intensity oscillates, reaching zero at points $z = z_n$ where $\cos(\pi R^2/z_n\lambda) = 1$, i.e. at $\pi R^2/z_n\lambda = 2\pi n$ (with $n = 1, 2, \ldots$), so

$$z_n = \frac{R^2}{2\lambda n}.$$

Between any two adjacent minima, the intensity reaches a maximum (twice larger than the incident wave's intensity!) at points $z = z'_n$ where $\cos(\pi R^2/z'_n\lambda) = -1$, i.e. $\pi R^2/z'_n\lambda = 2\pi n - \pi$, so[111]

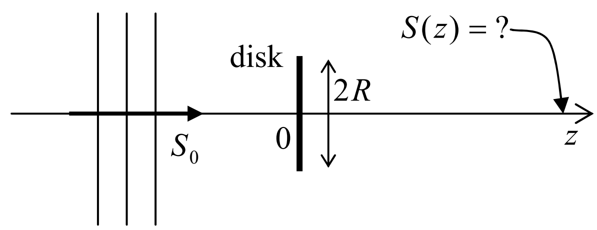
$$z'_n = \frac{R^2}{2\lambda(n-1/2)}.$$

Note that the first few of these maxima and minima (the ones most distant from the screen) correspond to the border between the Fresnel and Fraunhofer diffraction limits: $z/R \sim R/\lambda$, and that in the latter limit, the intensity does not oscillate but rather gradually vanishes at $z \to \infty$:

$$S(z) \to S_0\left(\frac{\pi R^2}{z\lambda}\right)^2 \propto \frac{1}{z^2}, \qquad \text{at } z >> \frac{R^2}{\lambda}. \qquad (*)$$

This decrease may be readily explained as the wave's taking a spherical form – the behavior natural for very large distances, from which the orifice looks like a point source.

Problem 8.18. A plane monochromatic wave is now normally incident on an opaque circular disk of radius $R >> \lambda$. Use the Huygens principle to calculate the wave's intensity at distances $z >> R$ behind the disk's center – see the figure on the right. Discuss the result.



Solution: According to Eq. (8.91) of the lecture notes, the complex amplitude of the wave on the system's symmetry axis is

$$f_\omega(z) = f_0\frac{k}{2\pi i}\int_R^\infty \rho' d\rho' \int_0^{2\pi} d\varphi' \frac{\exp\left\{ik\left(\rho'^2 + z^2\right)^{1/2}\right\}}{\left(\rho'^2 + z^2\right)^{1/2}}.$$

Using the axial symmetry of the problem, and the Huygens principle conditions (8.83), $z >> R >> \lambda$, this expression may be simplified just as this has been done in Sec. 8.5 – see, e.g., Eq. (8.86):

$$f_\omega(z) \approx f_0\frac{k}{iz}e^{ikz}\int_R^\infty \exp\left\{i\frac{k\rho'^2}{2z}\right\}\rho' d\rho' = f_0\frac{k}{iz}e^{ikz}\frac{z}{k}\int_{kR^2/2z}^\infty \exp\{i\xi\}d\xi = -f_0 e^{ikz}\exp\{i\xi\}\Big|_{kR^2/2z}^\infty.$$

Since $\exp\{i\xi\} \equiv \cos\xi + i\sin\xi$ is an oscillating rather than a decaying function, the result of the upper-limit substitution may not be immediately apparent. However, since this is an analytical function, we can always replace $i\xi$ with $(i - \delta)\xi$, where δ is a small real positive number. As a result, the upper-limit substitution disappears, and now we can make, in the lower-limit substitution, the limit $\delta \to 0$, getting

$$f_\omega(z) = f_0 e^{ikz}\exp\left\{i\frac{kR^2}{2z}\right\}.$$

This result leads to a somewhat counter-intuitive answer

[111] Note that these results are quantitatively valid only at $z >> R$, i.e. for $n << R/\lambda$; however, since that last ratio has to be large for the Huygens principle to be valid, the prediction of many maxima and minima is a sound one, and may be readily verified numerically or experimentally.

$$S = S_0 .$$

Curiously, this result was first obtained in 1818 by Poisson who used it as an argument *against* the theory of diffraction that had been developed by Fresnel – and against the wave theory of light as a whole. The following optical experiments have shown, however, that the pattern of light diffraction on a disk with $R << z$ indeed features a light spot at the center.

Note also that in the Fraunhofer diffraction limit (i.e. at $z >> kR^2$), this result may be obtained directly from Eq. (*) in the model solution of the previous problem, and the Babinet principle (8.129).

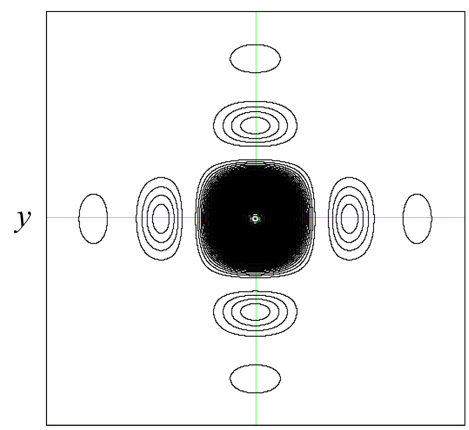
Problem 8.19. Use the Huygens principle to analyze the Fraunhofer diffraction of a plane wave normally incident on a square-shaped hole, of size $a \times a$, in an opaque screen. Sketch the diffraction pattern you would observe at a sufficiently large distance, and quantify the expression "sufficiently large" for this case.

Solution: As was discussed in Sec. 8.8 of the lecture notes, in the Fraunhofer limit, the diffraction pattern is just a 2D Fourier transform of the initial wave amplitude distribution. In our case, this means the complex amplitude of the scattered wave may be calculated as

$$f(x,y) = \text{const} \times \int_{-a/2}^{+a/2} dx' \int_{-a/2}^{+a/2} dy' \exp\{-i(\kappa_x x' + \kappa_y y')\},$$

with $\kappa_x = k(x/z)$, $\kappa_y = k(y/z)$, $k = 2\pi/\lambda$. This is merely a product of two similar 1D elementary integrals of complex-exponential (i.e. sinusoidal) functions, giving finally

$$f(x,y) = \text{const} \times \frac{\sin(\kappa_x a/2)}{(k_x a/2)} \frac{\sin(\kappa_y a/2)}{(k_y a/2)}$$
$$\equiv \text{const} \times \text{sinc}\left(\frac{kax}{2z}\right) \text{sinc}\left(\frac{kay}{2z}\right).$$



The figure on the right shows lines of equal intensity (proportional to f^2), drawn at each 1% of its largest value. Note that though the secondary maxima, centered at points $\{k_x = n\pi/a,\ k_y = m\pi/a\}$ do exist for all integer numbers n and m, they are extremely weak unless one of these numbers equals 1, while the other one is not much larger than 1.

This result is valid for "sufficiently large" distances z in the sense that they are well beyond the crossover from the Fresnel to Fraunhofer diffraction (see Sec. 8.6 of the lecture notes): $z >> a^2/\lambda \sim ka^2$.

Problem 8.20. Use the Huygens principle to analyze the propagation of a monochromatic Gaussian beam, described by Eq. (7.181) of the lecture notes, with the initial characteristic width $a_0 >> \lambda$, in a uniform isotropic medium. Use the result for a semi-quantitative derivation of the so-called *Abbe limit* for the spatial resolution of an optical system,

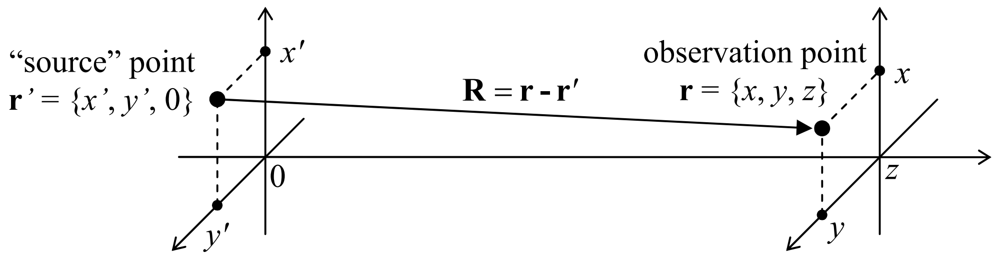
$$w_{\min} = \frac{\lambda}{2\sin\theta},$$

where θ is the half-angle of the wave cone propagating from the object and captured by the system.

Solution: Plugging Eq. (7.181) into Eq. (8.91), and using the Cartesian coordinates similar to those shown in Fig. 8.11 of the lecture notes (see the figure below), we get

$$f_\omega(\mathbf{r}) = \frac{k}{2\pi i} f_0 \int_{z'=0} \exp\left\{-\frac{\zeta\rho'^2}{2}\right\}\frac{e^{ikR}}{R} d^2r' \equiv \frac{k}{2\pi i} f_0 \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \exp\left\{-\frac{\zeta\left(x'^2 + y'^2\right)}{2}\right\}\frac{e^{ikR}}{R}, \qquad (*)$$

where $R = [(x - x')^2 + (y - y')^2 + z^2]^{1/2}$, and $\zeta \equiv 1/a_0^2$.



Since the Huygens principle, in its simple form (8.91), is only valid if $z^2 >> x^2, y^2, x'^2, y'^2 \sim a^2$, we may make the approximations used at the derivation of Eq. (8.87) and reduce Eq. (*) to

$$f_\omega(\mathbf{r}) = \frac{k}{2\pi i} f_0 \frac{e^{ikz}}{z} I_x I_y,$$

where I_x and I_y are two similar integrals, e.g.,

$$I_x \equiv \int_{-\infty}^{+\infty} \exp\left\{-\frac{\zeta x'^2}{2} + \frac{ik}{2z}(x - x')^2\right\} dx'.$$

This is a typical *Gaussian integral*;[112] it may be readily worked out by representing the exponent's argument as a full square of the sum (x' + const), plus another term independent of x':

$$-\frac{\zeta x'^2}{2} + \frac{ik}{2z}(x - x')^2 \equiv -\frac{\zeta x'^2}{2} + \frac{ik}{2z}x^2 - \frac{ik}{z}xx' + \frac{ik}{2z}x'^2$$
$$\equiv -\left[\left(\frac{\zeta}{2} - \frac{ik}{2z}\right)^{1/2} x' + \frac{ik}{2z}\left(\frac{\zeta}{2} - \frac{ik}{2z}\right)^{-1/2} x\right]^2 + \left[\frac{ik}{2z} - \left(\frac{k}{2z}\right)^2\left(\frac{\zeta}{2} - \frac{ik}{2z}\right)^{-1}\right]x^2.$$

Now we may argue that though the coefficient before x' is a complex number, since its real part is positive and the function under the integral is analytical, it may be worked out exactly as that of a real argument:[113]

[112] Such integrals are very frequently met in various fields of physics (and other qualitative sciences). For example, they naturally arise at the analysis of the dispersion of 1D wave packets – see, e.g., QM Sec. 2.2.

[113] Since the integration limits are infinite, the constant offset of x', expressed by the second term inside the first square brackets, does not affect the integral.

$$I_x = \exp\left\{\left[\frac{ik}{2z} - \left(\frac{k}{2z}\right)^2 \left(\frac{\zeta}{2} - \frac{ik}{2z}\right)^{-1}\right] x^2\right\} \int_{-\infty}^{+\infty} \exp\left\{-\left[\left(\frac{\zeta}{2} - \frac{ik}{2z}\right)^{1/2} x' - \frac{ik}{2z}\left(\frac{\zeta}{2} - \frac{ik}{2z}\right)^{-1/2} x\right]^2\right\} dx'$$
$$\equiv \exp\left\{\left[\frac{ik}{2z} - \left(\frac{k}{2z}\right)^2 \left(\frac{\zeta}{2} - \frac{ik}{2z}\right)^{-1}\right] x^2\right\} \left(\frac{\zeta}{2} - \frac{ik}{2z}\right)^{-1/2} \int_{-\infty}^{+\infty} \exp\left\{-\xi^2\right\} d\xi.$$

The last integral[114] equals $\pi^{1/2}$, so forming the absolutely similar integral I_y, we finally get[115]

$$f_\omega(\mathbf{r}) = \frac{k}{2i} f_0 \frac{e^{ikz}}{z} \left(\frac{\zeta}{2} - \frac{ik}{2z}\right)^{-1} \exp\left\{\left[\frac{ik}{2z} - \left(\frac{k}{2z}\right)^2 \left(\frac{\zeta}{2} - \frac{ik}{2z}\right)^{-1}\right] \left(x^2 + y^2\right)\right\}$$
$$\equiv f_0 e^{ikz} \left(1 + \frac{i\zeta z}{k}\right)^{-1} \exp\left\{\left(\frac{i\zeta^2 z}{2k} - \frac{\zeta}{2}\right)\rho^2 \Big/ \left(1 + \frac{\zeta^2 z^2}{k^2}\right)\right\}.$$

Hence, the wave's intensity is proportional to

$$\left|f_\omega(\mathbf{r})\right|^2 = \left|f_0\right|^2 \left(1 + \frac{\zeta^2 z^2}{k^2}\right)^{-1} \exp\left\{-\zeta\rho^2 \Big/ \left(1 + \frac{\zeta^2 z^2}{k^2}\right)\right\}$$
$$\equiv \left|f_0\right|^2 \left(1 + \frac{z^2}{k^2 a_0^4}\right)^{-1} \exp\left\{-\frac{\rho^2}{a_0^2} \Big/ \left(1 + \frac{z^2}{k^2 a_0^4}\right)\right\} \equiv \left|f_0\right|^2 \left(\frac{a}{a_0}\right)^2 \exp\left\{-\frac{\rho^2}{a^2}\right\}, \qquad (**)$$

where a is the beam's effective radius at the distance z from the initial plane:[116]

$$a^2 = a_0^2 + \frac{z^2}{k^2 a_0^2} \equiv a_0^2 + \frac{z^2 \lambda^2}{4\pi^2 a_0^2}. \qquad (***)$$

So, the beam retains its Gaussian shape but gradually broadens. The most interesting feature of this effect is that the narrower the initial beam, the faster it broadens.[117] Actually, this is not quite surprising, and is just the standard feature of the Fraunhofer diffraction, as discussed in Sec. 8.6 of the lecture notes. Indeed, the finite width a_0 of the initial beam plays the same role as the finite width of the orifice (or slit) at the usual diffraction – see Fig. 8.12. At relatively small distances, $z/a_0 << a_0/\lambda$, the beam is the subject of the Fresnel diffraction. As we know from the lecture notes, this diffraction is limited to small vicinities of sharp edges of the wave distribution. The Gaussian beam does not have such edges and hence remains virtually intact. On the other hand, at large distances, where $z/a_0 >> a_0/\lambda$, the beam is affected by the Fraunhofer diffraction, at which the effective half-angle $\theta \equiv a/z$ of the pattern's divergence is independent of z – see, e.g., Eq. (8.110). Similarly, in this limit, Eq. (***) yields $a = z\lambda/2\pi a_0$, i.e.

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114 See, e.g., MA Eq. (6.9b).

115 Alternatively, this result may be obtained from the *paraxial* (or "parabolic") *wave equation* $\partial f_\omega/\partial z = (1/2ik)\nabla_\perp^2 f_\omega$, asymptotically valid in the limit $|\nabla f_\omega|/|kf_\omega| \to 0$, which was discussed in Chapter 7.

116 Note also the pre-exponential factor in Eq. (**), which ensures that the total power of the beam, proportional to $|f_\omega|^2 a^2$, is independent of z – as it has to be in a lossless medium.

117 A similar behavior, described by a very similar math, is pertinent to the 1D wave packets propagating in dispersive media – see, e.g., QM Sec. 2.2.

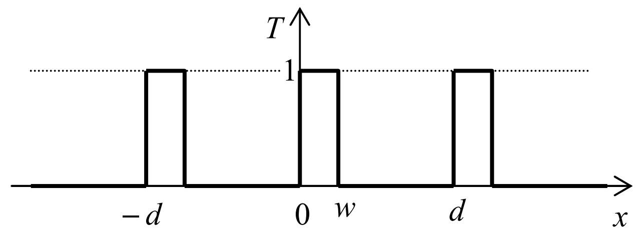
$$\theta \equiv \frac{a}{z} = \frac{\lambda}{2\pi a_0}.$$

Note that the smaller a_0 the larger this angle – though the Huygens approximation used in this solution is valid only if $a_0 >> \lambda$, i.e. if the diffraction angle is small, $\theta << 1$.

Now rewriting the same result as

$$a_{\min} \equiv a_0 = \frac{\lambda}{4\pi\theta},$$

we see that at $\theta << 1$ (when our calculation is valid), it has the same functional form as the Abbe limit's formula.[118] (Some difference in the numerical coefficients is due to the common definition of w, which is larger than a.)



Problem 8.21. Within the Fraunhofer approximation, analyze the pattern produced by a diffraction grating with the 1D-periodic transparency profile shown in the figure on the right, for the normal incidence of a monochromatic plane wave.

Solution: As was discussed in Sec. 8.8 of the lecture notes, within the Fraunhofer approximation (valid when the observer's distance z from the lattice is much larger than d^2/λ), the diffracted wave's amplitude $f_\omega(\boldsymbol{\rho})$ is just the 2D-spatial Fourier transform of the incident wave's amplitude $f_\omega(\boldsymbol{\rho}')$ just behind the lattice, i. e. of its transparency $T(\boldsymbol{\rho}') \equiv f_\omega(\boldsymbol{\rho}')/f_0$. In our 1D case, we have

$$f_\omega(\boldsymbol{\rho}) \to f_\omega(x) = \text{const} \times \int_{-\omega}^{+\infty} T(x') \exp\{i\kappa_t x'\} dx' \equiv \text{const} \times \int_{-\omega}^{+\infty} T(x') \exp\left\{i\frac{kx}{z}x'\right\} dx'.$$

Since in our case, the transparency is periodic: $T(x' + d) = T(x')$, the diffracted wave's intensity is nonvanishing only for a discrete set of equidistant values

$$\kappa_n = \frac{2\pi}{d}n, \qquad \text{i.e. } x_n = \frac{z}{k}\kappa_n = \frac{2\pi z}{kd}n, \quad \text{with } n = 0, \pm 1, \pm 2, ...$$

which show up on a screen as a set of thin "interference stripes" (or "fringes") parallel to those on the diffraction lattice. Their intensities are

$$f_n = \text{const} \times \int_0^d T(x') \exp\left\{2\pi i \frac{x'}{d} n\right\} dx' \propto \int_0^w \exp\left\{2\pi i \frac{x'}{d} n\right\} dx'$$

$$= \frac{d}{2\pi i n}\left(\exp\left\{2\pi i \frac{x'}{d} n\right\}\right)_{x'=0}^{x'=w} = \frac{d}{2\pi i n}\left(\exp\left\{2\pi i \frac{w}{d} n\right\} - 1\right).$$

For analysis, it is convenient to represent this complex amplitude as a product of a real function by a phase factor (which does not affect the wave's intensity):

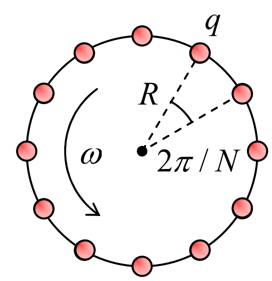
[118] Note that usually the Abbe limit is represented as $w_{\min} = \lambda_0/2n\sin\theta$, where n is the refraction index defined by Eq. (7.84), and $\lambda_0 = n\lambda$ is the free-space wavelength.

$$f_n \propto \frac{d}{\pi n} \sin \frac{\pi w n}{d} \exp\left\{ i\pi \frac{w}{d} n \right\} \propto \operatorname{sinc} \frac{\pi w n}{d} \exp\left\{ i\pi \frac{w}{d} n \right\}.$$

Thus the stripe intensity follows the square of the sinc function:

$$\left| f_n \right|^2 \propto \operatorname{sinc}^2 \frac{\pi w n}{d},$$

– see the solid red line in Fig. 8.8 of the lecture notes. Note that if d is a multiple of w, some interference stripes disappear; a convenient choice is $d = 2w$, leading to the suppression of every other interference fringe and thus doubling the wavelength range available for unperturbed spectroscopic analysis of the diffracting wave.



Problem 8.22. N equal point charges are attached, at equal intervals, to a circle rotating with a constant angular velocity about its center – see the figure on the right. For what values of N the system emits:

(i) the electric dipole radiation?
(ii) the magnetic dipole radiation?
(iii) the electric quadrupole radiation?

Solution: Let us number the charges sequentially by index $k = 1, 2, \ldots N$; then the angular position of the k^{th} charge is

$$\varphi_k = \frac{2\pi}{N} k + \varphi_0, \qquad \text{where } \varphi_0 = \omega t + \text{const},$$

so that in the reference frame with the origin in the circle's center, and the z-axis normal to its plane, the Cartesian coordinates of the charges are

$$x_k = R\cos\varphi_k = R\cos\left(\frac{2\pi}{N}k + \varphi_0\right), \qquad y_k = R\sin\varphi_k = R\sin\left(\frac{2\pi}{N}k + \varphi_0\right), \qquad z_k = 0. \quad (*)$$

With this analytical representation, we are ready to address all particular tasks of the problem.

(i) Per the definition (3.6) of the electric dipole moment $\mathbf{p}$, its Cartesian components, for this system, are as follows:

$$p_x = qR\sum_{k=1}^{N} \cos\left(\frac{2\pi}{N}k + \varphi_0\right) \equiv qR\left[\Sigma_c(N)\cos\varphi_0 - \Sigma_s(N)\sin\varphi_0\right],$$

$$p_y = qR\sum_{k=1}^{N} \sin\left(\frac{2\pi}{N}k + \varphi_0\right) \equiv qR\left[\Sigma_s(N)\cos\varphi_0 + \Sigma_c(N)\sin\varphi_0\right], \qquad p_z = 0,$$

where

$$\Sigma_c(N) \equiv \sum_{k=1}^{N} \cos\frac{2\pi}{N}k, \qquad \Sigma_s(N) \equiv \sum_{k=1}^{N} \sin\frac{2\pi}{N}k. \quad (**)$$

Mathematics tells us[119] that both these sums equal 0 for any integer $N > 0$, with the sole exception of $\Sigma_c(1) = \cos 2\pi = 1$. Hence only the system with one rotating charge can emit the electric dipole radiation. (Indeed, its calculation was the task of Problem 3.) This result may be interpreted as a destructive interference of coherent radiation by the elementary dipole moments $q\mathbf{r}_k$ of $N > 1$ charges.

(ii) For a system of N point charges q moving with velocities $\mathbf{v}_k$, the definition (5.91) of the magnetic dipole moment takes the form

$$\mathbf{m} = \frac{q}{2}\sum_{k=1}^{N}\mathbf{r}_k \times \mathbf{v}_k \ .$$

For our system, the velocity vector $\mathbf{v}_k$ of each charge is normal to both its radius vector $\mathbf{r}_k$ and the z-axis, so all vector products $\mathbf{r}_k\times\mathbf{v}_k$, and hence the net moment $\mathbf{m}$, are directed along that axis:

$$\mathbf{m} = m\mathbf{n}_z, \qquad \text{with } m = \frac{q}{2}\sum_{k=1}^{N} r_k v_k = \frac{qN}{2}R^2\omega .$$

Though this moment is not equal to zero for any $N > 0$, its time derivative is, so according to Eqs. (8.137)-(8.139), the system does not emit the magnetic-dipole radiation at any N. (Note that the answer might be different if the angular velocity ω depended on time.)

(iii) Plugging Eqs. (*) into Eq. (8.142) for the electric quadrupole tensor, with all $q_k = q$, we get

$$\mathcal{Q} = qR^2\sum_{k=1}^{N}\begin{pmatrix} 3\cos^2\varphi_k - 1 & 3\cos\varphi_k \sin\varphi_k & 0 \\ 3\sin\varphi_k\cos\varphi_k & 3\sin^2\varphi_k - 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\equiv qR^2\sum_{k=1}^{N}\begin{pmatrix} (1+3\cos 2\varphi_k)/2 & (3/2)\sin 2\varphi_k & 0 \\ (3/2)\sin 2\varphi_k & (1-3\cos 2\varphi_k)/2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Besides time-independent terms (which, according to Eqs. (8.143) and (8.147), do not provide any radiation), the tensor elements are proportional to the following sums:

$$\sum_{k=1}^{N}\cos 2\varphi_k \equiv \Sigma'_c(N)\cos 2\varphi_0 - \Sigma'_s(N)\sin 2\varphi_0, \qquad \sum_{k=1}^{N}\sin 2\varphi_k \equiv \Sigma'_s(N)\cos 2\varphi_0 + \Sigma'_c(N)\sin 2\varphi_0,$$

where the trigonometric function sums differ from those defined by Eq. (**) by an additional factor 2 before the functions' argument:

$$\Sigma'_c(N) \equiv \sum_{k=1}^{N}\cos\frac{4\pi}{N}k, \qquad \Sigma'_s(N) \equiv \sum_{k=1}^{N}\sin\frac{4\pi}{N}k .$$

Due to this difference, the first sum does not vanish not only for $N = 1$, but also for $N = 2$:

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[119] See, e.g., MA Eq. (3.6) with $n = N$ and $\xi = 2\pi/N$. A simple alternative way to prove this property is to represent the vectors $\mathbf{R}_k$ connecting the adjacent vertices of a regular N-side polygon (so R_k = const), by complex numbers $R_k\exp\{i(2\pi k/N + \text{const})\}$, and plug this representation into the condition that the polygon is a closed line: $\Sigma_k\mathbf{R}_k = 0$. (Evidently, this trick does not work for the special case $N = 1$.)

$$\Sigma_c'(1) = \cos 4\pi = 1, \qquad \Sigma_c'(2) = \cos 2\pi + \cos 4\pi = 2,$$

so the electric-quadrupole radiation is emitted not only by a single rotating charge (for which it is much weaker than the electric-dipole radiation) but also by a pair of equal charges on the ends of the same circle's diameter (which does not emit the electric-dipole radiation).

Completing the calculation of the electric-quadrupole radiation for the latter case, by using the above expression for the tensor $\mathcal{Q}$ together with Eqs. (8.143) and (8.147) of the lecture notes is a simple but useful additional exercise, highly recommended to the reader.

Problem 8.23. What general statements can you make about:

(i) the electric dipole radiation, and
(ii) the magnetic dipole radiation,

due to a collision of an arbitrary number of similar non-relativistic classical particles?

Solutions:

(i) According to Eq. (8.26) of the lecture notes, the electric dipole radiation power is proportional to the square of the second time derivative of the net electric dipole moment of the system,

$$\mathbf{p} \equiv \sum_k q_k \mathbf{r}_k .$$

But for similar particles with equal masses and charges, the first derivative of the moment is proportional to the total linear momentum of the system:

$$\dot{\mathbf{p}} = q\sum_k \dot{\mathbf{r}}_k \equiv \frac{q}{m}\sum_k m\dot{\mathbf{r}}_k .$$

Classical mechanics says[120] that the momentum cannot change in time as a result of internal interactions between the components of the system, and hence $d^2\mathbf{p}/dt^2 = 0$ as well, so the collision cannot induce any electric dipole radiation.

(ii) Very similarly, according to Eq. (8.139), the magnetic dipole radiation power is proportional to the square of the second derivative of the net magnetic dipole moment $\mathbf{m}$ of the system. But according to Eq. (5.95), for a system of particles with the same masses and electric charges, $\mathbf{m}$ is proportional to the angular momentum $\mathbf{L}$, and according to classical mechanics,[121] cannot change in time as a result of internal interactions, i.e. $d\mathbf{m}/dt = 0$, and hence $d^2\mathbf{m}/dt^2 = 0$, so the collision cannot induce any magnetic dipole radiation either.

Note that at such conditions, the electric quadrupole radiation can still take place – see, e.g., the two-particle example given at the end of Sec. 8.9 of the lecture notes.

Problem 8.24. Calculate the angular distribution and the total power radiated by a small planar loop antenna of radius R, fed with ac current with frequency ω and amplitude I_0, into free space.

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[120] See, e.g., CM Eq. (1.10).
[121] See, e.g., CM Eq. (1.38).

Solution: If the antenna's radius is sufficiently small, $R << \lambda \equiv 2\pi/k = 2\pi c/\omega$, the multipole expansion of the loop's magnetic field is dominated by the effect of its dipole moment $\mathbf{m}(t)$, directed normally to the loop's plane, with the magnitude given by Eq. (5.97) of the lecture notes:

$$m(t) = \pi R^2 I(t) = \pi R^2 I_0 \cos(\omega t + \varphi_0), \qquad \text{so } \ddot{m}(t) = \pi R^2 \ddot{I}(t) = -\pi\omega^2 R^2 I_0 \cos(\omega t + \varphi_0),$$

where φ_0 is a constant. Hence the angular distribution of the instant power of radiation into free space may be calculated using Eq. (8.139) with $Z = Z_0$ and $v = c$:

$$\frac{d\mathscr{P}}{d\Omega} = r^2 S_r = \frac{Z_0}{(4\pi c^2)^2}\ddot{m}^2(t)\sin^2\Theta = \frac{Z_0}{(4\pi c^2)^2}\left(\pi\omega^2 R^2 I_0\right)^2 \cos^2(\omega t + \varphi_0)\sin^2\Theta,$$

where Θ is the angle between the direction of the vector $\ddot{\mathbf{m}}$ (in our case, the symmetry axis of the antenna's loop) and the direction toward the observation point.

Averaging this expression over time, we get the following average power per unit solid angle:

$$\frac{d\overline{\mathscr{P}}}{d\Omega} = r^2 \overline{S}_r = \frac{1}{2}\frac{Z_0}{(4\pi c^2)^2}\left(\pi\omega^2 R^2 I_0\right)^2 \sin^2\Theta \equiv \frac{1}{32} Z_0 I_0^2 (kR)^4 \sin^2\Theta,$$

and an easy integration (similar to that carried out for the electric dipole radiation in Sec. 8.2) yields the total average power

$$\overline{\mathscr{P}} = \frac{1}{32} Z_0 I_0^2 (kR)^4 \oint_{4\pi} \sin^2\Theta d\Omega = \frac{1}{32} Z_0 I_0^2 (kR)^4 \frac{8\pi}{3} \equiv \frac{\pi}{12} Z_0 I_0^2 (kR)^4.$$

Just as in the electric-dipole case analyzed in Sec. 8.2 of the lecture notes, this expression may be used to calculate the effective impedance of the antenna:

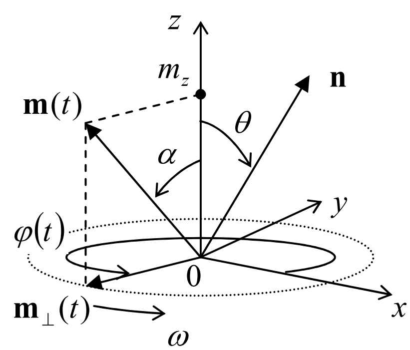
$$Z_A \equiv \frac{\overline{\mathscr{P}}}{I_0^2/2} = \frac{\pi}{6} Z_0 (kR)^4 << Z_0.$$

The impedance grows very fast (even faster than in the electric-dipole case) with the loop's radius R until that becomes comparable with the wavelength $\lambda = 2\pi/k$ of the emitted radiation.

Problem 8.25. The orientation of a magnetic dipole, with a constant magnitude m of its moment, is rotating about a certain axis with an angular velocity ω, with the angle α between them staying constant. Calculate the angular distribution and the average power of its radiation into free space.

Solution: The rotating dipole moment vector $\mathbf{m}(t)$ may be represented as a sum of two components: the time-independent component $m_z = m\cos\alpha$ along the rotation axis, and the perpendicular component

$$\mathbf{m}_\perp(t) = m\sin\alpha\,(\mathbf{n}_x \cos\varphi + \mathbf{n}_y \sin\varphi), \qquad \text{with } \varphi \equiv \omega t + \text{const},$$



rotating about this axis (see the figure on the right), so

$$\ddot{\mathbf{m}}(t) = \ddot{\mathbf{m}}_\perp(t) = -m\omega^2 \sin\alpha\,(\mathbf{n}_x \cos\varphi + \mathbf{n}_y \sin\varphi).$$

This expression is absolutely similar to the formula for $\ddot{\mathbf{p}}(t)$, used in the model solution of Problem 3, with the following replacement:

$$eR \to -m \sin\alpha \,.$$

Now we may use the fact that Eq. (8.139) for the magnetic dipole radiation is similar to Eq. (8.26) for the electric dipole radiation, with the following replacement:

$$\ddot{\mathbf{p}} \to \frac{1}{c}\ddot{\mathbf{m}} \,.$$

Hence, the angular distribution of the average radiation power in these two problems is also similar:

$$S_r \propto \cos^2\theta + \frac{1}{2}\sin^2\theta \,,$$

where θ is the angle between the axis of the moment's rotation and the direction $\mathbf{n}$ toward the observation point – see the figure above. Performing both these replacements in the expression for the total average power obtained in the solution of Problem 3, we get

$$\overline{\mathscr{P}}_{\text{magnetic}} = \frac{Z_0 \omega^4 m^2 \sin^2\alpha}{6\pi c^4} \,. \qquad (*)$$

Note that such a rotation of a magnetic dipole moment may be caused, in particular, by a constant external magnetic field $\mathbf{B}_{\text{ext}}$ directed along the z-axis, due to the torque $\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}_{\text{ext}}$ it exerts on the magnetic dipole – see Eq. (5.101) and a discussion of the torque-induced precession.[122] In this case, the radiation characterized above leads to a gradual reduction of the potential energy (5.100), $U = -\mathbf{m}\cdot\mathbf{B}_{\text{ext}} = -mB_{\text{ext}}\cos\alpha$ of the dipole-field interaction via a gradual increase of the angle α, eventually leading to the dipole's alignment with the field ($\alpha \to \pi$, $\sin\alpha \to 0$) and, according to Eq. (*), the corresponding expiration of the radiation's emission.

<u>Problem 8.26</u>. Solve Problem 12 (also in the low-frequency limit $kR << 1$), for the case when the sphere's material has a frequency-independent Ohmic conductivity σ, and $\varepsilon_{\text{opt}} = \varepsilon_0$, in two limits:

(i) of a very large skin depth ($\delta_s >> R$), and

(ii) of a very small skin depth ($\delta_s << R$).

Solutions:

(i) In the limit $\delta_s >> R$, the eddy currents, which are responsible for the skin effect, may be neglected – and hence their magnetic moment as well. For the electric dipole moment, we may use Eq. (3.64) of the lecture notes with the replacements $\mathbf{p} \to \mathbf{p}_\omega$, $\mathbf{E}_0 \to \mathbf{E}_\omega$, and $\kappa \to \varepsilon(\omega)/\varepsilon_0$:

$$\mathbf{p}_\omega = 4\pi\varepsilon_0 R^3 \frac{\varepsilon(\omega)/\varepsilon_0 - 1}{\varepsilon(\omega)/\varepsilon_0 + 2}\mathbf{E}_\omega \,.$$

For an Ohmic conductor, the complex electric permittivity $\varepsilon(\omega)$ may be calculated using Eq. (7.46), so with $\varepsilon_{\text{opt}} = \varepsilon_0$, we get[123]

[122] See, e.g., CM Sec. 4.5.

[123] See also the solution of Problem 15.

$$\mathbf{p}_\omega = 4\pi\varepsilon_0 R^3 \frac{(\varepsilon_0 + i\sigma/\omega)/\varepsilon_0 - 1}{(\varepsilon_0 + i\sigma/\omega)/\varepsilon_0 + 2}\mathbf{E}_\omega \equiv 4\pi\varepsilon_0 R^3 \frac{1}{1 - 3i\omega\tau_\text{r}}\mathbf{E}_\omega,$$

where $\tau_\text{r} \equiv \varepsilon_0/\sigma$ is the charge relaxation time constant of the material – see Eq. (4.10) of the lecture notes.[124] Now Eqs. (8.28) and (8.39) of the lecture notes yield

$$\sigma = \frac{8\pi}{3}(kR)^4 \left|\frac{1}{1 - 3i\omega\tau_\text{r}}\right|^2 R^2 \equiv \frac{8}{3}\frac{(kR)^4}{1 + 9\omega^2\tau_\text{r}^2}\sigma_0, \qquad \text{where } \sigma_0 \equiv \pi R^2.$$

This result shows that if $\omega\tau_\text{r} << 1$, i.e. if the conductivity of the sphere's material is relatively high, the total cross-section of scattering is the same as σ_max in the solution of Problem 12. It first grows fast (as ω^4) with the wave's frequency, but as ω reaches the reciprocal relaxation time, this increase is tempered by the factor ω^2 in the denominator.

(ii) In this limit, the skin effect does not allow the electromagnetic wave's fields to penetrate into the sphere's bulk. As we know from the discussions in Secs. 3.4 and 5.6 of the lecture notes (which are applicable to our current problem because of the low-frequency condition $kR << 1$), in this case, the electric and magnetic fields induced by the sphere outside it coincide exactly with those of the point electric and magnetic dipoles. Their moments are given, respectively, by Eq. (3.11):

$$\mathbf{p} = 4\pi\varepsilon_0 R^3 \mathbf{E},$$

and Eq. (5.125) in the ideal-diamagnetic limit $\mu \to 0$:

$$\mathbf{m} = -2\pi R^3 \mathbf{H},$$

where $\mathbf{E}$ and $\mathbf{H}$ are the unperturbed fields of the incident wave.[125] Since these fields change in time coherently (see Eqs. (7.6)-(7.8) of the lecture notes),

$$\mathbf{E}(t) = -Z_0 \mathbf{n}_0 \times \mathbf{H}(t),$$

so do the induced electric and magnetic moments, and their radiation fields – see Eqs. (8.24) and (8.137), both with $\varepsilon = \varepsilon_0$, $\mu = \mu_0$, and hence $v = c$:

$$\mathbf{B}_\text{e} = -\frac{\mu_0}{4\pi rc}\mathbf{n}\times\ddot{\mathbf{p}} = -\frac{\mu_0}{4\pi rc}4\pi\varepsilon_0 R^3 \mathbf{n}\times\ddot{\mathbf{E}} = \frac{\mu_0}{4\pi rc}4\pi\varepsilon_0 R^3 Z_0\, \mathbf{n}\times(\mathbf{n}_0\times\ddot{\mathbf{H}}) \equiv \frac{\mu_0 R^3}{c^2 r}\mathbf{n}\times(\mathbf{n}_0\times\ddot{\mathbf{H}}),$$

$$\mathbf{B}_\text{m} = \frac{\mu_0}{4\pi rc^2}\mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{m}}) = -\frac{\mu_0}{4\pi rc^2}2\pi R^3 \mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{H}}) \equiv -\frac{1}{2}\frac{\mu_0 R^3}{c^2 r}\mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{H}}).$$

(Here $\mathbf{n}_0 \equiv \mathbf{k}_0/k$ and $\mathbf{n} \equiv \mathbf{k}/k$ are the unit vectors directed along the propagation of, respectively, the incident and scattered waves, and the retarded time argument is implied.) These expressions show that the induced fields are not only coherent but also comparable in magnitude, so to calculate the total radiation power correctly, we have to add up the fields rather than their partial powers:

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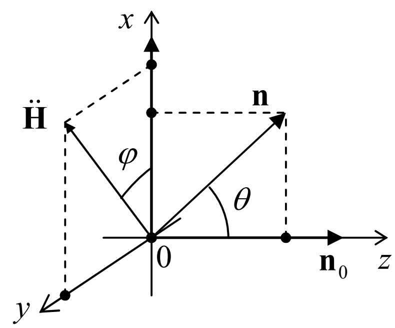
[124] The Ohmic conductivity σ participating in this expression should not be confused with the total cross-section of scattering. Let me hope that these notions may be readily distinguished from the context.

[125] As it follows from the derivation of these formulas in Secs. 3.4 and 5.6 of the lecture notes, their difference by the factor (-1/2) is due to the different conditions for the electric and magnetic fields outside a field-free object, on its boundary: $\mathbf{E}_\tau = 0$ vs. $H_n = 0$ – see also Eq. (7.104).

$$\mathbf{B} = \frac{\mu_0 R^3}{c^2 r}\left[\mathbf{n}\times\left(\mathbf{n}_0\times\ddot{\mathbf{H}}\right) - \frac{1}{2}\mathbf{n}\times\left(\mathbf{n}\times\ddot{\mathbf{H}}\right)\right]. \qquad (*)$$

To spell out the expression in the square brackets, let us use the coordinate system shown in the figure on the right, selected so that the unit vector $\mathbf{n}_0$ is directed along the z-axis (so the vector $\ddot{\mathbf{H}}$, normal to $\mathbf{n}_0$, is within the [x, y] plane), and the vector $\mathbf{n}$ is within the [x, z] plane. In these coordinates, each of the vectors has one zero component:



$$\mathbf{n} = \{\sin\theta, 0, \cos\theta\}, \qquad \mathbf{n}_0 = \{0,0,1\}, \qquad \ddot{\mathbf{H}} = \ddot{H}\{\cos\varphi, \sin\varphi, 0\},$$

so the result of the (straightforward) calculation of the double vector products participating in Eq. (*) is not too cumbersome:

$$\mathbf{n}\times\left(\mathbf{n}_0\times\ddot{\mathbf{H}}\right) - \frac{1}{2}\mathbf{n}\times\left(\mathbf{n}\times\ddot{\mathbf{H}}\right) = \ddot{H}\left\{-\cos\theta\cos\varphi\left(1-\frac{1}{2}\cos\theta\right), \sin\varphi\left(\frac{1}{2}-\cos\theta\right), \sin\theta\cos\varphi\left(1-\frac{1}{2}\cos\theta\right)\right\},$$

giving

$$\left[\mathbf{n}\times\left(\mathbf{n}_0\times\ddot{\mathbf{H}}\right) - \frac{1}{2}\mathbf{n}\times\left(\mathbf{n}\times\ddot{\mathbf{H}}\right)\right]^2 = \ddot{H}^2\left[\left(1-\frac{1}{2}\cos\theta\right)^2\left(\cos^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi\right) + \left(\frac{1}{2}-\cos\theta\right)^2\sin^2\varphi\right].$$

Now we may use Eq. (7.9b) to calculate the instant total power of the radiation:

$$\mathscr{P} = r^2\oint_{4\pi} S_r d\Omega = Z_0 r^2 \oint_{4\pi}\left(\frac{B}{\mu_0}\right)^2 d\Omega = \frac{Z_0 R^6}{c^4}\oint_{4\pi}\left[\mathbf{n}\times\left(\mathbf{n}_0\times\ddot{\mathbf{H}}\right) - \frac{1}{2}\mathbf{n}\times\left(\mathbf{n}\times\ddot{\mathbf{H}}\right)\right]^2 d\Omega.$$

Here the integration is nominally over all directions of the vector $\mathbf{n}$, at a fixed direction of the vector $\ddot{\mathbf{H}}$, but due to the spherical symmetry of the scatterer, it is equivalent to the usual integration over the spherical angles θ and φ (see the figure above):

$$\mathscr{P} = \frac{Z_0 R^6}{c^4}\ddot{H}^2\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\varphi\left[\left(1-\frac{1}{2}\cos\theta\right)^2\left(\cos^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi\right) + \left(\frac{1}{2}-\cos\theta\right)^2\sin^2\varphi\right]$$

$$= \frac{Z_0 R^6}{c^4}\ddot{H}^2\pi\int_0^\pi\left[\left(1-\frac{1}{2}\cos\theta\right)^2 + \left(\frac{1}{2}-\cos\theta\right)^2\right]\sin\theta d\theta \equiv \frac{Z_0 R^6}{c^4 r^2}\ddot{H}^2\pi\frac{10}{3}.$$

Finally, the time average of this power (at the sinusoidal time dependence of the incident field's H) is

$$\overline{\ddot{H}^2} = \frac{\omega^4}{2}\left|H_\omega\right|^2,$$

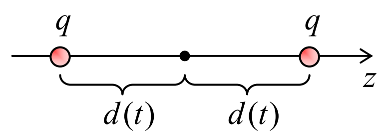
so by using the definition Eq. (8.39) of the total cross-section (with $E_\omega = H_\omega/Z_0$), we get

$$\sigma = \frac{10\pi}{3}\frac{R^6}{c^4}\omega^4 = \frac{10\pi}{3}k^4R^6 \equiv \frac{10}{3}(kR)^4\sigma_0, \qquad \text{where } \sigma_0 \equiv \pi R^2.$$

This result should be compared with that of Problem 12 for a non-conducting dielectric sphere, in the corresponding limit $\kappa\to\infty$ (when the sphere's interior is also fully screened from the penetration of the incident field):

$$\sigma = \frac{8}{3}(kR)^4 \sigma_0 .$$

The comparison shows that in contrast to electrostatics, where a conductor may be adequately described as a dielectric with an infinite dielectric constant, in dynamics, the ac eddy currents flowing on its surface add an extra 25% to the total cross-section of wave scattering. Note also that (as Eq. (*) clearly shows), the polarization of the scattered wave's component due to these currents is also different from its component originating from the purely electric polarization.



Problem 8.27. Complete the solution of the problem started in Sec. 8.9 of the lecture notes, by calculating the full power of radiation of the system of two charges oscillating in antiphase along the same straight line – see Fig. 8.16, partly reproduced on the right. Also, calculate the average radiation power for the particular case of harmonic oscillations, $d(t) = a\cos\omega t$, compare it with the case of a single charge performing similar oscillations, and interpret the difference.

Solution: Per Eq. (8.147), the full power of the electric-quadrupole radiation into free space is

$$\mathscr{P} = \frac{Z_0}{1440\pi c^4} \sum_{j,j'=1}^{3} \left(\dddot{\mathscr{Q}}_{jj'}\right)^2 .$$

By using Eqs. (8.144) for the components of the quadrupole tensor in our case, we get

$$\mathscr{P} = \frac{Z_0}{1440\pi c^4} \left\{ 2\left[\frac{d^3}{dt^3}\left(-2qd^2\right)\right]^2 + \left[\frac{d^3}{dt^3}\left(4qd^2\right)\right]^2 \right\} = \frac{Z_0 q^2}{60\pi c^4}\left[\frac{d^3}{dt^3} d^2(t)\right]^2 . \qquad (*)$$

For the particular case of harmonic oscillations, $d(t) = a\cos\omega t$, the time derivative is

$$\frac{d^3}{dt^3} d^2(t) = \frac{a^2}{2}\frac{d^3}{dt^3}\left(1+\cos 2\omega t\right) = 4a^2\omega^3 \sin 2\omega t , \qquad (**)$$

so the average radiation power is

$$\overline{\mathscr{P}} = \frac{Z_0 q^2}{60\pi c^4}\left(4a^2\omega^3\right)^2 \overline{\sin^2 2\omega t} = \frac{2Z_0 q^2 a^4 \omega^6}{15\pi c^4} .$$

For a similar motion of a single charge q, the electric dipole radiation dominates. For it, we may use Eq. (8.29) of the lecture notes:

$$\overline{\mathscr{P}_1} = \frac{Z_0 q^2 a^2 \omega^4}{12\pi c^2} .$$

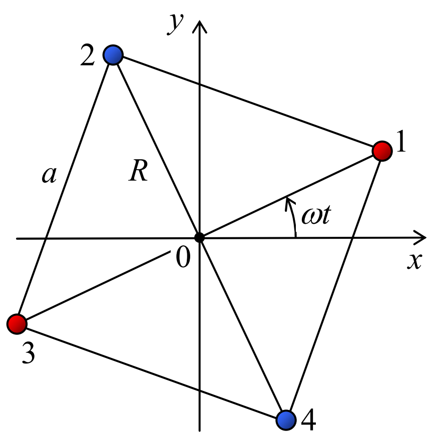
The ratio of these two values of power is

$$\frac{\overline{\mathscr{P}}}{\overline{\mathscr{P}_1}} = \frac{8}{5}\left(\frac{a\omega}{c}\right)^2 .$$

The fraction in the parentheses is just the ratio v_{max}/c, where v_{max} is the amplitude of velocity oscillations. Now we should recall that all expressions of Chapter 8, starting from Sec. 8.2, and hence the above results, are only valid for non-relativistic particles moving with velocity $v << c$. In this case, the calculated electric quadrupole radiation power is much smaller than that of the electric-dipole radiation.

This difference is natural, because the two charges moving coherently, in antiphase, kill the dipole component of each other's radiation by destructive interference, and the quadrupole radiation is just the main part of what remains of the total radiation after this cancellation. Note also that according to Eq. (**), the quadrupole system of two charges radiates at frequency 2ω, while the dipole radiation of each charge at the basic frequency ω is completely canceled by the interference.

Problem 8.28. The system of four alternating charges located at the angles of a square, considered in Problem 3.3(i), is now being rotated about the axis normal to their plane and passing through the square's center, with a constant angular frequency $\omega << v/a$. Calculate the time-averaged angular distribution and the total power of the resulting radiation.



Solution: Let us consider an arbitrary instant t of the square's rotation, placing the coordinate origin into its center and directing the x-axis to the initial position of one of the positive charges – see the figure on the right. Then basic trigonometry gives

$$\begin{aligned} x_1 &= R\cos\omega t, & y_1 &= R\sin\omega t,\\ x_2 &= -R\sin\omega t, & y_2 &= R\cos\omega t,\\ x_3 &= -R\cos\omega t, & y_3 &= -R\sin\omega t,\\ x_4 &= R\sin\omega t, & y_4 &= -R\cos\omega t, \end{aligned} \qquad (*)$$

where $R = a/\sqrt{2}$, with all $z_k = 0$. From Eq. (*), the Cartesian components of the electric dipole moment $\mathbf{p}$ of the system are

$$p_x \equiv \sum_{k=1}^{4} q_k x_k = 0, \qquad p_y \equiv \sum_{k=1}^{4} q_k y_k = 0, \qquad p_z \equiv \sum_{k=1}^{4} q_k z_k = 0,$$

so $\mathbf{p} = 0$, confirming the result obtained in the solution Problem 3.3(i) – or rather generalizing it for an arbitrary rotation angle.

The same is true for the magnetic dipole moment $\mathbf{m}$ of the system. Indeed, at the rotation of our rigid system about the z-axis, the individual charge velocities are[126]

$$\mathbf{v}_k = \boldsymbol{\omega}\times\mathbf{r}_k, \qquad \text{with } \boldsymbol{\omega} = \mathbf{n}_z\omega,$$

so by applying the *bac minus cab* rule[127] to the discrete-charge version of Eq. (5.93), and taking into account that in our system $\mathbf{r}_k\cdot\boldsymbol{\omega} \propto \mathbf{r}_k\cdot\mathbf{n}_z = 0$, we get

$$\mathbf{m} \equiv \frac{1}{2}\sum_k q_k \mathbf{r}_k \times \mathbf{v}_k = \frac{1}{2}\sum_k q_k \mathbf{r}_k \times (\boldsymbol{\omega}\times\mathbf{r}_k) = \frac{1}{2}\sum_k q_k\left[\boldsymbol{\omega} r_k^2 - \mathbf{r}_k(\mathbf{r}_k\cdot\boldsymbol{\omega})\right] = \frac{\boldsymbol{\omega}}{2}\sum_k q_k r_k^2 .$$

Since in our system, all r_k^2 are equal (from Eqs. (*), $r_k^2 \equiv x_k^2 + y_k^2 + z_k^2 = R^2 = a^2/2$), and the sum of all q_k vanishes, the last expression yields $\mathbf{m} = 0$.

[126] See, e.g., CM Eq. (4.9). Alternatively, in our simple case this result may be readily obtained by differentiating Eqs. (*) over time.
[127] If you still need a reminder, see MA Eq. (7.5).

Hence, the system's radiation is dominated by its electric quadrupole component. Indeed, as was shown in the solution of Problem 3.3(i), some components of the electric quadrupole tensor were different from zero even for the special position considered there (corresponding to our current $\omega t = \pi/4 + 2\pi n$). Using Eqs. (*) to generalize that result to an arbitrary rotation angle, i.e. arbitrary moment of time, we get

$$\mathscr{Q}_{xx} \equiv \sum_{k=1}^{4} q_k \left(3x_k^2 - r_k^2\right) = 3a^2 q \cos 2\omega t, \qquad \mathscr{Q}_{yy} \equiv \sum_{k=1}^{4} q_k \left(3y_k^2 - r_k^2\right) = -3a^2 q \cos 2\omega t,$$
$$\mathscr{Q}_{xy} = \mathscr{Q}_{yx} = 3\sum_{k=1}^{4} q_k x_k y_k = 3a^2 q \sin 2\omega t, \qquad \mathscr{Q}_{xz} = \mathscr{Q}_{zx} = \mathscr{Q}_{yz} = \mathscr{Q}_{zy} = \mathscr{Q}_{zz} = 0. \qquad (**)$$

Now, defining the direction toward the radiation's observer by the usual spherical angles θ and φ,

$$\mathbf{n} \equiv \frac{\mathbf{r}}{r} = \mathbf{n}_x \sin\theta\cos\varphi + \mathbf{n}_y \sin\theta\sin\varphi + \mathbf{n}_z \cos\theta\,, \qquad (***)$$

we may readily calculate the vector $\mathscr{Q}$ defined by the second of Eqs. (8.140), whose Cartesian components are related to $\mathscr{Q}_{jj'}$ by Eq. (8.141):

$$\mathscr{Q}_j = \sum_{j'=1}^{3} \mathscr{Q}_{jj'} n_{j'}\ .$$

A straightforward calculation of these components yields[128]

$$\begin{aligned}\mathscr{Q} &= 3a^2 q \sin\theta\left[\mathbf{n}_x\left(\cos 2\omega t\cos\varphi + \sin 2\omega t \sin\varphi\right) + \mathbf{n}_y\left(\sin 2\omega t\cos\varphi - \cos 2\omega t\sin\varphi\right)\right]\\ &\equiv 3a^2 q\sin\theta\left[\mathbf{n}_x \cos(2\omega t - \varphi) + \mathbf{n}_y \sin(2\omega t - \varphi)\right],\end{aligned}$$

so

$$\dddot{\mathscr{Q}} = 24a^2 q\omega^3 \sin\theta\left[\mathbf{n}_x \sin(2\omega t - \varphi) - \mathbf{n}_y \cos(2\omega t - \varphi)\right].$$

Plugging this expression, together with Eq. (***), into Eq. (8.143) of the lecture notes, we may calculate the magnetic field $\mathbf{H}_q \equiv \mathbf{B}_q/\mu$ of the quadrupole radiation at a distance $r >> a$ from its source:

$$\mathbf{H}_q(\mathbf{r},t) = \frac{1}{24\pi r v^2}\dddot{\mathscr{Q}}(t_r)\times\mathbf{n} = \frac{qa^2\omega^3}{\pi r v^2}\sin\theta\begin{vmatrix}\mathbf{n}_x & \mathbf{n}_y & \mathbf{n}_z\\ \sin(2\omega t_r - \varphi) & -\cos(2\omega t_r - \varphi) & 0\\ \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta\end{vmatrix}$$
$$= \frac{qa^2\omega^3}{\pi r v^2}\sin\theta\left[-\mathbf{n}_x\cos\theta\cos(2\omega t_r - \varphi) - \mathbf{n}_y\cos\theta\sin(2\omega t_r - \varphi) + \mathbf{n}_z\sin\theta\cos(2\omega t_r - 2\varphi)\right],$$

where $t_r \equiv t - r/v$ is the retarded time. This expression, accompanied by Eq. (7.6), $\mathbf{E}_q = Z\mathbf{H}_q\times\mathbf{n}$ for this locally-plane wave, determines all its characteristics. (In particular, note that the radiation frequency is 2ω rather than ω, and its polarization is generally elliptical.) However, for our task of calculating the time-averaged radiation power, we need just the sum of squares of the field's Cartesian component amplitudes:

[128] In hindsight, it is very plausible that the vector $\mathscr{Q}$ of this symmetric system should be a function of just one azimuthal angle, $2\omega t - \varphi$. If we foresaw this fact in advance, we could simplify this calculation a bit by referring time to the moment when this angle equals 0.

$$\overline{S}_r = Z\overline{H_{\rm q}^2} = Z\left(\frac{qa^2\omega^3}{\pi r v^2}\sin\theta\right)^2\left[\cos^2\theta\overline{\cos^2(2\omega t_{\rm r}-\varphi)}+\cos^2\theta\overline{\sin^2(2\omega t_{\rm r}-\varphi)}+\sin^2\theta\overline{\cos^2(2\omega t_{\rm r}-2\varphi)}\right]$$

$$= Z\left(\frac{qa^2\omega^3}{\pi r v^2}\right)^2\sin^2\theta\frac{\cos^2\theta+\cos^2\theta+\sin^2\theta}{2} \equiv Z\left(\frac{qa^2\omega^3}{\pi r v^2}\right)^2\frac{1-\cos^4\theta}{2}.$$

So, as we could expect, the average radiation power is axially symmetric and vanishes at $\theta = 0$, i.e. on the z-axis normal to the square's plane. Its total power is

$$\overline{\mathscr{P}_{\rm q}} = r^2\oint_{4\pi}\overline{S}_r d\Omega = Z\left(\frac{qa^2\omega^3}{\pi v^2}\right)^2 2\pi\int_0^\pi\frac{1-\cos^4\theta}{2}\sin\theta d\theta = \frac{8}{5\pi}Z\left(\frac{qa^2\omega^3}{v^2}\right)^2,$$

growing rapidly with both a and (especially) ω. As a sanity check, the last expression may be also calculated from the general Eq. (8.147):

$$\mathscr{P}_{\rm q} = \frac{Z}{720\pi v^4}\sum_{j,j'=1}^{3}\left(\dddot{\mathscr{Q}}_{jj'}\right)^2,$$

which, with our Eqs. (**) for the tensor elements, immediately gives a time-independent value, because

$$\sum_{j,j'=1}^{3}\left(\dddot{\mathscr{Q}}_{jj'}\right)^2 = \left(\dddot{\mathscr{Q}}_{xx}\right)^2+\left(\dddot{\mathscr{Q}}_{yy}\right)^2+\left(\dddot{\mathscr{Q}}_{xy}\right)^2+\left(\dddot{\mathscr{Q}}_{yx}\right)^2$$

$$= \left(24a^2q\omega^3\right)^2\left(\sin^2 2\omega t+\sin^2 2\omega t+\cos^2 2\omega t+\cos^2 2\omega t\right) \equiv 2\left(24a^2q\omega^3\right)^2,$$

and hence does not require further temporal averaging.

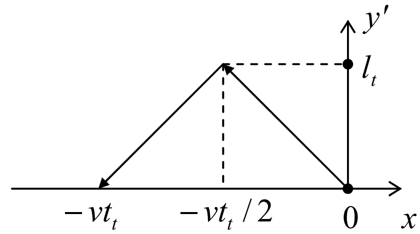
Chapter 9. Special Relativity

Problem 9.1. Use the pre-relativistic picture of light propagation with velocity c in a Sun-bound aether to derive Eq. (9.4) of the lecture notes.

Solution: As observed in the reference frame bound to a Michelson interferometer, in its longitudinal arm (i.e. the arm directed along the aether's velocity v) of length l_l, light moves with epy speed $(c - v)$ in one direction and $(c + v)$ in the opposite direction, so the roundtrip travel time is

$$t_l = \frac{l_l}{c-v} + \frac{l_l}{c+v} \equiv \frac{2l_l}{c}\frac{1}{1-v^2/c^2}.$$

The calculation of the roundtrip time t_t in the transverse arm, of length l_t, is easier in the aether-bound reference frame. In this frame, during a time interval t_t, the interferometer passes distance $(-vt_t)$ in the longitudinal direction, so the total length of the light's roundtrip (see the figure on the right) is



$$l = 2\left[l_t^2 + \left(\frac{vt_t}{2}\right)\right]^{1/2}.$$

In this frame, by the aether's definition, the light's velocity equals c, so we have to require that $l = ct_t$. Solving the resulting equation, we get

$$t_t = \frac{2l_t}{c}\frac{1}{\left(1-v^2/c^2\right)^{1/2}}.$$

For the roundtrip time difference $\Delta t \equiv t_l - t_t$, these expressions yield the first form of Eq. (9.4):

$$\Delta t = \frac{2}{c}\left(\frac{l_l}{1-v^2/c^2} - \frac{l_t}{\left(1-v^2/c^2\right)^{1/2}}\right).$$

Now by Taylor-expanding both fractions on the right-hand side in small parameter $(v/c)^2 \sim 10^{-8}$, in the linear approximation, we get

$$\Delta t \approx \frac{2}{c}\left[l_l\left(1+\frac{v^2}{c^2}\right) - l_t\left(1+\frac{v^2}{2c^2}\right)\right] \equiv \frac{2(l_l - l_t)}{c} + \frac{(2l_l - l_t)}{c}\left(\frac{v}{c}\right)^2.$$

Though at $l_l \neq l_t$ the first term may be much larger than the second one, it does not change as a result of the interferometer's rotation relative to the aether's motion direction and may be readily filtered out from the data. On the contrary, the second term, which does not vanish even at $l_l = l_t$, describes the effect we are looking for. For a nearly-symmetric instrument with $l_l = l_t \equiv l$, it yields the second form of Eq. (9.4):

$$\Delta t \approx \frac{l}{c}\left(\frac{v}{c}\right)^2.$$

Problem 9.2. Show that two successive Lorentz space/time transforms, with velocities u' and v in the same direction, are equivalent to the single transform with the velocity u given by Eq. (9.25) of the lecture notes.

Solution: Let us denote the normalized velocity u'/c as β_1 and v/c as β_2, with the corresponding indices of the Lorentz factor γ – see Eq. (9.17). Then per Eq. (9.19b), the first transform gives

$$x_1 = \gamma_1(x - \beta_1 ct), \qquad ct_1 = \gamma_1(ct - \beta_1 x),$$

and for the second transform, we get:

$$\begin{aligned} x_2 &= \gamma_2(x_1 - \beta_2 ct_1) = \gamma_1\gamma_2[(x - \beta_1 ct) - \beta_2(ct - \beta_1 x)] \equiv \gamma_1\gamma_2[(1+\beta_1\beta_2)x - (\beta_1 + \beta_2)ct], \\ ct_2 &= \gamma_2(ct_1 - \beta_2 x_1) = \gamma_1\gamma_2[(ct - \beta_1 x) - \beta_2(x - \beta_1 ct)] \equiv \gamma_1\gamma_2[(1+\beta_1\beta_2)ct - (\beta_1 + \beta_2)x]. \end{aligned} \qquad (*)$$

Let us spell out the product $\gamma_1\gamma_2$ involved in this result:

$$\gamma_1\gamma_2 = \frac{1}{(1-\beta_1^2)^{1/2}}\frac{1}{(1-\beta_2^2)^{1/2}} \equiv \frac{1}{[(1-\beta_1^2)(1-\beta_2^2)]^{1/2}} \equiv \frac{1}{(1+\beta_1^2\beta_2^2 - \beta_1^2 - \beta_2^2)^{1/2}}. \qquad (**)$$

On the other hand, in this notation, Eq. (9.25) takes the form

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2},$$

so the corresponding Lorentz factor is[129]

$$\gamma \equiv \frac{1}{(1-\beta^2)^{1/2}} = \frac{1}{\left[1 - \dfrac{(\beta_1+\beta_2)^2}{(1+\beta_1\beta_2)^2}\right]^{1/2}} \equiv \frac{1+\beta_1\beta_2}{(1+\beta_1^2\beta_2^2 - \beta_1^2 - \beta_2^2)^{1/2}},$$

and by using Eq. (**), we may reduce the last relation to

$$\gamma = \gamma_1\gamma_2(1+\beta_1\beta_2).$$

As a result, the direct transform from $\{x, t\}$ to $\{x_2, t_2\}$, using these values of β and γ, is

$$\begin{aligned} x_2 &= \gamma(x - \beta ct) = \gamma_1\gamma_2(1+\beta_1\beta_2)\left(x - \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}ct\right), \\ ct_2 &= \gamma(ct - \beta x) = \gamma_1\gamma_2(1+\beta_1\beta_2)\left(ct - \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}x\right), \end{aligned}$$

giving the same results as Eqs. (*) that follow from two sequential transforms.

[129] Note that γ is very different from $\gamma_1\gamma_2$, so the length contraction and time dilation formulas (9.20) and (9.21) cannot be applied sequentially! (Explain why.)

<u>Problem 9.3</u>. $N + 1$ reference frames numbered by index n (taking values 0, 1, …, N) move in the same direction as a particle. Express the particle's velocity in the frame number 0 via its velocity u_N in the frame number N and the velocities v_n of the frame number n relative to the frame number $(n - 1)$.

Solution: According to Eq. (9.25) of the lecture notes, applied to reference frames number $(n - 1)$ and number n:[130]

$$u_{n-1} = \frac{u_n + v_n}{1 + u_n v_n / c^2}, \quad \text{i.e. } \beta_{n-1} = \frac{\beta_n + \mathrm{\beta}_n}{1 + \beta_n \mathrm{\beta}_n}, \quad \text{where } \beta_n \equiv \frac{u_n}{c}, \text{ while } \mathrm{\beta}_n \equiv \frac{v_n}{c}, \tag{*}$$

where u_n is the particle's velocity measured in the n^{th} reference frame. A sequential application of N such formulas, starting from $n = N$ all the way down to $n = 1$, would give us the required answer for u_0, but at large N the formulas would be rather bulky. However, let us use the definition of *rapidity* φ, mentioned in Sec. 9.1:

$$\tanh \varphi \equiv \beta, \tag{**}$$

to rewrite Eq. (*) as

$$\tanh \varphi_{n-1} = \frac{\tanh \varphi_n + \tanh \phi_n}{1 + \tanh \varphi_n \tanh \phi_n}, \qquad \text{where } \tanh \varphi_n \equiv \beta_n \equiv \frac{u_n}{c}, \text{ while } \tanh \phi_n \equiv \mathrm{\beta}_n \equiv \frac{v_n}{c}.$$

But the first of these relations is just the algebraic identity for $\tanh(\varphi_n + \phi_n)$, so we may write simply

$$\varphi_{n-1} = \varphi_n + \phi_n.$$

(This is why the notion of rapidity is so useful: these parameters add up as non-relativistic velocities even in the relativistic case.) Now applying this formula N times, starting from $n = N$, we get

$$\begin{aligned} &\varphi_{N-1} = \varphi_N + \phi_N, \\ &\varphi_{N-2} = \varphi_{N-1} + \phi_{N-1} = (\varphi_N + \phi_N) + \phi_{N-1} \equiv \varphi_N + (\phi_N + \phi_{N-1}), \quad \ldots, \\ &\varphi_0 = \varphi_N + (\phi_N + \phi_{N-1} + \ldots + \phi_1) \equiv \varphi_N + \sum_{n-1}^{N} \phi_n. \end{aligned}$$

What remains is to use Eq. (**) to rewrite the last result in terms of the usual velocities:

$$u_0 = c \tanh\left(\tanh^{-1} \frac{u_N}{c} + \sum_{n=1}^{N} \tanh^{-1} \frac{v_n}{c} \right). \tag{***}$$

For relatively low (non-relativistic) velocities we may use the fact that in this limit, both involved functions, tanh and $\tanh^{-1}$, are well approximated by their arguments, so our general result is reduced to the obvious equality

$$u_0 \approx u_N + \sum_{n=1}^{N} v_n.$$

In the opposite limit, when either one or several of the involved velocities approach c, the corresponding $\tanh^{-1}$ functions become very large and the external tanh in Eq. (***) tends to 1, so we get a natural answer: $u_0 \to c$.

[130] Let me hope that the difference in fonts used for β_n and $\mathrm{\beta}_n$ is sufficient to distinguish these two parameters.

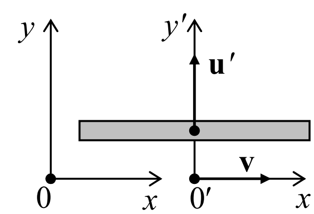
Problem 9.4. A spaceship moving with a constant velocity v directly from the Earth sends back brief flashes of light with a period Δt_S – as measured by the spaceship's clock. Calculate the period with which an Earth-based observer may receive these signals – as measured by their clock.

Solution: Due to the time dilation effect, described by Eq. (9.21) of the lecture notes, the period between the light flash emitting events, as measured by the Earth-bound clock, is $\Delta t_E = \gamma\Delta t_S$, where $\gamma \equiv (1 - v^2/c^2)^{1/2} \geq 1$. During such a time interval, the spaceship moves farther from the Earth by the space interval $\Delta x = v\Delta t_E = \gamma v\Delta t_S$ – again, as measured in the Earth's reference frame. Now, by the Earth's clock again, it will take the next signal, propagating with the speed of light, an additional time $\Delta t_A = \Delta x/c = \gamma v\Delta t_S/c \equiv \beta\gamma\Delta t_S$ (where $\beta \equiv v/c$) to return to the point where the previous flash was emitted, so the total interval between the received signals is

$$\Delta t_R = \Delta t_E + \Delta t_A = \gamma\Delta t_S + \beta\gamma\Delta t_S \equiv (1+\beta)\gamma\,\Delta t_S \equiv \left(\frac{1+\beta}{1-\beta}\right)^{1/2}\Delta t_S.$$

Note that the ratio $\Delta t_R/\Delta t_S$ is expressed by the fraction reciprocal to that describing the Doppler effect for frequencies – see Eq. (9.44) of the lecture notes, with the relevant (upper) sign, corresponding to the motion of the wave source. This is natural because $1/\Delta t$ is just the (cyclic) frequency of the periodic flashes. (In such a dispersion-free "medium" as free space, the group velocity of signal propagation coincides with the phase velocity of phase propagation.) In the non-relativistic limit, $v << c$, the ratio $\Delta t_R/\Delta t_S$ approaches $(1 + \beta) \to 1$, while at $v \to c$, the ratio grows as $2\gamma \to \infty$.

Problem 9.5. From the point of view of observers in a reference frame $0'$, a straight thin rod, parallel to the x'-axis, is moving without rotation with a constant velocity $\mathbf{u}'$ directed along the y'-axis. The reference frame $0'$ is itself moving relative to another ("lab") reference frame 0 with a constant velocity $\mathbf{v}$ along the x-axis, also without rotation – see the figure on the right. Calculate:



(i) the direction of the rod's velocity, and

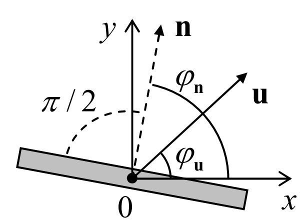
(ii) the orientation of the rod on the [x, y] plane,

– both as observed from the lab reference frame. Is the velocity, in this frame, perpendicular to the rod?

Solutions:

(i) To calculate the components of the rod's velocity $\mathbf{u}$ in frame 0, we may use Eqs. (9.23) of the lecture notes, for the particular case $u'_x = 0$ and $u'_y = u'$. The result is

$$u_x = v, \qquad u_y = \frac{u'}{\gamma}, \qquad \text{where } \gamma \equiv \frac{1}{\left(1 - v^2/c^2\right)^{1/2}},$$



so the velocity's direction may be characterized by the angle $\varphi_\mathbf{u}$ (see the figure on the right) with

$$\tan\varphi_\mathbf{u} \equiv \frac{u_y}{u_x} = \frac{u'}{\gamma v}. \qquad (*)$$

(ii) The simplest way to calculate the rod's orientation in frame 0 is to select the space and time origins of both frames to coincide with each other and with the middle of the rod, at $t = t' = 0$. Then we may use Eqs. (9.19) of the lecture notes, with $\{x', y' = u't'\}$ being the Cartesian coordinates of another point of the rod, displaced by x' from its center, as measured in the "moving" frame 0′. To find the rod's direction in the "lab" frame 0, we need to calculate the coordinates $\{x, y\}$ of that point in this frame at a certain fixed "lab" time t, for example, $t = 0$. With this substitution, Eqs. (9.19a) take the form

$$x = \gamma(x' + \beta ct'), \qquad y = u't', \qquad 0 = ct' + \beta x', \qquad \text{with } \beta \equiv \frac{v}{c},$$

so the elementary substitution of the last result rewritten in the form $t' = -\beta x'/c$ into the first two relations yields

$$x = \gamma(x' - \beta^2 x') = \frac{x'}{\gamma}, \qquad y = -\frac{u'}{c}\beta x'.$$

From this result, the angle describing the direction of the vector $\mathbf{n}$ normal to the rod (see the figure above) obeys the relation

$$\tan\varphi_{\mathbf{n}} \equiv -\frac{x}{y} = \frac{c}{\gamma\beta u'} \equiv \frac{c^2}{\gamma u'v}. \qquad (**)$$

A comparison of Eqs. (*) and (**) shows that, generally, the angles $\varphi_{\mathbf{u}}$ and $\varphi_{\mathbf{n}}$ are different even for non-relativistic velocities, and coincide (so the directions of the vectors **u** and **n** are the same) only asymptotically, in the limits when either $v \to 0$ (so $\tan\varphi_{\mathbf{u}} \to \tan\varphi_{\mathbf{n}} \to \infty$ and hence $\varphi_{\mathbf{u}} \to \varphi_{\mathbf{n}} \to \pi/2$), or $u' \to c$ (so $\tan\varphi_{\mathbf{u}} \to \tan\varphi_{\mathbf{n}} \to c/\gamma v \equiv 1/\beta\gamma$).

Problem 9.6. Starting from the rest at $t = 0$, a spaceship moves directly from the Earth, with a constant acceleration as measured in its *instantaneous rest frame*. Find its displacement $x(t)$ from the Earth, as measured in the Earth's reference frame, and interpret the result.

Hint: The instantaneous rest frame of a moving particle is the inertial reference frame that, at the considered moment of time, has the same velocity as the particle.

Solution: Let us apply Eq. (9.25) of the lecture notes,

$$u = \frac{u' + v}{1 + u'v/c^2}, \qquad (*)$$

to the velocities of the spaceship at the moment $(t + dt)$, as measured, respectively, in the Earth frame (u) and in the instantaneous rest frame of the spaceship (u'), if that frame's velocity, as measured in the lab frame at the moment t, is v. By the instantaneous frame's definition, at $dt = 0$, $u = v$, i.e. $u' = 0$, so for a non-zero but small dt, $u' \equiv du'$ is also small and we may Taylor-expand both parts of Eq. (*) with respect to small deviations proportional to dt, at constant $v = u(t)$. Keeping only the two leading terms of each expansion, we get

$$u(t) + du = \frac{du' + u(t)}{1 + du'u(t)/c^2} \equiv u(t)\frac{1 + du'/u(t)}{1 + du'u(t)/c^2} \approx u(t)\left[1 + du'\left(\frac{1}{u(t)} - \frac{u(t)}{c^2}\right)\right] \equiv u(t) + du'\left[1 - \frac{u^2(t)}{c^2}\right],$$

i.e.

$$du = du'\left[1 - \frac{u^2(t)}{c^2}\right] \equiv \frac{du'}{\gamma^2}, \qquad (**)$$

where γ is the Lorentz factor (9.17) of the spaceship at moment t. By definition, the acceleration a' measured in the instantaneous frame of the spaceship (frequently called the *proper acceleration*) is equal to $du'/d\tau$, where $d\tau = dt/\gamma$ is the proper time interval corresponding to dt – see Eq. (9.60) of the lecture notes. On the other hand, the acceleration a measured in the Earth's frame is evidently du/dt, so Eq. (**) yields

$$a \equiv \frac{du}{dt} = \frac{1}{\gamma^2}\frac{du'}{dt} = \frac{1}{\gamma^2}\frac{du'}{\gamma d\tau} = \frac{1}{\gamma^3}a' \equiv \left(1-\frac{u^2}{c^2}\right)^{3/2} a'.$$

Now we may integrate the resulting equation for the function $u(t)$,

$$\frac{du}{dt} = \left(1-\frac{u^2}{c^2}\right)^{3/2} a', \qquad \text{i.e. } \frac{d(u/c)}{\left(1-u^2/c^2\right)^{1/2}} = \frac{a'}{c}dt,$$

with a' = const, over time, in a straightforward way:

$$\int \frac{d(u/c)}{\left(1-u^2/c^2\right)^{3/2}} = \frac{a'}{c}\int dt + \text{const}.$$

Using the substitution $u/c \equiv \sin\xi$, so $d(u/c) = d(\sin\xi) \equiv \cos\xi\, d\xi$ and $(1-u^2/c^2)^{3/2} = \cos^3\xi$, for our initial condition $u(0) = 0$, we get

$$\int_0^{\sin^{-1}(u/c)} \frac{d\xi}{\cos^2\xi} \equiv \tan\left(\sin^{-1}\frac{u}{c}\right) = \frac{a't}{c},$$

giving us the following equation,

$$u \equiv \frac{dx}{dt} = c\sin\left(\tan^{-1}\frac{a't}{c}\right) \equiv \frac{a't}{\left[1+(a't/c)^2\right]^{1/2}},$$

for the distance $x(t)$ we are seeking. From here, with our initial condition, $x(0) = 0$, we get

$$x = a'\int_0^t \frac{tdt}{\left[1+(a't/c)^2\right]^{1/2}} \equiv \frac{c^2}{a'}\int_0^{a't/c}\frac{\xi d\xi}{\left(1+\xi^2\right)^{1/2}} = \frac{c^2}{a'}\left(1+\xi^2\right)^{1/2}\Bigg|_{\xi=0}^{\xi=a't/c} \equiv \frac{c^2}{a'}\left\{\left[1+\left(\frac{a't}{c}\right)^2\right]^{1/2} - 1\right\}. \qquad (***)$$

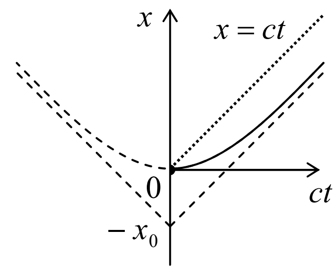
As a sanity check, expanding the right-hand side of this formula in the Taylor series in small time $t \ll c/a'$ (so $u \ll c$) and keeping only the leading term, we get the well-known non-relativistic result:

$$x = \frac{c^2}{2a'}\left(\frac{a't}{c}\right)^2 \equiv \frac{a't^2}{2}.$$

For arbitrary t, Eq. (***) may be represented in a simple space-time symmetric form,[131]

$$\left(x+x_0\right)^2 - (ct)^2 = x_0^2, \qquad \text{with } x_0 \equiv \frac{c^2}{a'}, \qquad (****)$$

[131] This result is frequently derived in a different, less direct way, by first integrating Eq. (**) multiplied by $\gamma^2 \equiv 1/(1-u^2/c^2)$, namely $du/(1-u^2/c^2) = du' \equiv a'd\tau$, to get $u = (c/a')\sinh(a'\tau/c)$. Now this expression may be plugged into the definition of γ, giving $\gamma = \cosh(a'\tau/c)$. Finally, the expressions for u and γ may be plugged into the relation $dx/d\tau \equiv (dx/dt)\,(dt/d\tau) \equiv u\gamma$, giving an easy differential equation for x as a function of τ. Its solution yields $x = (c^2/a')\cosh(a'\tau/c) - x_0$, so the identity $\cosh^2\xi - \sinh^2\xi = 1$, when applied to $\xi \equiv a'\tau/c$, coincides with Eq. (****). A useful byproduct of this alternative approach is an explicit relation between x and τ.



which describes, on the $[x, ct]$ plane, a hyperbola with $\pm\pi/4$-sloped asymptotes, offset by x_0 from the origin – see the figure on the right. From this plot, the constant x_0 may be clearly interpreted as the travel distance "lost" (at a fixed, large travel time $t >> x_0/c$) because of the final value of acceleration. For the Earth-surface gravity acceleration $a' = g \approx 9.8$ m/s^2 (that would create the best comfort for a human astronaut :-), $x_0 = c^2/g \approx 10^{16}$ m – almost exactly one light-year. Hence, for the hypothetical multi-light-year space missions, much higher accelerations would not create any substantial distance benefits, and just exacerbate the hardest problem of future interstellar travel – that of the initial mass of the rocket fuel, which is extremely challenging even for non-relativistic velocities.[132]

Problem 9.7. Analyze the twin paradox for the simplest case of 1D travel with a piecewise-constant acceleration.

Hint: You may use an intermediate result of the solution of the previous problem.

Solution: Let us send one of the twins to the space roundtrip consisting of the following four equal-length stages:

– the first stage, starting from the rest at Mother Earth, of some duration $\Delta t'$ with a certain constant acceleration a', both as measured in the instantaneous rest frame of the ship,

– the second one, of the same duration, with equal but oppositely directed acceleration, bringing the spaceship to a halt,

– the third stage of the same duration with the same acceleration as at the second stage (essentially extending it by accelerating the ship toward the Earth), and

– the final deceleration stage, with the same $\Delta t'$ and $|a'|$, to stop the ship at its return to the Earth.

Starting with the first stage, let us use the result for the ship's velocity u as a function of time t, both as measured in the "lab" (Earth's) reference frame, that was derived in the model solution of the previous problem:

$$u = \frac{a't}{\left[1 + (a't/c)^2\right]^{1/2}}.$$

The corresponding Lorentz factor is

$$\gamma \equiv \frac{1}{\left[1 - (u/c)^2\right]^{1/2}} = \left[1 + \left(\frac{a't}{c}\right)^2\right]^{1/2}.$$

Now we may use Eq. (9.60) of the lecture notes to calculate the proper time interval $d\tau$ of the spaceship, i.e. the interval dt' of the time measured in its instantaneous rest frame:

$$dt' = \frac{dt}{\gamma} = \frac{dt}{\left[1 + (a't/c)^2\right]^{1/2}}.$$

Integrating this relation over the first stage's duration, we get

[132] See, e.g., the solution of CM Problem 1.11.

$$\Delta t' \equiv \int_0^{\Delta t'} dt' = \int_0^{t} \frac{dt}{\left[1+(a't/c)^2\right]^{1/2}} = \frac{c}{a'}\int_0^{a'\Delta t/c} \frac{d\xi}{\left(1+\xi^2\right)^{1/2}} = \frac{c}{a'}\sinh^{-1}\frac{a'\Delta t}{c},$$

so, finally, the stage's duration as measured in the Earth's reference frame is

$$\Delta t = \frac{c}{a'}\sinh\frac{a'\Delta t'}{c}.$$

Now we may use the time-symmetry of the Lorentz transform, which is the origin of the key relation (9.60), to argue that the relation between Δt and $\Delta t'$ at the second stage of the trip is similar, and the similarity of its two last stages with the first two ones to conclude that the full lengths of the roundtrip, in both t and t', are just four times those of the first stage. As a result, we may write

$$\Delta t_{\text{full}} = \frac{4c}{a'}\sinh\frac{a'\Delta t'_{\text{full}}}{4c}, \qquad \text{i.e. } \frac{\Delta t_{\text{full}}}{\tau} = \sinh\frac{\Delta t'_{\text{full}}}{\tau}, \quad \text{where } \tau \equiv \frac{4c}{a'}.$$

The last form of the result shows especially clearly that, first, the roundtrip's time Δt_{full} measured by the Earth-bound clock is always longer than ($\Delta t'_{\text{full}}$) measured by the ship-bound clock, so the traveling twin is always older at the reunion. Second, the difference between the twins' ages is substantial only if the time is much longer than the scale τ. At the acceleration perhaps most comfortable for human travel, $a' = g \approx 9.81$ m/s^2, this time scale is close to 1.22×10^8 s ≈ 3.87 years. As a result if, for example, $\Delta t'_{\text{full}} = 5$ years, then the difference between the twin's ages is marginal: $\Delta t_{\text{full}} \approx 6.5$ years. However, if the traveling twin spends $\Delta t'_{\text{full}} = 25$ years (by their clock) for the roundtrip, then the difference is rather dramatic: $\Delta t_{\text{full}} \approx 1{,}230$ years.

Problem 9.8. Suggest a natural definition of the 4-vector of acceleration (commonly called the *4-acceleration*) of a point and calculate its components for a relativistic point moving with velocity $\mathbf{u} = \mathbf{u}(t)$.

Solution: In non-relativistic mechanics, the linear acceleration of a point is defined via its velocity exactly as the latter variable is defined via the point's radius vector:

$$\mathbf{u} \equiv \frac{d\mathbf{r}}{dt}, \qquad \mathbf{a} \equiv \frac{d\mathbf{u}}{dt}.$$

As was discussed in Sec. 9.3 of the lecture notes, the relativistic 4-velocity of a point is formed from its 4-component radius vector (9.48) similarly, "only" with the replacement of the non-relativistic time t with the (reference-frame-independent) proper time τ defined by Eqs. (9.59)-(9.60):[133]

$$u^\alpha \equiv \frac{dx^\alpha}{d\tau}, \quad \text{where } x^\alpha \equiv \{cdt, d\mathbf{r}\}, \quad \text{and } d\tau \equiv \frac{dt}{\gamma}, \quad \text{so } u^\alpha \equiv \gamma\{c, \mathbf{u}\}.$$

Hence it is natural to define the 4-acceleration of the point as[134]

[133] Here, for certainty, all 4-vectors are in their contravariant form; as was discussed in Sec. 9.4, the covariant forms differ only by the opposite sign before all their 3D-vector components.

[134] Note that with this definition, the key equation (9.145) of particle motion in an electromagnetic field takes the form $ma^\alpha = qF^{\alpha\beta}u_\beta$, very reminiscent of the non-relativistic 2$^{\text{nd}}$ Newton law.

$$a^{\alpha} \equiv \frac{du^{\alpha}}{d\tau} = \gamma \frac{d}{dt}\left(\gamma\{c, \mathbf{u}\}\right). \qquad (*)$$

Here we should be careful. If we spoke about observations from two inertial reference frames moving with a constant relative velocity **v**, as we did in the first two sections of Chapter 9, the Lorentz factor γ defined by Eq. (9.17) would be a function of that velocity (or rather its square) and hence a time-independent parameter. However, at the introduction of the proper time of a moving point in Sec. 9.3, we essentially used the *instantaneous frame* – an inertial reference frame moving, at the considered time instant, with the same velocity as the point itself, so the γ participating in Eq. (9.60) and hence in Eq. (*) is a function of **u**:

$$\gamma = \frac{1}{\left(1 - u^2/c^2\right)^{1/2}}.$$

Hence the differentiation in Eq. (*) should take this dependence into account. Representing a^{α} as $\{a_0, \mathbf{a}\}$ (and hence a_{α} as $\{a_0, -\mathbf{a}\}$),[135] we get

$$\begin{aligned} a_0 &= \gamma \frac{d}{dt}(\gamma c) = \gamma c \frac{d}{dt} \frac{1}{\left(1 - \mathbf{u}\cdot\mathbf{u}/c^2\right)^{1/2}} = \gamma c \frac{1}{2\left(1 - u^2/c^2\right)^{3/2}} \frac{1}{c^2} \frac{d}{dt}(\mathbf{u}\cdot\mathbf{u}) = \frac{\gamma^4}{c} \dot{\mathbf{u}}\cdot\mathbf{u}, \\ \mathbf{a} &= \gamma \frac{d}{dt}(\gamma \mathbf{u}) = \gamma \frac{d}{dt} \frac{\mathbf{u}}{\left(1 - \mathbf{u}\cdot\mathbf{u}/c^2\right)^{1/2}} = \gamma \left[\frac{\dot{\mathbf{u}}}{\left(1 - u^2/c^2\right)^{1/2}} + \mathbf{u} \frac{d}{dt} \frac{1}{\left(1 - \mathbf{u}\cdot\mathbf{u}/c^2\right)^{1/2}} \right] = \gamma^2 \dot{\mathbf{u}} + \frac{\gamma^4}{c^2} \mathbf{u}\left(\dot{\mathbf{u}}\cdot\mathbf{u}\right), \end{aligned} \qquad (**)$$

where

$$\dot{\mathbf{u}} \equiv \frac{d\mathbf{u}}{dt} = \frac{1}{\gamma} \frac{d\mathbf{u}}{d\tau}.$$

In the non-relativistic limit case when $u/c \to 0$ (and hence $\gamma \to 1$), the second of Eqs. (**) is naturally reduced to

$$\mathbf{a} = \dot{\mathbf{u}} \equiv \frac{d\mathbf{u}}{dt} = \frac{d\mathbf{u}}{d\tau}. \qquad (***)$$

One may wonder whether the last result may be valid in the reference frame that is bound to the point in question because, in such a frame, the point's acceleration has to vanish. The answer is: since all special relativity relations are only valid in inertial reference frames, Eq. (9.6) and hence Eq. (***) are only valid in the instantaneous rest frame, i.e. the inertial frame whose *time-independent* velocity coincides with the point's *time-dependent* velocity **u** at the considered moment of time. Relatively to such a frame, the point may indeed have a non-vanishing acceleration.

In the simplest particular case of a 1D motion when the vectors **u** and **a** are aligned: $\mathbf{u} = u\mathbf{n}$, $\mathbf{a} = a\mathbf{n}$, Eqs. (**) give

$$a_0 = \frac{\gamma^4}{c} u\dot{u} \equiv \frac{\gamma^4}{c} u \frac{du}{dt} = \gamma^3 \frac{u}{c} \frac{du}{d\tau}, \qquad a = \gamma^2 \dot{u} + \frac{\gamma^4}{c^2} u^2 \dot{u} \equiv \gamma^4 \dot{u} \equiv \gamma^4 \frac{du}{dt} = \gamma^3 \frac{du}{d\tau}.$$

[135] Note that in the relativistic case, **a** so defined is *not* the point's acceleration $d\mathbf{u}/dt$ as observed from the rest plane, just as the spatial component of the 4-velocity u^{α} is not the corresponding velocity **u** – see also the model solution of Problem 6.

Problem 9.9. Calculate the first relativistic correction to the frequency of a harmonic oscillator as a function of its amplitude.

Solution: Combining the potential energy of the oscillator,

$$U = \frac{\kappa}{2}x^2 \equiv \frac{m\omega_0^2}{2}x^2,$$

with the Taylor expansion of its kinetic energy in small u^2/c^2, and keeping just two first terms (see Eq. (9.74) of the lecture notes), we would describe the usual harmonic oscillator with frequency ω_0 independent of the oscillation amplitude A. To get the first relativistic correction to this result, we need to extend that expansion by one more term:

$$\frac{mc^2}{\left(1-u^2/c^2\right)^{1/2}} \approx mc^2 + \frac{mu^2}{2} + \frac{3mu^4}{8c^2},$$

so (ignoring the motion-independent term mc^2) we may approximate the full oscillator's energy $\mathscr{E}$ as

$$\mathscr{E} \approx \frac{mu^2}{2} + \frac{3mu^4}{8c^2} + \frac{m\omega_0^2 x^2}{2}.$$

This energy is an integral of motion,[136] and hence is always equal to its value at maximum deviations $x = \pm A$ from the equilibrium, when $u \equiv dx/dt = 0$:

$$\mathscr{E} = \frac{m\omega_0^2 A^2}{2}.$$

Equating these two expressions for $\mathscr{E}$ and dividing all terms by $m/2$, we get the following relation between u and x:

$$u^2\left(1+\frac{3u^2}{4c^2}\right) = \omega_0^2\left(A^2 - x^2\right).$$

With our assumption of the almost-non-relativistic motion, the second term in the first parentheses is small, so we may plug into it the velocity calculated in the non-relativistic limit: $(u^2)_0 = \omega_0^2(A^2 - x^2)$, and get the 1st relativistic approximation:

$$u^2 \approx \frac{\omega_0^2\left(A^2 - x^2\right)}{1+\left(3\omega_0^2/4c^2\right)\left(A^2 - x^2\right)}.$$

Plugging this approximation into the well-known expression for the period of oscillations in a symmetric confining potential,[137]

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136 This (almost evident) fact may be strictly proved using the full Lagrangian function $\mathscr{L}$ of the oscillator, which may be formed by subtracting U from the free-particle's Lagrangian given by Eq. (9.68). According to analytical mechanics (see, e.g., CM Sec. 2.3), since the total $\mathscr{L}$ does not depend on time explicitly ($\partial\mathscr{L}/\partial t = 0$), the Hamiltonian function $\mathscr{H}$ of the system is an integral of motion: $d\mathscr{H}/dt = 0$. Now calculating $\mathscr{H}$ exactly as it was done at the derivation of Eq. (9.72) of the lecture notes, we may see that in such a system (even for an arbitrary ratio u/c, and any time-independent U), $\mathscr{H}$ and $\mathscr{E}$ coincide, so $d\mathscr{E}/dt = 0$ as well.

137 See, e.g., CM Sec. 3.3.

$$\mathcal{T} = \int_{\text{all period}} dt \equiv \int_{\text{all period}} \frac{dx}{dx/dt} \equiv \int_{\text{all period}} \frac{dx}{u} = 4\int_0^A \frac{dx}{|u|},$$

and then introducing a new, dimensionless integration variable $\xi \equiv x/A$, we get

$$\mathcal{T} \approx \frac{4}{\omega_0}\int_0^A \frac{\left[1+\left(3\omega_0^2/4c^2\right)\left(A^2-x^2\right)\right]^{1/2}}{\left(A^2-x^2\right)^{1/2}}dx \equiv \frac{4}{\omega_0}\int_0^1 \frac{\left[1+\left(3\omega_0^2A^2/4c^2\right)\left(1-\xi^2\right)\right]^{1/2}}{\left(1-\xi^2\right)^{1/2}}d\xi$$

$$\approx \frac{4}{\omega_0}\int_0^1 \frac{1}{\left(1-\xi^2\right)^{1/2}}\left[1+\frac{3\omega_0^2A^2}{8c^2}\left(1-\xi^2\right)\right]d\xi \equiv \frac{4}{\omega_0}\int_0^1 \frac{d\xi}{\left(1-\xi^2\right)^{1/2}} + \frac{4}{\omega_0}\frac{3\omega_0^2A^2}{8c^2}\int_0^1\left(1-\xi^2\right)^{1/2}d\xi.$$

Both integrals may be readily worked out by the same substitution $\xi \equiv \sin\alpha$ (with $d\xi = \cos\alpha d\alpha$, $(1-\xi^2)^{1/2} = \cos\alpha$), giving

$$\int_0^1 \frac{d\xi}{\left(1-\xi^2\right)^{1/2}} = \int_0^{\pi/2} d\alpha = \frac{\pi}{2}, \qquad \int_0^1\left(1-\xi^2\right)^{1/2}d\xi = \int_0^{\pi/2}\cos^2\alpha\, d\alpha = \frac{1}{2}\int_0^{\pi/2}\left(1+\cos 2\alpha\right)d\alpha = \frac{\pi}{4},$$

so, finally, the oscillation period is

$$\mathcal{T} \approx \frac{2\pi}{\omega_0} + \frac{2\pi}{\omega_0}\frac{3\omega_0^2A^2}{16c^2} \equiv \frac{2\pi}{\omega_0}\left(1+\frac{3\omega_0^2A^2}{16c^2}\right),$$

and the oscillation frequency

$$\omega \equiv \frac{2\pi}{\mathcal{T}} \approx \omega_0\left(1+\frac{3\omega_0^2A^2}{16c^2}\right)^{-1} \approx \omega_0\left(1-\frac{3\omega_0^2A^2}{16c^2}\right). \qquad (*)$$

This formula (which is quantitatively correct only if the second term in parentheses is much smaller than 1) shows that the oscillation frequency *reduces* with A – a natural result of the *growth* of the particle's relativistic mass $M = m/(1-u^2/c^2)^{1/2}$ with the increase of its velocity amplitude $u_{\max} \approx \omega_0 A$.

Note that alternatively, Eq. (*) may be obtained by using the Lagrangian function (9.183) of the system,

$$\mathscr{L} = -\frac{mc^2}{\gamma} - \frac{m\omega_0^2x^2}{2} \approx -mc^2 + \frac{mu^2}{2} + \frac{mu^4}{8c^2} - \frac{m\omega_0^2x^2}{2},$$

to derive its (approximate) Lagrange equation of motion,

$$\frac{d}{dt}\left(mu+\frac{mu^3}{2c^2}\right) + m\omega_0^2x = 0, \qquad \text{i.e. } \ddot{x}+\omega_0^2x = -\frac{1}{2c^2}\frac{d}{dt}\left(\dot{x}^3\right) \equiv -\frac{3}{2c^2}\dot{x}^2\ddot{x}, \qquad (**)$$

with the right-hand side considered small, and then solving this equation using the standard van der Pol method (alternatively called the "rotating-wave approximation" – RWA).[138] The result is exactly the same;[139] however, for our particular case, when the right-hand side of Eq. (**) is a function of $u \equiv dx/dt$ alone, the first approach used above is a bit simpler.

[138] See, e.g., CM Sec. 5.2.
[139] See, e.g., the solution of CM Problem 5.7.

Problem 9.10. An atom with an initial rest mass m has been excited to an internal state with an additional energy $\Delta\mathscr{E}$, while still being at rest. Next, it returns to its initial state, emitting a photon. Calculate the photon's frequency, taking into account the relativistic recoil of the atom.

Hint: In this problem, and also in Problems 13-15 below, you may treat photons as classical ultra-relativistic point particles with zero rest mass, energy $\mathscr{E} = \hbar\omega$, and momentum $\mathbf{p} = \hbar\mathbf{k}$.

Solution: Combining the energy conservation law,

$$mc^2 + \Delta\mathscr{E} = \hbar\omega + \left[(mc^2)^2 + p^2c^2\right]^{1/2},$$

and the momentum conservation law

$$0 = \frac{\hbar\omega}{c} - p$$

(both in the lab reference frame, which, in this problem, is also the center-of-mass frame), we readily get

$$\omega = \frac{\Delta\mathscr{E}}{\hbar}\,\frac{1+\Delta\mathscr{E}/2mc^2}{1+\Delta\mathscr{E}/mc^2} < \frac{\Delta\mathscr{E}}{\hbar}.$$

For real atoms ($m \sim 10^{-26}$ kg, $\Delta\mathscr{E} \sim 1$ eV), $\Delta\mathscr{E} \ll mc^2$, so the above expression is reduced to

$$\omega \approx \frac{\Delta\mathscr{E}}{\hbar}\left(1 - \frac{\Delta\mathscr{E}}{2mc^2}\right).$$

Hence the recoil correction of frequency is typically relatively small ($\sim 10^{-10}$), but still may be measurable for quantum transitions with small natural linewidth. This correction is dramatically reduced at the *Mössbauer effect*,[140] in which the role of m is played by the mass of the whole crystal lattice. This effect serves as the basis of the high-resolution *Mössbauer spectroscopy* capable of detecting minute changes in the chemical environment of the nucleus.

Problem 9.11. A particle of mass m, initially at rest, decays into two particles with rest masses m_1 and m_2. Calculate the total energy of the first product particle, in the c.o.m. reference frame.

Solution: In the rest frame of the initial particle, before the decay, its momentum p was zero, and energy was mc^2. Hence, due to the energy and momentum conservation, in this frame (which is the c.o.m. frame of the system), the 4-momenta of the decay products may be represented as

$$p_1^\alpha = \left\{\frac{\mathscr{E}_1}{c}, \mathbf{p}_1\right\}, \qquad p_2^\alpha = \left\{\frac{mc^2 - \mathscr{E}_1}{c}, -\mathbf{p}_1\right\}.$$

Due to the Lorentz invariance of the scalar products of 4-vectors, the norm of each of these 4-vectors has to be equal to its value in the corresponding rest frame, i.e. to $(m_1c)^2$ and $(m_2c)^2$, respectively – see Eq. (9.78) of the lecture notes. This gives us a system of two equations for $\mathscr{E}_1$ and p_1:

$$\frac{\mathscr{E}_1^2}{c^2} - p_1^2 = (m_1c)^2, \qquad \frac{\left(mc^2 - \mathscr{E}_1\right)^2}{c^2} - p_1^2 = (m_2c)^2.$$

[140] See, e.g., H. Fraunfelder, *The Mössbauer Effect,* W. A. Benjamin, 1963.

Now an elementary elimination of p_1 yields

$$\mathscr{E}_1 = \frac{m^2 + m_1^2 - m_2^2}{2m} c^2 .$$

As a sanity check, if both product particles are similar ($m_1 = m_2$), this result is reduced to $\mathscr{E}_1 = mc^2/2$ – very naturally, because in this case, the initial energy mc^2 is equally shared by the products. Also, Eq. (*) is duly symmetric with respect to the index swap $1 \leftrightarrow 2$.

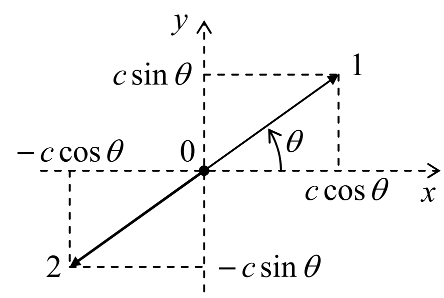
Problem 9.12. A relativistic particle with a rest mass m, moving with velocity u, decays into two particles with zero rest mass.

(i) Calculate the smallest possible angle between the decay product velocities (in the lab frame, in that the velocity u is measured).

(ii) What is the largest possible energy of a product particle?

Solutions:

(i) In the reference frame moving with the velocity of the initial particle (i.e. that of the center of mass of the system), that particle does not move, so the total momentum conservation requires the decay products to move in opposite directions, along the same straight line. Since per the basic Eq. (9.78), the speed of a massless particle with nonzero energy has to be equal to c, the Cartesian components of the product particle velocities, as measured in this c.o.m. frame, depend on just one free parameter, say the angle θ shown in the figure on the right. (Here the plane of the drawing is the common plane of the initial and final particle velocities, and the coordinate axes are oriented so that $\mathbf{u} = u\mathbf{n}_x$.)



Now we may use Eqs. (9.23) of the lecture notes, with $\mathbf{v} = \mathbf{u}$, to Lorentz-transform these velocity components into the lab frame:

$$v_{x1} = \frac{c\cos\theta + u}{1 + u\cos\theta / c}, \quad v_{y1} = \frac{1}{\gamma}\frac{c\sin\theta}{1 + u\cos\theta / c},$$
$$v_{x2} = \frac{-c\cos\theta + u}{1 - u\cos\theta / c}, \quad v_{y2} = -\frac{1}{\gamma}\frac{c\sin\theta}{1 - u\cos\theta / c}.$$

Since the speed of both particles in the lab system is still c, the trigonometric functions of the angles $\theta_{1,2}$ of their propagation in that frame (relative to the x-axis) may be calculated as $\cos\theta_1 = v_{x1}/c$, etc., giving

$$\cos\theta_1 = \frac{\cos\theta + \beta}{1 + \beta\cos\theta}, \quad \sin\theta_1 = \frac{1}{\gamma}\frac{\sin\theta}{1 + \beta\cos\theta}; \quad \cos\theta_2 = \frac{-\cos\theta + \beta}{1 - \beta\cos\theta}, \quad \sin\theta_2 = -\frac{1}{\gamma}\frac{\sin\theta}{1 - \beta\cos\theta}, \quad (*)$$

where $\beta \equiv u/c$. Now we can calculate the angle $\theta_- \equiv \theta_1 - \theta_2$ between the two particle velocities in the lab frame:

$$\cos\theta_- \equiv \cos(\theta_1 - \theta_2) \equiv \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 = \frac{\beta^2 - \cos^2\theta}{1 - \beta^2\cos^2\theta} - \frac{\sin^2\theta}{\gamma^2(1 - \beta^2\cos^2\theta)}$$
$$\equiv \frac{\beta^2\sin^2\theta - (1 - \beta^2)}{\beta^2\sin^2\theta + (1 - \beta^2)} \equiv \frac{\xi - 1}{\xi + 1}, \qquad \text{where } \xi \equiv \frac{\beta^2\sin^2\theta}{1 - \beta^2}.$$

Since the function $(\xi - 1)/(\xi + 1)$ grows monotonically with ξ at all its possible (non-negative) values, $\cos\theta_-$ is the largest, and hence θ_- (if defined on the segment $[0, \pi]$) is the smallest, at $\sin^2\theta = 1$, i.e., at the product particles' propagation along the y-axis, as observed in the c.o.m. frame. In this case, we have

$$\left(\cos\theta_-\right)_{\max} = \frac{\beta^2 - \left(1-\beta^2\right)}{\beta^2 + \left(1-\beta^2\right)} \equiv 2\beta^2 - 1, \qquad \left(\theta_-\right)_{\min} = \cos^{-1}\left(2\beta^2 - 1\right) \equiv 2\cos^{-1}\beta \equiv 2\cos^{-1}\frac{u}{c}.$$

So, at low velocities of the initial particle, the smallest angle θ_- tends to π, i.e. to the value this angle has in the c.o.m. frame, while at $u \to c$, it gradually approaches zero.

(ii) Evidently, the largest difference between the product energies is achieved at $\theta = 0$, when $\theta_1 = 0$ and $\theta_2 = \pi$, i.e. when the particles move along the same line (in the figure above, the x-axis), in the opposite directions.[141] In this case, we may employ the conservation of energy and of the x-component of the system's momentum in the lab frame, by using the basic Eqs. (9.70) and (9.73), with $p = \mathscr{E}/c$ for each of the massless product particles:

$$m\gamma u = \frac{\mathscr{E}_1}{c} - \frac{\mathscr{E}_2}{c}, \qquad m\gamma c^2 = \mathscr{E}_1 + \mathscr{E}_2.$$

Solving this simple system of equations, we get[142]

$$\mathscr{E}_{1,2} = \frac{m\gamma c^2}{2}\left(1 \pm \frac{u}{c}\right) \equiv \frac{mc^2}{2}\frac{1\pm\beta}{\left(1-\beta^2\right)^{1/2}} \equiv \frac{mc^2}{2}\left(\frac{1\pm\beta}{1\mp\beta}\right)^{1/2}.$$

The result shows that if the initial particle is non-relativistic ($\beta \to 0$), its energy (in this limit, close to the rest energy mc^2) is always equally divided between those of the decay products, but if it is ultra-relativistic ($\beta \to 1$), almost all its energy may go to just one product particle.

<u>Problem 9.13</u>. A relativistic particle flying in free space with velocity $\mathbf{u}$ decays into two photons.[143] Calculate the angular dependence of the photon detection probability, as measured in the lab frame.

Solution: Let θ be the angle between the direction of propagation of the detected photon and the vector of the velocity of the initial particle, as measured in the lab frame, while θ' be the similar angle measured in the reference frame moving with the initial particle, i.e. with the velocity $\mathbf{v} = \mathbf{u}$. Then they are related by the first of Eqs. (9.27) of the lecture notes, with $v = u$, $u_x = c\cos\theta$, and $u'_x = c\cos\theta'$:[144]

$$c\cos\theta = \frac{c\cos\theta' + u}{1 + uc\cos\theta'/c^2}, \qquad \text{i.e.} \cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}, \qquad \text{with } \beta \equiv \frac{u}{c}, \quad \gamma \equiv \frac{1}{\left(1-\beta^2\right)^{1/2}}.$$

[141] On the contrary, as it follows from Eq. (*), at $\theta = \pi/2$, i.e. at $\cos\theta = 0$, and $\sin\theta = 1$, the product particles propagate at equal angles to the initial direction ($\theta_1 = -\theta_2$), and hence have equal energies.

[142] Note a substantial similarity between the last expression and Eq. (9.44) describing the longitudinal Doppler effect.

[143] Such a decay may happen, for example, with a neutral pion.

[144] Actually, this relation, in a different notation, is just the first of Eqs. (*) of the solution of the previous problem, devoted to another aspect of a similar process.

because the photon's speed equals c in any reference frame. The reciprocal relation is

$$\cos\theta' = \frac{\cos\theta - \beta}{1-\beta\cos\theta}. \qquad (*)$$

Next, in the reference frame moving with the initial particle, it was at rest, so in the absence of any external polarizing field, the angular distribution of the resulting photons has to be isotropic. This means that of the total $N >> 1$ photons resulting from a large set (the *statistical ensemble*[145]) of similar but independent experiments, the number of photons going into any small solid angle interval $d\Omega' = \sin\theta' d\theta' d\varphi'$ is the same: $dN = N(d\Omega'/4\pi) \equiv N(\sin\theta' d\theta' d\varphi'/4\pi)$. From the point of view of the lab frame, these dN photons go into the interval $d\Omega = \sin\theta d\theta d\varphi$, with $\varphi = \varphi'$ due to the axial symmetry of the problem. Hence the probability density of the photon propagation at angle θ is

$$w(\theta) \equiv \frac{1}{N}\frac{dN}{d\Omega} = \frac{1}{4\pi}\frac{\sin\theta' d\theta'}{\sin\theta d\theta} = \frac{1}{4\pi}\frac{d(\cos\theta')}{d(\cos\theta)}.$$

This relation is general (for axially-symmetric systems); now using the particular Eq. (*) to calculate the involved derivative, we finally get

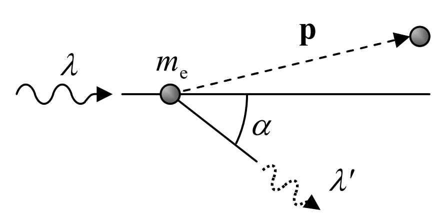
$$w(\theta) = \frac{1}{4\pi}\frac{1-\beta^2}{(1-\beta\cos\theta)^2}.$$

(As a sanity check, the integral of this function over the full solid angle, i.e. the total probability of propagation of a photon in *some* direction, equals 1.)

This result shows that the photon detection probability is the largest in the direction of the initial particle's propagation, $\theta = 0$:

$$w(0) = \frac{1}{4\pi}\frac{1+\beta}{1-\beta},$$

and that for ultra-relativistic particles ($\gamma >> 1$), this enhancement may be very significant, scaling as γ^2.



Problem 9.14. A photon with wavelength λ is scattered by an electron, initially at rest. Calculate the wavelength λ' of the scattered photon as a function of the scattering angle α – see the figure on the right.[146]

Solution: The problem may be most simply solved by writing the laws of conservation of the total relativistic momentum (9.70) and the relativistic energy (9.73), both in the lab reference frame:

$$\hbar\mathbf{k} = \hbar\mathbf{k}' + \mathbf{p}, \qquad (*)$$

$$\hbar\omega + m_e c^2 = \hbar\omega' + \left[(m_e c^2)^2 + p^2c^2\right]^{1/2}, \qquad (**)$$

[145] This notion is central not only for statistical physics, but for quantum mechanics as well, and is discussed in detail in the QM and SM parts of this series.

[146] This is the famous *Compton scattering* effect, whose discovery in 1923 was one of the major motivations for the development of quantum mechanics – see, e.g. QM Sec. 1.1.

where **p** is the electron's momentum after the scattering event (see the figure above), and $k = \omega/c = 2\pi/\lambda$. From Eq. (*), $\mathbf{p} = \hbar(\mathbf{k} - \mathbf{k}')$, so[147]

$$p^2 = \hbar^2(\mathbf{k}-\mathbf{k}')^2 = \hbar^2\left(k^2 + k'^2 - 2kk'\cos\alpha\right).$$

Plugging this expression into Eq. (**), and solving it for $\lambda' \equiv 2\pi/k'$, we get

$$\lambda' = \lambda + \frac{2\pi\hbar}{m_\mathrm{e}c}\left(1-\cos\alpha\right) \geq \lambda\ .$$

The only fundamental constant, with the dimension of length, participating in this result,

$$\frac{2\pi\hbar}{m_\mathrm{e}c} \approx 2.426\times 10^{-12}\,\mathrm{m},$$

is called the *Compton wavelength.*[148] Note the relation between this length and the classical radius r_c of the same particle, given by Eq. (8.41):

$$\frac{r_\mathrm{c}}{2\pi\hbar/m_\mathrm{e}c} \equiv \frac{e^2}{4\pi\varepsilon_0}\frac{1}{m_\mathrm{e}c^2}\frac{m_\mathrm{e}c}{2\pi\hbar} \equiv \frac{1}{2\pi}\left(\frac{1}{4\pi\varepsilon_0}\frac{e^2}{c\hbar}\right) \equiv \frac{\alpha}{2\pi} \sim 10^{-3},$$

where

$$\alpha \equiv \frac{1}{4\pi\varepsilon_0}\frac{e^2}{c\hbar} \approx \frac{1}{137}$$

is the fine structure constant,[149] which characterizes the strength (or rather the weakness :-) of electromagnetic interactions relative to the quantum and relativistic effects.

Problem 9.15. Calculate the threshold energy of a γ-photon for the reaction

$$\gamma + \mathrm{p} \rightarrow \mathrm{p} + \pi^0,$$

if the proton was initially at rest.

Hint: For protons, $m_\mathrm{p}c^2 \approx 938$ MeV, while for neutral pions, $m_\pi c^2 \approx 135$ MeV.

Solution: Similarly to the analysis of the $\mathrm{p}+\mathrm{p} \rightarrow \mathrm{p}+\mathrm{p}+\mathrm{p}+\overline{\mathrm{p}}$ reaction in Sec. 9.4 of the lecture notes, we may represent the conservation of the net 4-momentum of the system as[150]

$$(\gamma + p)_\alpha(\gamma + p)^\alpha = (p + \pi^0)_\alpha(p+\pi^0)^\alpha, \qquad (*)$$

where the particle names typeset in italics are used to denote the corresponding 4-momenta. After opening the parentheses, we can evaluate the Lorentz-invariant scalar product $p_\alpha p^\alpha$ on the left-hand side of Eq. (*) as $m_\mathrm{p}^2c^2$, and the similar product for the γ-photon as zero because photons do not have rest

[147] Note this useful (and frequently used) approach of avoiding the introduction of the direction of the vector **p**.
[148] This historic name is rather misleading: no particle in the problem has this wavelength, and (to the best of my knowledge :-) neither had Dr. A. Compton.
[149] See also the solutions of Problems 5.18 and 8.9.
[150] Here, just as in Eqs. (9.88)-(9.89) of the lecture notes, I am using the Italic font to denote the components of the 4-momentum of each particle, leaving the Roman font to denote the particle as such in the reaction's description.

masses. Similarly, the right-hand side of Eq. (*) is equal to $(m_p + m_\pi)^2c^2$ because, at the reaction's threshold, its products have zero velocity in the c.o.m. frame. As a result, Eq. (*) becomes

$$2\gamma_\alpha p^\alpha + m_p^2 c^2 = (m_p + m_\pi)^2 c^2, \qquad \text{giving } 2\gamma_\alpha p^\alpha = \left[(m_p + m_\pi)^2 - m_p^2\right]c^2 .$$

In the lab system, in which the initial proton rests, its 4-vector p^α has only one nonvanishing component, $p_0 = m_p c$, while the corresponding component γ_0 for the photon is just $\mathscr{E}_\gamma/c$. As a result, we get

$$\mathscr{E}_\gamma = \frac{1}{2m_p}\left[(m_p + m_\pi)^2 - m_p^2\right]c^2 \equiv m_\pi c^2\left(1 + \frac{m_\pi}{2m_p}\right).$$

The second term in the parentheses represents the energy price we have to pay for not having the center of mass of the whole initial system at rest. With the given rest masses of the involved particles, this correction is close to 7% (i.e., quite noticeable!), and we finally get $\mathscr{E}_\gamma \approx 145$ MeV.

Problem 9.16. Calculate the largest possible velocity of the electrons emitted by (initially, resting) neurons at their *β-decays*:

$$\mathrm{n} \rightarrow \mathrm{p} + \mathrm{e} + \bar{\nu}_\mathrm{e} .$$

Hint: Electron neutrinos ν_e and antineutrinos $\bar{\nu}_e$ are virtually massless (on the energy scale of this problem); the rest energies $\mathscr{E} \equiv mc^2$ of the other involved particles are as follows: 939.565 MeV for the neutron, 938.272 MeV for the proton, 0.511 MeV for the electron.

Solution: The initial energy of the system, $m_n c^2$, is shared by all decay products, and only that of the neutrino may be zero, due to its zero rest mass. Because of that, the largest energy (and hence velocity) of the electron corresponds to the neutrino's energy approaching zero. As a result, for the purposes of this problem, we may ignore the contributions of this particle to the energy and momentum of the finite system, and write their conservation laws as follows:[151]

$$\mathscr{E}_n = \left[\mathscr{E}_p^2 + (p_p c)^2\right]^{1/2} + \left[\mathscr{E}_e^2 + (p_e c)^2\right]^{1/2}, \qquad 0 = \mathbf{p}_p + \mathbf{p}_e .$$

The latter of these relations immediately yields $p_p^2 = p_e^2$. Denoting this magnitude as p^2, and plugging it into the former of the formulas, we get a single equation for $(pc)^2$:

$$\mathscr{E}_n = \left[\mathscr{E}_p^2 + (pc)^2\right]^{1/2} + \left[\mathscr{E}_e^2 + (pc)^2\right]^{1/2}.$$

This equation may be readily solved (as usual, by squaring both sides, and then singling out and squaring the remaining square root product), giving

$$pc = \frac{1}{\mathscr{E}_n}\left[\left(\frac{\mathscr{E}_n^2 - \mathscr{E}_p^2 - \mathscr{E}_e^2}{2}\right)^2 - \mathscr{E}_p^2\mathscr{E}_e^2\right]^{1/2}. \qquad (*)$$

In order to recalculate pc into the electron's velocity, we may use the fundamental Eq. (9.70) of the lecture notes:

[151] In this solution, for brevity, symbol $\mathscr{E}$ is used for the rest energies mc^2 of the particles, rather than their full energies. (High energy practitioners use the particle rest masses m for this purpose, implying units with $c = 1$.)

$$p = m_e \gamma u \equiv m_e \frac{u}{\left(1 - u^2 / c^2\right)^{1/2}}, \qquad \text{i.e. } pc = \mathscr{E}_e \frac{\beta}{\left(1-\beta^2\right)^{1/2}}.$$

Solving it for the reduced velocity $\beta \equiv u/c$, we get

$$\beta = \frac{pc/\mathscr{E}_e}{\left[1 + \left(pc/\mathscr{E}_e\right)^2\right]^{1/2}}. \qquad (**)$$

Plugging the numbers provided in the *Hint* into Eqs. (*) and (**), we finally get $pc \approx 1.187$ MeV, $\beta \approx 0.918$.

Problem 9.17. A relativistic particle with a rest mass m and an energy $\mathscr{E}$ collides with a similar particle, initially at rest in the laboratory reference frame. Calculate:

(i) the final velocity of the center of mass of the system, as measured in the lab frame,

(ii) the total energy of the system, in the center-of-mass frame, and

(iii) the final velocities of both particles (in the lab frame), provided that they move along the same direction.

Solutions:

(i) Before the collision, the total momentum p_s of the system in the lab frame equals that (p) of the only moving particle, which may be calculated from the basic Eq. (9.78) of the lecture notes:

$$p_s = p = \left[\left(\frac{\mathscr{E}}{c}\right)^2 - (mc)^2\right]^{1/2},$$

while the total energy of the system is evidently

$$\mathscr{E}_s = \mathscr{E} + mc^2.$$

The center of mass of a system may be interpreted as an imaginary particle having the same total momentum p_s and the total mass M_s as the whole system. (In relativity,[152] the latter is the dynamic mass $M_s = \mathscr{E}_s/c^2$.) Applying, to that particle, the general formula $\mathbf{p} = M\mathbf{u}$, giving $\boldsymbol{\beta} = \mathbf{p}/Mc = \mathbf{p}c/\mathscr{E}$, we get

$$\beta_{com} = \frac{\left[(\mathscr{E}/c)^2 - (mc)^2\right]^{1/2} c}{\mathscr{E} + mc^2} \equiv \left(\frac{\mathscr{E} - mc^2}{\mathscr{E} + mc^2}\right)^{1/2}, \quad \text{so } \gamma_{com} \equiv \frac{1}{\left(1 - \beta_{com}^2\right)^{1/2}} = \left(\frac{\mathscr{E} + mc^2}{2mc^2}\right)^{1/2}. \quad (*)$$

Note that although the particles are similar, $\beta_{com} \to \beta/2$ only in the non-relativistic limit; in the opposite, ultra-relativistic limit ($v \to c$, $\beta \to 1$, $\mathscr{E} >> mc^2$), our result yields $\beta_{com} \to 1$, i.e. the center-of-mass velocity is also close to c, while $\gamma_{com} \approx (\mathscr{E}/2mc^2)^{1/2} >> 1$.

(ii) Applying the Lorentz transform formula (see, e.g., Eqs. (9.50)-(9.51)), valid for any 4-vector, to the 0$^{\text{th}}$ component of the total system's 4-momentum $p_s^{\alpha} = \{\mathscr{E}_s/c, p_s, 0, 0\}$, we obtain

152 See the footnote at the end of Sec. 9.3 of the lecture notes.

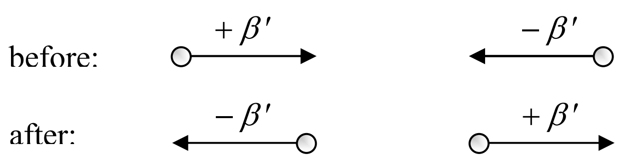
$$\frac{(\mathscr{E}_\mathrm{s})_\mathrm{com}}{c} = \gamma_\mathrm{com}\left(\frac{\mathscr{E}_\mathrm{s}}{c} - \beta_\mathrm{com} p_\mathrm{s}\right) = \gamma_\mathrm{com}\left\{\frac{\mathscr{E} + mc^2}{c} - \beta_\mathrm{com}\left[\left(\frac{\mathscr{E}}{c}\right)^2 - (mc)^2\right]^{1/2}\right\}.$$

Plugging in the expressions for β_com and γ_com from Eq. (*), we get[153]

$$(\mathscr{E}_\mathrm{s})_\mathrm{com} = 2\gamma_\mathrm{com} mc^2 = \left[2mc^2(\mathscr{E} + mc^2)\right]^{1/2}. \qquad (**)$$

Note that for ultra-relativistic particles, $\mathscr{E} >> mc^2$, this energy (which is the parameter crucial for new particle generation) grows only as the square root of that in the lab frame; this is why particle colliders are more popular nowadays than the resting-target accelerators. (The caveat is that the colliders should have a very high density of particle beams to ensure a sufficiently high collision rate.)

(iii) By the definition of the center-of-mass reference frame, the velocities of two similar particles, observed from the frame, should be equal and opposite, and since if the total kinetic energy is conserved at collision, their magnitudes β' should be the same both before and after it. If the particles, after the collision, move along the same direction as before it, this means that they just exchange their velocities – see the figure below.



But this particular fact (of the particles exchanging velocities) should be true in *any* reference frame because the Lorentz transform of a velocity does not depend on either the particle's number or whether the transformed velocity is the one before or after the collision. Hence, in the lab system, the final velocities of the particles are also the same as the initial ones:

$$\beta_+ = \beta = \frac{cp}{\mathscr{E}} = \frac{1}{\mathscr{E}}\left[\mathscr{E}^2 - (mc^2)^2\right]^{1/2} \equiv \left[1 - \left(\frac{mc^2}{\mathscr{E}}\right)^2\right]^{1/2}, \qquad \beta_- = 0.$$

<u>Problem 9.18</u>. A "primed" reference frame moves, relative to the "lab" frame, with a reduced velocity $\boldsymbol{\beta} \equiv \mathbf{v}/c = \mathbf{n}_x\beta$. Use Eq. (9.109) of the lecture notes to express the elements T'^{00} and T'^{0j} (with j = 1, 2, 3) of an arbitrary contravariant 4-tensor $T^{\gamma\delta}$ via its elements in the lab frame.

Solution: For the relative motion of the frames along the x-axis, the Lorentz transform equations have their traditional form (9.19), so the mixed Lorentz tensor has the form (9.98):

[153] Another (even simpler) way to derive Eq. (**) is to require the norm of the total 4-momentum of the system, $(p_\mathrm{s})^\alpha(p_\mathrm{s})_\alpha$, to be Lorentz invariant:

$$(\mathscr{E}_s)^2_\mathrm{com}/c^2 - (p_\mathrm{s})^2_\mathrm{com} = (\mathscr{E}_\mathrm{s})^2/c^2 - p_\mathrm{s}^2,$$

and then notice that the net momentum in the center-of-mass frame, $(p_\mathrm{s})_\mathrm{com}$, is zero.

$$\frac{\partial x'^{\alpha}}{\partial x^{\beta}} \equiv L^{\alpha}_{\beta} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{where } \gamma \equiv \frac{1}{\left(1-\beta^2\right)^{1/2}}.$$

Due to this abundance of zero elements, of the 16 terms in each implicit sum over the indices γ and δ in the second of Eqs. (9.109), giving the requested transform, there are only 4 non-zero terms in expressions for each of T'^{00} and T'^{01}:

$$T'^{00} = \sum_{\gamma,\delta=0}^{3} \frac{\partial x'^0}{\partial x^\gamma}\frac{\partial x'^0}{\partial x^\delta}T^{\gamma\delta} = \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^0}{\partial x^0}T^{00} + \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^0}{\partial x^1}T^{01} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^0}{\partial x^0}T^{10} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^0}{\partial x^1}T^{11}$$

$$= \gamma\gamma T^{00} + \gamma(-\beta\gamma)T^{01} + (-\beta\gamma)\gamma T^{01} + (-\beta\gamma)(-\beta\gamma)T^{11} \equiv \gamma^2\left[T^{00} - \beta\left(T^{01}+T^{10}\right) + \beta^2 T^{11}\right],$$

$$T'^{01} = \sum_{\gamma,\delta=0}^{3} \frac{\partial x'^0}{\partial x^\gamma}\frac{\partial x'^1}{\partial x^\delta}T^{\gamma\delta} = \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^1}{\partial x^0}T^{00} + \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^1}{\partial x^1}T^{01} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^1}{\partial x^0}T^{10} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^1}{\partial x^1}T^{11}$$

$$= \gamma(-\beta\gamma)T^{00} + \gamma\gamma T^{01} + (-\beta\gamma)(-\beta\gamma)T^{10} + (-\beta\gamma)\gamma T^{11} \equiv \gamma^2\left[T^{01} - \beta\left(T^{00}+T^{11}\right) + \beta^2 T^{10}\right],$$

and only 2 nonvanishing terms for each of T'^{02} and T'^{03}:

$$T'^{02} = \sum_{\gamma,\delta=0}^{3} \frac{\partial x'^0}{\partial x^\gamma}\frac{\partial x'^2}{\partial x^\delta}T^{\gamma\delta} = \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^2}{\partial x^2}T^{02} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^2}{\partial x^2}T^{12} = \gamma T^{02} + (-\beta\gamma)T^{12} \equiv \gamma\left(T^{02} - \beta T^{12}\right),$$

and absolutely similarly, with the replacement $2 \to 3$:

$$T'^{03} = \gamma\left(T^{03} - \beta T^{13}\right).$$

As the simplest sanity check, at $\beta \to 0$ (when $\gamma \to 1$), each tensor element is the same in both reference frames.

Problem 9.19. Prove that quantities $E^2 - c^2B^2$ and $\mathbf{E}\cdot\mathbf{B}$ are Lorentz-invariant.

Solution: Both facts may be proved in a straightforward way, by writing these expressions in the Cartesian components,

$$E^2 - c^2B^2 = \left(E_x^2 + E_y^2 + E_z^2\right) - c^2\left(B_x^2 + B_y^2 + B_z^2\right), \qquad \mathbf{E}\cdot\mathbf{B} = E_xB_x + E_yB_y + E_zB_z,$$

and then using Eqs. (9.134) of the lecture notes for the Lorentz transformation of each component.

A much shorter way to reach the same goal is to notice that according to Eqs. (9.125) and (9.131) of the lecture notes,[154]

$$F_{\alpha\beta}F^{\alpha\beta} = -\frac{2}{c^2}\left(E^2 - c^2B^2\right), \qquad F_{\alpha\beta}G^{\alpha\beta} = \frac{4}{c}\mathbf{E}\cdot\mathbf{B},$$

[154] Actually, the first of these facts was encountered in the lecture notes – see Eq. (9.217)

so both considered combinations are double scalar products (multiplied by constants) and hence are Lorentz-invariant.

Note that the first of these facts means that if $|E| > |cB|$ in some reference frame, the same relation holds in any other frame, while according to the second fact, if the fields **E** and **B** are mutually perpendicular (as they are, e.g., in a plane electromagnetic wave) in some reference frame, they are also perpendicular in any other frame.

Problem 9.20. Consider the situation when static fields **E** and **B** are uniform but arbitrary (both in magnitude and in direction). What should be the velocity of an inertial reference frame in order to have the vectors **E***'* and **B***'*, observed from that frame, parallel? Is this solution unique?

Solution: The only special direction **n** defined by both vectors **E** and **B** is the one perpendicular to both vectors:

$$\mathbf{n} = \frac{\mathbf{E}\times\mathbf{B}}{|\mathbf{E}\times\mathbf{B}|}. \qquad (*)$$

It is natural to guess that at least one of the reference frames 0', in which $\mathbf{E}\,||\,\mathbf{B}$, should move along that direction: $\mathbf{v} = \mathbf{n}v$. According to Eq. (*), for that direction, in the lab frame, $E_{||} = 0$, $\mathbf{E}_\perp = \mathbf{E}$, and $B_{||} = 0$, $\mathbf{B}_\perp = \mathbf{B}$, so per Eq. (9.135) of the lecture notes, the fields measured in a moving frame are

$$E'_{||} = E_{||} = 0, \qquad B'_{||} = B_{||} = 0,$$

$$\mathbf{E}'_\perp = \gamma(\mathbf{E} + \mathbf{v}\times\mathbf{B}), \qquad \mathbf{B}'_\perp = \gamma(\mathbf{B} - \mathbf{v}\times\mathbf{E}/c^2).$$

The requirement to have the vectors **E***'* and **B***'* parallel may be represented as $\mathbf{E}'\times\mathbf{B}' = 0$, so for the system's velocity **v**, we get the following equation:

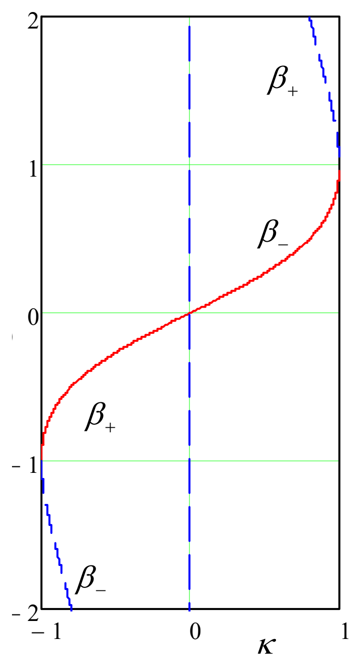
$$(\mathbf{E} + \mathbf{v}\times\mathbf{B})\times(\mathbf{B} - \mathbf{v}\times\mathbf{E}/c^2) = 0. \qquad (**)$$

Opening the parentheses, we can notice that all four operands of the result are vectors directed along the vector **n.** Also, $\mathbf{E}\times(\mathbf{v}\times\mathbf{E}) = vE^2\mathbf{n}$ because the vector $\mathbf{v}\times\mathbf{E}$ is perpendicular to **E**; similarly, $(\mathbf{v}\times\mathbf{B})\times\mathbf{B} = -vB^2\mathbf{n}$. Finally, $(\mathbf{v}\times\mathbf{B})\times(\mathbf{v}\times\mathbf{E}) = v^2\mathbf{B}\times\mathbf{E}$ because the angle between the vectors $\mathbf{v}\times\mathbf{B}$ and $\mathbf{v}\times\mathbf{E}$ is equal to that between **B** and **E**. As a result, Eq. (**) is reduced to a quadratic equation for v. This equation looks especially simple for the reduced velocity $\beta \equiv v/c$:

$$\beta^2 - 2\kappa^{-1}\beta + 1 = 0, \qquad \text{where } \kappa\mathbf{n} \equiv \frac{2(\mathbf{E}/c)\times\mathbf{B}}{(E/c)^2 + B^2}.$$

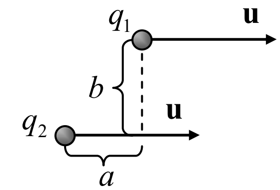
This quadratic equation has two solutions:

$$\beta_\pm = \kappa^{-1} \pm \left(\kappa^{-2} - 1\right)^{1/2}.$$



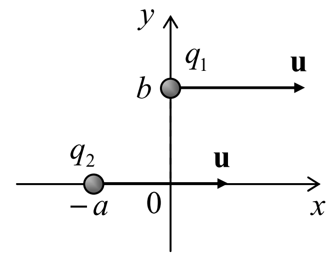
For any values and directions of the vectors **E** and **B**, the parameter κ, by its definition, is confined between –1 and +1, so one of the roots $\beta_\pm$ (plotted as functions of κ in the figure on the right) is always between –1 and + 1, and hence a physically acceptable reference frame with $|\beta| < 1$, providing the required property **E**' || **B**', exists for any pair of fields.

Now we may argue that since **E**' || **B**' in that frame, then according to the Lorentz transform (9.135), in *any* other reference frame 0" moving relative to the frame 0' with a velocity parallel to these two vectors (again, as measured in the frame 0', rather than in the lab frame!), the transverse fields would not appear, i.e. any such frame also is a solution to our problem.



Problem 9.21. Two charged particles moving with equal constant velocities **u** are offset by a constant vector **R** = {a, b} (see the figure on the right), as measured in the lab frame. Calculate the force of interaction between the particles – also in the lab frame.

Solution: Let us first calculate the electric and magnetic fields created by charge q_2 at the location of particle 1, as measured in the lab frame. Since these fields do not depend on whether particle 1 moves or not, this task is equivalent to the problem solved at the end of Sec. 9.5 of the lecture notes (see Fig. 9.11a), with the evident replacements $ut \to -a$ and $q \to q_2$. With these replacements, and with the same Cartesian coordinate choice (see the figure on the right), the final forms of Eqs. (9.139)-(9.140) become



$$E_x = \frac{q_2}{4\pi\varepsilon_0}\frac{\gamma a}{\left(\gamma^2 a^2 + b^2\right)^{3/2}}, \qquad E_y = \frac{q_2}{4\pi\varepsilon_0}\frac{\gamma b}{\left(\gamma^2 a^2 + b^2\right)^{3/2}}, \qquad E_z = 0,$$

$$B_x = 0, \qquad B_y = 0, \qquad B_z = \frac{u}{c^2}\frac{q_2}{4\pi\varepsilon_0}\frac{\gamma b}{\left(\gamma^2 a^2 + b^2\right)^{3/2}} \equiv \frac{u}{c^2}E_y, \qquad \text{where } \gamma \equiv \frac{1}{\left(1 - u^2/c^2\right)^{1/2}}.$$

Now we may calculate both components of the Lorentz force (5.10), $\mathbf{F}_{12} = q_1(\mathbf{E} + \mathbf{u}\times\mathbf{B})$, exerted by these fields on particle 1. Since at our choice of coordinates, the calculated magnetic field **B** has only one Cartesian component, B_z, and the velocity **u** also has only one component, $u_x = u$, so the vector product **u**×**B** also has just one component, $(\mathbf{u}\times\mathbf{B})_y = -uB_z = -(u^2/c^2)E_y$, the full Lorentz force has just two nonvanishing components:

$$F_x = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{\gamma a}{\left(\gamma^2 a^2 + b^2\right)^{3/2}}, \qquad F_y = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{\gamma b}{\left(\gamma^2 a^2 + b^2\right)^{3/2}}\left(1 - \frac{u^2}{c^2}\right) \equiv \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{b}{\gamma\left(\gamma^2 a^2 + b^2\right)^{3/2}}.$$

The reciprocal force $\mathbf{F}_{21}$ may be calculated absolutely similarly, with the replacements $a \to -a$ and $b \to -b$, giving the same result as above but with the opposite signs of both Cartesian components – thus complying with the 3rd Newton law. Note, however, that the ratio F_x/F_y equals $\gamma^2 a/b$, i.e. coincides with the ratio a/b only in the non-relativistic limit, so generally (in the lab frame), the forces between the charges are *not* directed along the line connecting them. This is a natural result of the finite speed of electromagnetic interactions' propagation: one particle "does not know" where exactly the other particle is in this particular instant.

Problem 9.22. Each of two thin, long, parallel particle beams of the same velocity **u**, separated by distance d, carries electric charges with a constant density λ per unit length, as measured in the reference frame moving with the particles.

(i) Calculate the distribution of the electric and magnetic fields in the system (outside the beams), as measured in the lab reference frame.

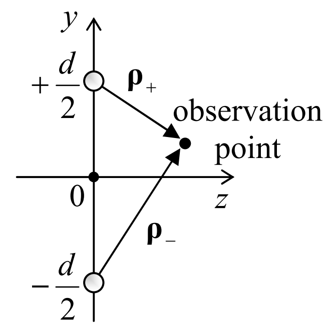
(ii) Calculate the interaction force between the beams (per particle) and the resulting acceleration, both in the lab reference frame and in the frame moving with the particles.

(iii) Compare the results and give a brief discussion of the comparison.

Solutions:

(i) In the reference frame moving with the particles, they are static, so there is no magnetic field: $\mathbf{B}' = 0$. The electric field observed in that frame may be represented as a sum of two fields (each created by one beam),

$$\mathbf{E}' = \mathbf{E}'_+ + \mathbf{E}'_- ,$$



and each of these components may be readily calculated, for example, from the Gauss theorem applied to a round cylinder of radius $\rho_\pm$ (see the figure on the right, in which the beams propagate along the x-axis, i.e. normally to the plane of the drawing), with the corresponding beam taken for the cylinder's axis:

$$\mathbf{E}'_\pm = \frac{\lambda \boldsymbol{\rho}_\pm}{2\pi\varepsilon_0 \rho_\pm^2},$$

i.e., with the coordinate choice shown in the figure above:

$$E'_{\pm x} = 0, \qquad E'_{\pm y} = \frac{\lambda (y \mp d/2)}{2\pi\varepsilon_0 \left[(y \mp d/2)^2 + z^2\right]}, \qquad E'_{\pm z} = \frac{\lambda z}{2\pi\varepsilon_0 \left[(y \mp d/2)^2 + z^2\right]}.$$

Now using the Lorentz transform (9.134), in the lab system we also can write $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-$, with[155]

$$E_{\pm x} = 0, \quad E_{\pm y} = \gamma E'_{\pm y} = \gamma \frac{\lambda (y \mp d/2)}{2\pi\varepsilon_0 \left[(y \mp d/2)^2 + z^2\right]}, \quad E_{\pm z} = \gamma E'_{\pm z} = \gamma \frac{\lambda z}{2\pi\varepsilon_0 \left[(y \mp d/2)^2 + z^2\right]},$$

$$B_{\pm x} = 0, \quad B_{\pm y} = -\frac{\gamma u}{c^2} E'_{\pm z} = -\frac{\gamma u}{c^2} \frac{\lambda z}{2\pi\varepsilon_0 \left[(y \mp d/2)^2 + z^2\right]}, \quad B_{\pm z} = \frac{\gamma u}{c^2} E'_{\pm y} = \frac{\gamma u}{c^2} \frac{\lambda (y \mp d/2)}{2\pi\varepsilon_0 \left[(y \mp d/2)^2 + z^2\right]}.$$

(ii) The Lorenz force exerted on one beam (say, the one located at $y = +d/2$ in the figure above) comes only from the fields created by the other beam (located at $y' = -d/2$).[156] As a result, in the frame moving with the particles,

$$F'_y = qE'_{-y}\Big|_{y=+d/2,\, z=0} = \frac{q\lambda}{2\pi\varepsilon_0 d}, \qquad F'_z = 0,$$

while in the lab frame, the magnetic field contributes to the force as well:

$$\mathbf{F} = q(\mathbf{E}_- + \mathbf{u} \times \mathbf{B}_-)_{y=+d/2,\, z=0},$$

so we get

[155] The Lorentz transform does not change the lengths perpendicular to the relative velocity, so $y' = y$, $z' = z$.

[156] Of course, charged particles within each beam do interact (repulse each other), but the forces of those interactions are directed along the beam. Also, these forces mutually cancel at their summation over the particles.

$$F_y = q(E_{-y} - uB_{-z}) = q\gamma \frac{\lambda}{2\pi\varepsilon_0 d}\left(1 - \frac{u^2}{c^2}\right) \equiv \frac{1}{\gamma}\frac{q\lambda}{2\pi\varepsilon_0 d}, \qquad F_z = q(E_{-z} + uB_{-y}) = 0\,.$$

The resulting vertical acceleration, as measured in the moving frame (where $M' = m$), is

$$a'_y = \frac{F'_y}{m} = \frac{q\lambda}{2\pi\varepsilon_0 dm},$$

while in the lab frame, where $M = \gamma m$, it is

$$a_y = \frac{F_y}{M} = \frac{1}{\gamma^2}\frac{q\lambda}{2\pi\varepsilon_0 dm} \equiv \frac{a'_y}{\gamma^2}.$$

(iii) So at $u \to c$, the results for F_y and a_y, obtained to the two reference frames, are very much different. However. they are actually consistent because time runs differently in the two frames. For example, if the acceleration produces a beam shift $\Delta y << d$, with the corresponding transverse velocity still much below c, we may write

$$\Delta y' = \frac{a'_y}{2}(\Delta t')^2 = \frac{a'_y}{2}(\Delta\tau)^2, \qquad \Delta y = \frac{a_y}{2}(\Delta t)^2 = \frac{a'_y}{2\gamma^2}(\Delta t)^2.$$

Since, per Eq. (9.21) of the lecture notes, the "proper" time interval $\Delta t' = \Delta\tau$ and the lab frame interval Δt are related as $\Delta t = \gamma\Delta\tau$ (describing the relativistic time dilation), we get that $\Delta y' = \Delta y$ – as it should be due to the Lorentz relation $y' = y$ between these transverse coordinates.

Note that alternatively, this problem may be solved by the summation (actually, integration) of the result of the previous problem over all particles of a beam, with the same $b = d$ but different a – the additional exercise highly recommended to the reader.

Problem 9.23.

(i) Spell out the Lorentz transform of the Cartesian components of the scalar potential and the vector potential of an arbitrary electromagnetic field.

(ii) Use this general result to calculate the potentials of the field created by a point charge q moving with a constant velocity $\mathbf{u}$, as measured in the lab reference frame.

Solutions:

(i) As was discussed in Sec. 9.5 of the lecture notes, the scalar potential ϕ (divided by c) and the three Cartesian components of the vector potential $\mathbf{A}$ form a legitimate 4-vector (9.116), whose contravariant version is

$$A^\alpha = \left\{\frac{\phi}{c}, \mathbf{A}\right\} \equiv \left\{\frac{\phi}{c}, A_x, A_y, A_z\right\}.$$

Hence the Lorentz transform of the elements of this tensor follow the regular rule (9.91):

$$A^\alpha = L^\alpha_\beta A'^\beta,$$

(with the usual implied summation over all four values of the repeated index β), where $L_\beta^{\ \alpha}$ is the mixed Lorentz tensor. If the x-axis is directed along the velocity $\mathbf{v}$ of the "primed" (i.e. "moving") frame – as shown, for example in Fig. 9.1 of the lecture notes, then the Lorentz tensor has the standard form (9.92):

$$L^{\alpha}_{\beta} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{with } \beta \equiv \frac{v}{c}, \text{ and } \gamma \equiv \frac{1}{\left(1-\beta^2\right)^{1/2}}.$$

Performing the usual matrix-by-vector multiplication, we get an explicit form of the required transform:

$$\frac{\phi}{c} = \gamma\frac{\phi'}{c} + \beta\gamma A'_x, \qquad A_x = \beta\gamma\frac{\phi'}{c} + \gamma A'_x, \qquad A_y = A'_y, \qquad A_z = A'_z.$$

Frequently, it is more convenient to use these relations rewritten in the semi-vector form similar to Eqs. (9.135) for the fields:

$$\frac{\phi}{c} = \gamma\frac{\phi'}{c} + \beta\gamma A'_{||}, \qquad A_{||} = \beta\gamma\frac{\phi'}{c} + \gamma A'_{||}, \qquad \mathbf{A}_{\perp} = \mathbf{A}'_{\perp}, \qquad (*)$$

where the indices || and $\perp$ mark the vector components that are, respectively, parallel and perpendicular to the vector **v**.[157]

(ii) Now let us use Eq. (*) to calculate ϕ and **A** of a point charge q moving with velocity **u** relative to the "lab" frame in that the potentials are measured. In the "primed" frame moving together with the charge in its origin, i.e. with velocity **u = v** relative to the lab frame, the charge's field is purely electrostatic – see Eq. (1.35):

$$\phi' = \frac{q}{4\pi\varepsilon_0 r'}, \qquad \mathbf{A}' = 0,$$

where **r**' is the observation point (as measured in the moving reference frame), so Eqs. (*) yield

$$\phi = \gamma\phi' = \gamma\frac{1}{4\pi\varepsilon_0}\frac{q}{r'}, \qquad A_{||} = \beta\gamma\frac{\phi'}{c} = \gamma\frac{v}{c^2}\frac{1}{4\pi\varepsilon_0}\frac{q}{r'}, \qquad \mathbf{A}_{\perp} = 0. \qquad (**)$$

In a certain sense, this is already the required solution, but for its applications, we may need to spell out the potentials as explicit functions of the observation point's position **r** and time t, as measured in the lab frame. For that, we may decompose the radius vector **r**' into the parallel and normal components as well: $\mathbf{r}' = \mathbf{r}'_{||} + \mathbf{r}'_{\perp}$, and then use the first three of Eqs. (9.19b), which may be rewritten in a more compact semi-vector form similar to that used in Eq. (*):

$$\mathbf{r}'_{||} = \gamma\left(\mathbf{r}_{||} - \boldsymbol{\beta}ct\right) \equiv \gamma\left(\mathbf{r}_{||} - \mathbf{u}t\right), \qquad \mathbf{r}'_{\perp} = \mathbf{r}_{\perp},$$

so, with $\mathbf{r}_{||} = \mathbf{n}_x x$ and $\mathbf{r}_{\perp} = \mathbf{n}_y y + \mathbf{n}_z z$, for the length r' of the vector **r**' we get

$$r' \equiv \left|\mathbf{r}'\right| = \left(r'^2_{||} + r'^2_{\perp}\right)^{1/2} = \left[\gamma^2\left(x-ut\right)^2 + y^2 + z^2\right]^{1/2}. \qquad (***)$$

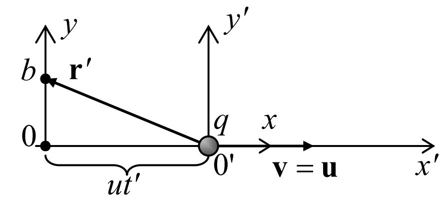
If we select the coordinate origin just as it was done in Sec. 9.5 of the lecture notes for the derivation of expressions (9.139)-(9.140) for the electric and magnetic fields in the same situation, namely to provide $x = 0$ for the observation point (for example, as in Fig. 9.11a), then Eq. (***) simplifies, and with its substitution, Eq. (**) yields

[157] The reciprocal transform may be obtained either by solving the system of equations (*) for the primed potentials, or by using Eq. (9.97b) with the reciprocal Lorentz matrix (9.98), or just by reversing the velocity sign.

$$\phi = \gamma \frac{1}{4\pi\varepsilon_0} \frac{q}{\left(\gamma^2 u^2 t^2 + b^2\right)^{1/2}}, \qquad A_{||} = \gamma \frac{u}{c^2} \frac{1}{4\pi\varepsilon_0} \frac{q}{\left(\gamma^2 u^2 t^2 + b^2\right)^{1/2}}, \qquad \mathbf{A}_\perp = 0,$$

where $b \equiv (y^2 + z^2)^{1/2}$ is the "impact parameter", i.e. the closest distance of the observation point from the charge's trajectory.

Comparing this result with Eqs. (9.139)-(9.140), we see that they involve the same denominator, but the expressions for the potentials are substantially simpler than those for the fields; in particular, they do not depend on the exact choice of the y- and z-axes within the plane normal to the vector $\mathbf{n}_x$ (and hence $\mathbf{u}$). However, we would need to make such a choice (for example, taking $y = b$, $z = 0$, as in the figure on the right) in order to derive the field formulas (9.139)-(9.140) from our result.[158]



Problem 9.24. Calculate the scalar and vector potentials created by a time-independent electric dipole $\mathbf{p}$, as measured in a reference frame that moves relative to the dipole with a constant velocity $\mathbf{v}$, with the shortest distance ("impact parameter") equal to b.

Solution: In the lab frame, in whose origin the dipole rests, its scalar potential is given by Eq. (3.7) of the lecture notes, while its magnetic field (and hence the vector potential) equals zero:

$$\phi = \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}, \qquad \mathbf{A} = 0.$$

Applying to these potentials the Lorenz transform reciprocal to Eq. (*) of the previous problem's solution,

$$\frac{\phi'}{c} = \gamma \frac{\phi}{c} - \beta\gamma A_{||}, \qquad A'_{||} = -\beta\gamma \frac{\phi}{c} + \gamma A_{||}, \qquad \mathbf{A}'_\perp = \mathbf{A}_\perp,$$

we get

$$\phi' = \gamma\phi = \gamma \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}, \qquad A'_{||} = -\beta\gamma \frac{\phi}{c} = -\gamma \frac{v}{c^2} \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}, \qquad \mathbf{A}'_\perp = 0. \qquad (*)$$

Just as Eq. (**) in the previous problem's solution, this is already an answer, but it is natural to express the potentials explicitly via the time t' and the observation point's position $\mathbf{r}'$, as measured in the moving system – especially since the vector $\mathbf{r}$ in the above formulas is an implicit function of time. For that, we may use the standard coordinate choice shown in Fig. 9.1 of the lecture notes, and hence the coordinate transform given by Eq. (9.19a):

$$x = \gamma(x' + vt'), \qquad y = y', \qquad z = z'.$$

From here,

$$r^2 \equiv x^2 + y^2 + z^2 = \gamma^2 (x' + vt')^2 + y'^2 + z'^2, \quad \mathbf{p} \cdot \mathbf{r} \equiv p_x x + p_y y + p_z z = p_x \gamma (x' + vt') + p_y y' + p_z z',$$

[158] For that, we would need to plug Eqs. (**), with r' given by Eq. (***), into Eqs. (9.121) of the lecture notes, spell them out in Cartesian components, perform the partial differentiation over x, y, z, and t, and only after that set $x = 0$, $y = b$, and $z = 0$. This calculation is more math than physics, but it may still be recommended to the reader as an additional exercise.

so at the observation point with $x' = 0$,[159] $y' = b$, and $z' = 0$ (see Fig. 9.11 of the lecture notes), we get

$$\phi' = \gamma \frac{1}{4\pi\varepsilon_0} \frac{p_x \gamma v t' + p_y b}{\left(\gamma^2 v^2 t'^2 + b^2\right)^{3/2}}, \qquad A'_x = -\beta\gamma \frac{\phi}{c} = -\gamma \frac{v}{c^2} \frac{1}{4\pi\varepsilon_0} \frac{p_x \gamma v t' + p_y b}{\left(\gamma^2 v^2 t'^2 + b^2\right)^{3/2}}, \qquad A'_y = A'_z = 0\,.$$

Note that the Cartesian components p_x and p_y of the dipole moment, which participate in this expression, are still as measured in the lab frame rather than in the moving frame. (Remember that in contrast to the electric charge, the dipole moment is not Lorentz-invariant!)

One may run into claims that in the limit $v \ll c$ when $\gamma \approx 1$, this result is reduced to the sum of the statically calculated potentials of the initial electric dipole $\mathbf{p}$ and a co-located magnetic dipole $\mathbf{m} = \mathbf{v}\times\mathbf{p}$. However, this claim is incorrect. This may be readily proved by applying the *bac minus cab* rule of the vector algebra[160] to the double vector product $(\mathbf{v}\times\mathbf{p})\times\mathbf{r}$ that would participate in that case in Eq. (5.90) of the lecture notes:

$$\mathbf{A}'_{\mathrm{m}} \approx \mathbf{A}_{\mathrm{m}} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3} = \frac{\mu_0}{4\pi}\frac{(\mathbf{v}\times\mathbf{p})\times\mathbf{r}}{r^3} = -\frac{\mu_0}{4\pi}\frac{\mathbf{v}(\mathbf{p}\cdot\mathbf{r}) - \mathbf{p}(\mathbf{r}\cdot\mathbf{v})}{r^3}\,.$$

Comparing this result with the two last Eqs. (*) which, in this limit, may be represented in a single vector form:

$$\mathbf{A}' \approx -\frac{\mathbf{v}}{c^2}\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} \equiv -\frac{\mu_0}{4\pi}\frac{\mathbf{v}(\mathbf{p}\cdot\mathbf{r})}{r^3}, \qquad (**)$$

we see that $\mathbf{A}' \neq \mathbf{A}'_{\mathrm{m}}$ besides the particular nearest-passage moment $t = t' = 0$ when $\mathbf{r} \perp \mathbf{v}$.

Note, however, that the magnetic *field* $\mathbf{B}'$ of the dipole in the moving reference frame, even in this linear approximation in $v/c \ll 1$, is different from that statically calculated from the potential $\mathbf{A}'$ given by Eq. (**).[161] Such a calculation (a very good additional exercise, highly recommended to the reader) shows[162] that the resulting $\mathbf{B}'(\mathbf{r}', t')$ may be decomposed into a sum of the statically calculated field of a magnetic dipole with the moment $\mathbf{m}' = \mathbf{v}\times\mathbf{p}/2$ (i.e. twice smaller than the $\mathbf{m}$ mentioned above) with a the displacement-current contribution of the field of a co-located electric quadrupole moment and higher-multipole terms.

Problem 9.25. Solve the previous problem, in the limit $v \ll c$, for a time-independent magnetic dipole $\mathbf{m}$.

Solution: In the lab frame, in whose origin the magnetic dipole rests, its electric field and hence the scalar potential equal zero, while its vector potential is given by Eq. (5.90) of the lecture notes:

$$\phi = 0, \qquad \mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3}\,. \qquad (*)$$

[159] As in Sec. 9.5 of the lecture notes and in the previous problem (see the figure in its model solution), this choice of x' implies such a setup of the moving clock that at $t' = 0$, the distance between the observation point and the dipole is the smallest (equal to b).
[160] See, e.g., MA Eq. (7.5).
[161] Performing the calculation, one has to use the basic Eq. (6.7) in the form $\mathbf{B}'(\mathbf{r}', t') = \nabla'\times\mathbf{A}'(\mathbf{r}', t')$, i.e. take into account the spatial dependence of the potential $\mathbf{A}'$ at an *arbitrary* observation point $\mathbf{r}' = \{x', y', z'\}$.
[162] See, e.g., V. Hnizdo, *Am. J. Phys*. **80**, 645 (2012) and references therein.

In the leading, linear approximation in $\beta << 1$, the Lorenz factor $\gamma = 1/(1 - v^2/c^2)^{1/2}$ may be taken for 1, so the Lorentz transform relations used in the model solution of the previous problem,

$$\frac{\phi'}{c} = \gamma\frac{\phi}{c} - \beta\gamma A_{||}, \qquad A'_{||} = -\beta\gamma\frac{\phi}{c} + \gamma A_{||}, \qquad \mathbf{A}'_{\perp} = \mathbf{A}_{\perp},$$

are reduced to just

$$\frac{\phi'}{c} = \frac{\phi}{c} - \beta A_{||}, \qquad A'_{||} = -\beta\frac{\phi}{c} + A_{||}, \qquad \mathbf{A}'_{\perp} = \mathbf{A}_{\perp}. \qquad (**)$$

Since in our current problem, $\phi = 0$, the two last Eqs. (**) may be summarized as $\mathbf{A}' = \mathbf{A}$, i.e. the vector potential is described by the second of Eqs. (*) even in the moving system. On the contrary, as the first of Eqs. (**) shows, the scalar potential ϕ' does not vanish even in the linear approximation in v:

$$\phi' = -c\beta A_{||} = -v\frac{\mu_0}{4\pi}\frac{(\mathbf{m}\times\mathbf{r})_{||}}{r^3}.$$

Let us use the fact that the velocity vector $\mathbf{v}$ has only one (longitudinal) component, to rewrite the last expression as

$$\phi' = -\frac{\mu_0}{4\pi}\frac{\mathbf{v}\cdot(\mathbf{m}\times\mathbf{r})}{r^3},$$

and then use the operand rotation rule of vector algebra[163] to transform it to

$$\phi' = -\frac{\mu_0}{4\pi}\frac{\mathbf{r}\cdot(\mathbf{v}\times\mathbf{m})}{r^3} \equiv -\frac{1}{4\pi\varepsilon_0}\frac{\left[\left(-\mathbf{v}/c^2\right)\times\mathbf{m}\right]\cdot\mathbf{r}}{r^3}.$$

Comparing this expression with that for an electric dipole, in the same linear approximation in $v << c$:

$$\phi'_{\mathrm{e}} \approx \phi_{\mathrm{e}} = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3},$$

we see that they coincide if we take

$$\mathbf{p} = -\frac{\mathbf{v}\times\mathbf{m}}{c^2}. \qquad (***)$$

Hence, in the low-velocity approximation, the electric potential of a static magnetic dipole $\mathbf{m}$ equals, for a moving observer, to that of a co-located electric dipole with the moment (***). As was discussed in the model solution of the previous problem, the reciprocal statement for an electric dipole potential is incorrect. However, as in that problem, the electric *field* of a magnetic dipole, observed in the moving reference frame (which should be calculated as $\mathbf{E}' = -\nabla'\phi' - \partial\mathbf{A}'/\partial t'$, i.e. includes the Faraday-induced component) is more complex and may be decomposed into a sum of the statically calculated field of a twice smaller electric dipole $\mathbf{p}' = \mathbf{p}/2$ and an electric-quadrupole field.[164]

<u>Problem 9.26</u>. Review the solution of Problem 6.23 (on the hypothetical magnetic monopole passing through a superconducting ring) for the case when this particle moves with an arbitrary constant velocity $\mathbf{u}$.

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[163] See, e.g., MA Eq. (7.6).

[164] See. e.g., http://kirkmcd.princeton.edu/examples/movingdipole.pdf.

Solution: As was discussed in Sec. 5.6 of the lecture notes (and in the model solution of Problem 6.23), the magnetic point charge q_m is usually defined so that if it is at rest at the origin, its magnetic field at point **r** in the surrounding free space is

$$\mathbf{B} = \frac{q_\mathrm{m}}{4\pi r^3}\mathbf{r}. \qquad (*)$$

If the monopole moves with a constant velocity **u**, then Eq. (*), with the replacements $\mathbf{B} \to \mathbf{B}'$ and $\mathbf{r} \to \mathbf{r}'$, is valid in the "primed" reference frame moving with it. Now let us see how the monopole's field looks from the lab frame. Directing the x-axis along the vector **u**, and the y-axis in such a way that the observation point has coordinates $\{0, b, 0\}$, where b is its distance from the monopole's trajectory (see, e.g., either Fig. 9.11a in the lecture notes, or its reproduction in the model solution of Problem 6.23), and using the Lorentz transform (9.135), we get the following analog of Eqs. (9.139):

$$B_x = -\frac{q_\mathrm{m}}{4\pi}\frac{u\gamma t}{\left(u^2\gamma^2 t^2 + b^2\right)^{3/2}}, \quad B_y = \frac{q_\mathrm{m}}{4\pi}\frac{\gamma b}{\left(u^2\gamma^2 t^2 + b^2\right)^{3/2}}, \quad B_z = 0, \qquad (**)$$

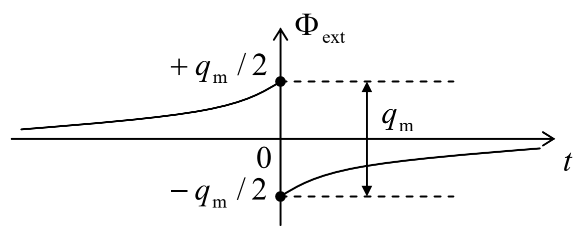
where the moment $t = 0$ corresponds to the closest approach of the monopole to the observation point.

Now we may use the first of these results to calculate the monopole-induced magnetic flux through a circular area with $b \leq R$:

$$\Phi_\mathrm{ext}(t) = 2\pi\int_0^R B_x b db = -\frac{q_\mathrm{m}}{2}u\gamma t\int_0^R \frac{bdb}{\left(u^2\gamma^2 t^2 + b^2\right)^{3/2}} = -\frac{q_\mathrm{m}}{4}u\gamma t\int_{b=0}^{b=R}\frac{d(u^2\gamma^2 t^2 + b^2)}{\left(u^2\gamma^2 t^2 + b^2\right)^{3/2}}$$

$$= \frac{q_\mathrm{m}}{2}u\gamma t\frac{1}{\left(u^2\gamma^2 t^2 + b^2\right)^{1/2}}\Bigg|_{b=0}^{b=R} = \frac{q_\mathrm{m}}{2}\left[\frac{u\gamma t}{\left(u^2\gamma^2 t^2 + R^2\right)^{1/2}} - \mathrm{sgn}(t)\right].$$

This is the same result as was obtained in the solution of Problem 6.23, with the proper notation replacement $z' \to u\gamma t$ (see the schematic plot in the figure on the right), creating the same contradiction (with the same Dirac-suggested resolution) as was discussed in that solution. Besides the instantaneous jump at $t = 0$, the lab-frame time scale of the changes of the monopole's flux (and the induced supercurrent) is $\Delta t = R/\gamma u$, reflecting the relativistic time dilation (9.21b) by the factor $\gamma \equiv 1/(1 - u^2/c^2)^{1/2} > 1$. For a typical size (~10 cm) of the superconducting loops used in the 1980s for the magnetic monopole search, and $u \sim c$, this time scale is well below a nanosecond, so the experiments were looking for practically instant step-like changes of the supercurrent induced in the loop, not trying to time-resolve their dynamics.



Problem 9.27. Re-derive Eq. (9.161) of the lecture notes for the simplest case $\mathbf{p}(0) = 0$, by using the 4-vector form (9.145) of the equation of motion and the notion of rapidity $\varphi \equiv \tanh^{-1}\beta$ that was briefly discussed in Sec. 9.2.

Solution: To use Eq. (9.145):

$$\frac{dp^\alpha}{d\tau} = qF^{\alpha\beta}u_\beta, \qquad (*)$$

let us direct the x-axis along the electric field, so $\mathbf{E} = E\mathbf{n}_x$; then we may use the particular form (9.125a) of the tensor $F^{\alpha\beta}$, with only two non-zero elements:

$$F^{\alpha\beta} = \begin{pmatrix} 0 & E/c & 0 & 0 \\ -E/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For the two non-zero components of the 4-momentum $p^\alpha = m\{u_0, u_x, 0, 0\}$, Eq. (*) gives:[165]

$$m\frac{du_0}{d\tau} = q\frac{E}{c}u_x, \qquad m\frac{du_x}{d\tau} = q\frac{E}{c}u_0. \qquad (**)$$

Admittedly, this system of two linear differential equations may be readily solved by elementary means, but let us do this by using the rapidity φ – just to illustrate some elegant (and useful) math this notion enables.[166] Since, by definition,

$$\tanh\varphi \equiv \beta \equiv \frac{\gamma\beta}{\gamma}, \qquad \text{and } \gamma^2 \equiv \frac{1}{1-\beta^2}, \quad \text{so } \gamma^2 - (\gamma\beta)^2 = 1,$$

we can immediately see that

$$\gamma = \cosh\varphi, \qquad \gamma\beta = \sinh\varphi.$$

As a result, the two equations resulting from plugging the definitions of the 4-velocity components, $u_0 \equiv c\gamma$ and $u_x \equiv \gamma u \equiv c(\gamma\beta)$, into Eqs. (**),

$$\frac{d\gamma}{d\tau} = \frac{qE}{mc}(\gamma\beta), \quad \frac{d(\gamma\beta)}{d\tau} = \frac{qE}{mc}\beta,$$

are equivalent to just one simple equation for the rapidity:

$$\frac{d\varphi}{d\tau} = \frac{qE}{mc},$$

which may be very useful for the solution of problems with time-dependent electric fields. In our simple case of a time-independent field and the motion starting from the rest (for $\tau = 0$, $u = 0$ and hence $\beta = 0$, i.e. $\varphi = 0$), the integration of this equation is elementary:

$$\varphi = \frac{qE}{mc}\tau, \qquad \text{so } \gamma = \cosh\frac{qE\tau}{mc}, \quad \gamma\beta = \sinh\frac{qE\tau}{mc}, \quad \beta = \tanh\frac{qE\tau}{mc}.$$

Now we may use the above result for $\gamma\beta$ to calculate the particle's coordinate x as a function of the proper time τ. Taking its initial position for the origin, we get

$$x(\tau) = \int_0^\tau u_x(\tau')d\tau' = c\int_0^\tau (\gamma\beta)d\tau' = c\int_0^\tau \sinh\frac{qE\tau'}{mc}d\tau' = c\frac{mc}{qE}\left(\cosh\frac{qE\tau}{mc} - 1\right). \qquad (***)$$

[165] In the second of Eqs. (**), the negative sign in the relation $F^{10} = -E/c$ is compensated by that in the covariant form of the 4-velocity $u_\beta = \{c, -u_x, 0, 0\}$.

[166] Also note the model solution of Problem 3 where the rapidity is used for analyzing a series of several (possibly, many) sequential Lorentz transforms.

What remains is to express x via the lab-frame time t. This may be done by integrating the basic Eq. (9.60), $dt = \gamma d\tau$, for our particular function $\gamma(\tau)$:

$$t = \int_0^t dt' = \int_0^\tau \gamma(\tau')d\tau' = \int_0^\tau \cosh\frac{qE\tau'}{mc}d\tau' = \frac{mc}{qE}\sinh\frac{qE\tau}{mc}, \qquad \text{i.e. } \sinh\frac{qE\tau}{mc} = \frac{qEt}{mc},$$

and hence

$$\cosh\frac{qE\tau}{mc} = \left[\left(\frac{qEt}{mc}\right)^2 + 1\right]^{1/2}.$$

With this substitution, Eq. (***) gives the final result:

$$x = \frac{mc^2}{qE}\left\{\left[\left(\frac{qEt}{mc}\right)^2 + 1\right]^{1/2} - 1\right\}.$$

With the proper notation replacement $x \to z$, it coincides with Eq. (9.161), provided that the initial energy $\mathscr{E}_0$ is equal to mc^2, as it has to be for a particle at rest.

Problem 9.28.* Calculate the trajectory of a relativistic particle in a uniform electrostatic field **E**, for an arbitrary direction of its initial velocity **u**(0), by using two different ways – at least one of them different from the approach described in Sec. 9.6 of the lecture notes for the case **u**(0) ⊥ **E**.

Solution: An elegant way to solve this problem is to integrate the 4-vector equation (9.145),

$$\frac{dp^\alpha}{d\tau} = qF^{\alpha\beta}u_\beta,$$

directly, by taking the proper time τ of the particle (rather than the lab-frame time t) for the argument. For the three non-zero components of the 4-velocity u_β,[167] this gives us three equations, which may be conveniently rewritten as follows:

$$\frac{d(\gamma c)}{d(\Gamma\tau)} = \gamma u_z, \qquad \frac{d(\gamma u_x)}{d(\Gamma\tau)} = 0, \qquad \frac{d(\gamma u_z)}{d(\Gamma\tau)} = \gamma c,$$

where $\Gamma \equiv qE/mc$ is a constant parameter with the dimensionality of the reciprocal time (s^{-1}). The integration of the middle equation is elementary, and yields

$$\gamma u_x = \text{const} = \gamma u_x\big|_{\tau=0} \equiv \frac{cu_x(0)}{\left[c^2 - u_x^2(0) - u_z^2(0)\right]^{1/2}} \equiv C.$$

The remaining two equations may be combined (by an additional differentiation of any of them over τ and plugging the complementary equation into the result) to give two similar second-order linear differential equations

$$\frac{d^2}{d\tau^2}(\gamma c) = \Gamma^2(\gamma c), \qquad \frac{d^2}{d\tau^2}(\gamma u_z) = \Gamma^2(\gamma u_z),$$

with the solutions

[167] I am using the same coordinate system choice as in Sec. 9.6 of the lecture notes, with the z-axis directed along the electric field, and the x-axis within the plane of the motion, so $u_y = 0$ for any τ.

$$\gamma c = A\cosh\Gamma(\tau-\tau_0), \qquad \gamma u_z = A\sinh\Gamma(\tau-\tau_0), \qquad (*)$$

where τ_0 is the proper time at that $u_z = 0$, i.e. the particle reaches the lowest value of z.

The constants A and τ_0 participating in these solutions may be found from the initial conditions

$$\gamma(0)c = A\cosh\Gamma\tau_0, \qquad \gamma(0)u_z(0) = -A\sinh\Gamma\tau_0, \qquad \text{so } \left[\frac{\gamma(0)c}{A}\right]^2 - \left[\frac{\gamma(0)u_z(0)}{A}\right]^2 = 1, \quad (**)$$

giving

$$A = c\left[\frac{c^2-u_z^2(0)}{c^2-u_x^2(0)-u_z^2(0)}\right]^{1/2} \equiv c\frac{\gamma(0)}{\gamma_z(0)}, \quad \text{with } \gamma_z(0) \equiv \frac{1}{\left[1-u_z^2(0)/c^2\right]^{1/2}}; \quad \tau_0 = -\frac{1}{\Gamma}\tanh^{-1}\frac{u_z(0)}{c}.$$

From here, we can calculate the tangent to the particle's trajectory:

$$\frac{dz}{dx} = \frac{u_z}{u_x} = \frac{A}{C}\sinh\Gamma(\tau-\tau_0),$$

but to integrate this equation, we still need to express its right-hand side as a function of either x or z. For example, we can find $z(\tau)$ by integrating the second of Eqs. (*):

$$\gamma u_z \equiv \gamma\frac{dz}{dt} \equiv \frac{dz}{d\tau} = A\sinh\Gamma(\tau-\tau_0),$$

yielding

$$z = A\int_0^\tau \sinh\Gamma(\tau-\tau_0)d\tau = \frac{A}{\Gamma}\left[\cosh\Gamma(\tau-\tau_0) - \cosh\Gamma\tau_0\right].$$

From this relation and Eqs. (**):

$$\cosh\Gamma(\tau-\tau_0) = \frac{\Gamma z}{A} + \cosh\Gamma\tau_0 = \frac{\Gamma z}{A} + \frac{\gamma(0)c}{A} = \frac{\Gamma z}{A} + \gamma_z(0), \quad \sinh\Gamma(\tau-\tau_0) = \left[\left(\frac{\Gamma z}{A}+\gamma_z(0)\right)^2 - 1\right]^{1/2},$$

and now we can finally express the trajectory's tangent via z:

$$\frac{dz}{dx} = \frac{A}{C}\sinh\Gamma(\tau-\tau_0) = \frac{1}{C}\left[\left(\Gamma z + A\gamma_z(0)\right)^2 - A^2\right]^{1/2},$$

and then find the trajectory itself:

$$x = C\int_0^z \frac{dz}{\left[\left(\Gamma z + A\gamma_z(0)\right)^2 - A^2\right]^{1/2}} = \frac{C}{\Gamma}\left[\cosh^{-1}\left(\frac{\Gamma z}{A}+\gamma_z(0)\right) - \cosh^{-1}\gamma_z(0)\right]. \qquad (***)$$

This is a natural generalization of the result derived in Sec. 9.6 for the particular case $u_z(0) = 0$. Indeed, in that case, $\gamma_z(0) = 1$, so the above expressions yield $A = c\gamma(0)$, $A/\Gamma = \gamma(0)mc^2/qE \equiv \mathscr{E}(0)/qE$, $C/\Gamma = p(0)c/qE$, and Eq. (***) is reduced to the form

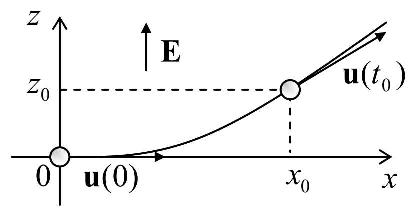
$$x = \frac{p(0)c}{qE}\left[\cosh^{-1}\left(\frac{qEz}{\mathscr{E}_0}+1\right)\right],$$

equivalent to Eq. (9.165) of the lecture notes.

An alternative way to solve this problem is to notice that Eq. (9.165),

$$z = \frac{\mathscr{E}(0)}{qE}\left(\cosh\frac{qEx}{cp(0)} - 1\right),$$

obtained for the case $\mathbf{u}(0) \perp \mathbf{E}$, may be used in the general case as well, provided that we shift the origins of x, z, and t (see the figure on the right),



$$z = z_0 + \frac{\mathscr{E}(0)}{qE}\left(\cosh\frac{qE(x - x_0)}{cp(0)} - 1\right),$$

and the "only" remaining thing to do is to express the parameters $\mathscr{E}(0)$, $p(0)$, and z_0 via the components $u_x(t_0)$ and $u_z(t_0)$ of the new initial velocity, which in this notation replace $u_x(0)$ and $u_z(0)$ However, the recalculation (leading to the same result as above) makes this approach not much easier than the first, more elegant way. Still, it is recommended to the reader as a useful additional exercise.

Problem 9.29. A charged relativistic particle with the rest mass m performs a planar cyclotron rotation, with velocity u, in a uniform external magnetic field of magnitude B. How much would the velocity and the orbit's radius change at a slow change of the field to a new magnitude B'?

Solution: The initial orbit's radius may be found from Eq. (9.153) of the lecture notes:

$$R = \frac{p}{qB} = \frac{\gamma m u}{qB} \equiv \frac{m}{qB}\frac{u}{\left(1 - u^2/c^2\right)^{1/2}}.$$

Thus, the initial magnetic field flux through the orbit's area $A = \pi R^2$ is

$$\Phi = \pi R^2 B = \pi \frac{m^2}{q^2 B}\frac{u^2}{1 - u^2/c^2}.$$

For the new field value B', the flux is described by a similar formula:

$$\Phi' = \pi R'^2 B' = \pi \frac{m^2}{q^2 B'}\frac{u'^2}{1 - u'^2/c^2},$$

where u' is the new velocity of the particle.

As was argued in Sec. 9.7 (in the context of slow *spatial* variations of the magnetic field), the magnetic flux through the orbit is an adiabatic invariant of the particle's motion. These arguments remain valid for sufficiently slow *temporal* variations of the field as well, so we may write $\Phi' = \Phi$, giving us the following equation for finding u':

$$\frac{1}{B'}\frac{u'^2}{1 - u'^2/c^2} = \frac{1}{B}\frac{u^2}{1 - u^2/c^2}.$$

This is a linear equation for u'^2, easy to solve. The result,

$$u' = u\left[\frac{B}{B'} + \frac{u^2}{c^2}\left(1 - \frac{B}{B'}\right)\right]^{-1/2},$$

becomes simpler in both the non-relativistic limit: $u' \approx u(B'/B)^{1/2}$, and the ultra-relativistic limit: $u' \approx u \approx c$. At arbitrary u, the field's increase results in an increase of the particle's velocity. Physically, this change of the particle's velocity, and hence of its energy, is the result of Faraday's induction e.m.f. (6.2) appearing during the field's change.

For the orbit's radius, the same adiabatic invariance of the flux, $\Phi' = \Phi$, provides the equation

$$B'R'^2 = BR^2,$$

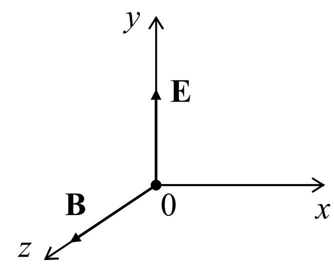
which gives a simpler result independent of the particle's velocity:

$$R' = R\left(\frac{B}{B'}\right)^{1/2}.$$

It shows that a field's increase leads to a shrinkage of the orbit.

Problem 9.30.* Analyze the motion of a relativistic particle in uniform, mutually perpendicular fields **E** and **B**, for the particular case when E is *exactly* equal to cB.

Solution: Evidently, for this particular case, both velocities defined by Eqs. (9.168) and (9.174) equal c, so the introduction of a reference frame moving with this velocity (the trick used in Sec. 9.6(iii) of the lecture notes) is not too helpful, so we better start from scratch.



Directing the coordinate axes as in Fig. 9.12 of the lecture notes (partly reproduced on the right), so that $E_y = E$ and $B_z = B = E/c$ are the only non-zero field components, for an arbitrary velocity $\mathbf{u}$ of the particle we get $\mathbf{u}\times\mathbf{B} = \mathbf{n}_x u_y E/c - \mathbf{n}_y u_x E/c$, so Eq. (9.144), decomposed into three scalar equations, gives

$$\frac{dp_x}{dt} = q\frac{u_y}{c}E, \qquad \frac{dp_y}{dt} = qE\left(1-\frac{u_x}{c}\right), \qquad \frac{dp_z}{dt} = 0, \qquad (*)$$

while Eq. (9.148) gives the following equation for energy $\mathscr{E}$:

$$\frac{d\mathscr{E}}{dt} = qEu_y. \qquad (**)$$

Combining the first of Eqs. (*) with Eq. (**), we get

$$\mathscr{E} - cp_x = \text{const} \equiv a, \qquad (***)$$

while the last of Eqs. (*) yields p_z = const as well, so we may form from it another (positive) constant of motion, $(mc^2)^2 + c^2p_z^2 \equiv b \geq 0$. Plugging these two constants (a and b) into the basic Eq. (9.78) for the relativistic energy of the particle, rewritten as

$$\mathscr{E}^2 - c^2p_x^2 \equiv (\mathscr{E} - cp_x)(\mathscr{E} + cp_x) = (mc^2)^2 + c^2p_y^2 + c^2p_z^2,$$

we get

$$a(\mathscr{E} + cp_x) = c^2p_y^2 + b. \qquad (****)$$

Now considering Eqs. (***) and (****) a system of two linear equations for $\mathscr{E}$ and p_x, we may express these variables via p_y (and the constants of motion):

$$\mathscr{E} = \frac{c^2 p_y^2 + b}{2a} + \frac{a}{2}, \qquad cp_x = \frac{c^2 p_y^2 + b}{2a} - \frac{a}{2}.$$

The first of these relations may be plugged into the product of $\mathscr{E}$ by the derivative dp_y/dt, expressed from the differential equation (*) for p_y, and then transformed by using the universal relativistic relation $\mathbf{p} = (\mathscr{E}/c^2)\mathbf{u}$, which was mentioned in Sec. 9.3:

$$\mathscr{E}\frac{dp_y}{dt} = \mathscr{E}qE\left(1 - \frac{u_x}{c}\right) \equiv qE\left(\mathscr{E} - \mathscr{E}\frac{u_x}{c}\right) \equiv qE\left(\mathscr{E} - cp_x\right) \equiv qEa\,. \qquad (*****)$$

The result of this substitution,

$$\left(\frac{c^2 p_y^2 + b}{2a} + \frac{a}{2}\right)\frac{dp_y}{dt} \equiv \frac{d}{dt}\left[\frac{c^2}{6a}p_y^3 + \left(\frac{b}{2a} + \frac{a}{2}\right)p_y\right] = qEa\,.$$

may be readily integrated over time to obtain the time dependence of p_y in an implicit form:

$$\frac{c^2}{6a}p_y^3 + \left(\frac{b}{2a} + \frac{a}{2}\right)p_y = qEa\left(t - t_0\right),$$

where t_0 is the time moment (determined by initial conditions) when $p_y = 0$.

Even without solving this cubic equation, we may use the relation $dt/\mathscr{E} = dp_y/qEa$ (which follows from Eq. (*****) above) to express the particle's coordinates as functions of p_y:

$$x \equiv \int u_x dt = \int \frac{c^2 p_x}{\mathscr{E}}dt = \frac{c}{qEa}\int cp_x dp_y = \frac{c}{qEa}\int\left(\frac{c^2 p_y^2 + b}{2a} - \frac{a}{2}\right)dp_y = \frac{c}{qEa}\left[\frac{c^2}{6a}p_y^3 + \left(\frac{b}{2a} - \frac{a}{2}\right)p_y\right] + \text{const},$$

$$y \equiv \int u_y dt = \int \frac{c^2 p_y}{\mathscr{E}}dt = \frac{c}{qEa}\int cp_y dp_y = \frac{c^2}{2qEa}p_y^2 + \text{const},$$

$$z = \int u_z dt = \int \frac{c^2 p_z}{\mathscr{E}}dt = \frac{c}{qEa}\int cp_z dp_y = \frac{c^2 p_z}{qEa}p_y + \text{const}.$$

Since, according to the last formula, z is a linear function of p_y, the first two of these relations yield the particle's trajectory in the form $x(z)$, $y(z)$. Note that at large times, the magnitude of p_y (and hence of z) grows at $(t - t_0)^{1/3}$, the coordinate y grows faster, as $(t - t_0)^{2/3}$, while the motion along the x-axis (i.e. in the direction perpendicular to both vectors **E** and **B**) is the fastest one:

$$x \to \frac{c^3}{6qEa^2}p_y^3 \to c(t - t_0)\,.$$

This is essentially the drift that was discussed in Sec. 9.6(iii), whose velocity formally obeys both Eqs. (9.168) and (9.174), but since in the reference frame moving with this velocity ($v = c$), both the electric and magnetic fields disappear, the frequency (9.172) vanishes, so the particle does not perform the simultaneous cyclotron motion.

Problem 9.31. Find the law of motion of a relativistic particle in uniform static fields **E** and **B** parallel to each other.

Solution: If **E** || **B**, then even for an arbitrary direction of the initial momentum **p**(0) of the particle, we can always select such a reference frame that

$$\mathbf{E} = \{0, 0, E\}, \qquad \mathbf{B} = \{0, 0, B\}, \qquad \mathbf{p}(0) = \{p_x(0), 0, p_z(0)\}.$$

Then the 4-vector equation (9.145) of the particle's motion,

$$\frac{du^\alpha}{d\tau} = \frac{q}{m} F^{\alpha\beta} u_\beta ,$$

where $F^{\alpha\beta}$ is given by Eq. (9.125a) of the lecture notes, may be separated into four scalar components as

$$\frac{d(\gamma c)}{d\tau} = \Gamma(\gamma u_z), \quad \frac{d(\gamma u_x)}{d\tau} = \omega(\gamma u_y), \quad \frac{d(\gamma u_y)}{d\tau} = -\omega(\gamma_u u_x), \quad \frac{d(\gamma u_z)}{d\tau} = \Gamma(\gamma c), \qquad (*)$$

where $\Gamma \equiv qE/mc$, and $\omega \equiv qB/m$.[168] The first and the last Eqs. (*) form an independent system, which may be readily reduced to one second-order equation (by the usual trick of differentiation of one of the initial equations over τ, and then plugging the counterpart equation into the result).[169] As a result, we get two similar equations,

$$\frac{d^2}{d\tau^2}(\gamma c) = \Gamma^2(\gamma c), \qquad \frac{d^2}{d\tau^2}(\gamma u_z) = \Gamma^2(\gamma u_z),$$

which may be readily integrated, with integration constants determined by the initial values of γ and u_z. The result is

$$\gamma(\tau) = \gamma(0)\left[\cosh\Gamma\tau + \beta_z(0)\sinh\Gamma\tau\right], \qquad u_z(\tau) = c\frac{\beta_z(0)\cosh\Gamma\tau + \sinh\Gamma\tau}{\cosh\Gamma\tau + \beta_z(0)\sinh\Gamma\tau}, \qquad (**)$$

where $\gamma(0) = [1 - \beta_x^2(0) - \beta_z^2(0)]^{-1/2}$; it describes an accelerated motion of the particle in the z-direction.

The second and third of Eqs. (*) form a similar system, in which the electric field does not participate. It is formally the same as for the non-relativistic cyclotron motion and also may be readily integrated. With the appropriate choice of origins of the proper time τ and of the x-axis (the last one in the plane of the initial velocity), we get

$$\gamma u_x = \frac{p_x(0)}{m}\cos\omega\tau, \qquad \gamma u_y = -\frac{p_x(0)}{m}\sin\omega\tau .$$

Since $\gamma u_x \equiv \gamma dx/dt = dx/d\tau$ (and similarly for y), these equations may be readily integrated again, giving a simple cyclotron orbit[170]

$$x = R\sin\omega\tau + \text{const}, \qquad y = -R\cos\omega\tau + \text{const}, \qquad (***)$$

[168] Note that the ω so defined is *not* the usual relativistic cyclotron frequency (9.151), $\omega_c = qB/M \equiv qB/\gamma m$. Indeed, the denominator in the definition of ω does not include the Lorentz factor $\gamma = (1 - \beta^2)^{-1/2}$, as it would if we dealt not with the proper time τ of the particle, but with the lab time t.

[169] Note that since this system of two equations does not depend on the magnetic field, it gives an alternative way of solving the problem discussed in Sec. 9.6(ii) of the lecture notes.

[170] Note that if $qB_z > 0$, i.e. $\omega > 0$, the particle rotates in the "negative" (clockwise) direction.

whose radius $R = p_x(0)/qB$ is not affected by either the initial momentum or the acceleration of the particle in the z-direction.

Note, however, that at $E \neq 0$, the proper time τ is not proportional to the lab time t. Indeed, their relation may be found by plugging the first of Eqs. (**) into the fundamental relation $dt = \gamma d\tau$, and then integrating it. With the natural choice $t = 0$ at $\tau = 0$, this simple integration gives

$$t = \frac{\gamma(0)}{\Gamma}\left[\sinh \Gamma\tau + \beta_z(0)(\cosh \Gamma\tau - 1)\right].$$

As a result, as viewed from the lab frame, the frequency of the particle's rotation is not constant. In particular, at large times ($\tau >> 1/\Gamma$), the proper time grows only as a logarithm of the lab time, so the cyclotron motion gradually slows down. Physically, this happens due to the gradual increase of the relativistic mass $M = \gamma m$ of the particle – because of a continuing increase of its energy by the electric field.

<u>Problem 9.32</u>. An external Lorentz force $\mathbf{F}$ is exerted on a relativistic particle with an electric charge q and a rest mass m, moving with velocity $\mathbf{u}$, as observed from some inertial "lab" frame. Calculate its acceleration as observed from that frame.

Solution: According to Eqs. (9.70) and (9.144) of the lecture notes, we may write

$$m\frac{d}{dt}(\gamma\mathbf{u}) = \mathbf{F}, \qquad \text{where } \mathbf{F} = q(\mathbf{E} + \mathbf{u}\times\mathbf{B}). \qquad (*)$$

Let us spell out the derivative on the left-hand side of this equation:

$$\frac{d}{dt}(\gamma\mathbf{u}) \equiv c\frac{d}{dt}(\gamma\boldsymbol{\beta}) \equiv c\frac{d}{dt}\frac{\boldsymbol{\beta}}{\left(1-\beta^2\right)^{1/2}} = c\left[\frac{\dot{\boldsymbol{\beta}}}{\left(1-\beta^2\right)^{1/2}} + \frac{\boldsymbol{\beta}}{2\left(1-\beta^2\right)^{3/2}}\frac{d}{dt}\left(\beta^2\right)\right].$$

For what follows, it is convenient to differentiate β^2 in the last term as the scalar product $\boldsymbol{\beta}\cdot\boldsymbol{\beta}$, getting

$$\frac{d}{dt}(\gamma\mathbf{u}) = c\left[\frac{\dot{\boldsymbol{\beta}}}{\left(1-\beta^2\right)^{1/2}} + \frac{\boldsymbol{\beta}}{2\left(1-\beta^2\right)^{3/2}}\left(2\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)\right] \equiv c\gamma^3\left[\left(1-\beta^2\right)\dot{\boldsymbol{\beta}} + \boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)\right],$$

so the first of Eqs. (*) becomes

$$mc\gamma^3\left[\left(1-\beta^2\right)\dot{\boldsymbol{\beta}} + \boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)\right] = \mathbf{F}. \qquad (**)$$

To solve this equation for $\dot{\boldsymbol{\beta}}$, let us scalar-multiply both sides by $\boldsymbol{\beta}$, getting

$$mc\gamma^3\left[\left(1-\beta^2\right)\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right) + \beta^2\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)\right] \equiv mc\gamma^3\,\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}} = \boldsymbol{\beta}\cdot\mathbf{F}, \qquad \text{so } \boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}} = \frac{\boldsymbol{\beta}\cdot\mathbf{F}}{mc\gamma^3}. \qquad (***)$$

Plugging the last relation back into Eq. (**), we get a simple equation,

$$mc\gamma^3\left[\left(1-\beta^2\right)\dot{\boldsymbol{\beta}} + \boldsymbol{\beta}\frac{\boldsymbol{\beta}\cdot\mathbf{F}}{mc\gamma^3}\right] \equiv mc\gamma\,\dot{\boldsymbol{\beta}} + \boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{F}) = \mathbf{F},$$

which yields an explicit formula for the acceleration:

$$\dot{\boldsymbol{\beta}} = \frac{\mathbf{F} - \boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{F})}{mc\gamma}, \qquad \text{i.e. } \mathbf{a} \equiv \dot{\mathbf{u}} \equiv c\dot{\boldsymbol{\beta}} = \frac{\mathbf{F} - \boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{F})}{m\gamma}. \qquad (****)$$

In the non-relativistic limit $\boldsymbol{\beta} \to 0$, the second term in the numerator is negligible and $\gamma \approx 1$, so this result is reduced (as it should) to the usual 2nd Newton law $\mathbf{a} = \mathbf{F}/m$, with the rest mass m. In the relativistic case, the acceleration depends on the applied force's direction: if it is normal to the particle's velocity ($\mathbf{F} \perp \boldsymbol{\beta}$), then $(\boldsymbol{\beta}\cdot\mathbf{F}) = 0$, so $\mathbf{a} = \mathbf{F}/m\gamma \equiv \mathbf{F}/M$, where M is the relativistic mass γm. However, for aligned vectors $\boldsymbol{\beta}$ and $\mathbf{F}$, Eq. (****) yields the result $\mathbf{a} = (1-\beta^2)\mathbf{F}/m\gamma \equiv \mathbf{F}/m\gamma^3 \equiv \mathbf{F}/M\gamma^2$, showing that due to the Lorentz transform, in the ultra-relativistic case $\gamma >> 1$, the longitudinal acceleration (as observed from the lab frame) is even much lower.

Problem 9.33. Neglecting relativistic kinetic effects, calculate the lowest voltage V that has to be applied between the anode and cathode of a magnetron (see Fig. 9.13 and its discussion in Sec. 9.6 of the lecture notes) to enable electrons to reach the anode, at negligible electron-electron interactions (including the space charge effects) and collisions with the residual gas molecules. You may:

(i) model the cathode and anode as two coaxial round cylinders, of radii R_1 and R_2, respectively;

(ii) assume that the magnetic field $\mathbf{B}$ is uniform and directed along their common axis; and

(iii) neglect the initial velocity of the electrons emitted by the cathode.

After the solution, estimate the validity of the last assumption, and of the non-relativistic approximation, for reasonable values of parameters.

Solution: In the non-relativistic limit, we may restrict the Taylor expansion

$$\frac{1}{\gamma} \equiv \left(1 - \frac{u^2}{c^2}\right)^{1/2} = 1 - \frac{u^2}{2c^2} + ...$$

to the two leading terms spelled up above, and hence approximate Eq. (9.183) of the lecture notes as

$$\mathscr{L} = mc^2 + \frac{m}{2}u^2 - q\phi + q\mathbf{u}\cdot\mathbf{A}. \qquad (*)$$

In our current problem, in the cylindrical coordinates $\mathbf{r} = \{\rho, \varphi, z\}$ (with the z-axis coinciding with the common axis of the two cylindrical electrodes), the magnetic field has only a z-component, $\mathbf{B} = \mathbf{n}_z B$, while the electric field has only a radial component: $\mathbf{E} = -\mathbf{n}_\rho d\phi(\rho)/d\rho$. Since the emitted electrons have negligible initial velocities, neither component of the Lorentz force (5.10) drives them in the z-direction at any time. As a result, each electron moves only in a plane normal to the common axis, and its velocity has only two components:

$$\mathbf{u} = \mathbf{n}_\rho \dot{\rho} + \mathbf{n}_\varphi \rho\dot{\varphi},$$

so

$$u^2 = \dot{\rho}^2 + \rho^2\dot{\varphi}^2.$$

Selecting the vector potential's gauge so that it has the natural axially-symmetric form $\mathbf{A} = \mathbf{n}_\varphi A(\rho)$, and using the well-known expression for the curl of such a vector in the cylindrical coordinates,[171] $\nabla\times\mathbf{A} = \mathbf{n}_z[\partial(\rho A)/\partial\rho]/\rho$, we see that the potential's definition, $\nabla\times\mathbf{A} = \mathbf{B}$, with $\mathbf{B} = \mathbf{n}_z B =$ const, is satisfied if $A(\rho) = B\rho/2$. So the scalar product participating in the Lagrangian function (*) is

$$\mathbf{u}\cdot\mathbf{A} = \rho\dot{\varphi}\frac{B\rho}{2},$$

and the whole function (excluding its inconsequential rest-energy part mc^2) becomes

$$\mathscr{L} = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2\right) - q\phi(\rho) + q\rho\dot{\varphi}\frac{B\rho}{2} + \text{const},$$

with $m = m_\mathrm{e}$ and $q = -e$. Now using this function in a regular way[172] to derive the Lagrange equations of motion of two generalized coordinates, ρ and φ, we get

$$\frac{d}{dt}(m\dot{\rho}) - m\rho\dot{\varphi}^2 + q\frac{d\phi}{d\rho} - q\rho\dot{\varphi}B = 0, \qquad \frac{d}{dt}\left(m\rho^2\dot{\varphi} + q\frac{B\rho^2}{2}\right) = 0, \qquad (**)$$

with the function $\phi(\rho)$ determined from the system's electrostatics.[173] (In our simple case of unconstrained motion, Eqs. (**) may be also obtained directly from the non-relativistic version of Eq. (9.144):

$$\frac{d}{dt}(m\mathbf{u}) = q(\mathbf{E} + \mathbf{u}\times\mathbf{B}),$$

spelled out in the polar coordinates, but technically, this pedestrian way is a bit more cumbersome.)

The second of Eqs. (**) is the simplest of the two to solve, immediately giving us the following first integral of motion:[174]

$$m\rho^2\dot{\varphi} + q\frac{B\rho^2}{2} = \text{const}.$$

The constant on the right-hand side of this relation may be evaluated from the condition of negligible velocity (including its azimuthal component) on the cathode's surface, i.e. at $\rho = R_1$. The result may be represented as

$$\dot{\varphi} = -\frac{qB}{2m}\left(1 - \frac{R_1^2}{\rho^2}\right). \qquad (***)$$

This equation may be combined with the law of conservation of the electron's energy:[175]

$$\frac{m}{2}u^2 + q\phi(\rho) \equiv \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2\right) + q\phi(\rho) = \text{const}.$$

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171 See, e.g., MA Eq. (10.5) with $\mathbf{f} = \mathbf{A}$, $A_\rho = A_z = 0$, and $\partial/\partial\varphi = \partial/\partial z = 0$.

172 If you need a reminder (I hope you do not :-), please see CM Eq. (2.19).

173 See, e.g., Eq. (2.45) of the lecture notes – which we (somewhat counter-intuitively) will not need in this solution.

174 This integral might also be derived by applying Eq. (9.201) to a circular contour of radius ρ, pierced by a constant, normal magnetic field $\mathbf{B}$, so $\Phi = \pi\rho^2 B$.

175 It may be also readily obtained by the integration of the first of Eqs. (**) over ρ, and then plugging into the result the second of these equations, integrated over t.

Applying this relation to points $\rho = R_1$ (where both velocity components are negligible) and $\rho = R_2$, and taking $\phi(R_1) = 0$ and $\phi(R_2) = V$, where V is the voltage between the anode and the cathode, we get

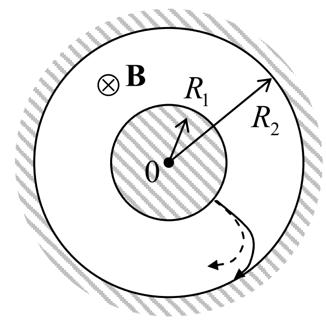
$$\frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2\right)_{\rho=R_2} + qV = 0 \ .$$

Plugging Eq. (***), written for $\rho = R_2$, into this equality, we may readily solve it for the voltage V:

$$V = -\frac{m}{2q}\left[\dot{\rho}^2\Big|_{\rho=R_2} + R_2^2\left(\frac{qB}{2m}\right)^2\left(1-\frac{R_1^2}{R_2^2}\right)^2\right]. \qquad (****)$$

(For electrons, with their negative charge $q = -e$, this voltage is positive.)

If V is lower than some threshold value V_t, the particles move along loop trajectories not reaching the anode – see the (very schematic) dashed line in the figure on the right. In this case, the radial velocity $\dot{\rho}$ is vanishing at some point, most distant from the axis but still smaller than R_2. The threshold voltage V_t we are looking for may be defined as the value of V at which this most distant point of the trajectory exactly equals R_2 – see the solid line in the same figure. For this value, Eq. (****), with $q = -e$, and $m = m_e$, yields



$$V_t = \frac{eB^2}{8m_e}R_2^2\left(1-\frac{R_1^2}{R_2^2}\right)^2 .$$

For typical magnetrons (say, those used in kitchen microwave ovens) with $R_2 \sim 1$ cm, $R_1 \sim R_2/2$, and $B \sim 0.1$ T,[176] this expression yields realistic values of V_t of the order of 10 kV. Note that on the 10-keV scale of the energy provided by this field, the random thermal energies of the order of $k_BT \sim 0.1$ eV of the electrons emitted by heated cathodes (with temperature $T \sim 10^3$ K) are indeed negligible. On the other hand, such kinetic energy is still much lower than the rest energy $m_ec^2 \approx 0.5$ MeV of an electron, so the *kinetic*[177] relativistic effects are negligible as well.

Problem 9.34. A charged relativistic particle has been injected into a region with a uniform electric field whose magnitude oscillates in time with frequency ω. Calculate the time dependence of the particle's velocity, as observed from the lab reference frame.

Solution: Directing the x-axis along the electric field, and the y-axis normally to it but within the plane of the initial velocity of the particle (so that $\beta_z \equiv 0$), we can rewrite the relativistic equation (9.144) of its motion, in the lab frame, as

$$\frac{d(\gamma\beta_x)}{d(\omega t)} = \kappa\cos\omega t, \qquad \frac{d(\gamma\beta_y)}{d(\omega t)} = 0,$$

[176] This value is chosen so that the average frequency $\omega = \overline{\dot{\varphi}}$ of the electron motion, following from Eq. (***), would be close to the own frequency of the resonating cavities of the device (see Fig. 9.13 of the lecture notes), and hence to the frequency of the microwaves it generates. This frequency is government-mandated to be within a narrow range around 2.45 GHz, to avoid microwave spectrum contamination.

[177] This qualifier is intended to remind the reader that the magnetic field (which *cannot* be neglected in this problem) is itself an essentially relativistic effect – see, e.g., the discussion in Sec. 5.1 of the lecture notes.

with the dimensionless parameter:

$$\kappa \equiv \frac{qE_\omega}{mc\omega},$$

E_ω being the electric field's amplitude. These equations may be readily integrated, giving

$$\gamma\beta_x = \gamma_0\beta_{x0} + \kappa \sin \omega t, \qquad \gamma\beta_y = \gamma_0\beta_{y0}. \qquad (*)$$

Since

$$\gamma^2 \equiv \frac{1}{1-\beta^2}, \qquad \text{i.e. } \gamma^2\left(1-\beta^2\right) = 1, \qquad \text{so } \gamma^2 = 1 + (\gamma\beta)^2 = 1 + (\gamma\beta_x)^2 + (\gamma\beta_y)^2,$$

we may use Eqs. (*) to calculate the Lorentz factor γ as an explicit function of time:

$$\gamma = \left[1 + (\gamma\beta_x)^2 + (\gamma\beta_y)^2\right]^{1/2} = \left[1 + \left(\gamma_0\beta_{x0} + \kappa \sin \omega t\right)^2 + \left(\gamma_0\beta_{y0}\right)^2\right]^{1/2}. \qquad (**)$$

Now we can combine Eqs. (*) and (**) to represent both reduced velocities as functions of time:

$$\beta_x = \frac{\gamma_0\beta_{x0} + \kappa \sin \omega t}{\left[1 + \left(\gamma_0\beta_{x0} + \kappa \sin \omega t\right)^2 + \left(\gamma_0\beta_{y0}\right)^2\right]^{1/2}}, \qquad \beta_y = \frac{\gamma_0\beta_{y0}}{\left[1 + \left(\gamma_0\beta_{x0} + \kappa \sin \omega t\right)^2 + \left(\gamma_0\beta_{y0}\right)^2\right]^{1/2}},$$

$$\beta \equiv \left(\beta_x^2 + \beta_y^2\right)^{1/2} = \left[\frac{\left(\gamma_0\beta_{x0} + \kappa \sin \omega t\right)^2 + \left(\gamma_0\beta_{y0}\right)^2}{1 + \left(\gamma_0\beta_{x0} + \kappa \sin \omega t\right)^2 + \left(\gamma_0\beta_{y0}\right)^2}\right]^{1/2}.$$

For the simplest particular case of no initial velocity ($\beta_{x0} = \beta_{y0} = 0$), the last expression is reduced to

$$\beta = \frac{\kappa \sin \omega t}{\left(1 + \kappa^2 \sin^2 \omega t\right)^{1/2}}. \qquad (***)$$

The results show that if the field is relatively weak,

$$\kappa << 1, \qquad \text{i.e. } qE_\omega a << mc^2,$$

where $a \equiv c/\omega$ is the relativistic scale of the particle's oscillation amplitude, its oscillations are virtually sinusoidal – natural for a free non-relativistic particle driven by a sinusoidal force.[178] However, in the opposite limit $\kappa >> 1$, Eq. (***) shows that $|\beta| \approx 1$ (i.e. $|u| \approx c$) during most of the oscillation period, except for very narrow time intervals during which the velocity "switches" from $+c$ to $-c$ and back.

Problem 9.35.* A linearly-polarized plane electromagnetic wave of frequency ω is incident on an otherwise free relativistic particle with electric charge q. Analyze the dynamics of the particle's momentum and compare the result with those of the previous problem and Problem 7.5.

Solution: Directing the z-axis along the direction of the wave's propagation and the x-axis along its electric field's polarization, and per of Eq. (7.6), we may represent the field distribution is

$$\mathbf{E} = \mathbf{n}_x E_0 \cos \Psi, \qquad \mathbf{B} = \mathbf{n}_y \left(E_0 / c\right) \cos \Psi, \qquad \text{where } \Psi \equiv kz - \omega t + \text{const},$$

[178] A useful sanity check, in this non-relativistic limit, the velocity's amplitude $u_{\max} \equiv \beta_{\max}c = \kappa c \equiv qE_\omega/m\omega$ coincides with the corresponding intermediate result of the model solution of Problem 7.5.

where the last constant may be always made zero by the appropriate choice of the time's origin. Let us use these expressions to spell out the Cartesian components of the basic equation of the particle's motion – see Eq. (9.144) of the lecture notes:

$$\frac{dp_x}{dt} = qE_0 \cos\Psi \left(1 - \frac{u_z}{c}\right), \qquad \frac{dp_y}{dt} = 0, \qquad \frac{dp_z}{dt} = qE_0 \cos\Psi \frac{u_x}{c}. \tag{*}$$

The second of these equations may be immediately integrated, giving $p_y(t) = p_y(0)$. The integration of the first equation may be simplified by noticing that the expression in the parentheses on its right-hand side may be simply expressed via the full derivative of the total phase Ψ of the wave, taking into account the particle's motion along the z-axis:

$$\frac{d\Psi}{dt} = k\frac{dz}{dt} - \omega \equiv ku_z - \omega \equiv \omega\left(\frac{u_z}{c} - 1\right),$$

so the equation takes the simple form

$$\frac{dp_x}{dt} = -\frac{qE_0}{\omega}\cos\Psi \frac{d\Psi}{dt} \equiv \frac{qE_0}{\omega}\frac{d(\sin\Psi)}{dt},$$

and now may be readily integrated over time:

$$p_x = \frac{qE_0}{\omega}\sin\Psi + \text{const} \equiv \frac{qE_0}{\omega}\sin(kz - \omega t) + \text{const}, \tag{**}$$

where the constant may be always selected to equal zero, by a proper choice of the z-axis' origin.

Next, let us by notice that for our problem, the scalar product $\mathbf{u}\cdot\mathbf{E}$ is merely $u_x E_0 \cos\Psi$, so Eq. (9.148) of the lecture notes is reduced to

$$\frac{d\mathscr{E}}{dt} = qE_0 \cos\Psi\, u_x,$$

where $\mathscr{E}$ is the energy (9.78) of the particle (not including its interaction with the wave's field). As a result, the last of Eqs. (*) takes the form $dp_z/dt = (d\mathscr{E}/dt)/c$ and may be also readily integrated, giving

$$p_z = \frac{\mathscr{E}}{c} - C \equiv \left[(mc)^2 + p_x^2 + p_y^2(0) + p_z^2\right]^{1/2} - C, \tag{***}$$

where p_x is given by Eq. (**) and the constant C may be found from the initial condition:

$$C = \frac{\mathscr{E}(0)}{c} - p_z(0) \equiv \left[(mc)^2 + p_x^2(0) + p_y^2(0) + p_z^2(0)\right]^{1/2} - p_z(0).$$

The algebraic equation (***) may be simplified by bringing the term C over to the left-hand side and squaring the resulting relation. After the cancellation of the terms p_z^2, we get an apparently explicit expression for this component:

$$p_z = \frac{p_x^2}{2C} + C' = \frac{1}{2C}\left(\frac{qE_0}{\omega}\right)^2 \sin^2(kz - \omega t) + C', \tag{****}$$

where C' is another constant:[179]

$$C' = \frac{(mc)^2 + p_y^2(0) - C^2}{2C}.$$

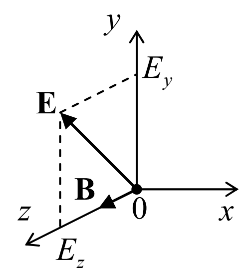
The simplicity of our final formulas (**) and (****) is somewhat deceiving: they express the particle momentum components not explicitly via z and t, but only via a linear combination of these independent arguments. In order to find their relation $z(t)$, i.e. the z-component of the particle's motion $\mathbf{r}(t)$, we need to integrate an explicit relation for the corresponding Cartesian component u_z of the particle's velocity $\mathbf{u} = \mathbf{p}/\gamma(u)m$. Since the Lorentz factor γ depends on all three components of $\mathbf{u}$, even obtaining such an explicit expression from our results runs into some prohibitive algebra.

In the non-relativistic limit, which requires, in particular, the wave's field to be sufficiently weak:

$$\left|p_x\right|_{\max} = \frac{qE_0}{\omega} << mc,$$

Eq. (**) reduces to the results of both the previous problem (formally corresponding to the low-frequency limit $k \to 0$) and that of Problem 7.5, describing sinusoidal oscillations of the particle in the direction of the wave's electric field. In this limit, Eq. (****) shows that the particle's oscillations along the z-axis, i.e. the direction of the wave's propagation, are much smaller, and their fundamental frequency is 2ω (rather than ω as for the x-motion).[180]

Problem 9.36. Analyze the motion of a non-relativistic particle in a region where the electric and magnetic fields are both uniform and constant in time, but not necessarily parallel or perpendicular to each other.



Solution: Let us select the direction of the vector $\mathbf{B}$ for the z-axis, and direct the y-axis so that the vector $\mathbf{E}$ resides in the $[y, z]$ plane – see the figure on the right. Then in the non-relativistic case, Eq. (9.144) of the lecture notes (with $\gamma_u = 1$, i.e. $\mathbf{p} = m\mathbf{u}$) has the following Cartesian components:

$$m\ddot{x} = q\dot{y}B, \qquad m\ddot{y} = q\left(E_y - \dot{x}B\right), \qquad m\ddot{z} = qE_z. \qquad (*)$$

The last equation is independent of its counterparts and may be easily integrated to give the usual "free fall" motion along the z-axis, with the constant acceleration $a_z = qE_z/m$. The first two equations may be merged by the introduction of the rotational 2D velocity

$$\boldsymbol{u} \equiv \mathbf{n}_x\left(\dot{x} - u_{\rm d}\right) + \mathbf{n}_y\dot{y}, \qquad \text{with } u_{\rm d} \equiv \frac{E_y}{B}.$$

Differentiating this vector over time, and then using Eqs. (*), we get

[179] The constants C and C' are especially simple in the particular case when the particle is at rest at least at one (say, initial) point: $p_y(0) = p_z(0) = 0$; then $C = (mc)^2$ and $C' = 0$.

[180] Note also that according to Eq. (****), the motion in the z-direction may also include some average drift, though the exact calculation of its velocity also requires the function $z(t)$.

$$\dot{\boldsymbol{u}} = \mathbf{n}_x \ddot{x} + \mathbf{n}_y \ddot{y} = \mathbf{n}_x \frac{q\dot{y}B}{m} + \mathbf{n}_y \frac{(E_y - \dot{x}B)}{m} = \frac{qB}{m}\left[\mathbf{n}_x \dot{y} - \mathbf{n}_y (\dot{x} - u_\mathrm{d})\right] = \omega_\mathrm{c}\left(\mathbf{n}_x \boldsymbol{u}_y - \mathbf{n}_y \boldsymbol{u}_x\right),$$

i.e. a first-order differential equation similar to Eq. (9.150) of the lecture notes:

$$\dot{\boldsymbol{u}} = \omega_\mathrm{c} \boldsymbol{u} \times \mathbf{n}_z ,$$

with the cyclotron frequency (9.151): $\omega_\mathrm{c} = qB/m$. This means that the particle's projection on the [x, y]-plane performs a circular motion around a point that moves along the x-axis with a constant drift velocity $u_\mathrm{d} = E_y/B$, i.e. essentially the same motion (along a trochoidal trajectory) as for the case of mutually perpendicular fields ($E_y = E$, $E_z = 0$), which was analyzed in the lecture notes – see Fig. 9.12 and its discussion.

Thus, the only essentially new feature resulting from the arbitrary angle between the fields is an accelerated motion of the particle along the z-direction of the vector **B**. This is qualitatively true for a relativistic particle as well but quantitatively, in that case, the motion is more complex because all velocity components become interrelated via their common Lorentz factor γ_u.

Problem 9.37. A static distribution of electric charge in otherwise free space has created a time-independent distribution $\mathbf{E}(\mathbf{r})$ of the electric field. Use two different approaches to express the field energy density u' and the Poynting vector $\mathbf{S}'$, as observed from a reference frame moving with a constant velocity $\mathbf{v}$, via the Cartesian components of the vector $\mathbf{E}$. In particular, is $\mathbf{S}'$ equal to $(-\mathbf{v}u')$?

Solution:

Approach 1. In the lab frame where the charges rest, their magnetic field equals zero, so the Lorentz transform formulas (9.135) are reduced to

$$\mathbf{E}'_{||} = \mathbf{E}_{||}, \qquad \mathbf{B}'_{||} = 0,$$
$$\mathbf{E}'_{\perp} = \gamma\mathbf{E}_{\perp}, \qquad \mathbf{B}'_{\perp} = -\gamma\frac{\mathbf{v}\times\mathbf{E}_{\perp}}{c^2},$$

where the indices || and $\perp$ mark the field components parallel and normal to the velocity vector **v.** As a result, the field energy density (6.113),[181] as observed in the moving frame, is

$$\begin{aligned} u' &= \frac{\varepsilon_0}{2}E'^2 + \frac{1}{2\mu_0}B'^2 = \frac{\varepsilon_0}{2}\left(E'^2_{||} + E'^2_{\perp}\right) + \frac{1}{2\mu_0}\left(B'^2_{||} + B'^2_{\perp}\right) = \frac{\varepsilon_0}{2}\left(E^2_{||} + \gamma^2 E^2_{\perp}\right) + \frac{1}{2\mu_0}\left(\gamma\frac{vE_\perp}{c^2}\right)^2 \\ &\equiv \frac{\varepsilon_0}{2}E^2_{||} + \left(\frac{\varepsilon_0}{2} + \frac{1}{2\mu_0 c^2}\beta^2\right)\gamma^2 E^2_{\perp} \equiv \frac{\varepsilon_0}{2}\left(E^2_{||} + \frac{1+\beta^2}{1-\beta^2}E^2_{\perp}\right), \end{aligned} \qquad (*)$$

while the corresponding Poynting vector (6.114) is

$$\begin{aligned} \mathbf{S}' &= \frac{1}{\mu_0}\mathbf{E}'\times\mathbf{B}' = \frac{1}{\mu_0}\left(\mathbf{E}'_{||} + \mathbf{E}'_{\perp}\right)\times\left(\mathbf{B}'_{||} + \mathbf{B}'_{\perp}\right) = \frac{1}{\mu_0}\left(\mathbf{E}_{||} + \gamma\mathbf{E}_{\perp}\right)\times\left(0 - \gamma\frac{\mathbf{v}\times\mathbf{E}_{\perp}}{c^2}\right) \\ &= -\varepsilon_0\gamma\left[\mathbf{E}_{||}\times\left(\mathbf{v}\times\mathbf{E}_{\perp}\right) + \gamma\mathbf{E}_{\perp}\times\left(\mathbf{v}\times\mathbf{E}_{\perp}\right)\right]. \end{aligned}$$

[181] See also Eq. (9.225).

Applying the *bac minus cab* rule[182] to both double vector products, and then using the mutual parallelism of the vectors **v** and $\mathbf{E}_{||}$, and their normality to the vector $\mathbf{E}_\perp$, we may reduce the last expression to a simpler one:

$$\mathbf{S}' = -\varepsilon_0\gamma\left[\mathbf{v}\left(\mathbf{E}_{||}\cdot\mathbf{E}_\perp\right) - \mathbf{E}_\perp\left(\mathbf{v}\cdot\mathbf{E}_{||}\right) + \gamma\mathbf{v}\left(\mathbf{E}_\perp\cdot\mathbf{E}_\perp\right) - \gamma\mathbf{E}_\perp\left(\mathbf{E}_\perp\cdot\mathbf{v}\right)\right] = \varepsilon_0\gamma\left(\mathbf{E}_\perp vE_{||} - \mathbf{v}\gamma E_\perp^2\right). \qquad (**)$$

<u>Approach 2</u>. Let us use the standard choice of coordinates (see, e.g., Fig. 9.1 of the lecture notes), so $\mathbf{E}_{||} \equiv \mathbf{n}_x E_{||} = \mathbf{n}_x E_x$, and $\mathbf{E}_\perp = \mathbf{n}_y E_y + \mathbf{n}_z E_z$, i.e. $E_\perp^2 = E_y^2 + E_z^2$. According to Eqs. (9.225)-(9.229) of the lecture notes, in our case, the symmetric 4-tensor (9.221) of the energy-momentum of the field in the lab reference frame (in which the charges are at rest and hence $\mathbf{B} = 0$) is

$$\theta = \frac{\varepsilon_0}{2}\begin{pmatrix} E_{||}^2 + E_\perp^2 & 0 & 0 & 0 \\ 0 & -E_{||}^2 + E_\perp^2 & -2E_{||}E_y & -2E_{||}E_z \\ 0 & -2E_{||}E_y & -E_y^2 + E_{||}^2 + E_z^2 & -2E_yE_z \\ 0 & -2E_{||}E_z & -2E_yE_z & -E_z^2 + E_{||}^2 + E_y^2 \end{pmatrix}.$$

Applying to this tensor the Lorentz transform (9.109), just as was spelled out in the model solution of Problem 18, we get

$$u' \equiv \theta'^{00} = \gamma^2\left[\theta^{00} - \beta\left(\theta^{01} + \theta^{10}\right) + \beta^2\theta^{11}\right] = \gamma^2\frac{\varepsilon_0}{2}\left[\left(E_{||}^2 + E_\perp^2\right) - \beta\cdot 0 + \beta^2\left(-E_{||}^2 + E_\perp^2\right)\right]$$
$$\equiv \frac{1}{1-\beta^2}\frac{\varepsilon_0}{2}\left[\left(1-\beta^2\right)E_{||}^2 + \left(1+\beta^2\right)E_\perp^2\right]$$

– the expression equivalent to Eq. (*) that was obtained using Approach 1. Similarly, the same Lorentz transform yields

$$\frac{S'_x}{c} = \theta'^{01} = \gamma^2\left[\theta^{01} - \beta\left(\theta^{00} + \theta^{11}\right) + \beta^2\theta^{10}\right]$$
$$= \gamma^2\frac{\varepsilon_0}{2}\left\{0 - \beta\left[\left(E_{||}^2 + E_\perp^2\right) + \left(-E_{||}^2 + E_\perp^2\right)\right] + \beta^2\cdot 0\right\} \equiv -\varepsilon_0\gamma^2\beta E_\perp^2,$$
$$\frac{S'_y}{c} = \theta'^{02} = \gamma\left(\theta^{02} - \beta\theta^{12}\right) = \gamma\left[0 - \beta\frac{\varepsilon_0}{2}\left(-2E_{||}E_y\right)\right] \equiv \varepsilon_0\gamma\beta E_{||}E_y,$$
$$\frac{S'_z}{c} = \theta'^{03} = \gamma\left(\theta^{03} - \beta\theta^{13}\right) = \gamma\left[0 - \beta\frac{\varepsilon_0}{2}\left(-2E_{||}E_z\right)\right] \equiv \varepsilon_0\gamma\beta E_{||}E_z.$$

The set of these three scalar relations is obviously equivalent to the vector equality (**), obtained using Approach 1. These formulas show that in the general case when $E_{||} \neq 0$, the Poynting vector **S***'* has not only a component directed along the velocity vector **v**, as the apparent result ($-\mathbf{v}u'$) would give, but also a component directed along the vector $\mathbf{E}_\perp$, i.e. normally to the vector **v**.

<u>Problem 9.38</u>. A traveling plane wave of frequency ω and intensity S is normally incident on a perfect mirror moving with velocity v in the same direction as the wave.

[182] See, e.g., MA Eq. (7.5).

(i) Calculate the reflected wave's frequency, and

(ii) use the Lorentz transform of the fields to calculate the reflected wave's intensity

– both as observed from the lab reference frame.

Solutions:

(i) Per Eq. (9.44) of the lecture notes, the incident radiation frequency, as perceived in the reference frame moving with the mirror, is

$$\omega' = \omega\left(\frac{1-\beta}{1+\beta}\right)^{1/2}, \qquad \text{with } \beta \equiv \frac{v}{c}.$$

From the point of view of a mirror-bound observer, the frequency does not change upon reflection, so the mirror may be treated as a source of reflected radiation of the frequency ω'. From the point of view of the lab-frame observer, this source moves from the lab-frame observer with the velocity v. Hence, we may apply Eq. (9.44) again to calculate the reflected wave's frequency as perceived by the observer:

$$\omega_\mathrm{r} = \omega'\left(\frac{1-\beta}{1+\beta}\right)^{1/2} \equiv \omega\frac{1-\beta}{1+\beta} \equiv \omega\frac{c-v}{c+v}. \qquad (*)$$

(ii) According to Eqs. (7.6) and (7.8), in the incident plane wave propagating in free space along some direction $\mathbf{n}$,

$$\mathbf{B} = \mu_0\mathbf{H} = \mu_0\frac{\mathbf{n}\times\mathbf{E}}{Z_0} \equiv \frac{\mathbf{n}\times\mathbf{E}}{c}. \qquad (**)$$

Now applying Eqs. (9.135) to this transverse wave (with $E_\| = 0$, $B_\| = 0$, $\mathbf{E}_\perp = \mathbf{E}$, and $\mathbf{B}_\perp = \mathbf{B}$), we may calculate its fields as perceived in the reference frame moving with the mirror:

$$\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{v}\times\mathbf{B}) = \gamma\left[\mathbf{E} + v\frac{\mathbf{n}\times(\mathbf{n}\times\mathbf{E})}{c}\right] = \gamma\mathbf{E}\left(1-\frac{v}{c}\right) \equiv \mathbf{E}\left(\frac{1-\beta}{1+\beta}\right)^{1/2},$$

$$\mathbf{B}' = \gamma\left(\mathbf{B} - \frac{\mathbf{v}\times\mathbf{E}}{c^2}\right) = \gamma\left[\frac{\mathbf{n}\times\mathbf{E}}{c} - \frac{\mathbf{v}\times\mathbf{E}}{c^2}\right] = \gamma\frac{\mathbf{n}\times\mathbf{E}}{c}\left(1-\frac{v}{c}\right) \equiv \frac{\mathbf{n}\times\mathbf{E}}{c}\left(\frac{1-\beta}{1+\beta}\right)^{1/2},$$

so the relation between the vectors $\mathbf{E}'$ and $\mathbf{B}'$ is also given by Eq. (**) – as could be expected. Calculating the Poynting vector (6.114), we get

$$\mathbf{S}' \equiv \mathbf{E}'\times\mathbf{H}' = \frac{\mathbf{E}'\times\mathbf{B}'}{\mu_0} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0}\frac{1-\beta}{1+\beta} = \mathbf{E}\times\mathbf{H}\frac{1-\beta}{1+\beta} = \mathbf{S}\frac{1-\beta}{1+\beta} \equiv \mathbf{S}\frac{c-v}{c+v}.$$

the last factor giving the relation between the wave intensities in the two reference frames.

As was discussed in Sec. 7.4, for the observer moving with the mirror, at a wave's reflection from a perfect mirror, its intensity does not change, so the last relation (with the opposite sign of $\mathbf{S}$ because of the change of the wave's direction) is also valid for the reflected wave – as observed from the moving reference frame. Now repeating the above calculation, we can find the intensity of the reflected wave as observed from the lab frame:[183]

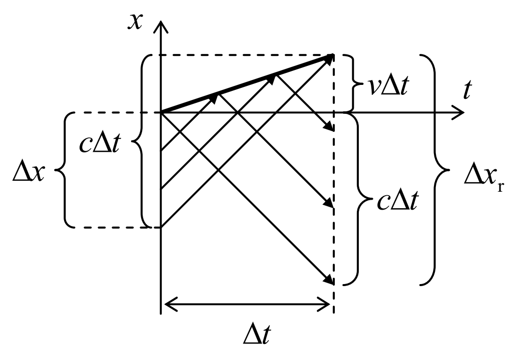
[183] This result may be also obtained by a completely different approach, using Eq. (9.237) of the lecture notes – see the next problem.

$$S_r = S' \frac{c-v}{c+v} = S\left(\frac{c-v}{c+v}\right)^2 . \qquad (***)$$

Note that all the relations in this solution, including the final results (*) and (***), are valid for any sign of the mirror's velocity v. In particular, Eq. (***) shows that the wave's intensity is decreased if the mirror moves in the same direction as the incident wave ("trying to run away" from it) and is increased in the opposite case. Since the intensity increase in the latter case (the mirror moving toward the wave's source) is frequency-insensitive, it may be interpreted as an amplification of the power of an arbitrary waveform. However, this system is rather different from the usual amplifiers, because the power gain is accompanied by an upward shift of all frequency components of the signal – see Eq. (*).

Problem 9.39. Perform the second task of the previous problem by using general relations between the wave's energy, power, and momentum.

Hint: As a byproduct, this approach should also give you the pressure exerted by the wave on the moving mirror.



Solution: Let us direct the x-axis along the incident wave's propagation, and plot the positions of several representative fronts (i.e. the planes of equal total phase) of that wave and the reflected wave, as well as of the mirror, as functions of time – see the figure on the right. As the plots show, during an arbitrary time interval Δt, the mirror is reached by the incident wave that was initially spread over the following volume:

$$\Delta V \equiv A\Delta x = A(c\Delta t - v\Delta t) \equiv A(c-v)\Delta t ,$$

where A is the wave front's area. On the other hand, by the end of the period, the reflected wave is spread over a different volume:

$$\Delta V_{\rm r} \equiv A\Delta x_{\rm r} = A(c\Delta t + v\Delta t) \equiv A(c+v)\Delta t .$$

(Though the figure above is drawn for $v > 0$, it is straightforward to verify that the above formulas are valid for any mirror's velocity with $|\, v \,| \leq c$.)

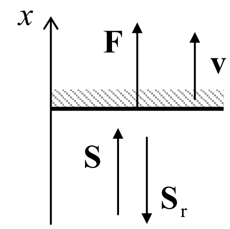
Let u be the energy density of the incident wave, and $u_{\rm r}$ be that of the reflected wave – both as measured in the lab frame. Then the initial wave's energy within the volume ΔV is $u\Delta V = uA(c-v)\Delta t$, and that of the reflected wave, within volume $\Delta V_{\rm r}$, is $u_{\rm r}\Delta V_{\rm r} = u_{\rm r}A(c+v)\Delta t$. Due to the energy conservation, the difference between these two energies has to be equal to the work, $\tau A\Delta x_{\rm m} = \tau A v\Delta t$, performed by the wave's pressure force τA at the mirror's displacement $\Delta x_{\rm m} = v\Delta t$ – see the figure above again. This balance gives us, after the cancellation of A and Δt, the following equation for $u_{\rm r}$ and τ:

$$u(c-v) - u_{\rm r}(c+v) = \tau v . \qquad (*)$$

Another equation for the same two quantities may be obtained from the conservation of the total momentum of the system: the force impulse $\mathbf{F}\Delta t = \tau A\Delta t\mathbf{n}_x$ given to the mirror by the wave, i.e. the change of the mirror's momentum during a time interval Δt, should be equal and opposite to the change of the wave's momentum, resulting from its reflection, during the same time interval. But according to

Eq. (9.237), and (7.9) of the lecture notes (the latter one with $v = c$), for a plane wave propagating in free space along an axis $\mathbf{n}$, the momentum per unit volume is just

$$\mathbf{g} = \frac{\mathbf{S}}{c^2} = \frac{u}{c}\mathbf{n}\,.$$



Hence, for our system's geometry (see the figure on the right), the momenta balance yields

$$\tau A \Delta t\, \mathbf{n}_x = \frac{u}{c}\mathbf{n}_x \Delta V - \frac{u_r}{c}(-\mathbf{n}_x)\Delta V_r\,.$$

Now plugging in the above expressions for ΔV and ΔV_r, and canceling $\mathbf{n}_x$, A, and Δt, we get

$$\tau = \frac{u(c-v) + u_r(c+v)}{c}. \qquad (**)$$

Solving the simple system of two equations (*) and (**), we get

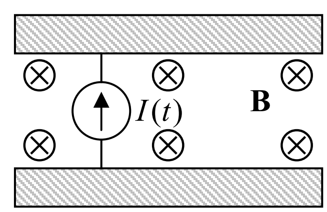
$$u_r = \left(\frac{c-v}{c+v}\right)^2 u, \qquad \text{so that } S_r = \left(\frac{c-v}{c+v}\right)^2 S\,,$$

i.e. the same result as by the Lorentz-transform approach used in the model solution of the previous problem. Besides that, the same system of equations yields the following expression for the wave's pressure:

$$\tau = 2u\frac{c-v}{c+v} \equiv \frac{2S}{c}\frac{c-v}{c+v}.$$

As a sanity check, for an immobile mirror ($v = 0$), this formula, with the account of Eq. (7.9b), gives the result (9.245)-(9.246).

Problem 9.40. For the simple model of capacitor charging by a lumped source of current $I(t)$, shown in the figure on the right, prove that the momentum given by a uniform, stationary, external magnetic field $\mathbf{B}$ to the current-carrying conductor is equal and opposite to the momentum of the electromagnetic field that this current builds up in the capacitor. (You may assume that the capacitor is planar and very broad, and hence neglect the fringe field effects.)



Solution: According to Eq. (5.8) of the lecture notes, in the geometry shown in the figure above, with a positive current $I(t)$, the total magnetic force exerted on the wire is directed to the left and has the magnitude $F(t) = I(t)Bd$, where d is the wire's length, i.e. the distance between the interior surfaces of the capacitor's plates. Integrating this expression over time, for the momentum transferred from the field to the wire, we get

$$\Delta p_{\text{wire}} = \int F(t)dt = Bd\int I(t)dt \equiv Bd\,\Delta Q, \qquad (*)$$

where $\Delta Q \equiv \int I(t)dt$ is the capacitor's charge built up by the current.

Since $I(t) > 0$, the charge ΔQ is also positive, and the electric field it creates is directed, in our figure, from the top plate down, so the electromagnetic field's momentum density vector (9.237) is directed to the right. The total field momentum has the magnitude

$$\Delta p_{\text{field}} = \Delta\int g d^3r = \Delta g A d = \frac{\Delta S}{c^2} A d = \frac{\Delta(EH)}{c^2} A d = \frac{(\Delta V/d)(B/\mu_0)}{c^2} A d \equiv \frac{\varepsilon_0 A}{d}\Delta V B d\,, \qquad (**)$$

where A is the capacitor's area, and $\Delta V = \Delta E d$ is the final change of the voltage between its plates. But the fraction in the last form of Eq. (**) is just the capacitance C, so its product by ΔV equals the same ΔQ that participates in Eq. (*). Thus the two momenta, p_{wire} and p_{field}, are indeed equal and opposite, so the total momentum of the whole system (capacitor + field) is conserved.

Problem 9.41. Consider an electromagnetic plane wave packet propagating in free space, with its electric field represented as the Fourier integral

$$\mathbf{E}(\mathbf{r},t) = \text{Re}\int_{-\infty}^{+\infty} \mathbf{E}_k e^{i\Psi_k} dk, \qquad \text{with } \Psi_k \equiv kz - \omega_k t, \quad \text{and } \omega_k \equiv c|k|.$$

Express the full linear momentum (per unit area of wave's front) of the packet via the complex amplitudes $\mathbf{E}_k$ of its Fourier components. Does the momentum depend on time? (In contrast with Problem 7.9, the wave packet is not necessarily narrow.)

Solution: Per Eq. (9.237) of the lecture notes, we need to calculate the time dependence of

$$\frac{p}{A} \equiv \int_{-\infty}^{+\infty} g_z(z,t)dz = \frac{1}{c^2}\int_{-\infty}^{+\infty} S_z(z,t)dz = \frac{1}{c^2}\int_{-\infty}^{+\infty} [\mathbf{E}(z,t)\times\mathbf{H}(z,t)]_z\, dz\,, \qquad (*)$$

where A is the wave front's area. According to Eqs. (7.6) and (7.7), the magnetic field of the packet may be represented as

$$\mathbf{H}(z,t) = \text{Re}\int_{-\infty}^{+\infty} \mathbf{H}_{k'} e^{i\Psi_{k'}} dk' \equiv \frac{1}{2}\left[\int_{-\infty}^{+\infty} \mathbf{H}_{k'} e^{i\Psi_{k'}} dk' + \text{c.c.}\right],$$

$$\text{with } \mathbf{H}_k = \frac{\mathbf{n}_z \times \mathbf{E}_k}{Z_0}\text{sgn}(k), \quad \text{and } Z_0 \equiv \left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2}.$$

Plugging the expressions for $\mathbf{E}$ and $\mathbf{H}$ into Eq. (*), taking into account that $1/cZ_0 \equiv \varepsilon_0$, and changing the order of integration, we get[184]

$$\frac{p}{A} = \frac{\varepsilon_0}{4c}\int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dk' \,\text{sgn}(k)\left[\mathbf{E}_k \times \left(\mathbf{n}_z \times \mathbf{E}_{k'}\right)\int_{-\infty}^{+\infty} e^{i\left(\Psi + \Psi'\right)} dz + \mathbf{E}_k \times \left(\mathbf{n}_z \times \mathbf{E}^*_{k'}\right)\int_{-\infty}^{+\infty} e^{i\left(\Psi - \Psi'\right)} dz + \text{c.c.}\right]. \quad (**)$$

Now we may readily work out both internal integrals, which do not depend on $\mathbf{E}_k$:[185]

[184] Cf. the derivation of Eq. (7.42) in Sec. 7.2.
[185] See, e.g. MA Eq. (14.4).

$$\int_{-\infty}^{+\infty} e^{i(\Psi_k \pm \Psi_{k'})} dz \equiv \exp\{-i(\omega_k \pm \omega_{k'})t\} \int_{-\infty}^{+\infty} e^{i(k \pm k')z} dz$$
$$= 2\pi \exp\{-i(\omega_k \pm \omega_{k'})t\}\delta(k \pm k') = 2\pi \times \begin{cases} \exp\{-2i\omega_k t\}\delta(k+k'), & \text{for the upper sign,} \\ \delta(k-k'), & \text{for the lower sign,} \end{cases}$$

so the integral over k' in Eq. (**) is also elementary, and that formula reduces to

$$\frac{p}{A} = \frac{\pi\varepsilon_0}{2c} \int_{-\infty}^{+\infty} \left[-\mathbf{E}_k \times \left(\mathbf{n}_z \times \mathbf{E}_{-k} \right) \exp\{-2i\omega_k t\} + \mathbf{E}_k \times \left(\mathbf{n}_z \times \mathbf{E}_k^* \right) + \text{c.c.} \right] \text{sgn}(k) dk \ .$$

Taking also into account that in a plane wave propagating along the z-axis, both vectors $\mathbf{E}_k$ and $\mathbf{E}_{-k}$ are perpendicular to the vector $\mathbf{n}_z$,[186] we may rewrite this expression in a simpler form:

$$\frac{p}{A} = -\frac{\pi\varepsilon_0}{2c} \int_{-\infty}^{+\infty} \left[\mathbf{E}_k \cdot \mathbf{E}_{-k} \exp\{-2i\omega_k t\} + \text{c.c.} \right] \text{sgn}(k) dk + \frac{\pi\varepsilon_0}{2c} \int_{-\infty}^{+\infty} \left[\mathbf{E}_k \cdot \mathbf{E}_k^* + \text{c.c.} \right] \text{sgn}(k) dk \ . \quad (***)$$

Since with our definition of ω_k, it is an even function of k,[187] the first term on the right-hand side is an integral, in symmetric limits, of an odd function of k, and hence is equal to zero. For the second term, this is not necessarily true.[188] In that term, the scalar product $\mathbf{E}_k \cdot \mathbf{E}_k^*$ equals just $E_k E_k^* \equiv |E_k|^2$,[189] so taking into account the (equal) complex conjugate term, we finally get

$$\frac{p}{A} = \frac{\pi\varepsilon_0}{c} \int_{-\infty}^{+\infty} E_k E_k^* \ \text{sgn}(k) dk \ .$$

Note that in contrast with the wave packet's energy (whose calculation was the subject of Problem 7.9), the cancellation of the rapidly oscillating terms in Eq. (***) is exact for the packet of any form.

Problem 9.42. Calculate the forces exerted on well-conducting walls of a waveguide with a rectangular ($a \times b$) cross-section, by a wave propagating along it in the fundamental (H_{10}) mode. Give an interpretation of the results.

Solution: According to Eq. (7.131) of the lecture notes (with $n = 1$ and $m = 0$), and Eqs. (7.137) – (7.138), the broader walls of the waveguide (in Fig. 7.22, with $y = 0$, b) are exposed to both the electric and magnetic fields of the wave, with complex amplitudes

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[186] A word of caution: the directions of the vectors $\mathbf{E}_k$ and $\mathbf{E}_{-k}$ (in the plane normal to $\mathbf{n}_z$) do not necessarily coincide (see, e.g., the last part of Sec. 7.1 of the lecture notes), so using their scalar product is important.

[187] An additional question for the reader: what would the initial expression for $\mathbf{E}(\mathbf{r}, t)$ describe if we took $\omega_k = ck$, without the modulus sign?

[188] For example, if the wave packet is narrow in the momentum space, and propagates in the direction of larger z, the only significant amplitudes $|\mathbf{E}_k|$ correspond to values of k that are close to a certain positive value k_0.

[189] If there is any doubt in this statement: since the complex-number vector $\mathbf{E}_k$ is normal to the propagation axis z, we may represent it as $\mathbf{n}_x E_x + \mathbf{n}_y E_y = \mathbf{n}_x A_x \exp\{i\varphi_x\} + \mathbf{n}_y A_y \exp\{i\varphi_y\}$, where $E_{x,y}$ may be still complex but $A_{x,y}$ and $\varphi_{x,y}$ are real. Now scalar-multiplying this vector by its complex conjugate, we get $A_x^2 + A_y^2$, i.e. the same result if we calculated $|E_k|^2 \equiv |E_x|^2 + |E_y|^2$.

$$E_x = 0, \quad E_y = i\frac{ka}{\pi} Z H_l \sin\frac{\pi x}{a}, \quad E_z = 0; \quad H_x = -i\frac{k_z a}{\pi} H_l \sin\frac{\pi x}{a}, \quad H_y = 0, \quad H_z = H_l \cos\frac{\pi x}{a};$$

while at the narrower walls ($x = 0$, a), only the longitudinal magnetic field is nonvanishing, with the complex amplitude equal to $\pm H_l$. The wave vector components participating in these formulas are related by Eqs. (7.132); for the H_{10} mode (with $k_x = \pi/a$, $k_y = 0$), they may be rewritten as

$$\left(\frac{ka}{\pi}\right)^2 - \left(\frac{k_z a}{\pi}\right)^2 = 1 .$$

Plugging these relations into Eqs. (9.239) and (9.241), for the average pressure on the walls in terms of the longitudinal field's amplitude H_l, we get

$$\overline{\frac{dF}{dA}}\bigg|_{y=0,b} = \frac{\mu_0}{4}\left|H_l\right|^2\left(\cos^2\frac{\pi x}{a} - \sin^2\frac{\pi x}{a}\right), \qquad \overline{\frac{dF}{dA}}\bigg|_{x=0,a} = \frac{\mu_0}{4}\left|H_l\right|^2,$$

where the extra factors ½ came from the averaging over time.

So, the pressure on the narrower walls is always positive, due to the magnetic field's effect, while that on the broader walls is negative in their middle parts (for example, at $x = a/2$ where $\sin^2(\pi x/a) = 1$, $\cos^2(\pi x/a) = 0$), due to the electric field effect (attraction). On average over x, these two pressures on the broader wall compensate each other, so the net forces exerted on them are equal to zero.

These results may be readily explained in terms of the fundamental relation between the flux $c\mathbf{g} = \mathbf{S}/c$ of a plane wave's momentum and its pressure on the mirror that reflects it. Indeed, as was discussed in the model solution of Problem 7.25, the H_{10} wave may be represented as a sum of two plane waves with wave vectors $\mathbf{k}_\pm = \{\pm\pi/a, 0, k_z\}$, which are reflected from the narrower walls of the waveguide. These reflections create an outward pressure on the walls, similar to that at the normal incidence – see Eq. (9.245) of the lecture notes and its discussion. On the other hand, since the H_{10} waves have $k_y = 0$, they are not reflected from the broader walls of the waveguide, and as a result, do not exert a net force on these walls.

Chapter 10. Radiation by Relativistic Charges

Problem 10.1. Derive Eqs. (10.10) of the lecture notes from Eqs. (10.1) by a direct (but careful!) integration.

Solution: Let us first recognize the error that led to the wrong results (10.4). A direct substitution of Eq. (10.1b) into Eq. (10.1a) yields

$$\phi(\mathbf{r},t)=\frac{q}{4\pi\varepsilon_0}\int d^3r''\frac{1}{\left|\mathbf{r}-\mathbf{r}''\right|}\delta(\mathbf{r}''-\mathbf{r}'). \tag{*}$$

The naïve integration over $\mathbf{r}''$, giving the first of Eqs. (10.4), ignores the fact that according to Eq. (10.1), the particle's position $\mathbf{r}'$ in the integral has to be taken at the retarded time $t - R/c \equiv t - |\mathbf{r} - \mathbf{r}'|/c$, i.e. at the point $\mathbf{r}'$ that depends on the integration variable $\mathbf{r}''$ as well.

In order to overcome this difficulty, let us rewrite the first of Eqs. (10.1a) in a different but equivalent form by introducing an additional integration over the auxiliary time argument t'', and compensating it with a delta function at the appropriate point $t'' = t - |\mathbf{r} - \mathbf{r}''|/c$:

$$\phi(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int d^3r''\int dt''\frac{\rho(\mathbf{r}'',t'')}{\left|\mathbf{r}-\mathbf{r}''\right|}\delta\left(t''-t+\frac{\left|\mathbf{r}-\mathbf{r}''\right|}{c}\right).$$

Now changing the integration order, and plugging in the first of Eqs. (10.1b), we get

$$\phi(\mathbf{r},t)=\frac{q}{4\pi\varepsilon_0}\int dt''\int d^3r''\frac{\delta\left[\mathbf{r}''-\mathbf{r}'(t'')\right]}{\left|\mathbf{r}-\mathbf{r}''\right|}\delta\left(t''-t+\frac{\left|\mathbf{r}-\mathbf{r}''\right|}{c}\right).$$

The internal integral may be limited to the vicinity of the point $\mathbf{r}'(t'')$, where it includes only a standard 3D delta function (multiplied by a smooth[190] function of $\mathbf{r}''$), and may be taken as usual, leaving us with

$$\phi(\mathbf{r},t)=\frac{q}{4\pi\varepsilon_0}\int dt''\frac{1}{\left|\mathbf{r}-\mathbf{r}'(t'')\right|}\delta\left(t''-t+\frac{\left|\mathbf{r}-\mathbf{r}'(t'')\right|}{c}\right). \tag{**}$$

This is still not a standard integral of a delta function but at least it is now one-dimensional. To work it out, we may use the following property of the 1D delta function: if a monotonic scalar function $f(\xi)$ at some point ξ_0 has a non-zero derivative $df/d\xi$, then

$$\delta(f-f_0)=\frac{\delta(\xi-\xi_0)}{\left|df/d\xi\right|_{\xi=\xi_0}},\qquad \text{where } f_0\equiv f(\xi_0). \tag{***}$$

The proof of this relation is easy: it is sufficient to integrate the right-hand side of Eq. (***) over any interval of f, by using the standard chain rule:

[190] The function is singular only if $\mathbf{r} = \mathbf{r}'$, i.e. if the field observation point exactly coincides with the charge's position – the case we are not interested in.

$$\int \frac{\delta(\xi-\xi_0)}{\left|df/d\xi\right|_{\xi=\xi_0}}df = \int \frac{\delta(\xi-\xi_0)}{\left|df/d\xi\right|_{\xi=\xi_0}}\frac{df}{d\xi}d\xi\ .$$

The last form of this expression shows that if the interval of integration over ξ contains the point ξ_0 (i.e. the corresponding interval over f contains the point f_0), then the integral equals 1,[191] while if the interval does not include this point, then the integral is equal to zero. Since this property is essentially the definition of the delta function $\delta(f-f_0)$,[192] it proves Eq. (***).

Let us apply this identity to our case, taking $\xi \equiv t''$, $f(\xi) \equiv f(t'') \equiv t'' - t + |\mathbf{r} - \mathbf{r}'(t'')|/c$, and the point $\xi_0 \equiv (t'')_0$ to correspond to the retarded time t_{ret}, i.e. to the physically-meaningful root (with $t_{\text{ret}} < t$) of Eq. (10.5):

$$t_{\text{ret}} - t + \frac{\left|\mathbf{r} - \mathbf{r}'(t_{\text{ret}})\right|}{c} = 0\ .$$

By our definition of the function $f(t'')$, at this point, it turns to zero, i.e. $f_0 = 0$. The derivative df/dt'' at this point may be calculated exactly as was done in Eqs. (10.11)-(10.16) of the lecture notes (just replacing the notation t_{ret} with t'', and returning to t_{ret} after the calculation), giving

$$\left|\frac{df}{dt''}\right|_{t''=t_{\text{ret}}} = \frac{1}{\partial t_{\text{ret}}/\partial t} = (1-\boldsymbol{\beta}\cdot\mathbf{n})_{\text{ret}} > 0\ .$$

Plugging Eq. (***), with these specifications, into Eq. (**), we get

$$\phi(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\int dt'' \frac{\delta\left[t''-t+\left|\mathbf{r}-\mathbf{r}'(t'')\right|/c\right]}{\left|\mathbf{r}-\mathbf{r}'(t'')\right|} \equiv \frac{q}{4\pi\varepsilon_0}\int dt'' \frac{\delta(f-f_0)}{\left|\mathbf{r}-\mathbf{r}'(t'')\right|}$$

$$= \frac{q}{4\pi\varepsilon_0\left|df/dt''\right|_{t''=t_{\text{ret}}}}\int dt'' \frac{\delta(t''-t_{\text{ret}})}{\left|\mathbf{r}-\mathbf{r}'(t'')\right|} = \frac{q}{4\pi\varepsilon_0(1-\boldsymbol{\beta}\cdot\mathbf{n})_{\text{ret}}}\int dt'' \frac{\delta(t''-t_{\text{ret}})}{\left|\mathbf{r}-\mathbf{r}'(t'')\right|}.$$

Since t_{ret} does not depend on t'', this is a standard integral of a delta function multiplied by a smooth function,[193] so it may be evaluated in a regular way, giving

$$\phi(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0(1-\boldsymbol{\beta}\cdot\mathbf{n})_{\text{ret}}}\frac{1}{\left|\mathbf{r}-\mathbf{r}'(t_{\text{ret}})\right|} \equiv \frac{q}{4\pi\varepsilon_0(1-\boldsymbol{\beta}\cdot\mathbf{n})_{\text{ret}}}\frac{1}{R_{\text{ret}}}\ .$$

This is Eq. (10.10a) of the lecture notes; the derivation of Eq. (10.10b) may be carried out absolutely similarly.

<u>Problem 10.2</u>. Derive the radiation-related parts of Eqs. (10.19)-(10.20) of the lecture notes from the Liénard-Wiechert potentials (10.10) by direct differentiation.

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[191] The sign of the derivative $df/d\xi$ at the point ξ_0 does not affect the result, because the delta function behaves as a symmetric one (i.e. may be considered as an approximation of a very short symmetric pulse).

[192] See, e.g., MA Eq. (14.1).

[193] Actually, the same approach might be applied directly to Eq. (*), but due to the 3D character of the involved delta function, the necessary calculations are somewhat longer.

Solution: We need to plug Eqs. (10.10),

$$\phi(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\text{ret}}, \qquad \mathbf{A}(\mathbf{r},t) = \frac{\mu_0 qc}{4\pi}\left[\frac{\boldsymbol{\beta}}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\text{ret}},$$

into the general Eqs. (6.7)

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi, \qquad \mathbf{B} = \nabla\times\mathbf{A}, \tag{*}$$

and perform these temporal and spatial differentiations. However, since we need to calculate only the radiative parts of the fields, which dominate at $R \to \infty$, at these operations we may skip the derivatives of the $(1/R_{\text{ret}})$ factor. Indeed, let us consider, for example, the derivative[194]

$$\frac{\partial\phi(\mathbf{r},t)}{\partial t} = \frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial t}\left[\frac{1}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\text{ret}}.$$

As was discussed in Sec. 10.1 of the lecture notes, at the fixed radius vector $\mathbf{r}$ of the observation point, the retarded time t_{ret} is a unique function of t, so for this partial differentiation we may use the standard chain rule:

$$\frac{\partial\phi}{\partial t} = \frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial t_{\text{ret}}}\left[\frac{1}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\text{ret}}\frac{\partial t_{\text{ret}}}{\partial t}, \tag{**}$$

where the last derivative is given by Eq. (10.16) of the lecture notes:

$$\frac{\partial t_{\text{ret}}}{\partial t} = \left(\frac{1}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\right)_{\text{ret}}.$$

Now, by taking the first derivative in Eq. (**) exactly, at the second step using Eq. (10.13), we get

$$\frac{\partial}{\partial t_{\text{ret}}}\left[\frac{1}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\text{ret}} = -\frac{\partial(1-\boldsymbol{\beta}\cdot\mathbf{n})_{\text{ret}}/\partial t_{\text{ret}}}{\left[R(1-\boldsymbol{\beta}\cdot\mathbf{n})^2\right]_{\text{ret}}} - \frac{\partial R_{\text{ret}}/\partial t_{\text{ret}}}{\left[R^2(1-\boldsymbol{\beta}\cdot\mathbf{n})\right]_{\text{ret}}} = \left[\frac{\dot{\boldsymbol{\beta}}\cdot\mathbf{n}}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} + \frac{c\boldsymbol{\beta}\cdot\mathbf{n}}{R^2(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\text{ret}}. \tag{***}$$

This expression shows that at $\mathbf{r} \to \infty$, i.e. at $R_{\text{ret}} \equiv |\mathbf{r} - \mathbf{r}'(t_{\text{ret}})| \to \infty$, namely at

$$R_{\text{ret}} >> c\left|\frac{\boldsymbol{\beta}\cdot\mathbf{n}}{\dot{\boldsymbol{\beta}}\cdot\mathbf{n}}(1-\boldsymbol{\beta}\cdot\mathbf{n})\right|_{\text{ret}} \sim c\tau,$$

where τ is the time scale of the particle's acceleration in the observer's direction,[195] the second term in Eq. (**), which was the result of differentiation of $(1/R_{\text{ret}})$, may be neglected. Similar estimates are valid for all partial derivatives contributing to the fields (*). So, at the differentiations required for the radiation field calculation, the fraction $1/R_{\text{ret}}$ may be indeed treated as a constant.

[194] Per Eq. (*), we do not need this particular derivative for the field calculation, but it is the simplest of all the derivatives of this type while being conceptually similar to them.

[195] Note that this condition is never fulfilled at zero acceleration. This conclusion is in accordance with the result of the analysis, in Sec. 9.5 of the lecture notes, of the fields induced by a uniformly moving particle, which are essentially its Lorentz-transformed Coulomb field rather than radiation – see Eqs. (9.139)-(9.140).

Upon this simplification, the required differentiation of Cartesian components of the Liénard-Wiechert potentials,

$$\phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\text{ret}}, \qquad A_j = \frac{\mu_0 qc}{4\pi}\left[\frac{\beta_j}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\text{ret}} \equiv \frac{q}{4\pi\varepsilon_0 c}\left[\frac{\beta_j}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\text{ret}}, \qquad \text{with } j = 1, 2, 3,$$

becomes easy. For example,

$$\frac{\partial A_j}{\partial t} = \frac{q}{4\pi\varepsilon_0 c}\left\{\frac{\partial(\beta_j)_{\text{ret}}/\partial t_{\text{ret}}}{[R(1-\boldsymbol{\beta}\cdot\mathbf{n})]_{\text{ret}}} - \frac{(\beta_j)_{\text{ret}}\partial(1-\boldsymbol{\beta}\cdot\mathbf{n})_{\text{ret}}/\partial t_{\text{ret}}}{[R(1-\boldsymbol{\beta}\cdot\mathbf{n})^2]_{\text{ret}}}\right\}\frac{\partial t_{\text{ret}}}{\partial t} = \frac{q}{4\pi\varepsilon_0 c}\left[\frac{\dot{\beta}_j}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} + \frac{\beta_j(\dot{\boldsymbol{\beta}}\cdot\mathbf{n})}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\text{ret}}.$$

Similar spatial differentiations are enabled by the relation similar to Eq. (10.16), cited in a footnote in Sec. 10.1 of the lecture notes, whose j^{th} Cartesian component is

$$\frac{\partial t_{\text{ret}}}{\partial r_j} = -\left[\frac{n_j}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\text{ret}}.$$

For example,

$$\frac{\partial\phi}{\partial r_j} = \frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial t_{\text{ret}}}\left[\frac{1}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\text{ret}}\frac{\partial t_{\text{ret}}}{\partial r_j} = -\frac{q}{4\pi\varepsilon_0 c}\left[\frac{n_j(\dot{\boldsymbol{\beta}}\cdot\mathbf{n})}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\text{ret}}.$$

From these results, for the j^{th} component of the electric field, we get

$$E_j = -\frac{\partial A_j}{\partial t} - \frac{\partial\phi}{\partial r_j} = \frac{q}{4\pi\varepsilon_0 c}\left[-\frac{\dot{\beta}_j}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} + \frac{(n_j-\beta_j)\dot{\boldsymbol{\beta}}\cdot\mathbf{n}}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\text{ret}} \equiv \frac{q}{4\pi\varepsilon_0 c}\left[\frac{(\mathbf{n}-\boldsymbol{\beta})_j(\dot{\boldsymbol{\beta}}\cdot\mathbf{n}) - \dot{\beta}_j(1-\boldsymbol{\beta}\cdot\mathbf{n})}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\text{ret}}.$$

But this is exactly the result given by the radiative part of Eq. (10.19):

$$\mathbf{E} \equiv \frac{q}{4\pi\varepsilon_0 c}\left[\frac{\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\text{ret}}.$$

This fact may be verified either by the Cartesian components or (even simpler) by using, for the numerator of the last fraction, the *bac minus cab* rule,[196] and then taking into account that **n** is a unit vector, so $\mathbf{n}\cdot\mathbf{n} = 1$:

$$\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\} = (\mathbf{n}-\boldsymbol{\beta})(\mathbf{n}\cdot\dot{\boldsymbol{\beta}}) - \dot{\boldsymbol{\beta}}\,[\mathbf{n}\cdot(\mathbf{n}-\boldsymbol{\beta})] \equiv (\mathbf{n}-\boldsymbol{\beta})(\dot{\boldsymbol{\beta}}\cdot\mathbf{n}) - \dot{\boldsymbol{\beta}}\,(1-\boldsymbol{\beta}\cdot\mathbf{n}).$$

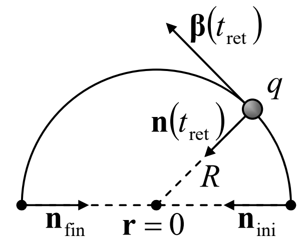
Evidently, the j^{th} component of this vector is exactly the numerator in our result for E_j.

An absolutely similar calculation of the derivatives $\partial A_j/\partial r_{j'}$ and then the Cartesian components of the vector $\mathbf{B} = \nabla\times\mathbf{A}$,

$$B_{j''} = \left(\frac{\partial A_j}{\partial r_{j'}} - \frac{\partial A_{j'}}{\partial r_j}\right)\varepsilon_{jj'j''},$$

[196] If you still do not remember it, please consult MA Eq. (7.5) again.

shows that the radiative part of the magnetic field is indeed related to the similar part of the electric field by Eq. (10.20): $\mathbf{B} = \mathbf{n}_{\text{ret}}\times\mathbf{E}/c$, i.e. by the same equality as in a plane wave.



Problem 10.3. A point charge q that was in a stationary position on a circle of radius R is carried over, along the circle, to the opposite position on the same diameter (see the figure on the right) as fast as only physically possible, and then is kept steady at this new position. Calculate and sketch the time dependence of the electric field $\mathbf{E}$ at the center of the circle.

Solution: In this simple case, the distance R between the field observation point (which we may take for $\mathbf{r} = 0$) and the charge's position $\mathbf{r}'$ remains constant (equal to the circle's radius), so the delay between the retarded time t_{ret} (defined by Eq. (10.5) of the lecture notes) and the field observation time t remains constant:

$$t - t_{\text{ret}} = \frac{R}{c}.$$

Let us take the retarded time of the beginning of the charge's move for zero (with this and all other times as measured in the lab reference frame). The length of the semi-circle's arc is πR, and the highest possible speed of charge motion is $u \approx c$; hence the time of its arrival at the final point is $\pi R/u \approx \pi R/c$. The corresponding interval of the field observation times t is

$$t_{\text{ini}} < t < t_{\text{fin}}, \qquad \text{with } t_{\text{ini}} = \frac{R}{c}, \qquad t_{\text{fin}} = \frac{\pi R}{c} + \frac{R}{c} \equiv (\pi + 1)\frac{R}{c}. \tag{*}$$

Since the charge does not move at $t_{\text{ret}} < 0$ and at $t_{\text{ret}} > \pi R/c$, the field observed outside of the interval (*) may be calculated just from the Coulomb law:

$$\mathbf{E}(0,t) = \frac{q}{4\pi\varepsilon_0 R^2} \times \begin{cases} \mathbf{n}_{\text{ini}}, & \text{for } t < t_{\text{ini}}, \\ \mathbf{n}_{\text{fin}}, & \text{for } t > t_{\text{fin}}, \end{cases} \tag{**}$$

where $\mathbf{n} \equiv \mathbf{R}/R$ is the unit vector pointing from the charge's position to the observation point – see the figure above. Within the interval (*), $\mathbf{E}(0, t)$ has to be found from the retarded-field formula (10.19):[197]

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{n} - \boldsymbol{\beta}}{\gamma^2 (1 - \boldsymbol{\beta}\cdot\mathbf{n})^3 R^2} + \frac{\mathbf{n}\times\{(\mathbf{n} - \boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}}{(1 - \boldsymbol{\beta}\cdot\mathbf{n})^3 cR}\right]_{\text{ret}}, \tag{***}$$

where the index "ret" marks the variable values at the retarded time $t_{\text{ret}} = t - R/c$. Since the Lorentz factor $\gamma \equiv (1 - \beta^2)^{-1/2}$ diverges at $\beta \to 1$, in our case, the first term in the square brackets is negligible. In the second term, the scalar product $\boldsymbol{\beta}\cdot\mathbf{n}$ in the denominator also vanishes, because at any instant t_{ret} of the circular motion of the charge, the vectors $\boldsymbol{\beta}(t_{\text{ret}})$ and $\mathbf{n}(t_{\text{ret}})$ are mutually perpendicular – see the figure above. To calculate the numerator of that term, we may use the well-known formula for the acceleration at a uniform circular motion:

$$\dot{\boldsymbol{\beta}} \equiv \frac{\dot{\mathbf{u}}}{c} = \frac{\mathbf{n}u^2/R}{c} \approx \frac{\mathbf{n}c}{R}.$$

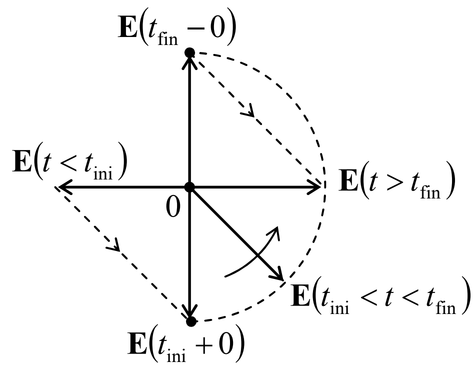
[197] This formula, with $\boldsymbol{\beta} \equiv 0$ and $\gamma = 1$, of course, yields Eqs. (**) as well.

With this, and taking into account (twice) that $\mathbf{n}\times\mathbf{n} = 0$, and also that $\mathbf{n}\cdot\mathbf{n} = 1$, the double vector product in the second term may be simplified as follows:

$$\mathbf{n}\times\left\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}=\frac{c}{R}\mathbf{n}\times\left\{(\mathbf{n}-\boldsymbol{\beta})\times\mathbf{n}\right\}=\frac{c}{R}\mathbf{n}\times\left\{(-\boldsymbol{\beta})\times\mathbf{n}\right\}=-\frac{c}{R}\boldsymbol{\beta},$$

so Eq. (***) is reduced to

$$\mathbf{E}(0,t)=-\frac{q}{4\pi\varepsilon_0 R^2}\boldsymbol{\beta}(t_{\text{ret}})\equiv-\frac{q}{4\pi\varepsilon_0 R^2}\boldsymbol{\beta}\left(t-\frac{R}{c}\right),\qquad \text{for } t_{\text{ini}}<t<t_{\text{fin}}. \qquad (****)$$



Since in our approximation $u = c$, the length of the vector $\boldsymbol{\beta}$ equals 1, the magnitude of this field stays constant, equal to that of the static Coulomb values (**), but its direction rotates in time as the figure on the right shows. The instant leaps of the field vector at moments t_{ini} and t_{fin}, shown by dashed arrows, are not quite surprising, taking into account that the problem assignment implies an instant (and hence not quite physical) acceleration/deceleration of the charge at these moments.

Note also that while Eq. (****) results from the second ("radiative") term of Eq. (***), the field it gives in this particular case, turns out to be proportional to $1/R^2$ – in contrast to the real radiation fields, which are proportional to $1/R$. This is also not quite surprising, because according to the discussion in Sec. 10.3 of the lecture notes (see, e.g., Fig. 10.7a), the radiation of an ultra-relativistic particle at its circular motion is concentrated within a very narrow cone centered to the vector $\boldsymbol{\beta}(t_{\text{ret}})$, and is always directed out of the circle – not toward its center.

Problem 10.4. Express the instantaneous power of electromagnetic radiation by a relativistic particle with electric charge q and rest mass m, moving with velocity $\mathbf{u}$, via the Lorentz force $\mathbf{F}$ providing its acceleration.

Solution: In the solution of Problem 9.32, we derived the following two formulas relating $\mathbf{F}$ and the reduced velocity $\boldsymbol{\beta}\equiv\mathbf{u}/c$:

$$\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}=\frac{\boldsymbol{\beta}\cdot\mathbf{F}}{mc\gamma^3},\qquad \dot{\boldsymbol{\beta}}=\frac{\mathbf{F}-\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{F})}{mc\gamma}.$$

Plugging them into the last form of the Liénard extension (10.37), we get:[198]

$$\begin{aligned}\mathscr{P}&=\frac{Z_0q^2}{6\pi}\frac{\gamma^4}{(mc\gamma)^2}\left\{\left[\mathbf{F}-\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{F})\right]^2+\frac{1}{\gamma^2}(\boldsymbol{\beta}\cdot\mathbf{F})^2\right\}\equiv\frac{Z_0q^2}{6\pi}\left(\frac{\gamma}{mc}\right)^2\left\{\begin{matrix}\left[F^2+\beta^2(\boldsymbol{\beta}\cdot\mathbf{F})^2-2\mathbf{F}\cdot\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{F})\right]\\+\left(1-\beta^2\right)(\boldsymbol{\beta}\cdot\mathbf{F})^2\end{matrix}\right\}\\&\equiv\frac{Z_0q^2}{6\pi}\left(\frac{\gamma}{mc}\right)^2\left[F^2-(\boldsymbol{\beta}\cdot\mathbf{F})^2\right].\end{aligned}\qquad (*)$$

This result may be expressed directly via the fields driving the particle. Since the magnetic part of the Lorentz force is always normal to the vector $\boldsymbol{\beta}\equiv\mathbf{u}/c$, it drops out of the second term, and we get

[198] An additional exercise: obtain this result by transforming Eq. (10.39) of the lecture notes.

$$\mathscr{P} = \frac{Z_0 q^4}{6\pi}\left(\frac{\gamma}{mc}\right)^2 \left[(\mathbf{E} + \mathbf{u}\times\mathbf{B})^2 - (\mathbf{u}\cdot\mathbf{E}/c)^2\right].$$

Alternatively, this expression may be obtained by plugging Eqs. (9.125) and (9.145) into the first (4-vector) form of Eq. (10.36) – a useful exercise, highly recommended to the reader.

Problem 10.5. A relativistic particle with rest mass m and electric charge q, initially at rest, is accelerated by a constant force $\mathbf{F}$ until it reaches a certain velocity u and then is left to move by inertia. Calculate the total energy radiated during the acceleration.

Solution: Evidently, the particle so accelerated moves along a straight line. The electromagnetic wave power radiated at this linear acceleration is given by Eq. (10.40) of the lecture notes:

$$\mathscr{P} = \frac{Z_0 q^2}{6\pi m^2 c^2}\left(\frac{dp}{dt_{\text{ret}}}\right)^2,$$

where the particle's momentum p and the radiation emission time t_{ret} are both measured in the lab reference frame. According to Eq. (9.144), $dp/dt_{\text{ret}} = F$, and the power may be expressed simply as

$$\mathscr{P} = \frac{Z_0 q^2 F^2}{6\pi m^2 c^2},$$

so it is time-independent through the acceleration process if F remains constant.[199] Hence the total radiated energy is

$$\mathscr{E}_{\text{rad}} = \frac{Z_0 q^2 F^2}{6\pi m^2 c^2}\Delta t, \qquad (*)$$

where $\Delta t = p_{\max}/F$ is the acceleration time. The maximum value $p_{\max}$ of the particle's momentum may be expressed via the given final value u of its velocity by using Eq. (9.70)

$$p_{\max} = m\gamma u \equiv m\frac{u}{(1-u^2/c^2)^{1/2}},$$

so Eq. (*) yields

$$\mathscr{E}_{\text{rad}} = \frac{Z_0 q^2 F}{6\pi m c^2}\frac{u}{(1-u^2/c^2)^{1/2}}.$$

This result shows that at a fixed final velocity, the radiated energy increases proportionally to the applied force F, i.e. a faster acceleration yields more radiation.

Problem 10.6. A charged relativistic particle with an initial momentum $\mathbf{p}_0$ flies ballistically from a free-space region into a region of a constant, uniform electric field $\mathbf{E}$, whose force is directed opposite to $\mathbf{p}_0$. Calculate the energy radiated by the particle during its motion in the field, assuming that it is small in comparison with the particle's initial kinetic energy.

[199] The same result readily follows from the last form of Eq. (*) in the previous problem's solution because for the 1D motion, $F^2 - (\boldsymbol{\beta}\cdot\mathbf{F})^2 = (1-\beta^2)F^2 \equiv F^2/\gamma^2$.

Solution: Neglecting the radiation energy loss, we may use Eq. (9.144) of the lecture notes to write the following scalar equation of motion for the only non-zero Cartesian component of the particle's momentum (in the lab reference frame):

$$\frac{dp}{dt} = -|qE| = \text{const},$$

giving $p(t) = p_0 - |qE|(t - t_0)$. From here, the time interval during which the initial momentum p_0 is exactly reversed is

$$\Delta t = \frac{2p_0}{|qE|}.$$

Since the motion of the particle is symmetric with respect to the momentum-reversal instant, the particle leaves the field-filled region of space at the end of this interval. Now using the last form of Eq. (10.40) for the radiation power, we get

$$\mathscr{P} = \frac{Z_0 q^2}{6\pi m^2 c^2}\left(\frac{dp}{dt}\right)^2 = \frac{Z_0 q^2}{6\pi m^2 c^2}(qE)^2.$$

(Alternatively, this expression follows from the last formula of the model solution of Problem 4, with **B** = 0 and **u**||**E**.)

Due to the constancy of this expression in time, we may calculate the full radiated energy (i.e. the particle's energy loss) just by multiplying the power by the interval Δt:

$$-\Delta\mathscr{E} = \mathscr{P}\Delta t = \frac{Z_0 q^2}{6\pi m^2 c^2}(qE)^2 \frac{2p_0}{|qE|} \equiv \frac{Z_0 q^2 p_0 |qE|}{3\pi m^2 c^2}.$$

As in all situations with linearly-accelerated particles, for all realistic values of parameters, this energy loss is much smaller than the initial energy of the particle – see the discussion at the end of Sec. 10.2 of the lecture notes.

Note that as a matter of principle, this problem could be solved using the Liénard extension (10.37). However, since we are discussing a relativistic particle, the factors β and γ participating in that formula are somewhat involved functions of time – see Sec. 9.6(ii) of the lecture notes. So, we are lucky that Eqs. (9.144) and (10.40) both involve the relativistic momentum **p** in such a simple form.

Problem 10.7. Calculate

(i) the instantaneous power, and
(ii)* the power spectrum

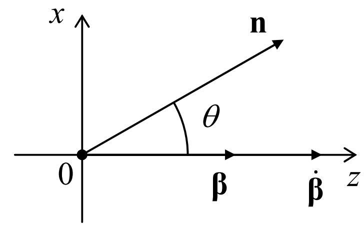
of the radiation emitted, into a unit solid angle, by a relativistic particle with charge q, performing 1D harmonic oscillations with frequency ω_0 and displacement amplitude a.

Solutions:

(i) For this task, we may directly use Eq. (10.30) of the lecture notes:

$$\frac{d\mathscr{P}}{d\Omega} = \frac{Z_0 q^2}{(4\pi)^2}\frac{\left|\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right|^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^5},$$

where all variables have to be evaluated at the retarded time t_{ret}. Selecting the reference frame as shown in Fig. 10.4a of the lecture notes (reproduced on the right), we may write



$$\mathbf{n} = \{\sin\theta,\, 0,\, \cos\theta\}, \qquad \mathbf{r}'(t_{\text{ret}}) = \{0,\, 0,\, a\cos\omega_0 t_{\text{ret}}\},$$
$$\boldsymbol{\beta}(t_{\text{ret}}) = \{0,\, 0, -\beta_0 \sin\omega_0 t_{\text{ret}}\}, \qquad \dot{\boldsymbol{\beta}}(t_{\text{ret}}) = \{0,\, 0, -\beta_0\omega_0 \cos\omega_0 t_{\text{ret}}\},$$

where β_0 is the oscillation amplitude of the particle's reduced velocity,

$$\beta_0 \equiv \frac{a\omega_0}{c}.$$

(Note that a and ω_0 have to satisfy the condition $\beta_0 \equiv a\omega_0/c \leq 1$.) Calculating the needed vector products,

$$\mathbf{n}\cdot\boldsymbol{\beta} = -\beta_0 \cos\theta \sin\omega_0 t_{\text{ret}}, \qquad (\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}} = \mathbf{n}\times\dot{\boldsymbol{\beta}} = \{0, \beta_0\omega_0 \sin\theta\cos\omega_0 t_{\text{ret}}, 0\},$$
$$\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right] = \beta_0\omega_0 \sin\theta\cos\omega_0 t_{\text{ret}}\{-\cos\theta, 0, \sin\theta\},$$
$$\left|\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right|^2 = (\beta_0\omega_0 \sin\theta\cos\omega_0 t_{\text{ret}})^2,$$

we get

$$\frac{d\mathscr{P}}{d\Omega} = \frac{Z_0 q^2}{(4\pi)^2}\frac{(\beta_0\omega_0 \sin\theta\cos\omega_0 t_{\text{ret}})^2}{(1+\beta_0\cos\theta\sin\omega_0 t_{\text{ret}})^5}.$$

For a non-relativistic oscillator ($\beta_0 << 1$), the denominator of the last fraction is close to 1, and our result, after integration over the full solid angle, is reduced to the Larmor law – see Eq. (8.27) of the lecture notes, with $Z = Z_0$, $v = c$, and

$$\ddot{p} = q\ddot{r}' = q\dot{u} = qc\dot{\beta} = qc\omega_0\beta_0\cos\omega_0 t\,.$$

In particular, the radiation intensity is proportional to $\sin^2\theta$, and its time dependence ($\mathscr{P} \propto \cos^2\omega_0 t$) shows that the fields of the radiated wave are sinusoidal functions of the observation time t, with the frequency ω_0. (Indeed, in the non-relativistic limit, the difference (10.5) between the times t and t_{ret} is virtually a constant.)

However, in case $\beta_0 \to 1$, our result describes periodic radiation peaks (two per the oscillation period, at $\sin\omega_0 t_{\text{ret}} = \pm 1$), very narrow both in time and space (i.e. in angle θ), emitted at the moments when the particle's speed approaches the speed of light. The angular structure of such a peak is the same as at the linear acceleration (see Sec. 10.2 of the lecture notes, in particular, Fig. 10.4b), while its exact temporal shape depends on whether we speak of the intensity as a function of t_{ret} or t, because exactly at their emission moments, the difference between these times is especially significant – see Eq. (10.16). Hence, this difference needs to be taken into account in the analysis of the radiation spectrum – see below.

(ii) Since the radiation is periodic, its spectrum may only consist of discrete harmonics of the motion frequency ω_0. Their amplitudes may be recovered from Eq. (10.61) of the lecture notes (which has been derived with the continuous spectrum in mind) by careful juggling delta functions,[200] but it is more prudent (and instructive) to recast all basic equations of the spectral theory outlined in Sec. 10.3

[200] Following this way may be recommended to the reader as an additional exercise.

for a periodic time dependence of the radiated field. In particular, Eq. (10.51) needs to be replaced with a sum over integer numbers (say, k – not to be confused with the wave number, which does not explicitly participate in this solution):

$$\mathbf{E}(t) = \sum_{k=-\infty}^{+\infty} \mathbf{E}_k \exp\{-ik\omega_0 t\}, \qquad \text{with } \mathbf{E}_{-k} = \mathbf{E}_k^*.$$

The magnetic field of the radiated wave (which, at a sufficiently large distance from the source, is locally planar) may be expanded into a similar series,

$$\mathbf{H}(t) = \sum_{k'=-\infty}^{+\infty} \mathbf{H}_{k'} \exp\{-ik'\omega_0 t\},$$

with the complex amplitudes $\mathbf{H}_{k'} = \mathbf{n}\times\mathbf{E}_{k'}/Z_0$, so at the time averaging over the oscillation period $\mathcal{T} = 2\pi/\omega_0$, the Poynting vector falls apart into a sum of independent harmonic components:

$$\overline{S_n} = \overline{E(t)H(t)} = \frac{1}{Z_0}\sum_{k,k'=-\infty}^{+\infty} E_k E_{k'} \overline{\exp\{-i(k+k')\omega_0 t\}} \equiv \frac{1}{Z_0}\sum_{k,k'=-\infty}^{+\infty} E_k E_{k'} \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} \exp\{-i(k+k')\omega_0 t\} dt$$

$$= \frac{1}{Z_0}\sum_{k,k'=-\infty}^{+\infty} E_k E_{k'} \delta_{k+k',0} = \frac{1}{Z_0}\sum_{k=-\infty}^{+\infty} E_k E_{-k} = \frac{1}{Z_0}\sum_{k=-\infty}^{+\infty} E_k E_k^* = \frac{2}{Z_0}\sum_{k=1}^{+\infty} |\mathbf{E}_k|^2.$$

Thus instead of Eq. (10.54) of the lecture notes, we can write the following expression for the average radiation power per unit solid angle:

$$\frac{d\overline{\mathscr{P}}}{d\Omega} = \sum_{k=1}^{\infty} \frac{d\overline{\mathscr{P}_k}}{d\Omega}, \qquad \text{with } \frac{d\overline{\mathscr{P}_k}}{d\Omega} = R^2 \overline{S_n} = \frac{2R^2}{Z_0}|\mathbf{E}_k|^2 . \tag{*}$$

The Fourier amplitudes $\mathbf{E}_k$ should be calculated from the reciprocal Fourier transform, for example over a symmetric time interval:

$$\mathbf{E}_k = \frac{1}{\mathcal{T}}\int_{-\mathcal{T}/2}^{+\mathcal{T}/2} \mathbf{E}(t)\exp\{ik\omega_0 t\} dt = \frac{\omega_0}{2\pi}\int_{-\pi/\omega_0}^{+\pi/\omega_0} \mathbf{E}(t)\exp\{ik\omega_0 t\} dt,$$

so

$$\frac{d\overline{\mathscr{P}_k}}{d\Omega} = \frac{R^2\omega_0^2}{2\pi^2 Z_0}\left|\int_{-\pi/\omega_0}^{+\pi/\omega_0} \mathbf{E}(t)\exp\{ik\omega_0 t\} dt\right|^2 .$$

Plugging in the $\mathbf{E}(t)$ given by the second (radiative) term of Eq. (10.19), and repeating the transformation from integration over t to that over t_{ret} discussed in Sec. 10.3 of the lecture notes, instead of Eq. (10.61), we get

$$\frac{d\overline{\mathscr{P}_k}}{d\Omega} = \frac{Z_0 q^2 k^2 \omega_0^4}{32\pi^4}\left|\int_{-\pi/\omega_0}^{+\pi/\omega_0} \left[\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\exp\left\{ik\omega_0\left(t - \frac{\mathbf{n}\cdot\mathbf{r}'}{c}\right)\right\}\right]_{\text{ret}} dt_{\text{ret}}\right|^2 . \tag{**}$$

As a sanity check, let us use this result to calculate the particle's energy loss per one period of motion, due to the radiation into a unit solid angle:

$$-\frac{d\mathscr{E}}{d\Omega} = \mathcal{T}\frac{d\overline{\mathscr{P}}}{d\Omega} = \frac{2\pi}{\omega_0}\frac{d\overline{\mathscr{P}}}{d\Omega} = \frac{2\pi}{\omega_0}\sum_{k=1}^{\infty}\frac{d\overline{\mathscr{P}_k}}{d\Omega}.$$

We may use this formula for a non-periodic motion as well, by following the limit $\mathcal{T} \to \infty$, i.e. $\omega_0 \to 0$, while keeping the observation frequency $\omega \equiv k\omega_0$ finite. In this limit, the sum over $k = \omega/\omega_0$ may be replaced with an integral, and we get

$$-\frac{d\mathscr{E}}{d\Omega} \to \frac{2\pi}{\omega_0}\int_0^{\infty}\frac{d\overline{\mathscr{P}_k}}{d\Omega}dk = \frac{2\pi}{\omega_0}\int_0^{\infty}\frac{d\overline{\mathscr{P}_{\omega/\omega_0}}}{d\Omega}d\left(\frac{\omega}{\omega_0}\right) \equiv \int_0^{\infty} I(\omega)d\omega, \qquad \text{where } I(\omega) \equiv \frac{2\pi}{\omega_0^2}\frac{d\overline{\mathscr{P}_{\omega/\omega_0}}}{d\Omega}.$$

By using Eq. (**), for the function $I(\omega)$ defined just like in the first of Eqs. (10.54), we get the expression,

$$I(\omega) \equiv \frac{2\pi}{\omega_0^2}\frac{d\overline{\mathscr{P}_{\omega/\omega_0}}}{d\Omega} = \frac{Z_0 q^2\omega^2}{16\pi^3}\left|\int_{-\pi/\omega_0}^{+\pi/\omega_0}\left[\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\exp\left\{i\omega\left(t - \frac{\mathbf{n}\cdot\mathbf{r}'}{c}\right)\right\}\right]_{\text{ret}}dt_{\text{ret}}\right|^2,$$

Which, at $\omega_0 \to 0$, duly tends to Eq. (10.61).

Now plugging into Eq. (**) the expressions for $\mathbf{n}$ and $\boldsymbol{\beta}$ discussed in Task (i) of this solution, performing the vector multiplications, and introducing the dimensionless integration variable $\xi \equiv \omega_0 t_{\text{ret}}$, we get

$$\frac{d\overline{\mathscr{P}_k}}{d\Omega} = \frac{Z_0 q^2\beta_0^2 k^2\omega_0^2\sin^2\theta}{32\pi^4}|I_k|^2,$$

where I_k is the following dimensionless integral:

$$I_k \equiv \int_{-\pi}^{+\pi}\sin\xi\exp\{ik(\xi - \alpha\cos\xi)\}d\xi, \quad \text{with} \quad \alpha \equiv \beta_0\cos\theta < 1.$$

Transforming this integral as

$$I_k = \frac{1}{2i}\int_{-\pi}^{+\pi}\left(e^{i\xi} - e^{-i\xi}\right)e^{ik\xi}e^{-ik\alpha\cos\xi}d\xi \equiv \frac{1}{2i}\int_{-\pi}^{+\pi}\left[e^{i(k+1)\xi} - e^{i(k-1)\xi}\right]e^{-ik\alpha\cos\xi}d\xi,$$

we may notice that the integrals of the imaginary parts of $\exp\{i(k \pm 1)\xi\} \equiv \cos(k \pm 1)\xi + i\sin(k \pm 1)\xi$, in our symmetric limits for ξ, vanish, so

$$\begin{aligned} I_k &= \frac{1}{2i}\int_{-\pi}^{+\pi}[\cos(k+1)\xi - \cos(k-1)\xi]\,e^{-ik\alpha\cos\xi}d\xi \\ &\equiv \frac{1}{i}\int_0^{+\pi}\cos(k+1)\xi\; e^{-ik\alpha\cos\xi}d\xi - \frac{1}{i}\int_0^{+\pi}\cos(k-1)\xi\; e^{-ik\alpha\cos\xi}d\xi. \end{aligned}$$

Apart from constant coefficients, these integrals are the standard integral representations of the Bessel functions of the first kind,[201] and we get

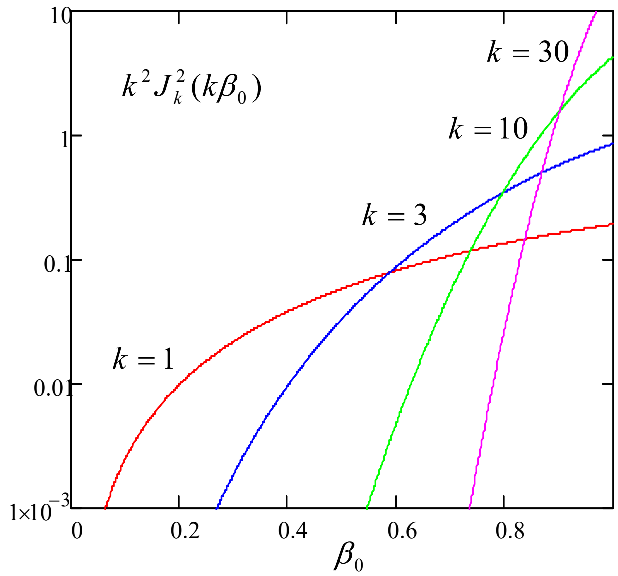
[201] See, e.g., MA Eq. (6.15a).

$$I_k = \frac{\pi}{i}\left[i^{k+1}J_{k+1}(k\alpha) - i^{k-1}J_{k-1}(k\alpha)\right] = i^k\pi\left[J_{k+1}(k\alpha) + J_{k-1}(k\alpha)\right] = i^k 2\pi\frac{k}{\alpha}J_k(k\alpha),$$

where, at the last step, the recurrence relation (2.142a) has been used. So, we get

$$\frac{d\overline{\mathscr{P}_k}}{d\Omega} = \frac{Z_0 q^2\omega_0^2}{8\pi^2}k^2 J_k^2(k\beta_0\cos\theta)\tan^2\theta, \qquad \text{for } k \geq 1. \qquad (***)$$

The figure on the right shows, on the appropriate semi-log scale, the dependence of the product $k^2 J_k^2(k\beta_0)$ on the particle's reduced velocity amplitude $\beta_0 = a\omega_0/c$, for several representative harmonics. As it follows from Eq. (***) and the plots, in the non-relativistic limit ($\beta_0 \to 0$), the first harmonic ($k = 1$) dominates, i.e. the radiation takes place only at the frequency of the particle's motion, and our result for it,



$$\frac{d\overline{\mathscr{P}_1}}{d\Omega} = \frac{Z_0 q^2\omega_0^2}{8\pi^2}J_1^2(\beta_0\cos\theta)\tan^2\theta$$

$$\to \frac{Z_0 q^2\omega_0^2}{8\pi^2}\left(\frac{\beta_0\cos\theta}{2}\right)^2\tan^2\theta \equiv \frac{Z_0 q^2\omega_0^2}{32\pi^2}\beta_0^2\sin^2\theta,$$

coincides with the time-averaged result of the first part of the problem, and hence complies with the non-relativistic Larmor formula (8.28).

On the other hand, when the maximum value β_0 of the normalized velocity of the particle approaches 1, i.e. the particle becomes ultra-relativistic during two short parts of each period of its motion, the intensity of high harmonics going in the direction of motion ($\cos\theta \approx 1$) grows rapidly, reflecting the narrow radiation cones generated during these small time intervals – see Fig. 10.4 and its discussion. (The additional factor $\tan^2\theta$ in Eq. (***) shows that these cones are hollow, just as in the constant-acceleration case.)

A good additional exercise: review these results for the case when the force driving the particle oscillation (rather than the particle displacement as above) is sinusoidal. The results should be different only in the relativistic case, due to the increase of the effective mass (9.71) of the particle at the periods of its fastest motion, and the corresponding reduction of its acceleration – cf. Problem 9.34.

<u>Problem 10.8</u>. Calculate and analyze the time dependence of the energy of a charged relativistic particle rotating in a constant and uniform magnetic field **B** and as a result, emitting the synchrotron radiation. Qualitatively, what is the particle's trajectory?

Hint: You may assume that the energy loss is relatively slow ($-d\mathscr{E}/dt \ll \omega_c\mathscr{E}$) but should spell out the condition of validity of this assumption.

Solution: With Eq. (9.153) of the lecture notes for the cyclotron orbit's radius R,

$$R = \frac{m\gamma u}{qB} \equiv \frac{mc}{qB}\beta\gamma, \qquad (*)$$

and Eq. (9.73) for the particle's energy, Eq. (10.45) for the radiative energy loss yields

$$-\frac{d\mathscr{E}}{dt} = \frac{Z_0 q^4 B^2}{6\pi m^2}\beta^2\gamma^2 \equiv \frac{Z_0 q^4 B^2}{6\pi m^2}\left(\gamma^2 - 1\right) = \frac{\mathscr{E}^2 - \left(mc^2\right)^2}{2mc^2\tau}, \qquad \text{where } \tau \equiv \frac{3\pi m^3 c^2}{Z_0 q^4 B^2}. \qquad (**)$$

(Note that this constant τ was already introduced and discussed in the model solution of Problem 8.4.) Separating the variables of this differential equation,

$$\frac{d\varepsilon}{\varepsilon^2 - 1} = -\frac{dt}{2\tau}, \qquad \text{where } \varepsilon \equiv \frac{\mathscr{E}}{mc^2} \geq 1, \qquad (***)$$

we can readily integrate it:[202]

$$\frac{1}{2}\ln\frac{\varepsilon - 1}{\varepsilon + 1} = -\frac{t}{2\tau} + \text{const}.$$

Solving this simple equation for ε, we finally get

$$\mathscr{E} = mc^2 \coth\left(\frac{t}{2\tau} + \text{const}\right). \qquad (****)$$

This expression shows that the kinetic energy $T \equiv \mathscr{E} - mc^2$ of the particle is gradually lost on the time scale given by the constant τ. (As was already discussed in the model solution of Problem 8.4, for an electron in a typical magnetic field of 1 T, this constant is close to 3 seconds.) At the finite stage of this process, at $t >> \tau$, the argument of coth becomes large, and this function may be approximated as

$$\coth\xi \equiv \frac{e^{\xi} + e^{-\xi}}{e^{\xi} - e^{-\xi}} \equiv \frac{1 + e^{-2\xi}}{1 - e^{-2\xi}} \approx 1 + 2e^{-2\xi}, \qquad \text{for } \xi >> 1,$$

so Eq. (****) is reduced to the exponential decay law obtained in the solution of Problem 8.4:

$$T \equiv \mathscr{E} - mc^2 = mc^2\left[\coth\left(\frac{t}{2\tau} + \text{const}\right) - 1\right] \approx \text{const} \times \exp\left\{-\frac{t}{\tau}\right\}.$$

If the initial value of T is much smaller than mc^2, i.e. the particle is non-relativistic to start with, this exponential law is obeyed at all times.

In the opposite limit when the particle is initially ultra-relativistic, the decay starts from a law that differs from the exponential one. Indeed, in this limit, the function coth in Eq. (***) has to be much larger than 1, so it may be approximated as

$$\coth\xi \equiv \frac{\cosh\xi}{\sinh\xi} \approx \frac{1}{\xi}, \qquad \text{for } \xi << 1,$$

and the solution is reduced to

[202] As a math reminder, the integral on the left-hand side of Eq. (***) may be reduced to two integrals of the type $\int d\xi/\xi = \ln\xi$, by representing the involved fraction as the following sum:

$$\frac{1}{\varepsilon^2 - 1} \equiv \frac{1}{2(\varepsilon - 1)} - \frac{1}{2(\varepsilon + 1)}.$$

$$\mathscr{E}(t) \approx mc^2 \frac{2\tau}{t + \text{const}}.$$

The constant participating in this expression may be readily related to the initial energy $\mathscr{E}(0)$ of the particle, finally giving

$$\mathscr{E}(t) = \frac{\mathscr{E}(0)}{1 + \mathscr{E}(0)t / 2\tau mc^2} \approx \mathscr{E}(0)\left[1 - \frac{\mathscr{E}(0)}{2mc^2}\frac{t}{\tau}\right], \qquad \text{for } \mathscr{E}(t) >> mc^2.$$

This result describes an even faster initial decrease of energy, on the time scale

$$\tau' = \frac{2mc^2}{\mathscr{E}(0)}\tau << \tau, \qquad \text{for } \mathscr{E}(0) >> mc^2.$$

The above simple theory, which assumes that each orbit is approximately circular, is only valid if $\omega_c\tau >> 1$. By using one of the forms of Eq. (9.151),

$$\omega_c = \frac{qB}{\gamma m},$$

we may represent the validity condition as

$$\gamma << \frac{m^2c^2}{Z_0 q^3 B}.$$

For electrons in the field of 1 T, the right-hand side of this relation is of the order of 10^{12} (for heavier particles, it is even larger), so in all practical systems, this condition is well satisfied.

As the kinetic energy of the particle goes down due to the radiative losses, so does the parameter $\beta\gamma$, so, as Eq. (*) shows, the cyclotron radius R decreases as well. Hence the particle's trajectory is a slowly converging spiral.

Problem 10.9. Analyze the polarization of the synchrotron radiation propagating within the particle's rotation plane.

Solution: The second term of Eq. (10.19) of the lecture notes shows that the instant electric field **E** of the radiated wave is oriented as the vector

$$\mathbf{n} \times \left[(\mathbf{n} - \boldsymbol{\beta}) \times \frac{d\boldsymbol{\beta}}{dt}\right] \qquad (*)$$

at the retarded point, i.e. at the charge's position. At the synchrotron radiation, the vectors $\boldsymbol{\beta}$ and $d\boldsymbol{\beta}/dt$ are always within the particle's rotation plane, and perpendicular to each other. This is why if the observer and hence the vector **n** are also in that plane, the inner product's vector $(\mathbf{n} - \boldsymbol{\beta})\times d\boldsymbol{\beta}/dt$ in Eq. (*) is normal to that plane, so the outer product vector (and hence the vector **E**) is within the rotation plane again.[203]

[203] **E** is of course also perpendicular to vector **n**, i.e. to the direction of the wave's propagation.

In hindsight, this conclusion could be also made from the evident mirror symmetry of the problem with respect to the rotation plane: if **E** had an out-of-the-plane component, what side would it be directed to?

Problem 10.10. Analyze the polarization and the spectral contents of the synchrotron radiation propagating in the direction normal to the particle's rotation plane. How do the results change if not one, but $N > 1$ similar particles move around the circle, at equal angular distances?

Solution: It is instrumental here to represent the double product in the basic Eq. (10.19),

$$\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right],$$

describing the wave's electric field, as the sum of two terms:

$$\mathbf{n}\times\left(\mathbf{n}\times\dot{\boldsymbol{\beta}}\right)-\mathbf{n}\times\left(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right), \qquad (*)$$

because in our current case, the second term vanishes. (Indeed, both vectors $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ rotate within the particle's rotation plane, so the inner vector product $\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}$ is normal to the plane, just as the vector **n**, i.e. the two vectors are parallel to each other, and their vector product equals zero.)

The double vector product in the remaining first term of Eq. (*) is opposite in direction to the instant vector $\dot{\boldsymbol{\beta}}$, i.e. it rotates, within the plane, with the angular velocity ω_c of the particle's motion. So, the vector **E** rotates (about the wave's propagation direction **n**) with the same angular velocity, i.e. the synchrotron radiation in the direction out of the rotation plane is circularly polarized, in the angular direction of the particle's rotation.

Now note that in our case, a different double vector product, $\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})$, also rotates similarly, with the same angular velocity ω_c, while the scalar product $\mathbf{n}\cdot\mathbf{r}'$ vanishes. According to Eq. (10.61) of the lecture notes, this means that the out-of-plane radiation has only one frequency component (ω_c) – even in the ultra-relativistic case when the in-plane radiation has $\sim\gamma^3 >> 1$ harmonics of this basic frequency!

If we have $N > 1$ similar particles moving around the same circle at equal angular distances $\Delta\varphi = 2\pi/N$, the net electric field is the vector sum of N vectors with equal magnitudes and the same angular distances $\Delta\varphi$. As was already discussed in the model solution of Problem 8.22,[204] such a sum equals zero, so the system's radiation in the direction normal to the particle rotation plane vanishes. (This is not true for the in-plane radiation.)

Problem 10.11.* The basic quantum theory of radiation shows that the electric dipole radiation by a particle is allowed only if the change of its angular momentum's magnitude L at the transition is of the order of Plank's constant $\hbar$.

(i) Estimate the change of L of an ultra-relativistic particle due to its emission of a typical single photon of the synchrotron radiation.

(ii) Do you think quantum mechanics forbid such radiation? If not, why?

[204] And as follows, for example, from MA Eq. (3.6).

Solutions:

(i) According to Eq. (9.70) of the lecture notes, the magnitude of the linear momentum of a particle is $p = \gamma mu$, so the orbital angular momentum due to its circular motion with radius R is $L = pR = \gamma muR$. From the basic geometry, the cyclotron orbit radius is $R = u/\omega_c$, so

$$L = \frac{\gamma mu^2}{\omega_c},$$

where the z-axis is normal to the rotation plane. On the other hand, according to the basic Eq. (9.73), the particle's energy is $\mathscr{E} = \gamma mc^2$, so in the ultra-relativistic limit ($u \to c$), we may write

$$\mathscr{E} = L\omega_c. \qquad (*)$$

This means that at the angular momentum change allowed by the usual quantum-mechanical selection rule, $|\Delta L| = \hbar$, the change of the particle's energy is

$$|\Delta\mathscr{E}| = \hbar\omega_c.$$

On the other hand, Eq. (*) means that the most typical frequency (10.72) of the synchrotron radiation, giving photons with energy

$$|\Delta\mathscr{E}| = \hbar\omega_{max} \sim \frac{\gamma^3}{2}\hbar\omega_c,$$

corresponds to a much larger angular momentum change:

$$|\Delta L| = \frac{|\Delta\mathscr{E}|}{\omega_c} \sim \frac{\gamma^3}{2}\hbar >> \hbar. \qquad (**)$$

(ii) The above result means that the selection rule $|\Delta L| = \hbar$, which is sometimes perceived as a universal law, is invalid. This apparent violation of quantum mechanics is only superficial because the standard selection rule is valid only in a non-relativistic spinless particle in the electric-dipole approximation. As we know from Sec. 8.2, this approximation (and, as a result, the Larmor formula (8.27) as well) is only valid if the size scale of the radiating system (e.g., a in Fig. 8.1) is much smaller than the radiation wavelength $\lambda = 2\pi/k = 2\pi c/\omega = 2\pi c\hbar/|\Delta\mathscr{E}|$. However, for the synchrotron radiation by an ultra-relativistic particle, the effective system's size scale is $R = c/\omega_c$, and the relation is opposite:

$$\lambda \sim \frac{2\pi c}{\omega_{max}} = \frac{2\pi c\hbar}{|\Delta\mathscr{E}|} \sim \frac{2}{\gamma^3}\frac{2\pi c}{\omega_c} = \frac{4\pi}{\gamma^3}R << R,$$

so the naïve selection rule is invalid.

<u>Problem 10.12</u>. A relativistic particle moves along the z-axis, with velocity u_z, through an undulator – a system of permanent magnets providing (in the simplest model) a perpendicular magnetic field, whose distribution near the axis is sinusoidal:[205]

[205] As the Maxwell equation for $\nabla\times\mathbf{H}$ shows, this stationary field distribution cannot be created in any nonvanishing volume of free space. However, it may be created on a line – e.g., on the particle's trajectory.

$$\mathbf{B} = \mathbf{n}_y B_0 \cos k_0 z \,.$$

Assuming that the field is so weak that it causes negligible deviations of the particle's trajectory from the straight line, calculate the angular distribution of the resulting radiation. What condition does the above assumption impose on the system's parameters?

Solution: Under the assumption of a nearly linear trajectory of the particle, $\mathbf{u} \approx \mathbf{n}_z u_z$, the Lorentz force acting on the particle has only one Cartesian component:

$$\mathbf{F} = q\mathbf{u}\times\mathbf{B} = (\mathbf{n}_z \times \mathbf{n}_y) q u_z B_0 \cos k_0 z = -\mathbf{n}_x q u_z B_0 \cos k_0 z \,,$$

so the equations of its motion (with all variables measured in the lab frame) are

$$\dot{p}_x = -q u_z B_0 \cos k_0 z, \quad \text{with } p_x = m\gamma u_x = \frac{m u_x}{\left[1-\left(u_x^2 + u_z^2\right)/c^2\right]^{1/2}},$$

$$\dot{p}_z = 0, \quad \text{with } p_z = m\gamma u_z = \frac{m u_z}{\left[1-\left(u_x^2 + u_z^2\right)/c^2\right]^{1/2}}.$$

In the first (linear) approximation in small u_x, the equations are simplified:

$$m\gamma \dot{u}_x \approx -q u_z B_0 \cos k_0 z, \qquad m\gamma \dot{u}_z \mathbf{n}_z \approx 0, \qquad \text{now with } \gamma \approx \frac{1}{\left(1-u_z^2/c^2\right)^{1/2}} = \text{const},$$

so, with an appropriate choice of time origin, $z \approx u_z t$, and hence

$$\dot{u}_x \approx -\frac{q u_z B_0}{m\gamma}\cos(k_0 u_z t), \qquad \text{i.e. } \dot{\boldsymbol{\beta}} \approx -\mathbf{n}_x \frac{q\beta_z B_0}{m\gamma}\cos(k_0 u_z t), \quad \text{and } \boldsymbol{\beta} \approx \mathbf{n}_z \beta_z \,. \qquad (*)$$

Hence the mutual orientation of the vectors $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ is similar to that at the synchrotron radiation – see Fig. 10.5 and Eq. (10.46) of the lecture notes. This means that for the instantaneous power of radiation we may reuse Eq. (10.47):

$$\frac{d\mathscr{P}}{d\Omega} = \frac{2Z_0 q^2}{\pi^2}\left|\dot{\boldsymbol{\beta}}\right|^2 \gamma^6 f(\theta,\varphi), \qquad (**)$$

where the angular distribution function,

$$f(\theta,\varphi) \equiv \frac{1}{8\gamma^6 (1-\beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\varphi}{\gamma^2(1-\beta\cos\theta)^2}\right], \qquad (***)$$

has been discussed in Sec. 10.3 of the notes and plotted (for the ultra-relativistic case) in Fig. 10.6. The only (but very substantial!) difference of our current problem is that the particle's acceleration's magnitude is not constant in time (as it is at the synchrotron radiation) but oscillates sinusoidally with frequency $\omega = k_0 u_z$. Time-averaging Eq. (**) and using the second of Eqs. (*), we get

$$\frac{d\overline{\mathscr{P}}}{d\Omega} = \frac{Z_0 q^4 B_0^2}{\pi^2 m^2}\beta_z^2 \gamma^4 f(\theta,\varphi).$$

It may look like this approximate analysis is valid if the amplitude

$$u_0 = \frac{qu_z B_0}{m\gamma\omega} \equiv \frac{qB_0}{m\gamma k_0}$$

of the transverse velocity $u_x(t)$ following from the above calculation is much lower than u_z. Using Eq. (9.151) for the cyclotron frequency, this condition may be recast in a very simple and natural form $\omega_c << \omega \equiv k_0 u_z$. However, since the angular distribution (***) has a width scale of the order of $1/\gamma$, the actual condition of the theory's validity is more strict $u_0 << u_z/\gamma$, i.e.

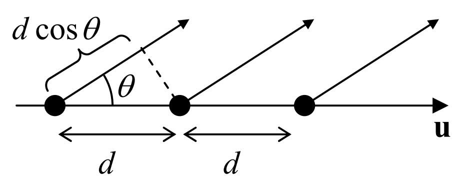
$$\omega_c << \frac{\omega}{\gamma}.$$

In modern undulators, this condition limits the magnet period $\lambda = 2\pi/k_0$ to a few centimeters – for typical numbers, see Sec. 10.3 of the lecture notes, in particular, Fig. 10.10.

Note, however, that the above solution, based on Eq. (10.30) of the lecture notes, neglects the possible coherent addition of the radiation induced by different periods of the undulator's magnetic field – see the next problem.

Problem 10.13. Discuss possible effects of the interference of the undulator radiation from different periods of its static field distribution. In particular, calculate the angular positions of the power density maxima.

Solution: As was discussed in detail in Sec. 8.4 of the lecture notes, the constructive interference of the radiation from two coherent, in-phase sources takes place when the difference Δt of times of propagation of the emitted waves toward the observer (in the figure on the right, $\Delta t = (d\cos\theta)/c$) equals the integer number of the wave periods $\mathcal{T} = \lambda/c$. However, in the case of the undulator with a period d (equal to $2\pi/k_0$ of the previous problem), with a particle propagating along it with velocity u, the radiation induced by adjacent periods of the system has an additional time shift d/u, due to the difference of the particle arrival times, so the constructive interference condition becomes



$$\frac{d}{u} - \frac{d\cos\theta}{c} = n\frac{\lambda}{c}, \tag{*}$$

with any integer n. On the other hand, the wavelength λ of the k^{th} harmonic of the undulator's radiation (where $k = 1, 2,\ldots$) equals $c\mathcal{T}_k = c\mathcal{T}_1/k = c(d/u)/k$. Combining these expressions, we see that the condition for the angles of the radiation maxima does not include d:

$$\cos\theta = \frac{1}{\beta}\left(1 - \frac{n}{k}\right), \qquad \text{where } \beta \equiv \frac{u}{c}. \tag{**}$$

Since $1/\beta$ cannot be smaller than 1, real angles θ with $-1 \leq \cos\theta \leq +1$ may satisfy this condition only for certain positive values of the integer number n.

In particular, for ultra-relativistic particles with $\gamma >> 1$ used in standard undulators, the similarity between the undulator radiation and the synchrotron radiation (see the model solution of the previous problem) allows us to use Eq. (10.72) of the lecture notes to conclude that the most intense radiation harmonics have numbers $k \sim \gamma^3/2 >> 1$. Since for such particles, $1/\beta \equiv 1/(1-1/\gamma^2)^{1/2} \approx 1 + 1/2\gamma^2$, the condition (**) becomes

$$\cos\theta_n \approx \left(1+\frac{1}{2\gamma^2}\right)\left(1-\frac{2n}{\gamma^3}\right) \approx 1-\frac{1}{2\gamma^2}\left(\frac{4n}{\gamma}-1\right).$$

Using the Taylor expansion $\cos\theta \approx 1 - \theta^2/2$, valid for small angles θ, we see that the lowest-order maxima correspond to the angles

$$\theta_n \approx \frac{1}{\gamma}\left(\frac{4n}{\gamma}-1\right)^{1/2} = \frac{2}{\gamma^{3/2}}\left(n-n_{\min}\right)^{1/2} << 1, \qquad \text{where } n_{\min} \equiv \frac{\gamma}{4} >> 1,$$

with a very small angular distance,

$$\Delta\theta_n \equiv \left(\theta_{n+1}-\theta_n\right)_{n\sim n_{\min}} \approx \frac{d\theta_n}{dn}\bigg|_{n\sim n_{\min}} \approx \frac{1}{\gamma^{3/2} n_{\min}^{1/2}} = \frac{2}{\gamma^2} << \frac{1}{\gamma},$$

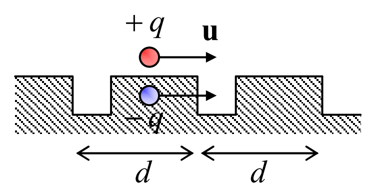
between the adjacent maxima, completely masked by the relativistic width $\Delta\theta \sim 1/\gamma$ of the radiation – again, see the model solution of the previous problem. In this case, all results of that solution, in which the interference has been neglected, are quantitatively valid.

On the other hand, for non- (or nearly-) relativistic particles, especially in "soft" (sinusoidal or nearly-sinusoidal) undulator field profiles, such as the sinusoidal profile of the previous problem, the radiation is nearly sinusoidal, i.e. the largest power corresponds to $k = 1$. In this case, according to Eq. (**), the distance between the adjacent interference maxima may be quite substantial:

$$\Delta\left(\cos\theta_n\right)_{k=1} = \frac{1}{\beta} \sim 1,$$

and they may be readily resolved. A quantitative calculation of the radiation power distribution in this case (a good additional exercise for the reader :-) should start from Eq. (10.19), rather than (10.30), of the lecture notes.

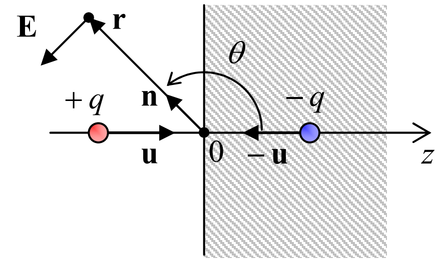
If the radiation is produced by a short bunch of particles with a length of the order of d, rather than by a single particle, the power spectrum of the emitted radiation depends on the bunch's shape. This fact enables using this effect for the experimental non-destructive measurements of the shape of very short (in the time domain, femtosecond) electron bunches. It is interesting that for such experiments the undulator is not really necessary: the needed space-periodic transverse force may be produced just by the flying charge's image in a periodic metallic structure – see the figure on the right. The standard diffraction gratings, with $d \sim 1$ μm, are quite convenient for this purpose because, with $\beta \sim 1$ (the condition which is easy to reach with electrons), the resulting radiation wavelengths $\lambda \sim d/\beta$ are in the visible light range, for which sensitive radiation detectors are readily available.[206]



[206] This is exactly the arrangement that was used by S. Smith and E. Purcell in 1953 for the first observation of this *Smith-Purcell effect* (which may be also considered as a specific form of the transition radiation discussed at the end of Sec. 10.5 of the lecture notes). Historically that experiment was the first step toward modern undulators and free-electron lasers, which use much stronger periodic fields and hence produce much more intense radiation.

Problem 10.14. An electron launched directly toward a plane surface of a perfect conductor is instantly absorbed by it at the impact. Calculate the angular distribution and the frequency spectrum of the electromagnetic waves radiated at this event, provided that the initial kinetic energy T of the particle is much larger than the conductor's workfunction ψ.[207] Is your result valid near the conductor's surface?

Solution: As was discussed in Sec. 10.5 of the lecture notes in the context of transition radiation, at not very high frequencies ($\omega \ll \omega_p$),[208] the field outside of the conductor may be viewed as a result of a head-on collision of the particle of the given charge q with its mirror image with the charge ($-q$), moving toward each other – see the figure on the right.



Due to the condition $\psi \ll T$, the change of the particle's velocity, due to its Coulomb interaction with the image, may be neglected. Hence for the pre-impact velocities of the particle and its image, we may write

$$\boldsymbol{\beta}_{\text{charge}} = \frac{u}{c}\mathbf{n}_z, \qquad \boldsymbol{\beta}_{\text{image}} = -\frac{u}{c}\mathbf{n}_z.$$

At collision, both the particle's charge and that of its image, disappear – in our approximation, instantly. Hence, for the calculation of the emitted radiation, we may use the low-frequency approximation described by Eq. (10.76), in which the term marked "fin" equals zero. Since the motion of these two charges is correlated, we have to sum up their electric fields rather than radiated powers – in Eq. (10.76), inside the modulus sign, taking into account the difference of their charge signs:

$$I(\omega) = \frac{1}{4\pi^2 c}\frac{1}{4\pi\varepsilon_0}\left|\sum_k q_k \frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}_k)}{1-\boldsymbol{\beta}_k\cdot\mathbf{n}}\right|^2 = \frac{1}{4\pi^2 c}\frac{1}{4\pi\varepsilon_0}\left| q_{\text{charge}}\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}_{\text{charge}})}{1-\boldsymbol{\beta}_{\text{charge}}\cdot\mathbf{n}} + q_{\text{image}}\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}_{\text{image}})}{1-\boldsymbol{\beta}_{\text{image}}\cdot\mathbf{n}}\right|^2$$

$$= \frac{1}{4\pi^2 c}\frac{q^2\beta^2}{4\pi\varepsilon_0}\left|\frac{\mathbf{n}\times(\mathbf{n}\times\mathbf{n}_z)}{1-\beta\,\mathbf{n}_z\cdot\mathbf{n}} + \frac{\mathbf{n}\times(\mathbf{n}\times\mathbf{n}_z)}{1+\beta\,\mathbf{n}_z\cdot\mathbf{n}}\right|^2 = \frac{1}{\pi^2 c}\frac{q^2}{4\pi\varepsilon_0}\frac{\beta^2\sin^2\theta}{(1-\beta^2\cos^2\theta)^2}.$$

So, the spectral density of the radiation is frequency-independent, corresponding to a short (delta-function-like) pulse radiated at the collision, though in reality its time duration is spread by $\sim 1/\omega_p$, i.e. the frequency spectrum is limited to frequencies below $\sim\omega_p$. Concerning the angular distribution, for non-relativistic particles, it follows the usual $\sin^2\theta$ pattern, but in the ultra-relativistic case is squeezed to a hollow cone of width $\Delta\theta \sim 1/\gamma$, qualitatively similar to that illustrated by Fig. 10.4b. Note that our result is only valid outside of the conductor (into which the waves do not penetrate), so the cone corresponds to the angles $\theta \approx \pi$.

Since the calculated function $I(\omega)$ does not vanish (and in the non-relativistic case, even reaches its maximum) at $\theta = \pi/2$, i.e. in directions along the conductor's surface, one may wonder whether this is also an artifact of some approximation. Actually (again, at frequencies $\omega \ll \omega_p$) it is not. Indeed, as we already know, the vector $\mathbf{E}$ of the electric field of the wave generated at any linear motion of the charge lies within the common plane of the observer and the line of the charge's motion – see, e.g., Eq.

[207] See Sec. 2.9, in particular Fig. 2.27a.

[208] Taking the (admittedly, too vague :-) characterization of the conductor as "perfect" in the assignment literally, we may assume that its free carrier density n is infinite, and so is the plasma frequency ω_p – see Eq. (7.37).

(10.19). On the other hand, it is perpendicular to the vector **n** of wave propagation – see the figure above. Hence at $\theta = \pi/2$, the wave's electric field is normal to the conductor's surface (while its magnetic field is parallel to the surface). As we know from the discussion in Sec. 7.6, this is fully consistent with the boundary conditions (7.104) on a perfect conductor's surface. Note that this means that the propagation of the electromagnetic pulse from the collision point is accompanied by ring-shaped pulses of the surface electric charge and the radially-directed current, propagating from the collision point on the conductor's surface.

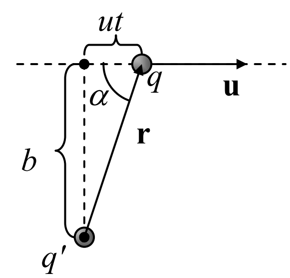
Problem 10.15. A relativistic particle, with a rest mass m and an electric charge q, flies ballistically with velocity u by an immobile point charge q', with an impact parameter b so large that the deviations of its trajectory from the straight line are negligible. Calculate the total energy loss due to the electromagnetic radiation during the passage. Quantify the conditions of validity of your result.

Solution: The easiest way to solve the problem is to use the formula obtained in the solution of Problem 4:

$$\mathscr{P} = \frac{Z_0 q^4}{6\pi}\left(\frac{\gamma}{mc}\right)^2\left[(\mathbf{E}+\mathbf{u}\times\mathbf{B})^2 - \left(\frac{\mathbf{u}\cdot\mathbf{E}}{c}\right)^2\right]. \qquad (*)$$

In the lab frame where the charge q' rests, it produces only the Coulomb electrostatic field,

$$\mathbf{E} = \frac{q'}{4\pi\varepsilon_0 r^2}\frac{\mathbf{r}}{r},$$



directed along the vector **r** that connects the charge positions, and hence is an implicit function of time – see the figure on the right. According to the conditions of our current problem, the velocity **u** of the charge q may be taken for a constant, so from this sketch, we get

$$r \equiv |\mathbf{r}| = \left(b^2 + u^2t^2\right)^{1/2}, \qquad \frac{\mathbf{u}}{c}\cdot\frac{\mathbf{r}}{r} = c\beta\cos\alpha, \qquad \cos\alpha = \frac{ut}{r} = \frac{ut}{\left(b^2+u^2t^2\right)^{1/2}},$$

where t is the time referred to the instant of the closest approach of the particles. As a result, the expression in the square brackets of Eq. (*) is reduced to

$$E^2 - \left(\frac{\mathbf{u}\cdot\mathbf{E}}{c}\right)^2 = \left(\frac{q'}{4\pi\varepsilon_0 r^2}\right)^2\left(1-\beta^2\cos^2\alpha\right) = \left(\frac{q'}{4\pi\varepsilon_0}\right)^2\frac{1}{\left(b^2+u^2t^2\right)^2}\left(1-\beta^2\frac{u^2t^2}{b^2+u^2t^2}\right).$$

Integrating the power (*) over the whole duration of the particle passage, we get the full energy loss:

$$-\Delta\mathscr{E} = \int_{-\infty}^{+\infty}\mathscr{P}dt = \frac{Z_0q^4}{6\pi}\left(\frac{\gamma}{mc}\right)^2\left(\frac{q'}{4\pi\varepsilon_0}\right)^2 I,$$

where

$$I \equiv \int_{-\infty}^{+\infty}\frac{1}{\left(b^2+u^2t^2\right)^2}\left(1-\frac{\beta^2u^2t^2}{b^2+u^2t^2}\right)dt \equiv \frac{2}{ub^3}\left[\int_0^{+\infty}\frac{d\xi}{\left(1+\xi^2\right)^2} - \beta^2\int_0^{+\infty}\frac{\xi^2 d\xi}{\left(1+\xi^2\right)^3}\right], \qquad \text{with } \xi \equiv \frac{ut}{b}.$$

The first of these dimensionless integrals is well-known end equal to $\pi/4$, while the second one may be readily reduced to a sum of the same integral and one more integral of the same type:[209]

$$\int_0^{+\infty}\frac{\xi^2 d\xi}{\left(1+\xi^2\right)^3}\equiv\int_0^{+\infty}\frac{\left[\left(1+\xi^2\right)-1\right]d\xi}{\left(1+\xi^2\right)^3}\equiv\int_0^{+\infty}\frac{d\xi}{\left(1+\xi^2\right)^2}-\int_0^{+\infty}\frac{d\xi}{\left(1+\xi^2\right)^3}=\frac{\pi}{4}-\frac{3\pi}{16}\equiv\frac{\pi}{16},$$

so

$$I=\frac{2}{ub^3}\left(\frac{\pi}{4}-\beta^2\frac{\pi}{16}\right)\equiv\frac{\pi}{2ub^3}\left(1-\frac{\beta^2}{4}\right),$$

and, finally, the radiative energy loss is[210]

$$-\Delta\mathscr{E}=\frac{Z_0 q^4}{6\pi}\left(\frac{\gamma}{mc}\right)^2\left(\frac{q'}{4\pi\varepsilon_0}\right)^2\frac{\pi}{2ub^3}\left(1-\frac{\beta^2}{4}\right)\equiv\frac{1}{12}\frac{Z_0 q^4}{m^2 b^3 c^3}\left(\frac{q'}{4\pi\varepsilon_0}\right)^2\frac{1-\beta^2/4}{\beta\left(1-\beta^2\right)}.$$

This result has been obtained by neglecting the change of the particle's trajectory and velocity (and hence its energy) due to its interaction with the Coulomb center. These assumptions are valid if the scales of the potential energy of interaction and the calculated energy loss are much smaller than the particle's kinetic energy:

$$-\Delta\mathscr{E},\ \frac{qq'}{4\pi\varepsilon_0 b}<<mc^2\left(\gamma-1\right).$$

[209] See, e.g., MA Eq. (6.5b) with, respectively, $n = 2$ and $n = 3$.

[210] Additional task for the reader: compare this result semi-qualitatively with the one that would follow from a combination of Eq. (10.79) with an interpretation of Eq. (10.94), that would be suitable for our current problem.