Chapter 10. Making Sense of Quantum Mechanics

This (rather brief) chapter addresses the conceptually important issues of quantum measurements and quantum state interpretation. Please note that some of these issues are still subjects of debate[1] *– fortunately not affecting the quantum mechanics' practical results discussed in the previous chapters.*

10.1. Quantum measurements

The knowledge base outlined in the previous chapters gives us a sufficient background for a (by necessity, very brief) discussion of *quantum measurements*.[2] Let me start by reminding the reader of the only quantum theory's postulate that relates it to experiment. In the simplest case when the system is in a coherent (pure) quantum state, its ket-vector may be represented as a linear superposition

$$\left|\alpha\right\rangle = \sum_j \alpha_j \left|a_j\right\rangle, \tag{10.1}$$

where a_j are the eigenstates of the operator of an observable A, related to its eigenvalues A_j by Eq. (4.68):

$$\hat{A}\left|a_j\right\rangle = A_j\left|a_j\right\rangle. \tag{10.2}$$

In such a state, the outcome of a single measurement (at this stage, meaning a *perfect* measurement) of the observable A may be uncertain but is restricted to the set of eigenvalues A_j, with the j^{th} outcome probability equal to

$$W_j = \left|\alpha_j\right|^2. \tag{10.3}$$

As was discussed in Chapter 7, the state of the system (or rather of the statistical ensemble of macroscopically similar systems we are using for this particular series of similar experiments) may be mixed rather than pure, and hence even more uncertain than the state described by Eq. (1). Hence, the measurement postulate means that even if the system is in its least uncertain state, the measurement outcomes are *still* probabilistic.[3]

If we believe that each particular measurement may be done perfectly, and do not worry too much about how exactly, we are subscribing to the *mathematical* notion of measurement, that was, rather reluctantly, used in these notes – up to this point. However, the actual (*physical*) measurements are *always* imperfect, first of all, because of the huge gap between the energy-time scale $\hbar \sim 10^{-34}$ J·s of the quantum phenomena in "microscopic" systems such as atoms, and the "macroscopic" scale of the direct human perception, so the role of the instruments bridging this gap (Fig. 1), is highly nontrivial.

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[1] For an excellent review of these controversies, as presented in a few leading textbooks, I highly recommend J. Bell's paper published in the collection by A. Miller (ed.), *Sixty-Two Years of Uncertainty*, Plenum, 1989.

[2] "Quantum measurements" is a rather unfortunate and misleading term; it would be more sensible to speak about "measurements of observables in quantum mechanical systems". However, the former term is so common and compact that I will use it – albeit rather reluctantly.

[3] The measurement outcomes become definite only in the trivial case when the system is definitely in one of the eigenstates a_j, say a_0; then $\alpha_j = \delta_{j,0}\exp\{i\varphi\}$, and $W_j = \delta_{j,0}$.

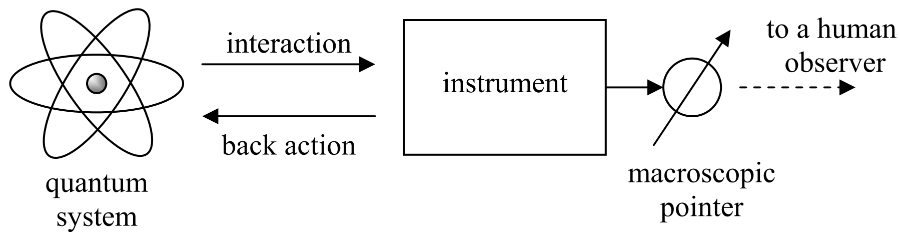

Fig.10.1. General scheme of a quantum measurement.

Besides the famous Bohr-Einstein discussion in the mid-1930s, which will be briefly reviewed in Sec. 3, the founding fathers of quantum mechanics have not paid much attention to these issues, apparently because of the following reason. At that time it looked like the experimental instruments (at least the best of them :-) were doing exactly what the measurement postulate was telling. For example, the z-oriented Stern-Gerlach experiment (Fig. 4.1) turns two complex coefficients $\alpha_\uparrow$ and $\alpha_\downarrow$ describing the spin state of the incoming particles, into a set of particle-counter clicks, with the rates proportional to, respectively, $|\alpha_\uparrow|^2$ and $|\alpha_\downarrow|^2$. The complex internal nature of these instruments makes more detailed questions unnatural. For example, each click of a Geiger counter involves an effective disappearance of the observed particle in a zillion-particle electric discharge avalanche it has triggered. A century ago, it looked much more important to extend the newborn quantum mechanics to more complex systems (such as atomic nuclei, etc.) than to think about the physics of such instruments.

However, since that time the experimental techniques, notably including high-vacuum and low-temperature systems, micro- and nano-fabrication, and low-noise electronics, have improved quite dramatically. In particular, we now may observe the quantum-mechanical behavior of more and more macroscopic objects – such as the mechanical oscillators mentioned in Sec. 2.9. Moreover, some "macroscopic quantum systems" (in particular, special systems of Josephson junctions, see below) have properties enabling their use as essential parts of measurement setups. Such developments are making the line separating the "micro" and "macro" worlds finer and finer, so more inquisitive questions about the physical nature of quantum measurements are not so hopeless now. In my personal scheme of these developments, the main questions may be grouped as follows:

(i) Does a quantum measurement involve any laws besides those of quantum mechanics? In particular, should it necessarily involve a human/intelligent observer? (The last question is not as laughable as it may look – see below.)

(ii) What is the state of the measured system *just after* a *single-shot measurement* – meaning a measurement process limited to a time interval much shorter than the time scale of the measured system's evolution? (This question is a necessary part of any discussion of *repeated measurements* and of their ultimate form – *continuous monitoring* of a certain observable.)

(iii) If a measurement of an observable A has produced a certain outcome A_j, what statements may be made about the state of the system *just before* the measurement? (This question is most closely related to various interpretations of quantum mechanics.)

Let me discuss these issues in the listed order. First of all, I am happy to report that there is a virtual consensus of physicists on some aspects of these issues. According to this consensus, any reasonable quantum measurement needs to result in a certain, distinguishable state of a *macroscopic* output component of the measurement instrument – see Fig. 1. (Traditionally, its component is called a *pointer*, though its role may be played by a printer or a plotter, an electronic circuit sending out the result as a number, etc.). This requirement implies that the measurement process should have the following features:

– provide a large "signal gain", i.e. some means of mapping the quantum state with its $\hbar$-scale of action (i.e. of the energy-by-time product) onto a macroscopic position of the pointer with a much larger action scale, and

– if we want to approach the fundamental limit of uncertainty, the instrument should introduce as little additional fluctuation ("noise") as permitted by the laws of physics.

Both these requirements *are* fulfilled in a well-designed Stern-Gerlach experiment – see Fig. 4.1 again. Indeed, the magnetic field gradient, splitting the particle beam, turns the minuscule (microscopic) energy difference (4.167) between two spin-polarized states into a macroscopic difference between the final positions of two output beams, where two particle detectors may be located. However, as was noted above, the internal physics of the particle detectors (say, Geiger counters) at this measurement is rather complicated, and would not allow us to discuss some aspects of the measurement, in particular. to answer the second question we are working on.

This is why let me describe the scheme of an almost similar single-shot measurement of a two-level quantum system, which shares the simplicity, high gain, and low internal noise of the Stern-Gerlach apparatus, but has an advantage that at its certain hardware implementations,[4] the measurement process allows a thorough, quantitative theoretical description. Let us consider a 1D particle confined in a double-well potential (Fig. 2), where x is some continuous generalized coordinate – not necessarily a mechanical displacement. Let the particle be initially in a pure quantum state, with the energy close to the wells' bottom. Then, as we know from the discussion of such systems in Secs. 2.6 and 5.1, the state may be represented by a ket-vector similar to that of spin-½:

$$\left|\alpha\right\rangle = \alpha_{\rightarrow}\left|\rightarrow\right\rangle + \alpha_{\leftarrow}\left|\leftarrow\right\rangle, \tag{10.4}$$

where the component states $\rightarrow$ and $\leftarrow$ are described by wavefunctions localized near the potential well bottoms at $x \approx \pm x_0$ – see the blue lines in Fig. 2. Our goal is to measure in which well the particle resides at a certain time instant, say at $t = 0$. For that, let us rapidly change, at that moment, the potential profile of the system, so at $t > 0$, near the origin, it may be well described by an inverted parabola:

$$U(x) \approx -\frac{m\lambda^2}{2}x^2, \qquad \text{for } t > 0, \quad |x| << x_{\rm f}. \tag{10.5}$$

[4] The scheme may be implemented, for example, using a simple Josephson-junction circuit called the *balanced comparator* – see, e.g., T. Walls *et al*., *IEEE Trans. on Appl. Supercond*. **17**, 136 (2007), and references therein. (Its ac versions, first demonstrated by I. Siddiqi *et al*., *Phys. Rev. Lett*. **94**, 027005 (2005), are currently called "bifurcation detectors". This term is applicable to the balanced comparator as well.) Experiments have demonstrated that this system may have a measurement variance dominated by the theoretically expected quantum-mechanical uncertainty, at quite practicable experimental conditions (temperatures of the order of 1K). A conceptual advantage of this system is that it is based on externally-shunted Josephson junctions, i.e. the devices whose quantum-mechanical model, including the coupling to the environment, is in a quantitative agreement with experiment – see, e.g., D. Schwartz *et al*., *Phys. Rev. Lett*. **55**, 1547 (1985). Colloquially, the balanced comparator is a high-gain instrument with a "well-documented Hamiltonian", eliminating the need for speculations about the environmental effects. In particular, the dephasing process in it, and its time T_2, are well described by Eqs. (7.89) and (7.142), with the coefficient η equal to the Ohmic conductance G of the shunt.

It is straightforward to verify that the Heisenberg equations of motion in such an inverted potential describe an exponential growth of the operator $\hat{x}$ in time (proportional to $\exp\{\lambda t\}$) and hence a similar, proportional growth of the expectation value $\langle x \rangle$ and its r.m.s. uncertainty δx.[5] At this "inflation" stage, the coherence between the two component states → and ← is still preserved, i.e. the time evolution of the system is, in principle, reversible.

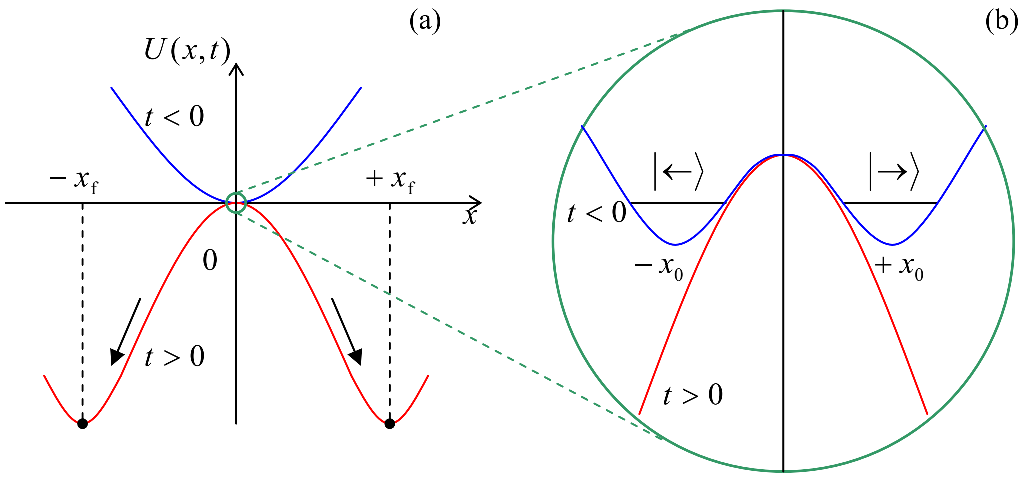

Fig. 10.2. The confining potential's inversion, as viewed on the (a) "macroscopic" and (b) "microscopic" scales of the generalized coordinate x.

Now let the system be weakly coupled, also at $t > 0$, to a dissipative (e.g., Ohmic) environment. As we know from Chapter 7, such coupling ensures the state's dephasing on some time scale T_2. If

$$x_0 << x_0 \exp\{\lambda T_2\},\ x_f\ , \tag{10.6}$$

then the process, after the potential inversion, consists of two stages, well separated in time:

– the already discussed "inflation" stage, preserving the state components' coherence, and

– the dephasing stage, at which the coherence of the component states → and ← is gradually suppressed as described by Eq. (7.89), i.e. the density matrix of the system is gradually reduced to the diagonal form describing a classical mixture of two probability packets with the probabilities (3) equal to, respectively, $W_\rightarrow = |\alpha_\rightarrow|^2$ and $W_\leftarrow = |\alpha_\leftarrow|^2 \equiv 1 - |\alpha_\rightarrow|^2$.

Besides dephasing, the environment gives the motion certain kinematic friction, with the drag coefficient η (7.141), so the system eventually settles to rest at one of the macroscopically separated minima $x = \pm x_f$ of the inverted potential (Fig. 2a), thus ensuring a high "signal gain" $x_f/x_0 >> 1$. As a result, the final probability density distribution $w(x)$ along the x-axis has two narrow, well-separated peaks. But this is just the situation that was discussed in Sec. 2.5 – see, in particular, Fig. 2.17. Since that discussion is very important, let me repeat – or rather rephrase it. The final state of the system is a classical mixture of two well-separated states, with the respective probabilities $W_\leftarrow$ and $W_\rightarrow$, whose sum equals 1. Now let us use some detector to test whether the system is in one of these states – say the right one. (If x_f is sufficiently large, the noise contribution of this detector to the measurement uncertainty is

[5] Somewhat counter-intuitively, the latter growth *improves* the measurement's fidelity. Indeed, it does not affect the intrinsic "signal-to-noise ratio" $\langle x \rangle/\delta x$, while making the intrinsic (say, quantum-mechanical) uncertainty much larger than the possible noise contribution by the later measurement stage(s).

negligible,[6] and its physics is unimportant.) If the system has been found at this location (again, the probability of this outcome is $W_{\rightarrow} = |\alpha_{\rightarrow}|^2$), the probability of finding it at the counterpart (left) location at a consequent detection turns to zero.

This probability "reduction" is a purely classical (or if you like, mathematical) effect of the statistical ensemble's re-definition: $W_{\leftarrow}$ equals zero not in the initial ensemble of all similar experiments (where is equals $|\alpha_{\leftarrow}|^2$), but only in the re-defined ensemble of experiments in that the system had been found at the right location. Of course, which ensemble to use, i.e. what probabilities to register/publish is a purely accounting decision, which should be made by a human (or otherwise intelligent :-) observer. If we are only interested in an objective recording of the results of a pre-fixed sequence of experiments (i.e. the members of a pre-defined, fixed statistical ensemble), there is no need to include such an observer in any discussion. In any case, this detection/registration process, very common in classical statistics, leaves no space for any mysterious "wave packet reduction" – understood as a hypothetical process that would not obey the regular laws of quantum mechanical evolution.

The state dephasing and ensemble re-definition at measurements are also at the core of several paradoxes, of which the so-called *quantum Zeno paradox* is perhaps the most spectacular.[7] Let us return to a two-level system with the unperturbed Hamiltonian given by Eq. (4.166), the quantum oscillation period $2\pi/\Omega$ much longer than the single-shot measurement time, and the system initially (at $t = 0$) definitely in one of the partial quantum states – for example, a certain potential well of the double-well potential. Then, as we know from Secs. 2.6 and 4.6, the probability to find the system in this initial state at time $t > 0$ is

$$W(t) = \cos^2\frac{\Omega t}{2} \equiv 1 - \sin^2\frac{\Omega t}{2}. \tag{10.7}$$

If the time is small enough ($t = dt << 1/\Omega$), we may use the Taylor expansion to write

$$W(dt) \approx 1 - \frac{\Omega^2 dt^2}{4}. \tag{10.8}$$

Now, let us use some good measurement scheme (say, the potential inversion discussed above) to measure whether the system is still in this initial state. If it is (as Eq. (8) shows, the probability of such an outcome is nearly 100%), then the system, after the measurement, is in the same state. Let us allow it to evolve again, following the same Hamiltonian. Then the evolution of W will follow the same law as in Eq. (7). Thus, when the system is measured again at time $2dt$, the probability to find it in the same state both times is

$$W(2dt) \approx W(dt)\left(1 - \frac{\Omega^2 dt^2}{4}\right) = \left(1 - \frac{\Omega^2 dt^2}{4}\right)^2. \tag{10.9}$$

After repeating this cycle N times (with the total time $t = Ndt$ still much less than $N^{1/2}/\Omega$), the probability that the system is still in its initial state is

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[6] At the balanced-comparator implementation mentioned above, the final state detection may be readily performed using a "SQUID" magnetometer based on the same Josephson junction technology – see, e.g., EM Sec. 6.5. In this case, the distance between the potential minima $\pm x_f$ is close to one superconducting flux quantum (3.38), while the additional uncertainty induced by the SQUID may be as low as a few millionths of that amount.

[7] This name, coined by E. Sudarshan and B. Mishra in 1997 (though the paradox had been discussed in detail by A. Turing in 1954) is due to its superficial similarity to the classical paradoxes by the ancient Greek philosopher Zeno of Elea.

$$W(Ndt) \equiv W(t) \approx \left(1 - \frac{\Omega^2 dt^2}{4}\right)^N = \left(1 - \frac{\Omega^2 t^2}{4N^2}\right)^N \approx 1 - \frac{\Omega^2 t^2}{4N}. \tag{10.10}$$

Comparing this result with Eq. (7), we see that the process of the system's transfer to the opposite partial state has been slowed down rather dramatically, and in the limit $N \to \infty$ (at fixed t), its evolution is virtually stopped by the measurement process. There is of course nothing mysterious here; the evolution slowdown is due to the state dephasing and the statistical ensemble re-definition at each measurement.

This may be the only acceptable occasion for me to mention, very briefly, one more famous – or rather infamous *Schrödinger cat paradox*, so much overplayed in popular publications.[8] For this thought experiment, there is no need to discuss the (rather complicated :-) physics of the cat. As soon as the charged particle produced at the radioactive decay reaches the Geiger counter, the initial coherent superposition of the two possible quantum states ("the decay has happened"/"the decay has not happened") of the system is rapidly dephased, i.e. reduced to their classical mixture, leading, correspondingly, to the classical mixture of the final macroscopic states "cat dead"/"cat alive". So, despite attempts by numerous authors lacking a proper physics background to represent this situation as a mystery whose discussion needs the involvement of professional philosophers, hopefully, the reader of these notes knows enough about dephasing from Chapter 7 to ignore all this babble.

10.2. QND measurements

I hope that the above discussion has sufficiently illuminated the issues of group (i), so let me proceed to question group (ii), in particular to the general issue of the *back action* of the instrument upon the system under measurement – symbolized with the back arrow in Fig. 1. In the instruments like the Geiger counter, such back action is large: the instrument essentially destroys ("demolishes") the state of the system under measurement. Even the "cleaner" potential-inversion measurement illustrated by Fig. 2 fully destroys the initial coherence of the system's states, i.e. perturbs it very substantially.

However, in the 1970s it was understood that this is not really necessary. For example, in Sec. 7.3, we have already discussed an example of a two-level system coupled with its environment and described by the Hamiltonian (7.68)-(7.70):

$$\hat{H} = \hat{H}_s + \hat{H}_{\text{int}} + \hat{H}_e\{\lambda\}, \quad \text{with } \hat{H}_s = c_z\hat{\sigma}_z, \text{ and } \hat{H}_{\text{int}} = -f\{\lambda\}\hat{\sigma}_z, \tag{10.11}$$

so

$$\left[\hat{H}_s, \hat{H}_{\text{int}}\right] = 0. \tag{10.12}$$

Comparing this equality with Eq. (4.199), applied to the explicitly-time-independent Hamiltonian $\hat{H}_s$,

$$i\hbar\dot{\hat{H}}_s = \left[\hat{H}_s, \hat{H}\right] \equiv \left[\hat{H}_s, \left(\hat{H}_s + \hat{H}_{\text{int}} + \hat{H}_e\{\lambda\}\right)\right] = \left[\hat{H}_s, \hat{H}_{\text{int}}\right] = 0, \tag{10.13}$$

we see that in the Heisenberg picture, the Hamiltonian operator (and hence the energy) of the system of our interest does not change in time. On the other hand, if the "environment" in this discussion is the instrument used for the measurement (see Fig. 1 again), then the interaction can change its state, so it

[8] I fully agree with S. Hawking who has been quoted to say, "When I hear about the Schrödinger cat, I reach for my gun." The only good aspect of this popularity is that the formulation of this paradox should be so well known to the reader that I do not need to waste time/space repeating it.

may be used to measure the system's energy – or another observable whose operator commutes with the interaction Hamiltonian. Such a trick is called the *quantum non-demolition* (QND), or sometimes "back-action-evading" measurements.[9] Due to the lack of back action of the instrument on the corresponding variable, such measurements allow its continuous monitoring. Let me present a fine example of an actual measurement of this kind – see Fig. 3.[10]

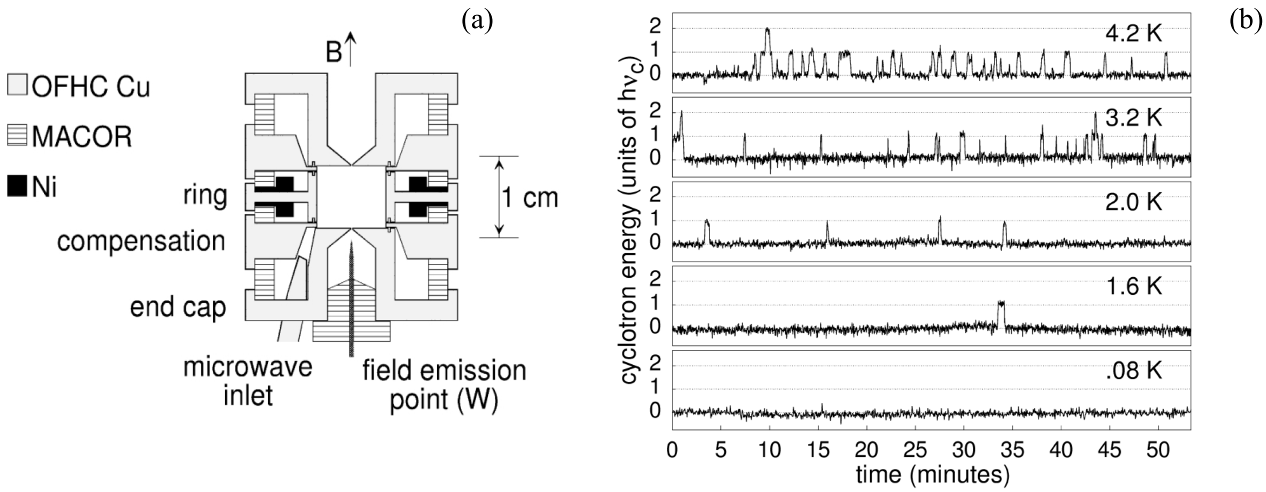

Fig. 10.3. QND measurements of single electron's energy by Peil and Gabrielse: (a) the experimental setup's core, and (b) a record of the thermal excitation and spontaneous relaxation of the Fock states. © 1999 APS; reproduced with permission.

In this experiment, a single electron is captured in a *Penning trap* – a combination of a (virtually) uniform magnetic field $\mathscr{B}$ and a quadrupole electric field.[11] This electric field stabilizes the cyclotron orbits but does not have any noticeable effect on electron motion in the plane normal to the magnetic field, and hence on its Landau level energies – see Eq. (3.50):

$$E_n = \hbar\omega_c\left(n+\frac{1}{2}\right), \quad \text{with} \quad \omega_c = \frac{e\mathscr{B}}{m_e}. \tag{10.14}$$

(In the cited work, with $\mathscr{B} \approx 5.3$ T, the cyclic frequency $\omega_c/2\pi$ was about 147 GHz, so the Landau level splitting $\hbar\omega_c$ was close to 10^{-22} J, i.e. corresponded to k_BT at $T \sim 10$ K, while the physical temperature of the system might be reduced well below that, down to 80 mK). Now note that the analogy between a Landau-level particle and a harmonic oscillator goes beyond the energy spectrum (14). Indeed, since the Hamiltonian of a 2D particle in a perpendicular magnetic field may be reduced to Eq. (3.47) similar to that of a 1D oscillator, we may repeat all procedures of Sec. 5.4 and rewrite this effective Hamiltonian in terms of the creation-annihilation operators – see Eq. (5.72):

$$\hat{H}_s = \hbar\omega_c\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right). \tag{10.15}$$

[9] For a more detailed discussion of this approach, including a review of its initial development, see V. Braginsky and F. Y. Khalili, *Rev. Mod. Phys*. **68**, 1 (1996).
[10] S. Peil and G. Gabrielse, *Phys. Rev. Lett.* **83,** 1287 (1999).
[11] It is similar to the 2D system discussed in EM Sec. 2.7 but with additional rotation about one of the axes.

In the Peil and Gabrielse experiment, the trapped electron had one more degree of freedom – along the magnetic field. The electric field of the Penning trap created a very soft confining potential along this direction (vertical in Fig. 3a; I will take it for the z-axis), so small electron oscillations along that axis could be well described as those of a 1D harmonic oscillator of a much lower frequency, in that particular experiment with $\omega_z/2\pi \approx 64$ MHz. This frequency could be measured very accurately (with an error of ~1 Hz) by sensitive electronics whose electric field does affect the z-motion of the electron, but not its motion in the perpendicular plane. In an exactly uniform magnetic field, the two modes of electron motion would be completely uncoupled. However, the experimental setup included two special superconducting rings made of niobium (see Fig. 3a), which slightly distorted the magnetic field and created an interaction between the modes, which might be well approximated by the Hamiltonian[12]

$$\hat{H}_{\text{int}} = \text{const} \times \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \hat{z}^2 , \tag{10.16}$$

so the main condition (12) of a QND measurement was very closely satisfied. At the same time, the coupling (16) ensured that a change of the Landau level number n by 1 changed the z-oscillation eigenfrequency by ~12.4 Hz. Since this shift was substantially larger than the used electronics' noise, rare spontaneous changes of n (due to a weak coupling of the electron to the environment) could be readily measured – moreover, continuously monitored – see Fig. 3b. The record shows spontaneous excitations of the electron to higher Landau levels, with its sequential relaxation, just as described by Eqs. (7.208)-(7.210). The detailed data statistics analysis showed that there was virtually no effect of the measuring instrument on these processes – at least on the scale of minutes, i.e. as many as $\sim 10^{13}$ cyclotron orbit periods.[13]

It is important, however, to emphasize that any measurement – QND or not – cannot avoid the uncertainty relations between incompatible variables; in the particular case described above, continuous monitoring of the Landau state number n does not allow the simultaneous monitoring of its quantum phase (which may be defined exactly as in the harmonic oscillator). In this context, it is natural to wonder whether the QND measurement concept may be extended from quadratic-form variables like energy to "usual" observables such as coordinates and momenta whose uncertainties are bound by the ordinary Heisenberg's relation (1.35). The answer is yes, but the required methods are a bit more tricky.

For example, let us place an electrically charged particle into a uniform electric field $\boldsymbol{\mathscr{E}} = \mathbf{n}_x \mathscr{E}(t)$ of an instrument, so their interaction Hamiltonian is

$$\hat{H}_{\text{int}} = -q\hat{\mathscr{E}}(t)\hat{x} . \tag{10.17}$$

Such interaction may certainly pass the information on the time evolution of the coordinate x to the instrument. However, in this case, Eq. (12) is *not* satisfied – at least for the kinetic-energy part of the particle's Hamiltonian; as a result, the interaction distorts its time evolution. Indeed, by writing the Heisenberg equation (4.199) for the x-component of the momentum, we get

[12] Here I am simplifying the real situation a bit. Actually, in that experiment, there was an electron spin's contribution to the interaction Hamiltonian as well, but since the used high magnetic field polarized the spins quite reliably, their only effect was a constant shift of the frequency ω_z, which is not important for our discussion.

[13] See also the conceptually similar experiments, performed by different means: G. Nogues *et al.*, *Nature* **400**, 239 (1999).

$$\dot{\hat{p}} - \dot{\hat{p}}\Big|_{\mathscr{E}=0} = q\hat{\mathscr{E}}(t)\,. \tag{10.18}$$

On the other hand, integrating Eq. (5.139) for the coordinate operator evolution,[14] we get the expression

$$\hat{x}(t) = \hat{x}(t_0) + \frac{1}{m}\int_{t_0}^{t} \hat{p}(t')dt'\,, \tag{10.19}$$

which shows that the perturbations (18) of the momentum eventually find their way to the coordinate evolution, not allowing its unperturbed sequential measurements.

However, for such an important particular system as a harmonic oscillator, the following trick is possible. For this system, Eqs. (5.139) with the addition (18) may be readily combined to give a second-order differential equation for the coordinate operator, that is absolutely similar to the classical equation of motion of the system, and has a similar solution:[15]

$$\hat{x}(t) = \hat{x}(t)\Big|_{\mathscr{E}=0} + \frac{q}{m\omega_0}\int_{-\infty}^{t} \hat{\mathscr{E}}(t')\sin\omega_0(t-t')dt'\,. \tag{10.20}$$

This formula confirms that generally, the external field $\mathscr{E}(t)$ (in our case, the sensing field of the measurement instrument) affects the time evolution law – of course. However, Eq. (20) shows that if the field is applied only at moments t'_n separated by intervals $\mathcal{T}/2$, where $\mathcal{T} \equiv 2\pi/\omega_0$ is the oscillation period, its effect on coordinate vanishes at similarly spaced observation instants $t_n = t_{n'} + (m+1/2)\mathcal{T}$. This is the idea of *stroboscopic* QND measurements. Of course, according to Eq. (18), even such measurement perturbs the oscillator momentum, so even if the values x_n are measured with high accuracy, Heisenberg's uncertainty relation is not violated.

For high-frequency systems, direct implementation of stroboscopic measurements is technically complicated, but this initial idea has opened ways to more practicable solutions. For example,[16] it is straightforward to use the Heisenberg equations of motion to show that if the coupling of two harmonic oscillators, with coordinates x and X, and unperturbed frequencies ω and Ω, is modulated in time as

$$\hat{H}_{\text{int}} \propto \hat{x}\hat{X}\cos\omega t\,\cos\Omega t\,, \tag{10.21}$$

then the coupling is reduced to non-reciprocal interaction between the *quadrature components* of the oscillations defined as:[17]

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[14] This simple relation is limited to 1D systems with Hamiltonians of the type (1.41), but by now the reader certainly knows enough to understand that this discussion may be readily generalized to many other systems.

[15] Note in particular that the function $\sin\omega_0\tau$ (with $\tau \equiv t - t'$) under the integral, divided by ω_0, is nothing more than the temporal Green's function $G(\tau)$ of a loss-free harmonic oscillator – see, e.g., CM Sec. 5.1.

[16] See E. Majorana *et al*., *Appl. Phys. B* **64**, 145 (1997). Note that this idea is substantially based on the prior work by K. Thorne *et al*., *Phys. Rev. Lett*. **40**, 667 (1978) – also the detailed review paper C. Caves *et al*., *Rev. Mod. Phys.* **52**, 41 (1980).

[17] The physical sense of these relations should be clear from Fig. 5.8: they define a system of coordinates rotating clockwise with the angular velocity equal to ω, so the point representing unperturbed classical oscillations with that frequency is at rest in this rotating frame. Note that according to the discussion at the end of Sec. 5.5, if an autonomous (stand-alone) oscillator was initially in a squeezed state, the degree of squeezing of the variables x_1 and x_2, i.e. their r.m.s. widths, would not change in time.

$$\hat{x}_1 \equiv \hat{x}\cos\omega t - \frac{\hat{p}}{m\omega}\sin\omega t, \qquad \hat{x}_2 \equiv \hat{x}\sin\omega t + \frac{\hat{p}}{m\omega}\cos\omega t, \tag{10.22}$$

and similarly for the counterpart oscillator. Specifically, $\hat{x}_1$ directly affects the dynamics of only $\hat{X}_2$, while $\hat{X}_1$ directly affects only $\hat{x}_2$. As a result, for example, if both oscillators are in their ground squeezed states with very narrow probability distributions of the quadrature components x_1 and X_2, their interaction (22) would not result in their broadening due to the (inevitably broad) distributions of x_2 and X_1. On this background, a useful "signal", i.e. an additional small deterministic shift of x_1, may be transferred to X_2 without such "quantum noise contamination". Now the oscillators may be decoupled, and X_2 measured (for example, by using the parameter inversion discussed in Sec. 1), without perturbing the initial oscillator's state. Upon restoration of the second oscillator to its initial state (with squeezed X_2), the systems may be reconnected again, and the measurement process repeated (if desired, again and again), without disturbing the squeezed quadrature component x_1.

So, periodic modulation of certain parameters of quantum systems in time may be used for repeated QND measurements. Such measurements were demonstrated first using nonlinear interactions of optical waves.[18] Similar experiments were carried out with optomechanical and electromechanical systems as well.[19] Note that such an approach is not limited to harmonic oscillators, and may be applied, with appropriate modifications, to other quantum objects – notably to two-level (i.e. spin-½-like) systems.[20]

However, if the only goal of a QND measurement is a sensitive measurement of a weak *classical force* acting on a *quantum probe system*, e.g. a 1D oscillator of eigenfrequency ω_0, it may be implemented simpler – just by modulating the oscillator's impedance with a frequency $\omega \approx 2\omega_0$. From the classical dynamics, we know that if the depth of such modulation exceeds a certain threshold value, it results in the excitation of the degenerate parametric oscillations of frequency $\omega/2 \approx \omega_0$, with one of two opposite phases.[21] Close to, but below the excitation threshold, the modulation boosts all fluctuations of the almost-excited quadrature component, including its quantum-mechanical uncertainty, and suppresses (squeezes) those of the counterpart component.

This fact may be conveniently formulated in electronic-engineering terms by using the notion of *noise parameter* Θ_N of a *linear amplifier* – the last term meaning any instrument for continuous monitoring of an input *signal* – e.g., a microwave or optical waveform.[22] Namely, the Θ_N of the system discussed above (called the *degenerate parametric amplifier*), which is sensitive to just one quadrature component of the signal, may have Θ_N well below $\hbar\omega/2$, due to its ground state's squeezing.[23] On the

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[18] See, e.g., the review by P. Grangier *et al.*, *Nature* **396**, 537 (1998).

[19] See, e.g., F. Lecocq *et al.*, *Phys. Rev. X* **5**, 041037 (2015).

[20] See, e.g., D. Averin, *Phys. Rev. Lett.* **88**, 207901 (2002); A. Lupaşcu *et al.*, *Nature Physics* **3**, 119 (2007).

[21] See, e.g., CM Sec. 5.5, and also Fig. 5.8 and its discussion in Sec. 5.6 of this course.

[22] For a quantitative definition of the latter parameter (with the dimensionality of energy), which is suitable for the quantum sensitivity range ($\Theta_N \sim \hbar\omega$), see, e.g., I. Devyatov *et al.*, *J. Appl. Phys*. **60**, 1808 (1986). In the classical noise limit ($\Theta_N >> \hbar\omega$), it coincides with $k_B T_N$, where T_N is a more popular measure of electronics' noise, called the *noise temperature*.

[23] See, e.g., the pioneering experiments with optical waves by R. Slusher *et al.*, *Phys. Rev. Lett.* **55**, 2409 (1985) and with microwaves by B. Yurke *et al.*, *Phys. Rev. Lett*. **60**, 764 (1988). Note also that the squeezed ground states of light are now used to improve the sensitivity of interferometers in gravitational wave detectors – see, e.g., the review by R. Schnabel, *Phys. Repts*. **684**, 1 (2017), and the later papers by F. Acernese *et al.*, *Phys. Rev. Lett.* **123**, 231108 (2019) and D. Ganapathy *et al.*, *Phys. Rev. X* **13**, 041021 (2023).

other hand, "usual" (say, transistor or laser/maser) amplifiers that are equally sensitive to both quadrature components of the signal, Θ_N has the minimum value $\hbar\omega/2$, due to the quantum uncertainty of the quantum state of the amplifier itself – the fact that was recognized already in the early 1960s.[24]

Finally, let me mention that the composite systems consisting of a quantum subsystem and a classical subsystem performing its continuous weakly-perturbing measurement and using its results for providing specially crafted feedback to the quantum subsystem, may have some curious properties, in particular mock a quantum system detached from the environment.[25]

10.3. Hidden variables and local reality

Now we are ready to proceed to the discussion of the last, hardest group (iii) of the questions posed in Sec. 1, namely on the state of a quantum system *just before* its measurement. After a very important but inconclusive discussion of this issue by Albert Einstein and his collaborators on one side, and Niels Bohr on the other side, in the mid-1930s, such discussions resumed in the 1950s.[26] They have led to a key contribution by John Stewart Bell in the early 1960s, summarized as so-called *Bell's inequalities*, and then to experimental work on better and better verification of these inequalities. (Besides that continuing work, the recent progress, in my humble view, has been rather marginal.)

The central question may be formulated as follows: what *had been* the "real" state of a quantum-mechanical system *just before* a virtually perfect single-shot measurement was performed on it, and gave a certain documented outcome? To be specific, let us focus again on the example of Stern-Gerlach measurements of spin-½ particles – because of their conceptual simplicity.[27] For a single-component system (in this case a single spin-½), the answer to the posed question may look evident. Indeed, as we know, if the spin is in a pure (least-uncertain) state α, i.e. its ket-vector may be expressed in the form similar to Eq. (4),

$$\left|\alpha\right\rangle = \alpha_\uparrow\left|\uparrow\right\rangle + \alpha_\downarrow\left|\downarrow\right\rangle, \tag{10.23}$$

where, as usual, $\uparrow$ and $\downarrow$ denote the states with definite spin orientations along the z-axis, the probabilities of the corresponding outcomes of the z-oriented Stern-Gerlach experiment are $W_\uparrow = |\alpha_\uparrow|^2$ and $W_\downarrow = |\alpha_\downarrow|^2$. Then it looks natural to suggest that if a particular experiment gave the outcome corresponding to the state $\uparrow$, the spin had been in that state just before the experiment. For a classical system such an answer would be certainly correct, and the fact that the probability $W_\uparrow = |\alpha_\uparrow|^2$ defined for the statistical ensemble of *all* experiments (regardless of their outcome), may be less than 1, would merely reflect our ignorance about the *real state* of this particular system before the measurement – which just *reveals* the real situation.

However, as was first argued in the famous *EPR paper* published in 1935 by A. Einstein, B. Podolsky, and N. Rosen, such an answer becomes impossible in the case of an entangled quantum

[24] See, e.g., H. Haus and J. Mullen, *Phys. Rev*. **128**, 2407 (1962).

[25] See, e.g., the monograph by H. Wiseman and G. Milburn, *Quantum Measurement and Control*, Cambridge U. Press (2009), more recent experiments by R. Vijay *et al*., *Nature* **490**, 77 (2012), and references therein.

[26] See, e.g., the collection by J. Wheeler and W. Zurek (eds.), *Quantum Theory and Measurement*, Princeton U. Press, 1983.

[27] As was discussed in Sec. 1, the Stern-Gerlach-type experiments may be readily made virtually perfect, provided that we do not care about the evolution of the system *after* the single-shot measurement.

system, if only one of its components is measured with an instrument. The original EPR paper discussed thought experiments with a pair of 1D particles prepared in a quantum state in that both the *sum* of their momenta and the *difference* of their coordinates simultaneously have definite values: $p_1 + p_2 = 0$, $x_1 - x_2 = a$.[28] However, usually this discussion is recast into an equivalent Stern-Gerlach experiment shown in Fig. 4a.[29] A source emits rare pairs of spin-½ particles propagating in opposite directions. The particle spin states are random, but with the net spin of the pair is definitely equal to zero. After the spatial separation of the particles has become sufficiently large (see below), the spin state of each of them is measured with a Stern-Gerlach detector, with one of them (in Fig. 1, SG_1) somewhat closer to the particle source, so it makes the measurement first, at a time $t_1 < t_2$.

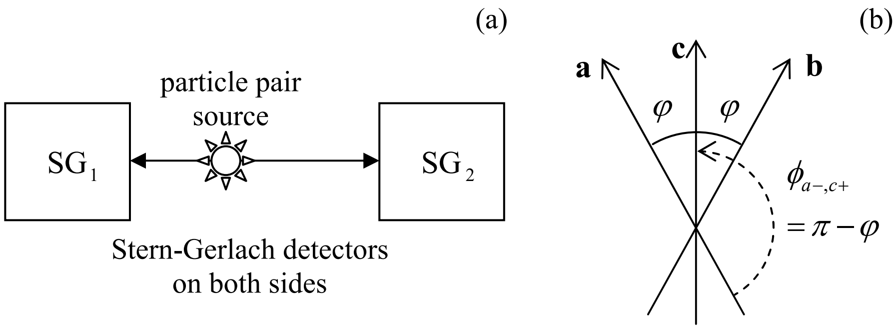

Fig. 10. 4. (a) General scheme of two-particle Stern-Gerlach experiments, and (b) the orientation of the detectors, assumed at Wigner's deviation of Bell's inequality (36).

First, let the detectors be oriented say along the same direction, say the z-axis. Evidently, the probability of each detector giving any of the values $s_z = \pm\hbar/2$ is 50%. However, if the first detector had given the result $S_z = -\hbar/2$, then even before the second detector's measurement, we know that it will give the result $S_z = +\hbar/2$ with 100% probability. So far, this situation still allows for a classical interpretation, just as for the single-particle measurements: we may fancy that the second particle has a definite spin before the measurement, and the first measurement just removes our ignorance about that reality. In other words, the change of the probability of the outcome $S_z = +\hbar/2$ at the second detection from 50% to 100% is due to the statistical ensemble re-definition: the 50% probability of this detection belongs to the ensemble of all experiments, while the 100% probability, to the sub-ensemble of experiments with the $S_z = -\hbar/2$ outcome of the first experiment.

However, let the source generate the spin pairs in the singlet state (8.18):

$$\left| s_{12} \right\rangle = \frac{1}{\sqrt{2}} \left(\left| \uparrow\downarrow \right\rangle - \left| \downarrow\uparrow \right\rangle \right). \tag{10.24}$$

As was discussed in Sec. 8.2, this state satisfies the above assumptions: the probability of each value of S_z of any particle is 50%, and the sum of both S_z is definitely zero, so if the first detector's result is $S_z = -\hbar/2$, then the state of the remaining particle is $\uparrow$, with zero uncertainty.[30] Now let us use Eqs. (4.123) to represent the same state (24) in a different form:

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[28] This is possible because the corresponding operators commute: $\left[\hat{p}_1 + \hat{p}_2, \hat{x}_1 - \hat{x}_2\right] = \left[\hat{p}_1, \hat{x}_1\right] - \left[\hat{p}_2, \hat{x}_2\right] = 0$.

[29] This version was first proposed by D. Bohm in 1951. Another equivalent and experimentally more convenient (and as a result, frequently used) technique is the degenerate parametric excitation of entangled optical photon pairs – see, e.g., the publications cited at the end of this section.

[30] Here we assume that both detectors are perfect in the sense of their readout fidelity. As was discussed in Sec. 1, this condition may be closely approached in practical SG experiments.

$$\left|s_{12}\right\rangle = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(\left|\rightarrow\right\rangle + \left|\leftarrow\right\rangle\right)\frac{1}{\sqrt{2}}\left(\left|\rightarrow\right\rangle - \left|\leftarrow\right\rangle\right) - \frac{1}{\sqrt{2}}\left(\left|\rightarrow\right\rangle - \left|\leftarrow\right\rangle\right)\frac{1}{\sqrt{2}}\left(\left|\rightarrow\right\rangle + \left|\leftarrow\right\rangle\right)\right]. \qquad (10.25)$$

Opening the parentheses (carefully, without swapping the ket-vector order, which encodes the particle numbers!), we get an expression similar to Eq. (24), but now for the x-basis:

$$\left|s_{12}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\rightarrow\leftarrow\right\rangle - \left|\leftarrow\rightarrow\right\rangle\right). \qquad (10.26)$$

Hence if we use the first detector (closest to the particle source) to measure S_x rather than S_z, then after it had given a certain result (say, $S_x = -\hbar/2$), we know for sure, before the second particle spin's measurement, that its S_x component definitely equals $+\hbar/2$.

So, depending on the experiment performed on the first particle, the second particle, before its measurement, may be in one of two states – either with a definite component S_z or with a definite component S_x, in each case with zero uncertainty. Evidently, this situation cannot be interpreted in classical terms – if the particles do not interact during the measurements. A. Einstein was deeply unhappy with this situation because it did not satisfy what, in his view, was the general requirement to any theory, which nowadays is called the *local reality*. His definition of this requirement was as follows: "The real factual situation of system 2 is independent of what is done with system 1 that is spatially separated from the former". (Here the term "spatially separated" is not defined, but from the context, it is clear that Einstein meant the detector separation by a *superluminal interval*, i.e. by distance

$$\left|\mathbf{r}_1 - \mathbf{r}_2\right| > c\left|t_1 - t_2\right|, \qquad (10.27)$$

where the difference between the measurement times on the right-hand side includes the measurement duration.) In Einstein's view, since quantum mechanics did not satisfy the local reality condition, it could not be considered a complete theory of Nature.

This situation naturally raises the question of whether *something* (usually called *hidden variables*) may be added to the quantum-mechanical description to enable it to satisfy the local reality requirement. The first definite statement in this regard was John von Neumann's "proof"[31] (first famous, then infamous :-) that such variables cannot be introduced; for a while, his work satisfied the quantum mechanics' practitioners, who apparently did not pay much attention. A major new contribution to the problem was made only in the 1960s by John Bell.[32] First of all, he has found an elementary (in his words, "foolish") error in von Neumann's logic, which voids his "proof". Second, he has demonstrated that Einstein's local reality condition is *incompatible* with conclusions of quantum mechanics – that had been, by that time, confirmed by too many experiments to be seriously questioned.

Let me describe a particular version of Bell's result (suggested by E. Wigner), using the same EPR pair experiment (Fig. 4a) where each SG detector may be oriented in any of three directions: **a**, **b**, or **c** – see Fig. 4b. As we already know from Chapter 4, if a fully-polarized beam of spin-½ particles is passed through a Stern-Gerlach apparatus forming angle ϕ with the polarization axis, the probabilities of two alternative outcomes of the experiment are

[31] In his very early book J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* [Mathematical Foundations of Quantum Mechanics], Springer, 1932. (The first English translation was published only in 1955.)
[32] See, e.g., either J. Bell, *Rev. Mod. Phys*. **38**, 447 (1966) or J. Bell, *Foundations of Physics* **12**, 158 (1982).

$$W(\phi_+) = \cos^2 \frac{\phi}{2}, \qquad W(\phi_-) = \sin^2 \frac{\phi}{2}. \tag{10.28}$$

Let us use this formula to calculate all joint probabilities of measurement outcomes, starting from the detectors 1 and 2 oriented, respectively, in the directions **a** and **c**. Since the angle between the *negative* direction of the **a**-axis and the *positive* direction of the **c**-axis is $\phi_{\mathbf{a}-,\mathbf{c}+} = \pi - \varphi$ (see the dashed arrow in Fig. 4b), we get

$$W(a_+ \wedge c_+) \equiv W(a_+)W(c_+ | a_+) = W(a_+)W(\phi_{a-,c+}) = \frac{1}{2}\cos^2 \frac{\pi - \varphi}{2} \equiv \frac{1}{2}\sin^2 \frac{\varphi}{2}, \tag{10.29}$$

where $W(x \wedge y)$ is the joint probability of both outcomes x and y, while $W(x | y)$ is the conditional probability of the outcome x, provided that the outcome y has happened.[33] Absolutely similarly,

$$W(c_+ \wedge b_+) \equiv W(c_+)W(b_+ | c_+) = \frac{1}{2}\sin^2 \frac{\varphi}{2}, \tag{10.30}$$

$$W(a_+ \wedge b_+) \equiv W(a_+)W(b_+ | a_+) = \frac{1}{2}\cos^2 \frac{\pi - 2\varphi}{2} \equiv \frac{1}{2}\sin^2 \varphi. \tag{10.31}$$

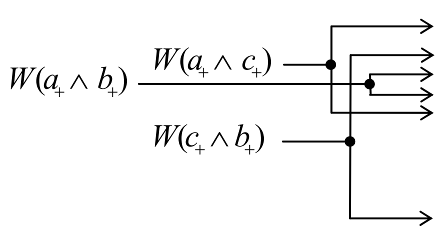
Now note that for any angle φ smaller than $\pi/2$ (as in the case shown in Fig. 4b), trigonometry gives

$$\frac{1}{2}\sin^2 \varphi \geq \frac{1}{2}\sin^2 \frac{\varphi}{2} + \frac{1}{2}\sin^2 \frac{\varphi}{2} \equiv \sin^2 \frac{\varphi}{2}. \tag{10.32}$$

(For example, for $\varphi \to 0$, the left-hand side of this inequality tends to $\varphi^2/2$, while the right-hand side, to $\varphi^2/4$.) Hence the quantum-mechanical result gives, in particular,

$$\boxed{W(a_+ \wedge b_+) \geq W(a_+ \wedge c_+) + W(c_+ \wedge b_+), \qquad \text{for } |\varphi| \leq \pi/2.} \tag{10.33}$$ Quantum-mechanical result

On the other hand, we can get a different inequality for these probabilities without calculating them from any particular theory, but using the local reality assumption. For that, let us prescribe some probability to each of $2^3 = 8$ possible outcomes of a set of three spin measurements. (Due to zero net spin of particle pairs, the probabilities of the sets shown in both columns of the table have to be equal.)



Detector 1	Detector 2	Probability
$a_+ \wedge b_+ \wedge c_+$	$a_- \wedge b_- \wedge c_-$	W_1
$a_+ \wedge b_+ \wedge c_-$	$a_- \wedge b_- \wedge c_+$	W_2
$a_+ \wedge b_- \wedge c_+$	$a_- \wedge b_+ \wedge c_-$	W_3
$a_+ \wedge b_- \wedge c_-$	$a_- \wedge b_+ \wedge c_+$	W_4
$a_- \wedge b_+ \wedge c_+$	$a_+ \wedge b_- \wedge c_-$	W_5
$a_- \wedge b_+ \wedge c_-$	$a_+ \wedge b_- \wedge c_+$	W_6
$a_- \wedge b_- \wedge c_+$	$a_+ \wedge b_+ \wedge c_-$	W_7
$a_- \wedge b_- \wedge c_-$	$a_+ \wedge b_+ \wedge c_+$	W_8

From the local-reality point of view, these measurement options are independent, so we may write (see the arrows on the left of the table):

[33] The first equality in Eq. (29) is the well-known identity of the basic probability theory.

$$W(a_+ \wedge c_+) = W_2 + W_4, \qquad W(c_+ \wedge b_+) = W_3 + W_7, \qquad W(a_+ \wedge b_+) = W_3 + W_4. \tag{10.34}$$

On the other hand, since no probability may be negative (by its very definition), we may always write

$$W_3 + W_4 \leq (W_2 + W_4) + (W_3 + W_7). \tag{10.35}$$

Plugging into this inequality the values of these two parentheses, given by Eq. (34), we get

Bell's inequality (local-reality theory)

$$W(a_+ \wedge b_+) \leq W(a_+ \wedge c_+) + W(c_+ \wedge b_+). \tag{10.36}$$

This is *Bell's inequality*, which has to be satisfied by *any* local-reality theory; it directly contradicts the quantum-mechanical result (33) – opening the issue to direct experimental testing. Such tests were started in the late 1960s, but the first results were vulnerable to two criticisms:

(i) The detectors were not fast enough and not far enough to have the relation (27) satisfied. This is why, as a matter of principle, there was a chance that information on the first measurement outcome had been transferred (by some, mostly implausible) means to particles before the second measurement – the so-called *locality loophole*.

(ii) The particle/photon detection efficiencies were too low to have sufficiently small error bars for both parts of the inequality – the *detection loophole*.

Gradually, these loopholes have been closed.[34] As expected, substantial violations of the Bell inequalities (36) (or their equivalent forms) have been proved, essentially rejecting any possibility to reconcile quantum mechanics with Einstein's local reality requirement.

10.4. Interpretations of quantum mechanics

The fact that quantum mechanics is incompatible with local reality, makes its reconciliation with our (classically bred) "common sense" rather challenging. Here is a brief list of the major interpretations of quantum mechanics, that try to provide at least a partial reconciliation of this kind.

(i) The so-called *Copenhagen interpretation* – to which most physicists adhere. This "interpretation" does not really interpret anything; it just accepts the intrinsic stochasticity of measurement results in quantum mechanics and the absence of local reality, essentially saying: "Do not worry; this is just how it is; live with it". I generally subscribe to this school of thought, with the following qualification. While the Copenhagen interpretation implies statistical ensembles (otherwise, how would you define the probability? – see Sec. 1.3), its most frequently stated formulations[35] do not put sufficient emphasis on their role, in particular on the ensemble re-definition as the only point of human observer's involvement in a nearly-perfect measurement process – see Sec. 1 above. The most famous objection to the Copenhagen interpretation belongs to A. Einstein: "God does not play dice."

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[34] Important milestones in that way were the experiments by A. Aspect *et al., Phys. Rev. Lett.* **49**, 91 (1982) and M. Rowe *et al.*, *Nature* **409**, 791 (2001). Detailed reviews of the experimental situation were given, for example, by M. Genovese, *Phys. Repts.* **413**, 319 (2005) and A. Aspect, *Physics* **8**, 123 (2015); see also the later paper by J. Handsteiner *et al.*, *Phys. Rev. Lett.* **118**, 060401 (2017). Presently, a high-fidelity demonstration of the Bell inequality violation has become a standard test in virtually every experiment with entangled qubits used for quantum encryption research – see Sec. 8.5, and in particular, the paper by J. Lin cited there.

[35] With certain pleasant exceptions – see, e.g. L. Ballentine, *Rev. Mod. Phys*. **42**, 358 (1970).

OK, when Einstein speaks, we all should listen, but perhaps when God speaks (through random results of the same experiment), we have to pay even more attention.

(ii) *Non-local reality*. After the dismissal of J. von Neumann's "proof" by J. Bell, to the best of my knowledge, there has been *no* proof that hidden parameters *could not* be introduced, provided that they do not imply the local reality. Of constructive approaches, perhaps the most notable contribution was made by David Bohm,[36] who developed the initial Louis de Broglie's interpretation of the wavefunction as a "pilot wave", making it quantitative. In the wave-mechanics version of this concept, the wavefunction governed by the Schrödinger equation just guides a "real", point-like classical particle whose coordinates serve as hidden variables. However, this concept does not satisfy the notion of local reality. For example, the measurement of the particle's coordinate at a certain point $\mathbf{r}_1$ has to *instantly* change the wavefunction everywhere in space, including the points $\mathbf{r}_2$ in the superluminal range (27). After A. Einstein's private criticism, D. Bohm essentially abandoned his theory.

(iii) The *many-world interpretation* that was introduced in 1957 by Hugh Everett and popularized in the 1960s and 1970s by Bruce de Witt. In this interpretation, *all* possible measurement outcomes *do* happen, splitting the Universe into the corresponding number of "parallel multiverses", so from one of them, other multiverses and hence other outcomes cannot be observed. Let me leave to the reader an estimate of the rate at which new parallel multiverses have to be constantly generated (say, per second), taking into account that such generation should take place not only at explicit lab experiments but at *every* irreversible process – such as fission of every atomic nucleus or an absorption/emission of every photon, everywhere in each multiverse – whether its result is formally recorded or not. Nicolaas van Kampen has called this a "mind-boggling fantasy".[37] Even the main proponent of this interpretation, B. de Witt has confessed: "The idea is not easy to reconcile with common sense." I agree.

To summarize, as far as I know, neither of these interpretations has yet provided a suggestion on how it might be tested experimentally to exclude the other ones. On the other hand, quantum mechanics makes correct (if sometimes probabilistic) predictions that do not contradict any reliable experimental results we are aware of. Maybe, this is not that bad for a scientific theory.[38]

10.5. Exercise problem

10.1.* The original (circa 1964) J. Bell's inequality was derived for the results of SG measurements performed on two non-interacting particles with zero net spin, by using the following local-reality-based assumption: the result of each single-particle measurement is uniquely determined (besides the experimental setup) by some *c*-number hidden parameter λ that may be random, i.e. change from experiment to experiment. Derive such inequality for the experiment shown in Fig. 4 and compare it with the corresponding quantum-mechanical result for the singlet state (24).

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[36] D. Bohm, *Phys. Rev.* **85**, 165; 180 (1952).

[37] N. van Kampen, *Physica A* **153**, 97 (1988). By the way, I highly recommend the very reasonable summary of the quantum measurement issues, given in this paper, though believe that the quantitative theory of dephasing, discussed in Chapter 7 of this course, might give additional clarity to some of van Kampen's statements.

[38] For the reader who is not satisfied with this "positivistic" approach and wants to improve the situation, my earnest advice is to start not from square one but from reading what other (including some very clever!) people thought about it. The review collection by J. Wheeler and W. Zurek, cited above, may be a good starting point.