Konstantin K. Likharev

Essential Graduate Physics

Lecture Notes and Problems

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Appendix MA

Selected Mathematical Formulas

that are used in this series but not always remembered by students
(and some instructors :-)

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1. Constants

– Euclidean circle's *length-to-diameter ratio*:

$$\pi = 3.141\,592\,653...; \qquad \pi^{1/2} \approx 1.77\,. \tag{1.1}$$

– *Natural logarithm base* (sometimes called the *Euler number* or the *Napier constant*):

$$e \equiv \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.718\,281\,828...\ ; \tag{1.2a}$$

from that value, the logarithm conversion factors are as follows (for any $\xi > 0$):

$$\frac{\ln\xi}{\log_{10}\xi} = \frac{1}{\log_{10}e} \equiv \ln 10 \approx 2.303, \qquad \frac{\log_{10}\xi}{\ln\xi} = \log_{10}e \approx 0.4343\,. \tag{1.2b}$$

– The *Euler* (or "Euler-Mascheroni") *constant*:

$$\gamma \equiv \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}-\ln n\right) = 0.5771566490\ldots; \qquad e^{\gamma} \approx 1.781\,. \tag{1.3}$$

2. Combinatorics, sums, and series

(i) Combinatorics

– The number of different *permutations*, i.e. *ordered* sequences of k elements selected from a set of n distinct elements ($n \geq k$), is

$${}^nP_k \equiv n\cdot(n-1)\cdot...\cdot(n-k+1) = \frac{n!}{(n-k)!}\,; \tag{2.1a}$$

in particular, the number of different permutations of *all* elements of the set ($n = k$) is

$$^{k}P_{k} = k\cdot(k-1)\cdot\ldots\cdot 2\cdot 1 = k! \tag{2.1b}$$

– The number of different *combinations*, i.e. *unordered* sequences of k elements from a set of $n \geq k$ distinct elements, is equal to the *binomial coefficient*

$$^{n}C_{k} \equiv \binom{n}{k} \equiv \frac{^{n}P_{k}}{^{k}P_{k}} = \frac{n!}{k!(n-k)!}. \tag{2.2}$$

In an alternative, very popular "ball/box language", $^{n}C_{k}$ is the number of different ways to put in a box, in an arbitrary order, k balls selected from n distinct balls.

– A generalization of the binomial coefficient notion is the *multinomial coefficient*,

$$^{n}C_{k_1,k_2,\ldots k_l} \equiv \frac{n!}{k_1!k_2!\ldots k_l!}, \qquad \text{with } n = \sum_{j=1}^{l} k_j, \tag{2.3}$$

which, in the standard mathematical language, is a number of different permutations in a multiset of l distinct element types from an n-element set which contains k_j (j = 1, 2,…l) elements of each type. In the less formal "ball/box language", the coefficient (2.3) is the number of different ways to distribute n distinct balls between l distinct boxes, each time keeping the number (k_j) of balls in the j^{th} box fixed, but ignoring their order inside the box. The binomial coefficient $^{n}C_{k}$, given by Eq. (2.2), is a particular case of the multinomial coefficient (2.3) for l = 2 – counting the explicit box for the first one, and the remaining space for the second box, so that if $k_1 \equiv k$, then $k_2 = n - k$.

– One more important combinatorial quantity is the number $M_n^{(k)}$ of different ways to place n *indistinguishable* balls into k distinct boxes. It may be readily calculated from Eq. (2.2) as the number of different ways to select (k – 1) partitions between the boxes in an imagined linear row of (k – 1 + n) "objects" (balls in the boxes *and* partitions between them):

$$M_n^{(k)} = {}^{n-1+k}C_{k-1} \equiv \frac{(k-1+n)!}{(k-1)!n!}. \tag{2.4}$$

(ii) <u>Sums and series</u>

– *Arithmetic progression*:

$$r + 2r + \ldots + nr \equiv \sum_{k=1}^{n} kr = \frac{n(r+nr)}{2}; \tag{2.5a}$$

in particular, at r = 1 it is reduced to the sum of n first natural numbers:

$$1 + 2 + \ldots + n \equiv \sum_{k=1}^{n} k = \frac{n(n+1)}{2}. \tag{2.5b}$$

– Sums of squares and cubes of n first natural numbers:

$$1^2 + 2^2 + \ldots + n^2 \equiv \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}; \tag{2.6a}$$

$$1^3 + 2^3 + \ldots + n^3 \equiv \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}. \tag{2.6b}$$

– The *Riemann zeta function*:

$$\zeta(s) \equiv 1 + \frac{1}{2^s} + \frac{1}{3^s} + ... \equiv \sum_{k=1}^{\infty} \frac{1}{k^s}\,; \tag{2.7a}$$

the particular values frequently met in applications are

$$\zeta\left(\frac{3}{2}\right) \approx 2.612, \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta\left(\frac{5}{2}\right) \approx 1.341, \quad \zeta(3) \approx 1.202, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(5) \approx 1.037. \tag{2.7b}$$

– Finite geometric progression (for real $\lambda \neq 1$):

$$1 + \lambda + \lambda^2 + ... + \lambda^{n-1} \equiv \sum_{k=0}^{n-1} \lambda^k = \frac{1-\lambda^n}{1-\lambda}\,; \tag{2.8a}$$

in particular, if $\lambda^2 < 1$, the progression has a finite limit at $n \to \infty$ (called the *geometric series*):

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \lambda^k \equiv \sum_{k=0}^{\infty} \lambda^k = \frac{1}{1-\lambda}. \tag{2.8b}$$

– *Binomial sum* (also called the "binomial theorem"):

$$(1+a)^n = \sum_{k=0}^{n} {}^nC_k a^k\,, \tag{2.9}$$

where ${}^nC_k$ are the binomial coefficients given by Eq. (2.2).

– The *Stirling formula*:

$$\lim_{n\to\infty} \ln(n!) = n(\ln n - 1) + \frac{1}{2}\ln(2\pi n) + \frac{1}{12n} - \frac{1}{360n^3} + ...\,; \tag{2.10}$$

for most applications in physics, the first term[1] is sufficient.

– The *Taylor* (or "Taylor-Maclaurin") *series*: for any infinitely differentiable function $f(\xi)$:

$$\lim_{\tilde{x}\to 0} f(\xi + \tilde{\xi}) = f(\xi) + \frac{df}{d\xi}(\xi)\,\tilde{\xi} + \frac{1}{2!}\frac{d^2 f}{d\xi^2}(\xi)\tilde{\xi}^2 + ... = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{d^k f}{d\xi^k}(\xi)\,\tilde{\xi}^k\,; \tag{2.11a}$$

note that for many functions this series converges only within a limited, sometimes small range of deviations $\tilde{\xi}$. For a function of several arguments, $f(\xi_1,\xi_2,\dots,\xi_N)$, the first terms of the Taylor series are

$$\lim_{\tilde{\xi}_k\to 0} f(\xi_1 + \tilde{\xi}_1, \xi_2 + \tilde{\xi}_2, ...) = f(\xi_1, \xi_2, ...) + \sum_{k=1}^{N} \frac{\partial f}{\partial \xi_k}(\xi_1, \xi_2, ...)\,\tilde{\xi}_k + \frac{1}{2!}\sum_{k,k'=1}^{N} \frac{\partial^2 f}{\partial_k \xi\, \partial \xi_{k'}}\tilde{\xi}_k\tilde{\xi}_{k'} + ... \tag{2.11b}$$

– The *Euler-Maclaurin formula*, valid for any infinitely differentiable function $f(\xi)$:

$$\sum_{k=1}^{n} f(k) = \int_0^n f(\xi)d\xi + \frac{1}{2}\left[f(n) - f(0)\right] + \frac{1}{6}\cdot\frac{1}{2!}\left[\frac{df}{d\xi}(n) - \frac{df}{d\xi}(0)\right]$$
$$- \frac{1}{30}\cdot\frac{1}{4!}\left[\frac{d^3 f}{d\xi^3}(n) - \frac{d^3 f}{d\xi^3}(0)\right] + \frac{1}{42}\cdot\frac{1}{6!}\left[\frac{d^5 f}{d\xi^5}(n) - \frac{d^5 f}{d\xi^5}(0)\right] + ...; \tag{2.12a}$$

[1] Actually, this leading term was conjectured by A. de Moivre in1733, before J. Stirling's proof of the series.

the coefficients participating in this formula are the so-called *Bernoulli numbers*:[2]

$$B_1 = \frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = \frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}, \quad B_7 = 0, \quad B_8 = \frac{1}{30}, \ldots \tag{2.12b}$$

3. Basic trigonometric functions

– Trigonometric functions of the sum and the difference of two arguments:[3]

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b\,, \tag{3.1a}$$

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b\,. \tag{3.1b}$$

– Sums of two functions of arbitrary arguments:

$$\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{b-a}{2}\,, \tag{3.2a}$$

$$\cos a - \cos b = 2\sin\frac{a+b}{2}\sin\frac{b-a}{2}\,, \tag{3.2b}$$

$$\sin a \pm \sin b = 2\sin\frac{a \pm b}{2}\cos\frac{\pm b - a}{2}\,. \tag{3.2c}$$

– Trigonometric function products:

$$2\cos a\,\cos b = \cos(a+b) + \cos(a-b)\,, \tag{3.3a}$$

$$2\sin a\,\cos b = \sin(a+b) + \sin(a-b)\,, \tag{3.3b}$$

$$2\sin a\,\sin b = \cos(a-b) - \cos(a+b)\,; \tag{3.3c}$$

for the particular case of equal arguments, $b = a$, these three formulas yield the following expressions for the squares of trigonometric functions, and their product:

$$\cos^2 a = \frac{1}{2}(1 + \cos 2a), \qquad \sin a \cos a = \frac{1}{2}\sin 2a, \qquad \sin^2 a = \frac{1}{2}(1 - \cos 2a)\,. \tag{3.3d}$$

– Cubes of trigonometric functions:

$$\cos^3 a = \frac{3}{4}\cos a + \frac{1}{4}\cos 3a, \qquad \sin^3 a = \frac{3}{4}\sin a - \frac{1}{4}\sin 3a. \tag{3.4}$$

– Trigonometric functions of a complex argument:

$$\begin{aligned} \sin(a + ib) &= \sin a\,\cosh b + i\cos a\,\sinh b, \\ \cos\,(a + ib) &= \cos a\,\cosh b - i\sin a\,\sinh b. \end{aligned} \tag{3.5}$$

[2] Note that definitions of B_k (or rather their signs and indices) vary even in the most popular handbooks.

[3] I am confident that the reader is quite capable of deriving the relations (3.1) by representing exponent in the elementary relation $e^{i(a \pm b)} = e^{ia}e^{\pm ib}$ as a sum of its real and imaginary parts, then Eqs. (3.3) directly from Eqs. (3.1), and then Eqs. (3.2) from Eqs. (3.3) by variable replacement; however, I am still providing these formulas to save their time. (Quite a few formulas below are included for of the same reason.)

– Sums of trigonometric functions of n equidistant arguments:

$$\sum_{k=1}^{n}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}k\xi = \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\left(\frac{n+1}{2}\xi\right)\sin\left(\frac{n}{2}\xi\right)\Big/\sin\left(\frac{\xi}{2}\right). \qquad (3.6)$$

4. General differentiation

– Full differential of a product of two functions:

$$d(fg) = (df)g + f(dg). \qquad (4.1)$$

– Full differential of a function of several independent arguments, $f(\xi_1, \xi_2,\ldots, \xi_n)$:

$$df = \sum_{k=1}^{n}\frac{\partial f}{\partial \xi_k}d\xi_k . \qquad (4.2)$$

– Curvature of the Cartesian plot of a smooth function $f(\xi)$:

$$\kappa \equiv \frac{1}{R} = \frac{\left|d^2 f / d\xi^2\right|}{\left[1+\left(df / d\xi\right)^2\right]^{3/2}} . \qquad (4.3)$$

5. General integration

– Integration *by parts*:[4]

$$\int_{g(A)}^{g(B)} f\, dg = fg\Big|_A^B - \int_{f(A)}^{f(B)} g\, df . \qquad (5.1)$$

– Numerical (approximate) integration of 1D functions: the simplest *trapezoidal rule*,

$$\int_a^b f(\xi)d\xi \approx h\left[f\left(a+\frac{h}{2}\right)+f\left(a+\frac{3h}{2}\right)+\ldots+f\left(b-\frac{h}{2}\right)\right] = h\sum_{n=1}^{N} f\left(a-\frac{h}{2}+nh\right), \qquad h\equiv\frac{b-a}{N}. \quad (5.2)$$

has a relatively low accuracy (error of the order of $(h^3/12)d^2f/d\xi^2$ per step), so that the following *Simpson formula*,

$$\int_a^b f(\xi)d\xi \approx \frac{h}{3}\left[f(a)+4f(a+h)+2f(a+2h)+\ldots+4f(b-h)+f(b)\right], \qquad h\equiv\frac{b-a}{2N}, \qquad (5.3)$$

whose error per step scales as $(h^5/180)d^4f/d\xi^4$, is used much more frequently.[5]

[4] This formula immediately follows from Eq. (4.1).

[5] Higher-order formulas (e.g., the *Bode rule*), and other guidance including ready-for-use codes for computer calculations may be found, for example, in the popular reference texts by W. H. Press *et al.*, cited in Sec. 16 below. Besides that, some advanced codes are used as subroutines in the software packages listed in the same section. In some cases, the Euler-Maclaurin formula (2.12) also may be useful for numerical integration.

6. A few 1D integrals[6]

(i) Indefinite integrals

– Integrals with $(1+\xi^2)^{1/2}$:

$$\int \left(1+\xi^2\right)^{1/2} d\xi = \frac{\xi}{2}\left(1+\xi^2\right)^{1/2} + \frac{1}{2}\ln\left|\,\xi + \left(1+\xi^2\right)^{1/2}\right|, \tag{6.1}$$

$$\int \frac{d\xi}{\left(1+\xi^2\right)^{1/2}} = \ln\left|\,\xi + \left(1+\xi^2\right)^{1/2}\right|, \tag{6.2a}$$

$$\int \frac{d\xi}{\left(1+\xi^2\right)^{3/2}} = \frac{\xi}{\left(1+\xi^2\right)^{1/2}}. \tag{6.2b}$$

– Miscellaneous indefinite integrals:

$$\int \frac{d\xi}{\xi\left(\xi^2+2a\xi-1\right)^{1/2}} = \cos^{-1}\frac{a\xi-1}{|\,\xi\,|\left(a^2+1\right)^{1/2}}, \tag{6.3a}$$

$$\int \frac{\left(\sin\xi-\xi\cos\xi\right)^2}{\xi^5}d\xi = \frac{2\xi\sin 2\xi+\cos 2\xi-2\xi^2-1}{8\xi^4}, \tag{6.3b}$$

$$\int \frac{d\xi}{a+b\cos\xi} = \frac{2}{\left(a^2-b^2\right)^{1/2}}\tan^{-1}\left[\frac{(a-b)}{\left(a^2-b^2\right)^{1/2}}\tan\frac{\xi}{2}\right], \qquad \text{for } a^2>b^2. \tag{6.3c}$$

$$\int \frac{d\xi}{1+\xi^2} = \tan^{-1}\xi. \tag{6.3d}$$

(ii) Semi-definite integrals:

– Integrals with $1/(e^{\xi}\pm 1)$:

$$\int_a^{\infty} \frac{d\xi}{e^{\xi}+1} = \ln\left(1+e^{-a}\right), \tag{6.4a}$$

$$\int_{a>0}^{\infty} \frac{d\xi}{e^{\xi}-1} = \ln\frac{1}{1-e^{-a}}. \tag{6.4b}$$

(iii) Definite integrals

– Integrals with $1/(1+\xi^2)$:[7]

$$\int_0^{\infty} \frac{d\xi}{1+\xi^2} = \frac{\pi}{2}, \tag{6.5a}$$

[6] A powerful (and free :-) interactive online tool for working out indefinite 1D integrals is available at http://integrals.wolfram.com/index.jsp.

[7] Eq. (6.5a) follows immediately from Eq. (6.3d), and Eq. (6.5c) from Eq. (6.2b) – more examples of the (intentional) redundancies in this list.

$$\int_0^\infty \frac{d\xi}{(1+\xi^2)^n} = \frac{\pi}{2}\frac{(2n-3)!!}{(2n-2)!!} \equiv \frac{\pi}{2}\frac{1\cdot 3\cdot 5\ldots(2n-3)}{2\cdot 4\cdot 6\ldots(2n-2)}, \qquad \text{for } n = 2,3,\ldots; \tag{6.5b}$$

$$\int_0^\infty \frac{d\xi}{(1+\xi^2)^{3/2}} = 1, \tag{6.5c}$$

$$\int_0^\infty \frac{d\xi}{(1+\xi^2)^{n+1/2}} = \frac{(2n-2)!!}{(2n-1)!!} \equiv \frac{2\cdot 4\cdot 6\ldots(2n-2)}{3\cdot 5\cdot 7\ldots(2n-1)}, \qquad \text{for } n = 2,3,\ldots. \tag{6.5d}$$

– Integrals with $(1-\xi^{2s})^{1/2}$:

$$\int_0^1 \frac{d\xi}{(1-\xi^{2s})^{1/2}} = \frac{\pi^{1/2}}{2s}\Gamma\left(\frac{1}{2s}\right)\Big/\Gamma\left(\frac{s+1}{2s}\right), \tag{6.6a}$$

$$\int_0^1 (1-\xi^{2s})^{1/2} d\xi = \frac{\pi^{1/2}}{4s}\Gamma\left(\frac{1}{2s}\right)\Big/\Gamma\left(\frac{3s+1}{2s}\right), \tag{6.6b}$$

where $\Gamma(s)$ is the *gamma function,* which is most often defined (for Re $s > 0$) by the following integral:

$$\int_0^\infty \xi^{s-1} e^{-\xi} d\xi = \Gamma(s). \tag{6.7a}$$

The key property of this function is the recurrence relation, which is valid for any $s \neq 0, -1, -2, \ldots$:

$$\Gamma(s+1) = s\Gamma(s). \tag{6.7b}$$

Since, according to Eq. (6.7a), $\Gamma(1) = 1$, Eq. (6.7b) for non-negative integers takes the form

$$\Gamma(n+1) = n!, \quad \text{for } n = 0, 1, 2,... \tag{6.7c}$$

(where $0! \equiv 1$). Because of this, for integer $s = n + 1 \geq 1$, Eq. (6.7a) reduces to

$$\int_0^\infty \xi^n e^{-\xi} d\xi = n!. \tag{6.7d}$$

Other frequently met values of the gamma function are those for positive semi-integer values:

$$\Gamma\left(\frac{1}{2}\right) = \pi^{1/2}, \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\pi^{1/2}, \quad \Gamma\left(\frac{5}{2}\right) = \frac{1}{2}\cdot\frac{3}{2}\pi^{1/2}, \quad \Gamma\left(\frac{7}{2}\right) = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\pi^{1/2}, \; \ldots \tag{6.7e}$$

– Integrals with $1/(e^\xi \pm 1)$:

$$\int_0^\infty \frac{\xi^{s-1} d\xi}{e^\xi + 1} = \left(1 - 2^{1-s}\right)\Gamma(s)\zeta(s), \quad \text{for } s > 0, \tag{6.8a}$$

$$\int_0^\infty \frac{\xi^{s-1} d\xi}{e^\xi - 1} = \Gamma(s)\zeta(s), \quad \text{for } s > 1, \tag{6.8b}$$

where $\zeta(s)$ is the Riemann zeta-function – see Eq. (2.6). Particular cases: for $s = 2n$,

$$\int_0^\infty \frac{\xi^{2n-1}d\xi}{e^\xi+1} = \frac{2^{2n-1}-1}{2n}\pi^{2n}B_{2n}, \tag{6.8c}$$

$$\int_0^\infty \frac{\xi^{2n-1}d\xi}{e^\xi-1} = \frac{(2\pi)^{2n}}{4n}B_{2n}. \tag{6.8d}$$

where B_n are the Bernoulli numbers – see Eq. (2.12). For the particular case $s = 1$ (when Eq. (6.8a) yields uncertainty),

$$\int_0^\infty \frac{d\xi}{e^\xi+1} = \ln 2. \tag{6.8e}$$

– Integrals with $\exp\{-\xi^2\}$:

$$\int_0^\infty \xi^s e^{-\xi^2}d\xi = \frac{1}{2}\Gamma\left(\frac{s+1}{2}\right), \quad \text{for } s > -1; \tag{6.9a}$$

for applications the most important particular values of s are 0 and 2:

$$\int_0^\infty e^{-\xi^2}d\xi = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\pi^{1/2}}{2}, \tag{6.9b}$$

$$\int_0^\infty \xi^2 e^{-\xi^2}d\xi = \frac{1}{2}\Gamma\left(\frac{3}{2}\right) = \frac{\pi^{1/2}}{4}, \tag{6.9c}$$

though we will also run into the cases $s = 4$ and $s = 6$:

$$\int_0^\infty \xi^4 e^{-\xi^2}d\xi = \frac{1}{2}\Gamma\left(\frac{5}{2}\right) = \frac{3\pi^{1/2}}{8}, \qquad \int_0^\infty \xi^6 e^{-\xi^2}d\xi = \frac{1}{2}\Gamma\left(\frac{7}{2}\right) = \frac{15\pi^{1/2}}{16}; \tag{6.9d}$$

for odd integer values $s = 2n + 1$ (with $n = 0, 1, 2,\ldots$), Eq. (6.9a) takes a simpler form:

$$\int_0^\infty \xi^{2n+1} e^{-\xi^2}d\xi = \frac{1}{2}\Gamma(n+1) = \frac{n!}{2}. \tag{6.9e}$$

– Integrals with cosine and sine functions:

$$\int_0^\infty \cos(\xi^2)d\xi = \int_0^\infty \sin(\xi^2)d\xi = \left(\frac{\pi}{8}\right)^{1/2}. \tag{6.10}$$

$$\int_0^\infty \frac{\cos\xi}{a^2+\xi^2}d\xi = \frac{\pi}{2|a|}e^{-|a|}. \tag{6.11}$$

$$\int_0^\infty \frac{\sin\xi}{\xi}d\xi = \int_0^\infty \left(\frac{\sin\xi}{\xi}\right)^2 d\xi = \frac{\pi}{2}. \tag{6.12}$$

– Integrals with logarithms:

$$\int_0^1 \ln\frac{a+(1-\xi^2)^{1/2}}{a-(1-\xi^2)^{1/2}}d\xi = \pi\left[a-(a^2-1)^{1/2}\right], \quad \text{for } a \geq 1. \tag{6.13}$$

$$\int_0^1 \ln \frac{1+(1-\xi)^{1/2}}{\xi^{1/2}} d\xi = 1 . \qquad (6.14)$$

– Integral representations of the Bessel functions of integer order:

$$J_n(\alpha) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{i(\alpha \sin \xi - n\xi)} d\xi \quad \text{and hence} \quad e^{i\alpha \sin \xi} = \sum_{k=-\infty}^{\infty} J_k(\alpha) e^{ik\xi} ; \qquad (6.15a)$$

$$I_n(\alpha) = \frac{1}{\pi} \int_0^{\pi} e^{\alpha \cos \xi} \cos n\xi \, d\xi . \qquad (6.15b)$$

7. 3D vector products

(i) Definitions:

– *Scalar* ("dot-") *product*:

$$\mathbf{a} \cdot \mathbf{b} = \sum_{j=1}^{3} a_j b_j , \qquad (7.1)$$

where a_j and b_j are vector components in any orthogonal coordinate system. In particular, the vector squared (the same as its norm squared) is the following scalar:

$$a^2 \equiv \mathbf{a} \cdot \mathbf{a} = \sum_{j=1}^{3} a_j^2 \equiv \| \mathbf{a} \|^2 . \qquad (7.2)$$

– *Vector* ("cross-") *product*:

$$\mathbf{a} \times \mathbf{b} \equiv \mathbf{n}_1 (a_2 b_3 - a_3 b_2) + \mathbf{n}_2 (a_3 b_1 - a_1 b_3) + \mathbf{n}_3 (a_1 b_2 - a_2 b_1) = \begin{vmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} , \qquad (7.3)$$

where $\{\mathbf{n}_j\}$ is the set of mutually perpendicular unit vectors[8] along the corresponding coordinate system axes.[9] In particular, Eq. (7.3) yields

$$\mathbf{a} \times \mathbf{a} = 0. \qquad (7.4)$$

(ii) Corollaries (readily verified by Cartesian components):

– Double vector product (the so-called *bac minus cab* rule):

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) . \qquad (7.5)$$

– Mixed scalar-vector product (the *operand rotation rule*):

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) . \qquad (7.6)$$

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[8] Other popular notations for this vector set are $\{\mathbf{e}_j\}$ and $\{\hat{\mathbf{r}}_j\}$.

[9] It is easy to use Eq. (7.3) to check that the direction of the product vector corresponds to the well-known "right-hand rule" and to the even more convenient *corkscrew rule*: if we rotate a corkscrew's handle from the first operand toward the second one, its axis moves in the direction of the product.

– Scalar product of vector products:

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}); \qquad (7.7a)$$

in the particular case of two similar operands (say, **a** = **c** and **b** = **d**), the last formula is reduced to

$$(\mathbf{a}\times\mathbf{b})^2 = (ab)^2 - (\mathbf{a}\cdot\mathbf{b})^2 . \qquad (7.7b)$$

8. Differentiation in 3D Cartesian coordinates

– Definition of the *del* (or "nabla") vector operator ∇: [10]

$$\nabla \equiv \sum_{j=1}^{3} \mathbf{n}_j \frac{\partial}{\partial r_j}, \qquad (8.1)$$

where r_j is a set of linear and orthogonal (*Cartesian*) coordinates along directions $\mathbf{n}_j$. In accordance with this definition, the operator ∇ acting on a *scalar* function of coordinates, $f(\mathbf{r})$,[11] gives its gradient, i.e. a new *vector*:

$$\nabla f \equiv \sum_{j=1}^{3} \mathbf{n}_j \frac{\partial f}{\partial r_j} \equiv \mathbf{grad}\, f . \qquad (8.2)$$

– The *scalar product* of del by a *vector* function of coordinates (a *vector field*),

$$\mathbf{f}(\mathbf{r}) \equiv \sum_{j=1}^{3} \mathbf{n}_j f_j(\mathbf{r}), \qquad (8.3)$$

compiled by formally following Eq. (7.1), is a *scalar* function – the *divergence* of the initial function:

$$\nabla\cdot\mathbf{f} \equiv \sum_{j=1}^{3} \frac{\partial f_j}{\partial r_j} \equiv \operatorname{div}\mathbf{f}, \qquad (8.4)$$

while the *vector product* of ∇ and **f**, formed in a formal accordance with Eq. (7.3), is a new vector – the **curl** (in European tradition, called *rotor* and denoted **rot**) of **f**:

$$\nabla\times\mathbf{f} \equiv \begin{vmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ \dfrac{\partial}{\partial r_1} & \dfrac{\partial}{\partial r_2} & \dfrac{\partial}{\partial r_3} \\ f_1 & f_2 & f_3 \end{vmatrix} = \mathbf{n}_1\left(\frac{\partial f_3}{\partial r_2} - \frac{\partial f_2}{\partial r_3}\right) + \mathbf{n}_2\left(\frac{\partial f_1}{\partial r_3} - \frac{\partial f_3}{\partial r_1}\right) + \mathbf{n}_3\left(\frac{\partial f_2}{\partial r_1} - \frac{\partial f_1}{\partial r_2}\right) \equiv \mathbf{curl}\,\mathbf{f}. \qquad (8.5)$$

– One more frequently met "product" is $(\mathbf{f}\cdot\nabla)\mathbf{g}$, where **f** and **g** are two arbitrary vector functions of **r**. This product should be also understood in the sense implied by Eq. (7.1), i.e. as a vector whose j^{th} Cartesian component is

$$[(\mathbf{f}\cdot\nabla)\mathbf{g}]_j = \sum_{j'=1}^{3} f_{j'} \frac{\partial g_j}{\partial r_{j'}}. \qquad (8.5)$$

[10] One can run into the following notation: $\nabla \equiv \partial/\partial\mathbf{r}$, which is convenient in some cases, but may be misleading in quite a few others, so it will be not used in this series.

[11] In this, and four next sections, all scalar and vector functions are assumed to be differentiable.

9. The Laplace operator $\nabla^2 \equiv \nabla \cdot \nabla$

– Expression in Cartesian coordinates – in the formal accordance with Eq. (7.2):

$$\nabla^2 = \sum_{j=1}^{3} \frac{\partial^2}{\partial r_j^2}. \tag{9.1}$$

– According to its definition, the Laplace operator acting on a *scalar* function of coordinates gives a new scalar function:

$$\nabla^2 f \equiv \nabla \cdot (\nabla f) = \text{div}(\textbf{grad } f) = \sum_{j=1}^{3} \frac{\partial^2 f}{\partial r_j^2}. \tag{9.2}$$

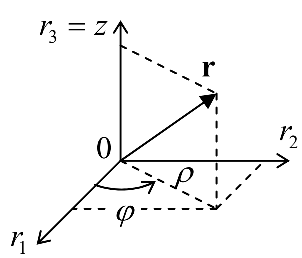
– On the other hand, acting on a *vector* function (8.3), the operator ∇^2 returns another *vector*:

$$\nabla^2 \mathbf{f} = \sum_{j=1}^{3} \mathbf{n}_j \nabla^2 f_j \ . \tag{9.3}$$

Note that Eqs. (9.1)-(9.3) are only valid in Cartesian (i.e. orthogonal and linear) coordinates, but generally not in other orthogonal coordinates – see, e.g., Eqs. (10.3), (10.6), (10.9) and (10.12) below.

10. Operators ∇ and ∇^2 in the most important systems of orthogonal coordinates[12]

(i) Cylindrical[13] coordinates $\{\rho, \varphi, z\}$ (see Fig. below) may be defined by their relations with the Cartesian coordinates:



$$r_1 = \rho \cos\varphi, \quad r_2 = \rho \sin\varphi, \quad r_3 = z. \tag{10.1}$$

– Gradient of a scalar function:

$$\nabla f = \mathbf{n}_\rho \frac{\partial f}{\partial \rho} + \mathbf{n}_\varphi \frac{1}{\rho}\frac{\partial f}{\partial \varphi} + \mathbf{n}_z \frac{\partial f}{\partial z}. \tag{10.2}$$

– The Laplace operator of a scalar function:

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}, \tag{10.3}$$

– Divergence of a vector function of coordinates ($\mathbf{f} = \mathbf{n}_\rho f_\rho + \mathbf{n}_\varphi f_\varphi + \mathbf{n}_z f_z$):

$$\nabla \cdot \mathbf{f} = \frac{1}{\rho}\frac{\partial(\rho f_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial f_\varphi}{\partial \varphi} + \frac{\partial f_z}{\partial z}. \tag{10.4}$$

[12] Some other orthogonal curvilinear coordinate systems are discussed in EM Sec. 2.3.
[13] In the 2D geometry with fixed coordinate z, these coordinates are called *polar*.

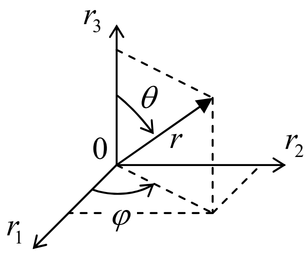
– Curl of a vector function:

$$\nabla \times \mathbf{f} = \mathbf{n}_\rho \left(\frac{1}{\rho} \frac{\partial f_z}{\partial \varphi} - \frac{\partial f_\varphi}{\partial z} \right) + \mathbf{n}_\varphi \left(\frac{\partial f_\rho}{\partial z} - \frac{\partial f_z}{\partial \rho} \right) + \mathbf{n}_z \frac{1}{\rho} \left(\frac{\partial (\rho f_\varphi)}{\partial \rho} - \frac{\partial f_\rho}{\partial \varphi} \right). \tag{10.5}$$

– The Laplace operator of a vector function:

$$\nabla^2 \mathbf{f} = \mathbf{n}_\rho \left(\nabla^2 f_\rho - \frac{1}{\rho^2} f_\rho - \frac{2}{\rho^2} \frac{\partial f_\varphi}{\partial \varphi} \right) + \mathbf{n}_\varphi \left(\nabla^2 f_\varphi - \frac{1}{\rho^2} f_\varphi + \frac{2}{\rho^2} \frac{\partial f_\rho}{\partial \varphi} \right) + \mathbf{n}_z \nabla^2 f_z . \tag{10.6}$$

(ii) <u>*Spherical coordinates*</u> $\{r, \theta, \varphi\}$ (see Fig. below) may be defined as:



$$\begin{aligned} r_1 &= r \sin\theta \cos\varphi, \\ r_2 &= r \sin\theta \sin\varphi, \\ r_3 &= r \cos\theta. \end{aligned} \tag{10.7}$$

– Gradient of a scalar function:

$$\nabla f = \mathbf{n}_r \frac{\partial f}{\partial r} + \mathbf{n}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} + \mathbf{n}_\varphi \frac{1}{r \sin\theta} \frac{\partial f}{\partial \varphi}. \tag{10.8}$$

– The Laplace operator of a scalar function:

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{(r \sin\theta)^2} \frac{\partial^2 f}{\partial \varphi^2} . \tag{10.9}$$

– Divergence of a vector function $\mathbf{f} = \mathbf{n}_r f_r + \mathbf{n}_\theta f_\theta + \mathbf{n}_\varphi f_\varphi$:

$$\nabla \cdot \mathbf{f} = \frac{1}{r^2} \frac{\partial (r^2 f_r)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial (f_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial f_\varphi}{\partial \varphi} . \tag{10.10}$$

– Curl of the similar vector function:

$$\nabla \times \mathbf{f} = \mathbf{n}_r \frac{1}{r \sin\theta} \left(\frac{\partial (f_\varphi \sin\theta)}{\partial \theta} - \frac{\partial f_\theta}{\partial \varphi} \right) + \mathbf{n}_\theta \frac{1}{r} \left(\frac{1}{\sin\theta} \frac{\partial f_r}{\partial \varphi} - \frac{\partial (r f_\varphi)}{\partial r} \right) + \mathbf{n}_\varphi \frac{1}{r} \left(\frac{\partial (r f_\theta)}{\partial r} - \frac{\partial f_r}{\partial \theta} \right). \tag{10.11}$$

– The Laplace operator of a vector function:

$$\begin{aligned} \nabla^2 \mathbf{f} = {} & \mathbf{n}_r \left(\nabla^2 f_r - \frac{2}{r^2} f_r - \frac{2}{r^2 \sin\theta} \frac{\partial}{\partial \theta} (f_\theta \sin\theta) - \frac{2}{r^2 \sin\theta} \frac{\partial f_\varphi}{\partial \varphi} \right) \\ & + \mathbf{n}_\theta \left(\nabla^2 f_\theta - \frac{1}{r^2 \sin^2\theta} f_\theta + \frac{2}{r^2} \frac{\partial f_r}{\partial \theta} - \frac{2 \cos\theta}{r^2 \sin^2\theta} \frac{\partial f_\varphi}{\partial \varphi} \right) \\ & + \mathbf{n}_\varphi \left(\nabla^2 f_\varphi - \frac{1}{r^2 \sin^2\theta} f_\varphi + \frac{2}{r^2 \sin\theta} \frac{\partial f_r}{\partial \varphi} + \frac{2 \cos\theta}{r^2 \sin^2\theta} \frac{\partial f_\theta}{\partial \varphi} \right). \end{aligned} \tag{10.12}$$

11. Products involving ∇

(i) Useful zeros:

– For any scalar function $f(\mathbf{r})$,

$$\nabla \times (\nabla f) \equiv \mathbf{curl}\,(\mathbf{grad}\, f) = 0. \tag{11.1}$$

– For any vector function $\mathbf{f}(\mathbf{r})$,

$$\nabla \cdot (\nabla \times \mathbf{f}) \equiv \mathrm{div}\,(\mathbf{curl}\,\mathbf{f}) = 0. \tag{11.2}$$

(ii) The Laplace operator expressed via the curl of a curl:

$$\nabla^2 \mathbf{f} = \nabla(\nabla \cdot \mathbf{f}) - \nabla \times (\nabla \times \mathbf{f}). \tag{11.3}$$

(iii) Spatial differentiation of a product of a scalar function by a vector function:

– The scalar 3D generalization of Eq. (4.1) is

$$\nabla \cdot (f\,\mathbf{g}) = (\nabla f) \cdot \mathbf{g} + f(\nabla \cdot \mathbf{g}). \tag{11.4a}$$

– Its vector generalization is similar:

$$\nabla \times (f\,\mathbf{g}) = (\nabla f) \times \mathbf{g} + f(\nabla \times \mathbf{g}). \tag{11.4b}$$

(iv) 3D spatial differentiation of products of two vector functions:

$$\nabla \times (\mathbf{f} \times \mathbf{g}) = \mathbf{f}(\nabla \cdot \mathbf{g}) - (\mathbf{f} \cdot \nabla)\mathbf{g} - (\nabla \cdot \mathbf{f})\mathbf{g} + (\mathbf{g} \cdot \nabla)\mathbf{f}, \tag{11.5}$$

$$\nabla(\mathbf{f} \cdot \mathbf{g}) = (\mathbf{f} \cdot \nabla)\mathbf{g} + (\mathbf{g} \cdot \nabla)\mathbf{f} + \mathbf{f} \times (\nabla \times \mathbf{g}) + \mathbf{g} \times (\nabla \times \mathbf{f}), \tag{11.6}$$

$$\nabla \cdot (\mathbf{f} \times \mathbf{g}) = \mathbf{g} \cdot (\nabla \times \mathbf{f}) - \mathbf{f} \cdot (\nabla \times \mathbf{g}). \tag{11.7}$$

12. Integro-differential relations

(i) For an arbitrary surface *S* limited by closed contour *C*:

– The *Stokes theorem*, valid for any differentiable vector field $\mathbf{f}(\mathbf{r})$:

$$\int_S (\nabla \times \mathbf{f}) \cdot d^2\mathbf{r} \equiv \int_S (\nabla \times \mathbf{f})_n d^2 r = \oint_C \mathbf{f} \cdot d\mathbf{r} \equiv \oint_C f_\tau dr, \tag{12.1}$$

where $d^2\mathbf{r} \equiv \mathbf{n} d^2 r$ is the elementary area vector (normal to the surface), and $d\mathbf{r}$ is the elementary contour length vector (tangential to the contour line).

(ii) For an arbitrary volume *V* limited by closed surface *S*:

– *Divergence* (or "Gauss") *theorem*, valid for any differentiable vector field $\mathbf{f}(\mathbf{r})$:

$$\int_V (\nabla \cdot \mathbf{f})\, d^3 r = \oint_S \mathbf{f} \cdot d^2\mathbf{r} \equiv \oint_S f_n d^2 r. \tag{12.2}$$

– *Green's theorem*, valid for two differentiable scalar functions $f(\mathbf{r})$ and $g(\mathbf{r})$:

$$\int_V \left(f\,\nabla^2 g - g\nabla^2 f\right) d^3 r = \oint_S \left(f\,\nabla g - g\nabla f\right)_n d^2 r. \tag{12.3}$$

– An identity valid for any two scalar functions *f* and *g*, and a vector field **j** with $\nabla\cdot\mathbf{j} = 0$ (all differentiable):

$$\int_V \left[f(\mathbf{j}\cdot\nabla g) + g(\mathbf{j}\cdot\nabla f) \right] d^3r = \oint_S fgj_n d^2r \,. \tag{12.3}$$

13. The Kronecker delta and Levi-Civita permutation symbols

– The *Kronecker delta symbol* (defined for integer indices):

$$\delta_{jj'} \equiv \begin{cases} 1, & \text{if } j' = j, \\ 0, & \text{otherwise.} \end{cases} \tag{13.1}$$

– The *Levi-Civita permutation symbol* for three integer indices (each taking one of the values 1, 2, or 3):

$$\varepsilon_{jj'j''} \equiv \begin{cases} +1, & \text{if the indices follow in any "correct" ("even") order}: 1 \to 2 \to 3 \to 1 \to 2..., \\ -1, & \text{if the indices follow in any "incorrect" ("odd") order}: 1 \to 3 \to 2 \to 1 \to 3..., \\ 0, & \text{if any two indices coincide.} \end{cases} \tag{13.2}$$

– Relation between the products of the Levi-Civita and Kronecker symbols:

$$\varepsilon_{jj'j''}\varepsilon_{kk'k''} = \sum_{l,l',l''=1}^{3} \begin{vmatrix} \delta_{jl} & \delta_{jl'} & \delta_{jl''} \\ \delta_{j'l} & \delta_{j'l'} & \delta_{j'l''} \\ \delta_{j''l} & \delta_{j''l'} & \delta_{j''l''} \end{vmatrix}; \tag{13.3a}$$

the summation of three such relations written for three different values of $j = k$ yields the so-called *contracted epsilon identity*:

$$\sum_{j=1}^{3} \varepsilon_{jj'j''}\varepsilon_{jk'k''} = \delta_{j'k'}\delta_{j''k''} - \delta_{j'k''}\delta_{j''k'} \,. \tag{13.3b}$$

14. The Dirac delta function, sign function, and step function

– Definition of 1D *delta function* (for real $a < b$):

$$\int_a^b f(\xi)\delta(\xi)d\xi = \begin{cases} f(0), & \text{if } a < 0 < b, \\ 0, & \text{otherwise,} \end{cases} \tag{14.1}$$

where $f(\xi)$ is any function continuous near $\xi = 0$. In particular (if $f(\xi) = 1$ near $\xi = 0$),

$$\int_a^b \delta(\xi)d\xi = \begin{cases} 1, & \text{if } a < 0 < b, \\ 0, & \text{otherwise.} \end{cases} \tag{14.2}$$

– Relation to the *Heaviside step function* $\theta(\xi)$ and the *sign function* $\text{sgn}(\xi)$

$$\delta(\xi) = \frac{d}{d\xi}\theta(\zeta) = \frac{1}{2}\frac{d}{d\xi}\text{sgn}(\xi) \,, \tag{14.3a}$$

where

$$\theta(\xi) \equiv \frac{\operatorname{sgn}(\xi)+1}{2} = \begin{cases} 0, & \text{if } \xi < 0, \\ 1, & \text{if } \xi > 1, \end{cases} \qquad \operatorname{sgn}(\xi) \equiv \frac{\xi}{|\xi|} = \begin{cases} -1, & \text{if } \xi < 0, \\ +1, & \text{if } \xi > 1. \end{cases} \tag{14.3b}$$

– An important integral:[14]

$$\int_{-\infty}^{+\infty} e^{is\xi} ds = 2\pi\delta(\xi) . \tag{14.4}$$

– 3D generalization: the delta function $\delta(\mathbf{r})$ of the radius-vector is defined as

$$\int_V f(\mathbf{r})\delta(\mathbf{r})d^3r = \begin{cases} f(0), & \text{if } 0 \in V, \\ 0, & \text{otherwise;} \end{cases} \tag{14.5}$$

it may be represented as a product of 1D delta functions of Cartesian coordinates:

$$\delta(\mathbf{r}) = \delta(r_1)\delta(r_2)\delta(r_3) . \tag{14.6}$$

(The 2D generalization is similar.)

15. The Cauchy theorem and integral

Let a complex function $f(z)$ be analytic within a part of the complex plane z, which is limited by a closed contour C and includes point z'. Then

$$\oint_C f(z)dz = 0 , \tag{15.1}$$

$$\oint_C f(z)\frac{dz}{z-z'} = 2\pi i f(z'). \tag{15.2}$$

The first of these relations is usually called the *Cauchy integral theorem* (or the "Cauchy-Goursat theorem"), and the second one, the *Cauchy integral* (or the "Cauchy integral formula").

16. References

(i) Properties of some *special functions* are briefly discussed at the relevant points of the lecture notes (in alphabetical order):

– *Airy functions*: QM Sec. 2.4;
– *Bessel functions*: EM Sec. 2.7;
– *Fresnel integrals*: EM Sec. 8.6;
– *Hermite polynomials*: QM Sec. 2.9;

[14] The coefficient in this relation may be readily recalled by considering its left-hand side as the Fourier-integral representation of the function $f(s) \equiv 1$, and applying Eq. (14.1) to the reciprocal Fourier transform:

$$f(s) \equiv 1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-is\xi}\left[2\pi\delta(\xi)\right]d\xi.$$

– *Laguerre polynomials* (both *simple* and *associated*): QM Sec. 3.7;
– *Legendre polynomials*, *associated Legendre functions*: EM Sec. 2.8 and QM Sec. 3.6;
– *Spherical Bessel functions*: QM Secs. 3.6 and 3.8;
– *Spherical harmonics*: QM Sec. 3.6.

(ii) For *more formulas* and their discussions, I can recommend the following handbooks (in alphabetical order):[15]

– M. Abramowitz and I. Stegun (eds.), *Handbook of Mathematical Formulas*, Dover, 1965;[16]
– I. Gradshteyn and I. Ryzhik, *Tables of Integrals, Series, and Products*, 5th ed., Acad. Press, 1980;
– G. Korn and T. Korn, *Mathematical Handbook for Scientists and Engineers*, 2nd ed., Dover, 2000;
– A. Prudnikov *et al*., *Integrals and Series*, vols. 1 and 2, CRC Press, 1986.

The popular textbook

– G. Arfken *et al., Mathematical Methods for Physicists*, 7th ed., Acad. Press, 2012

may be also used as a formula manual.

Many formulas are also available from the symbolic calculation parts of commercially available software packages listed in Sec. (iv) below.

(iii) Probably the most popular collection of *numerical calculation codes* are the twin manuals

– W. Press *et al.*, *Numerical Recipes in Fortran 77*, 2nd ed., Cambridge U. Press, 1992;
– W. Press *et al.*, *Numerical Recipes* [in C++ – KKL], 3rd ed., Cambridge U. Press, 2007.

These lecture notes include very brief introductions into numerical methods of differential equation solution:

– ordinary differential equations: CM Sec. 5.7, and
– partial differential equations: CM Sec. 8.5 and EM Sec. 2.11,

which include references to the literature for further reading.

(iv) The most popular *software packages* for numerical and symbolic calculations, all with plotting capabilities (in alphabetical order):

– *Maple* (http://www.maplesoft.com/);
– *MathCAD* (http://www.ptc.com/products/mathcad/);
– *Mathematica* (http://www.wolfram.com/products/mathematica/index.html);
– *MATLAB* (http://www.mathworks.com/products/matlab/);
– *Maxima* (https://maxima.sourceforge.io/index.html).

[15] On a personal note, perhaps 90% of all formula needs throughout my research career were satisfied by a wonderfully compiled old book: H. Dwight, *Tables of Integrals and Other Mathematical Data*, 4th ed., Macmillan, 1961. Its used copies, rather amazingly, are still available online.
[16] See also later printings; an updated version of this collection is now available online at http://dlmf.nist.gov/.